Competing orders and hidden duality symmetries in two-leg spin ladder systems

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(Received 3 July 2006; published 21 December 2006)

A unifying approach to competing quantum orders in generalized two-leg spin ladders is presented. Hidden relationship and quantum phase transitions among the competing orders are thoroughly discussed by means of a low-energy field theory starting from an SU(4) quantum multicritical point. Our approach reveals that the system has a relatively simple phase structure in spite of its complicated interactions. On top of the U(1) symmetry which is known from previous studies to mix up antiferromagnetic order parameter with that of the *p*-type nematic, we find an emergent U(1) symmetry which mixes order parameters dual to the above. On the basis of the field-theoretical and variational analysis, we give a qualitative picture for the global structure of the phase diagram. Interesting connection to other models (e.g., the bosonic *t*-*J* model) is also discussed.

DOI: 10.1103/PhysRevB.74.224426 PACS number(s): 75.10.Jm

I. INTRODUCTION

In the past two decades, quantum magnetism has been serving not only as effective theories describing insulating phases of strongly correlated electron systems but also as theoretical laboratories to look for and test new concepts. The discovery of high-temperature superconductors sparked the search for unconventional or exotic phases which are quite different from the ordinary ferromagnetic or antiferromagnetic phases. Despite the effort of many researchers in searching for novelty, it is by now well known that, in two or higher dimensions, antiferromagnetic phases are found provided that spin frustration is not very strong.[1] In order to suppress antiferromagnetism and stabilize exotic phases,[2] various mechanisms have been proposed. One realistic example of such mechanisms would be multispin-exchange interactions. Such interactions are expected to be crucial for explaining unusual magnetic behavior in ^{3}He absorbed on graphite.[3] Moreover, it was reported that a certain amount of four-spin ring exchange would be necessary to account for neutron-scattering experiments for the parent compound of high-temperature superconductor[4] La_2CuO_4 and for a spin-ladder compound[5,6] $La_6Ca_8Cu_{24}O_{41}$. Extensive numerical simulations carried out[7,8] for the two-dimensional Heisenberg antiferromagnets with four-spin ring exchange found various phases with unconventional orders: a spin-nematic phase[8] and a spin-liquid phase with topological ordering.[7]

On the other hand, various approaches have been proposed in electron systems to unify several (and sometimes quite different) competing orders[9–12] and succeeded in clarifying the nature of the quantum phase transitions among them.[13,14] Usually, in those approaches, extended symmetries are adopted so that mutually competing order parameters may be transformed to each other. A typical example would be the SO(5) theory[9,10] for the competition between *d*-wave superconductivity and antiferromagnetism, where the order parameters of *d*-wave superconductivity and those of antiferromagnetism are combined to form a unified order-parameter quintet. For a one-dimensional geometry (two-leg ladders), it is known that even larger symmetries SO(8) (Refs. 15 and 16) and SO(6) (Refs. 17–19) can emerge at low energies and be useful for the description of the electronic phases at half filling and away from half filling, respectively. In particular, the existence of an extended symmetry might provide a route to classify one-dimensional gapped phases.[20]

Unfortunately, no systematic approach based on extended symmetries is known for unconventional phases found in spin ladders and it is desirable to construct such theories. As the first step along this line, we investigate here two-leg spin ladders with four-spin interactions, since they possess high enough symmetry to unify various competing orders. Without the four-spin interactions, the two-leg spin ladder has a finite spin gap in magnetic excitations and short-ranged magnetic correlations.[21] Basic physics of the two-leg ladder can be understood by considering a ground state consisting of almost localized dimer singlets on antiferromagnetic rung bonds (see Fig. 1) and low-energy excitations over it. For this reason, the spin-gap phase in the usual two-leg spin lad-

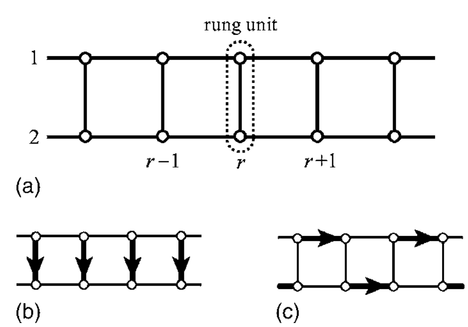

FIG. 1. (a) Two-leg spin ladder. The unit of constructing the model Hamiltonian (see Sec. II) is a pair of spin-1/2's enclosed in a broken line (called *rung* in the text). Any site can be labeled by the chain index (1,2) and the rung index *r*. (b) and (c): Two typical states appearing in two-leg ladders. The usual rung-singlet phase (b) and the staggered-dimer phase (c) to be discussed in the text. Arrows denote spin-singlet pairs.

1098-0121/2006/74(22)/224426(33)

der sometimes is dubbed as rung-singlet-or rung-dimer phase. If we change the rung coupling to ferromagnetic, the rung bonds will be dominantly occupied not by singlets but by triplets. Then, the system is effectively equivalent to the spin-1 systems and the knowledges in those systems may apply.[22–24] When the four-spin interactions (say, ring-exchange) are switched on, the size of the spin gap will be reduced and finally at a certain critical strength it even vanishes.[25–28] Large-scale numerical simulations[29,30] suggested that the model has a rather rich phase diagram. In particular, a spin liquid phase with scalar chirality ordering was found.[29] Such a phase breaks both time-reversal and parity symmetries and has been discussed previously in the context of anyon superconductivity.[31,32] A hallmark of the unconventional phases in the phase diagram of the two-leg spin ladder with ring-exchange interaction is that neither singlets nor triplets dominate over the others. Hence the conventional approaches starting from the limit of strong rung couplings (whether ferromagnetic or antiferromagnetic) is not very convenient to explore the nature of novel phases stabilized by four-spin interactions. The main goal of this paper is to fully describe the nature of the unconventional phases and quantum phase transitions among them by an approach based on an extended symmetry.

The organization of the present paper reads as follows. In Sec. II, we construct the ladder Hamiltonian by requiring rotational invariance and equivalence of two constituent chains. Our goal is to unify several unconventional phases stabilized by four-spin interactions. To this end, we shall use an enlarged symmetry SU(4) which contains the ordinary (spin) SU(2). We shall also pay particular attention to an interesting symmetry (*spin-chirality transformation*) which is a special case of the above SU(4) and commutes with the spin rotation. This symmetry, which has been introduced[30,33] in the context of the ring-exchange two-leg spin ladder, exchanges the Néel order parameter with the vector chirality (the order parameter of a *p*-type nematic[34]). It will play a crucial role in our analysis and give an important clue to understand the global structure of the phase diagram.

A low-energy field-theory analysis will be developed in Sec. III. Although various field-theory approaches are known for two-leg ladder systems,[21,35–37] most of them start from the limit of two decoupled chains and will not be suited to investigating the phase structure when four-spin interactions are by no means small. Instead of starting from two weakly coupled $S=1/2$ chains, we shall take the SU(4)-invariant point, which is a special case of the lattice Hamiltonian, as the starting point. The rotational- and the spin-chirality symmetry are beautifully incorporated into our low-energy effective action.

In Sec. IV, the phase structure and unexpected high symmetry among unconventional phases will be discussed with the help of one-loop renormalization group (RG) calculation. In particular, we shall find four dominant phases where SU(4) [SO(6), more precisely] symmetry is approximately restored in the low-energy limit. The quantum phase transitions among these dominant phases will be investigated in Secs. V and VI. The crucial role played by the spin-chirality symmetry in transitions among spin-singlet phases will be revealed. In this respect, a bosonization scheme based on $U(1)\times SU(3)$ symmetry will be introduced to extract relevant low-energy degrees of freedom which govern the phase transition. By using an effective theory for the low-energy fluctuations of a vector doublet composed of competing order parameters, we shall clarify how time-reversal symmetry is broken in one of these unconventional phases.

Readers who are not interested in the detail of the field-theoretical analysis may skip Secs. III, IV, and VI. Our main results have been already published in part as a short communication.[38]

In order to supplement the field-theoretical analysis, we shall carry out a variational- and a strong-coupling analysis in Sec. VII. As has been described above, the usual strong-coupling expansion starting from the limit of isolated rungs is not very convenient. Instead, we shall start from the limit of isolated *plaquettes* to successfully describe the competition among several quantum phases. On the basis of results obtained from these analyses, we map out the global phase diagram and discuss a connection to a spin-1 bosonic *t*-*J* model. This relationship might be useful for the possible experimental realization of the exotic gapped phases stabilized by four-spin exchange interactions. Indeed, in standard two-leg ladder compounds discovered so far, four-spin exchange interactions (10–20 % of leg-rung interactions) are not strong enough to stabilize the spin liquid phase with scalar chirality ordering in these systems. However, with the connection to the spin-1 bosonic *t*-*J* model presented in Sec. VII, one can expect that this phase (together with other unconventional ones) might be feasible in ultracold bosonic atomic gases in optical lattices.[39] Finally, Sec. VIII presents our concluding remarks and technical details will be presented in the appendixes.

II. MODEL AND ITS SYMMETRIES

A. Building blocks of a Hamiltonian

As has been discussed in the Introduction, enlarged symmetries are very powerful in unifying various competing orders. Therefore it is necessary first to identify the enlarged symmetry in our problem. Since we are considering spin-$1/2$ two-leg ladders, the maximal symmetry would be SU(4). The appearance of SU(4) in our problem is easily understood by noting that four states ($|\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle$) on a single rung [say, the rth rung in Fig. 1(a)] span the four-dimensional defining representation (**4**) of SU(4).[40,41] Let us denote $S=1/2$ spins on the first (upper) and the second (lower) chains by $\mathbf{S}_{1,r}$ and $\mathbf{S}_{2,r}$, respectively. Then, one of the standard choices of the 15 generators X_i on the rth rung is

$$X^i(r) = S^i_{1,r}, \quad X^{i+3}(r) = S^i_{2,r} \quad (i=1,2,3),$$

$$X^i(r) = 2S^a_{1,r}S^b_{2,r} \equiv G_{ab}(r) \quad (i=7,\ldots,15; a,b,=1,2,3). \tag{1}$$

The explicit matrix expressions of these generators are given in Appendix A.

Now we are at the point of constructing the Hamiltonian by requiring the invariance under (i) SU(2), (ii) $\mathbb{Z}_2$ ($1\leftrightarrow 2$: interchange of the two chains; see Fig. 1), and (iii) reflection with respect to horizontal links (*link parity*). Since we are considering $S=1/2$, we can safely restrict ourselves to second-order polynomials in S_1^a and S_2^b. We can divide them into (a) a scalar, (b) [SO(3)] vectors, and (c) rank-2 symmetric tensors. Of course, the scalar is given by $\mathbf{S}_{1,r}\cdot\mathbf{S}_{2,r}$ and the followings are all vectors:

$$\begin{aligned} &\mathbf{S}_{1,r}+\mathbf{S}_{2,r} \quad \ldots \mathbb{Z}_2 \text{ even},\\ &\mathbf{S}_{1,r}-\mathbf{S}_{2,r} \quad \ldots \mathbb{Z}_2 \text{ odd},\\ &\mathbf{S}_{1,r}\times\mathbf{S}_{2,r} \quad \ldots \mathbb{Z}_2 \text{ odd}. \end{aligned} \tag{2}$$

Although the last one seems an antisymmetric tensor, it behaves like a (pseudo)vector in the spin space and serves as the order parameter of the p-type spin nematic.[34] On top of them, we have a symmetric tensor whose components are given essentially by G_{ab}:[42]

$$Q_{ab}(r) \equiv S_{1,r}^a S_{2,r}^b + S_{1,r}^b S_{2,r}^a = \frac{1}{2}[G_{ab}(r)+G_{ba}(r)], \tag{3}$$

which reduces to a set of order parameters of the n-type spin nematic[34] when the dimer bonds are occupied by triplets. The point here is that all these operators are essentially the SU(4) generators (see Appendix A 2 for more details).

Now we proceed to constructing a Hamiltonian. If we consider only interactions involving two neighboring rungs (i.e., four spins), the SU(2) and $\mathbb{Z}_2$ invariance strongly restrict the possible interactions and we are left with the following seven ones:

$$\mathcal{H}_1 = \sum_r (\mathbf{S}_{1,r}+\mathbf{S}_{2,r})\cdot(\mathbf{S}_{1,r+1}+\mathbf{S}_{2,r+1}), \tag{4a}$$

$$\mathcal{H}_2 = \sum_r (\mathbf{S}_{1,r}-\mathbf{S}_{2,r})\cdot(\mathbf{S}_{1,r+1}-\mathbf{S}_{2,r+1}), \tag{4b}$$

$$\begin{aligned} \mathcal{H}_3 &= 2\sum_r \sum_{a,b=1}^{3} Q_{ab}(r)Q_{ab}(r+1)\\ &= 4\sum_r [(\mathbf{S}_{1,r}\cdot\mathbf{S}_{1,r+1})(\mathbf{S}_{2,r}\cdot\mathbf{S}_{2,r+1})\\ &\quad + (\mathbf{S}_{1,r}\cdot\mathbf{S}_{2,r+1})(\mathbf{S}_{2,r}\cdot\mathbf{S}_{1,r+1})], \end{aligned} \tag{4c}$$

$$\begin{aligned} \mathcal{H}_4 &= \sum_r 4(\mathbf{S}_{1,r}\times\mathbf{S}_{2,r})\cdot(\mathbf{S}_{1,r+1}\times\mathbf{S}_{2,r+1})\\ &= 4\sum_r [(\mathbf{S}_{1,r}\cdot\mathbf{S}_{1,r+1})(\mathbf{S}_{2,r}\cdot\mathbf{S}_{2,r+1})\\ &\quad - (\mathbf{S}_{1,r}\cdot\mathbf{S}_{2,r+1})(\mathbf{S}_{2,r}\cdot\mathbf{S}_{1,r+1})], \end{aligned} \tag{4d}$$

$$\begin{aligned} \mathcal{H}_5 &= 2\sum_r [(\mathbf{S}_{1,r}-\mathbf{S}_{2,r})\cdot(\mathbf{S}_{1,r+1}\times\mathbf{S}_{2,r+1})\\ &\quad + (\mathbf{S}_{1,r}\times\mathbf{S}_{2,r})\cdot(\mathbf{S}_{1,r+1}-\mathbf{S}_{2,r+1})], \end{aligned} \tag{4e}$$

$$\mathcal{H}_6 = \frac{1}{2}\sum_r (\mathbf{S}_{1,r}\cdot\mathbf{S}_{2,r}+\mathbf{S}_{1,r+1}\cdot\mathbf{S}_{2,r+1}), \tag{4f}$$

$$\mathcal{H}_7 = \sum_r (\mathbf{S}_{1,r}\cdot\mathbf{S}_{2,r})(\mathbf{S}_{1,r+1}\cdot\mathbf{S}_{2,r+1}), \tag{4g}$$

where the summation Σ_r is taken over all rungs of the ladder. Aside from the four-spin terms $\mathcal{H}_3$, $\mathcal{H}_4$, and $\mathcal{H}_7$, we have a three-spin term $\mathcal{H}_5$ which *explicitly* breaks time-reversal symmetry. If $S=1/2$ comes from the electron spin (more generally, magnetic moment of charged particles), the three-spin term $\mathcal{H}_5$ may result from the electron hopping on each plaquette.

It would be useful to rewrite $\mathcal{H}_1$ and $\mathcal{H}_3$ in terms of spin-1 operator $\mathbf{T}_r$ defined on the rth rungs:

$$\mathcal{H}_1 = \sum_r \mathbf{T}_r\cdot\mathbf{T}_{r+1},$$

$$\mathcal{H}_3 = \sum_r \left[\mathbf{T}_r\cdot\mathbf{T}_{r+1} + 2(\mathbf{T}_r\cdot\mathbf{T}_{r+1})^2 - 4\mathcal{H}_6 - \frac{3}{2}\right]. \tag{5}$$

In the above equations, the projection operators onto the triplet subspace $P_{\text{triplet}}(r)\equiv\mathbf{S}_{1,r}\cdot\mathbf{S}_{2,r}+3/4$ are implied, i.e.,

$$\mathbf{T}_r \equiv P_{\text{triplet}}(r)(\mathbf{S}_{1,r}+\mathbf{S}_{2,r})P_{\text{triplet}}(r).$$

These two blocks describe the interaction between effective spin-1 objects, that is, they dictate the *magnetic* part of the Hamiltonian. All the above seven interactions are used to construct the following general Hamiltonian:

$$\mathcal{H} = A\mathcal{H}_1 + B\mathcal{H}_2 + C\mathcal{H}_3 + D\mathcal{H}_4 + E\mathcal{H}_5 + F\mathcal{H}_6 + G\mathcal{H}_7. \tag{6}$$

Of course, another set of interactions could have been used. For example, basis $\mathcal{H}_1\pm\mathcal{H}_2$ and $\mathcal{H}_3\pm\mathcal{H}_4$ could have been chosen instead of $\mathcal{H}_1$, $\mathcal{H}_2$, $\mathcal{H}_3$, and $\mathcal{H}_4$. In fact, the latter choice is convenient when discussing the systems with SU(2)×SU(2) symmetry (e.g., the spin-orbital model[41,43,44]), while our choice here is useful when dealing with the spin-chirality transformation which will be introduced in the next subsection.

B. Spin-chirality transformation

1. Construction

As mentioned in the previous sections, the largest symmetry of the problem is SU(4), whose defining representation is spanned by the four states (singlet and triplet) on a single rung. As a subgroup of SU(4), there is an interesting U(1) symmetry[33] called the spin-chirality transformation.

Let us consider two spins ($\mathbf{S}_{1,r}$ and $\mathbf{S}_{2,r}$) on the rth rung and look for a local unitary transformation $\mathcal{U}_r(\theta)$ which commutes with the SU(2) rotation generated by $\mathbf{S}_{1,r}+\mathbf{S}_{2,r}$. The commutation relations

$$\left[\mathbf{S}_{1,r}+\mathbf{S}_{2,r}, \sum_{i=1}^{15} x_i X_r^i\right] = \mathbf{0} \tag{7}$$

satisfied by the generator of $\mathcal{U}_r(\theta)$ *uniquely* (up to a constant phase) determine the following form:

$$\mathcal{U}_r(\theta) = \exp[i\theta P_{\text{triplet}}(r)] \equiv \exp\left[i\theta\left(\frac{3}{4} + \mathbf{S}_{1,r}\cdot\mathbf{S}_{2,r}\right)\right]$$
$$= \frac{1}{4}(1+3e^{i\theta}) + (e^{i\theta}-1)\mathbf{S}_{1,r}\cdot\mathbf{S}_{2,r}. \tag{8}$$

In the above expression, we have adopted a slightly different definition from the original one in Ref. 33 for reasons which will become clear later. By construction, it is obvious that $\mathcal{U}_r(\theta)$ is the *only* U(1) transformation which commutes with the spin-rotation symmetry. The U(1) transformation $\mathcal{U}_r(\theta)$ has remarkable properties. By fully utilizing the properties of $S=1/2$, we can show that the following equations hold:

$$\mathcal{U}_r(\theta)(\mathbf{S}_{1,r}+\mathbf{S}_{2,r})\mathcal{U}_r^\dagger(\theta)$$
$$= \mathbf{S}_{1,r}+\mathbf{S}_{2,r} \quad \text{(total-spin conservation)}, \tag{9a}$$

$$\mathcal{U}_r(\theta)(\mathbf{S}_{1,r}-\mathbf{S}_{2,r})\mathcal{U}_r^\dagger(\theta)$$
$$= (\mathbf{S}_{1,r}-\mathbf{S}_{2,r})\cos\theta - 2(\mathbf{S}_{1,r}\times\mathbf{S}_{2,r})\sin\theta, \tag{9b}$$

$$\mathcal{U}_r(\theta)(\mathbf{S}_{1,r}\times\mathbf{S}_{2,r})\mathcal{U}_r^\dagger(\theta)$$
$$= \frac{1}{2}(\mathbf{S}_{1,r}-\mathbf{S}_{2,r})\sin\theta + (\mathbf{S}_{1,r}\times\mathbf{S}_{2,r})\cos\theta.$$

The first line (9a) is a trivial consequence of the requirement (7). The local U(1) transformation $\mathcal{U}_r(\theta)$ can be readily generalized to the whole lattice:

$$\mathcal{U}(\theta) \equiv \prod_{r=\text{rungs}} \mathcal{U}_r(\theta) \tag{10}$$

and all the above properties are preserved for $\mathcal{U}(\theta)$ as well. In what follows, we shall call $\mathcal{U}(\theta)$ *spin-chirality transformation*, since, as can be seen in the above equations (9b), it mixes up the antiferromagnetic order parameters $\mathbf{S}_1-\mathbf{S}_2$ and the vector chirality (or the order parameter of the *p*-type spin nematic[34]) $\mathbf{S}_1\times\mathbf{S}_2$.

For our purpose, it is helpful to view $\mathcal{U}(\theta)$ as an SU(4) transformation rather than as a nonlinear transformation for two spin operators $\mathbf{S}_1$ and $\mathbf{S}_2$. The latter two lines (9b) suggest that two antisymmetric (in $1\leftrightarrow 2$) quantities $\mathbf{S}_1-\mathbf{S}_2$ and $2(\mathbf{S}_1\times\mathbf{S}_2)$ behave as an O(2) doublet under the spin-chirality transformation $\mathcal{U}(\theta)$:

$$\begin{pmatrix} \tilde{\mathbf{S}}_1-\tilde{\mathbf{S}}_2 \\ 2\tilde{\mathbf{S}}_1\times\tilde{\mathbf{S}}_2 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \mathbf{S}_1-\mathbf{S}_2 \\ 2\mathbf{S}_1\times\mathbf{S}_2 \end{pmatrix}. \tag{11}$$

The remaining nine generators ($S_1^a+S_2^a$ and G_{ab}) are invariant under $\mathcal{U}(\theta)$.

2. Duality

Now we are at the position to discuss the effect of spin-chirality transformation on our building blocks. After some algebra, we obtain the following rules:

$$\mathcal{H}_1 \mapsto \mathcal{H}_1, \tag{12a}$$

$$\mathcal{H}_2 \mapsto \frac{1}{2}(\mathcal{H}_2+\mathcal{H}_4) + \frac{1}{2}\cos 2\theta(\mathcal{H}_2-\mathcal{H}_4) - \frac{1}{2}\sin 2\theta\mathcal{H}_5, \tag{12b}$$

$$\mathcal{H}_3 \mapsto \mathcal{H}_3, \tag{12c}$$

$$\mathcal{H}_4 \mapsto \frac{1}{2}(\mathcal{H}_2+\mathcal{H}_4) - \frac{1}{2}\cos 2\theta(\mathcal{H}_2-\mathcal{H}_4) + \frac{1}{2}\sin 2\theta\mathcal{H}_5, \tag{12d}$$

$$\mathcal{H}_5 \mapsto \sin 2\theta(\mathcal{H}_2-\mathcal{H}_4) + \cos 2\theta\mathcal{H}_5, \tag{12e}$$

$$\mathcal{H}_6 \mapsto \mathcal{H}_6, \tag{12f}$$

$$\mathcal{H}_7 \mapsto \mathcal{H}_7. \tag{12g}$$

Now the reason why we have decomposed the Hamiltonian into $\mathcal{H}_1,\ldots,\mathcal{H}_7$ can be easily understood from the above equations. The spin-chirality transformation for $\theta=\frac{\pi}{2}$ is particularly simple:

$$\mathcal{H}_2 \leftrightarrow \mathcal{H}_4, \quad \mathcal{H}_5 \mapsto -\mathcal{H}_5 \quad \text{(all the others are invariant)}. \tag{13}$$

Hereafter, this special case

$$\mathcal{D} \equiv \mathcal{U}(\theta=\pi/2) \tag{14}$$

will be called *duality transformation*,[30,33] although it is rather different from the standard "duality" which maps local objects onto nonlocal ones and vice versa. It readily follows from Eqs. (12a)–(12g) and (13) that a model with $B=D$ and $E=0$ is invariant (or *self-dual*) not only for $\theta=\pi/2$ but also for *arbitrary* values of θ.[33] In what follows, we will see that this enhanced U(1) symmetry at $B=D$ will play a crucial role.

C. Special cases

Before discussing the duality property of them, we mention interesting special cases of Hamiltonian (6).

Ordinary two-leg ladder. Much is known[21,35,36] about the ordinary two-leg ladder defined by the following Hamiltonian:

$$\mathcal{H} = J\sum_r (\mathbf{S}_{1,r}\cdot\mathbf{S}_{1,r+1} + \mathbf{S}_{2,r}\cdot\mathbf{S}_{2,r+1}) + J_\perp\sum_r \mathbf{S}_{1,r}\cdot\mathbf{S}_{2,r}$$
$$= \frac{1}{2}J(\mathcal{H}_1+\mathcal{H}_2) + J_\perp\mathcal{H}_6. \tag{15}$$

The basic picture of the ground state is provided by putting dimer singlets on rung (i.e., $J_\perp$) bonds and low-lying excitations may be understood as propagating dimer triplets.[45]

Spin-orbital model. The Hamiltonian of the spin-orbital model is defined by[41,43,44]

$$\mathcal{H}_{\text{SO}} = J\sum_r (\mathbf{S}_{1,r}\cdot\mathbf{S}_{1,r+1} + \mathbf{S}_{2,r}\cdot\mathbf{S}_{2,r+1}) + K\sum_r (\mathbf{S}_{1,r}\cdot\mathbf{S}_{1,r+1}) \times(\mathbf{S}_{2,r}\cdot\mathbf{S}_{2,r+1}) \tag{16}$$

and is obtained by choosing

$$A = B = \frac{1}{2}J, \quad C = D = \frac{K}{8}, \quad E = F = G = 0.$$

It was shown by weak-coupling analysis[25] and later by explicitly constructing the exact ground state[46] that the model displays a staggered dimer (SD) ordering in a certain region ($0<K<4J$) of the parameter space.

If we further impose the restriction

$$A = B = C = D\left(=\frac{1}{2}J\right),$$

the model reduces to the so-called *SU(4)-spin-orbital model*.[40,41,47] In this particular case, the Hamiltonian can be written as

$$\begin{aligned}\mathcal{H}_{\text{SU(4)}} &= J\sum_{r\in\text{rung}}\left[\left(\frac{1}{2}+2\mathbf{S}_{1,r}\cdot\mathbf{S}_{1,r+1}\right)\left(\frac{1}{2}+2\mathbf{S}_{2,r}\cdot\mathbf{S}_{2,r+1}\right)-\frac{1}{4}\right]\\ &= J\sum_{r\in\text{rung}}\left(P(\mathbf{S}_{1,r},\mathbf{S}_{1,r+1})P(\mathbf{S}_{2,r},\mathbf{S}_{2,r+1})-\frac{1}{4}\right)\\ &= J\sum_{a=1}^{15} X_r^a X_{r+1}^a,\end{aligned} \tag{17}$$

where $P(\mathbf{S},\mathbf{T})$ denotes a permutation operator for two $S=1/2$ modules (corresponding to $\mathbf{S}$ and $\mathbf{T}$). This is an SU(4) generalization of the $S=1/2$ Heisenberg model since $P(\mathbf{S}_{1,r},\mathbf{S}_{1,r+1})P(\mathbf{S}_{2,r},\mathbf{S}_{2,r+1})$ is nothing but the SU(4)-permutation operator if we regard the four states on a rung as **4**. The model $\mathcal{H}_{\text{SU(4)}}$ is integrable[47] and will be used as a starting point of the following analysis.

As shown in Ref. 48, a term $G\mathcal{H}_6$ can be added to $\mathcal{H}_{\text{SU(4)}}$ without spoiling integrability. According to the Bethe-ansatz results,[48] we have two critical values of G; when $G<G_{c,1}=\pi/(2\sqrt{3})+\ln 3/2$, all rungs are occupied by triplets and the system is described by two gapless bosons while for $G>G_{c,2}=4$ the system is in the so-called rung-singlet phase and all excitations are gapped.

Self-dual models. An important class of models is defined by the following choice of parameters:

$$B = D, \quad E = 0, \quad A,\ C,\ F,\ G = \text{arbitrary}.$$

This defines a family of models which are invariant under the full spin-chirality rotation $\mathcal{U}(\theta)$. Hereafter, we call this family of models *self-dual models* and the manifold characterized by the above set of parameters *self-dual manifold*. Obviously, the self-dual models have $\text{SU(2)}_{\text{spin}}\times\text{U(1)}_{\text{spin-chiral}}$ symmetry.

SU(3)×U(1)-models. On a special submanifold,

$$A = C = \frac{1}{2}J_1, \quad B = D = \frac{1}{2}J_2, \quad E = 0, \quad F,\ G = \text{arbitrary},$$

of the self-dual manifold (obtained by setting $A=C$ in the self-dual models), the spin SU(2) gets enlarged to SU(3) (see Appendix A 2). The conditions $A=C$ and $B=D$ are crucial for the SU(3) invariance. In fact, these SU(3) and U(1) are broken *simultaneously* [$\text{SU(3)}\mapsto\text{SU(2)}$, $\text{U(1)}\mapsto\mathbb{Z}_2$] when we move away from the self-dual manifold ($B=D$).

Composite-spin model. The so-called composite-spin model is defined by[49,50]

$$\mathcal{H}_{\text{composite}} = \sum_r \{(\mathbf{S}_{1,r}+\mathbf{S}_{2,r})\cdot(\mathbf{S}_{1,r+1}+\mathbf{S}_{2,r+1}) - \beta[(\mathbf{S}_{1,r}+\mathbf{S}_{2,r})\cdot(\mathbf{S}_{1,r+1}+\mathbf{S}_{2,r+1})]^2\}, \tag{18}$$

which can be rewritten as

$$\mathcal{H}_{\text{composite}} = \left(1+\frac{1}{2}\beta\right)\mathcal{H}_1 - \frac{1}{2}\beta\mathcal{H}_3 - 2\beta\mathcal{H}_6.$$

This preserves the total spin $\mathbf{S}_{1,r}+\mathbf{S}_{2,r}$ on each rung and the problem reduces essentially to that of a collection of finite chain segments.

Note that this is a special case ($B=D=0$) of the self-dual models. If we choose $\beta=-1$, we get $A=C=1/2$ and as a consequence we have SU(3) symmetry. This is in agreement with the well-known fact that the $\beta=-1$ bilinear-biquadratic chain is SU(3) invariant [the SU(3) Uimin-Lai-Sutherland model[47,51]].

Spin-ladder with four-body cyclic exchange. The four-body cyclic exchange on elementary plaquettes made up of two rungs r and $r+1$ can be recast as

$$\mathcal{H}_{\text{cyc}} \equiv \sum_r [P_4(r,r+1) + P_4^{-1}(r,r+1)] = \mathcal{H}_1 + \mathcal{H}_4 + 2\mathcal{H}_6 + 4\mathcal{H}_7 + \text{const}, \tag{19}$$

where $P_4(r,r+1)$ and $P_4^{-1}(r,r+1)$, respectively, make a cyclic permutation and its inverse on a plaquette formed by rungs r and $r+1$. Using this, the Hamiltonian for a two-leg ladder with four-spin exchange is given as[5,52]

$$\begin{aligned}\mathcal{H}_{\text{ladder+4-spin}} &= J\sum_r (\mathbf{S}_{1,r}\cdot\mathbf{S}_{1,r+1} + \mathbf{S}_{2,r}\cdot\mathbf{S}_{2,r+1}) + K_4\mathcal{H}_{\text{cyc}} + J_{\text{R}}\sum_r \mathbf{S}_{1,r}\cdot\mathbf{S}_{2,r}\\ &= \left(\frac{1}{2}J+K_4\right)\mathcal{H}_1 + \frac{1}{2}J\mathcal{H}_2 + K_4\mathcal{H}_4 + (J_{\text{R}} + 2K_4)\mathcal{H}_6 + 4K_4\mathcal{H}_7.\end{aligned} \tag{20}$$

Note that the model with $K_4=J/2$ is self-dual ($B=D$) and we can find not only the exact rung-singlet ground state but also the exact one-magnon state[26] (in notations used in Ref. 26, $J_{\text{ring}}\equiv 2K_4$) for certain choices of parameters. The phase diagram for $J_{\text{R}}=1$ has been mapped out in Refs. 26, 27, 29, and 30.

We summarize the relation between parameters and the symmetries of Hamiltonian (6) in Fig. 2. First of all, in most

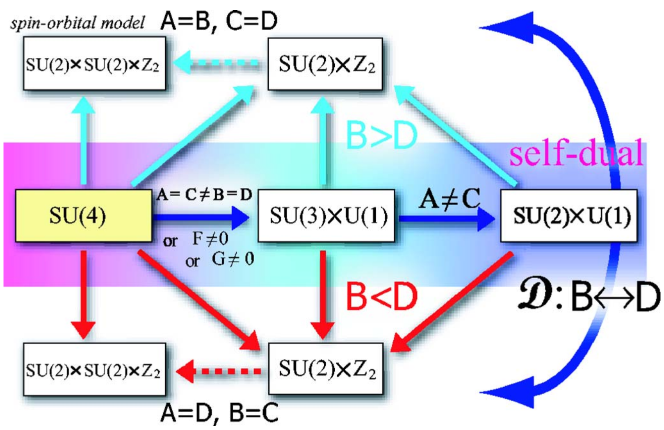

FIG. 2. (Color online) Changes in the symmetry as the parameters are varied. Our starting point denoted by "SU(4)" corresponds to the choice $A=B=C=D$, $E=F=G=0$. "Duality" $\mathcal{D}$ maps a model with $B>D$ onto another with $B<D$ and vice versa. Note that the full symmetry for generic cases (including the spin-orbital case) is $\mathrm{SU(2)}_{\mathrm{spin}}\times\mathbb{Z}_2$.

cases perturbation around the SU(4) point explicitly breaks the SU(4) symmetry. The fate of the system after the breaking of SU(4) is different according to whether or not the system is invariant under the spin-chirality rotation $\mathcal{U}(\theta)$; if the system is invariant (i.e., $B=D$), then we can have a high symmetry like $\mathrm{SU(3)}\times\mathrm{U(1)}$ (when $A=C$) or $\mathrm{SU(2)}\times\mathrm{U(1)}$ (when $A\neq C$). These cases will be treated in Sec. V. Otherwise, the system generically assumes the lowest possible symmetry $\mathrm{SU(2)}\times\mathbb{Z}_2$.

D. Useful analogies

1. Pseudospin-1/2 model

Although the Hamiltonian (6) looks complicated, there is a useful analogy to a more familiar model-$S=1/2$ *XXZ* model. To understand this, we note that the quantity

$$\mathcal{S}_r^z = \mathbf{S}_{1,r}\cdot\mathbf{S}_{2,r} + \frac{1}{4} \tag{21}$$

formally plays a role of S^z in the $S=1/2$ *XXZ* problem. Indeed, it is not difficult to show

$$[\mathcal{S}^z,(\mathbf{S}_1-\mathbf{S}_2)^a] = 2i(\mathbf{S}_1\times\mathbf{S}_2)^a,$$

$$[\mathcal{S}^z,2(\mathbf{S}_1\times\mathbf{S}_2)^a] = -i(\mathbf{S}_1-\mathbf{S}_2)^a \tag{22}$$

and

$$[\mathcal{S}^z,(\mathbf{S}_1+\mathbf{S}_2)^a]=0,$$

$$[\mathcal{S}^z,Q_{ab}]=0. \tag{23}$$

The first two equations imply that $\mathbf{S}_1-\mathbf{S}_2$ and $2(\mathbf{S}_1\times\mathbf{S}_2)$ rotate like S^x and S^y, respectively, as is expected from Eq. (11). Of course, $(\mathbf{S}_1-\mathbf{S}_2)^a\pm 2i(\mathbf{S}_1\times\mathbf{S}_2)^a$ $(a=x,y,z)$ give rise to singlet-triplet transitions and play a role of raising or lowering operators of the pseudospins.

Therefore two building blocks

$$\mathcal{H}_2 = \sum_r \sum_{a=x,y,z} (\mathbf{S}_{1,r}-\mathbf{S}_{2,r})_a(\mathbf{S}_{1,r+1}-\mathbf{S}_{2,r+1})_a,$$

$$\mathcal{H}_4 = \sum_r \sum_{a=x,y,z} 4(\mathbf{S}_{1,r}\times\mathbf{S}_{2,r})_a(\mathbf{S}_{1,r+1}\times\mathbf{S}_{2,r+1})_a$$

behave like $\Sigma\mathcal{S}_r^x\mathcal{S}_{r+1}^x$ and $\Sigma\mathcal{S}_r^y\mathcal{S}_{r+1}^y$ with respect to U(1) [SO(2)] generated by $\Sigma\mathcal{S}_r^z$ except that they have additional degeneracy coming from the SU(2) invariance.[53] Of course, as far as the pseudospin degrees of freedom are concerned, the roles of $-h\Sigma\mathcal{S}^z$ and $\Sigma\mathcal{S}_r^z\mathcal{S}_{r+1}^z$ are $\mathcal{H}_6$ and the rung-rung four-body term $\mathcal{H}_7$, respectively. Although the Hamiltonian $B(\mathcal{H}_2+\mathcal{H}_4)+F\mathcal{H}_6+G\mathcal{H}_7$ may formally look like a collection of three $S=1/2$ *XXZ* chains $(a=x,y,z)$ in a finite magnetic field, it is not true because $S_1^a-S_2^a$ and $S_1^b-S_2^b$ $(a\neq b)$ do *not* commute with each other. [In fact, they obey the SU(4) commutation relations; see Appendix A.] Nevertheless, an analogy to $S=1/2$ *XXZ* chain is still useful because the model Hamiltonian decomposes into (spin-chirality) pseudospin $S=1/2$ *XXZ* part and a (real) spin part. That is, the effective *XXZ* part $(\mathcal{H}_2+\mathcal{H}_4+F\mathcal{H}_6+G\mathcal{H}_7)$ has nonzero off-diagonal elements only for singlet-triplet transitions (pseudospin flipping), while the magnetic part ($\mathcal{H}_1$ and $\mathcal{H}_3$) gives rise only to triplet-triplet transitions. Therefore we may expect that, if for some reasons the spin sector gets gapped and the magnetic dynamics is frozen, the low-energy part of the full dynamics will be described by the above effective (pseudospin) *XXZ* model.

2. Spin-1 Bose system

Another formulation of our system facilitates us capturing the physical meaning of the spin-chirality transformation $\mathcal{U}(\theta)$. As a first step, we note that the asymmetric part ($\mathbf{S}_1-\mathbf{S}_2$ and $\mathbf{S}_1\times\mathbf{S}_2$) of the Hamiltonian $B\mathcal{H}_2+D\mathcal{H}_4$ can be written as a hopping term of spin-1 bosons:

$$\begin{aligned}B\mathcal{H}_2+D\mathcal{H}_4 &= (B+D)\sum_{r,a}(b_{r,a}^\dagger b_{r+1,a}+b_{r+1,a}^\dagger b_{r,a})\\ &\quad+(B-D)\sum_{r,a}(b_{r,a}^\dagger b_{r+1,a}^\dagger+b_{r+1,a}b_{r,a})\quad(a=x,y,z),\end{aligned} \tag{24}$$

where the operator $b_{r,a}^\dagger$ creates a spin-1 (triplet) boson with spin index a on the site r. Since, in the spin language, the rth rung is occupied either by a singlet or by a triplet, the bosons b_a should be thought of as hard-core particles. Namely, the following local constraints on the boson occupation numbers should be imposed:

$$n_r^{\mathrm{B}} \equiv \sum_{a=x,y,z} b_{r,a}^\dagger b_{r,a} = \mathbf{S}_{1,r}\cdot\mathbf{S}_{2,r}+3/4 = \mathcal{S}_r^z+1/2 = 0,1. \tag{25}$$

Note that these particles obey nonstandard commutation relations:

$$[b_{r,a},b^{\dagger}_{r',b}]=\{\delta_{ab}(1-n^{\mathrm{B}}_r)-b^{\dagger}_{r,a}b_{r,b}\}\delta_{r,r'}.$$

One of the greatest merits of this mapping is that the above bosons are directly related to the order parameters $\mathbf{S}_1-\mathbf{S}_2$ (antiferromagnetic) and $\mathbf{S}_1\times\mathbf{S}_2$ (p-type nematic) as

$$b^{\dagger}_{a,r}=\frac{1}{2}(\mathbf{S}_{1,r}-\mathbf{S}_{2,r})^a+i(\mathbf{S}_{1,r}\times\mathbf{S}_{2,r})^a \quad (a=x,y,z). \quad (26)$$

The spin-chirality transformation (10) is simply expressed as a gauge transformation of these bosons:

$$b^{\dagger}_{a,r}\mapsto e^{i\theta}b^{\dagger}_{a,r}. \quad (27)$$

As has been mentioned above, $\mathcal{H}_6$ and $\mathcal{H}_7$ have simple interpretation in terms of an effective spin $\mathcal{S}^z_r$:

$$F\mathcal{H}_6=F\sum_r n^{\mathrm{B}}_r+\text{const}, \quad (28)$$

$$G\mathcal{H}_7=G\sum_r\left(\mathcal{S}^z_r-\frac{1}{4}\right)\left(\mathcal{S}^z_{r+1}-\frac{1}{4}\right)=G\sum_r n^{\mathrm{B}}_r n^{\mathrm{B}}_{r+1}-\frac{3}{2}G\mathcal{H}_6+\text{const}. \quad (29)$$

Last, the symmetric part of the Hamiltonian $A\mathcal{H}_1+C\mathcal{H}_3$ can be recasted as [see Eq. (5)]

$$A\mathcal{H}_1+C\mathcal{H}_3=\sum_r[(A+C)\mathbf{T}_r\cdot\mathbf{T}_{r+1}+2C(\mathbf{T}_r\cdot\mathbf{T}_{r+1})^2-4C\mathcal{H}_6]+\text{const}. \quad (30)$$

In this equation, the spin-1 operators $(\mathbf{T}_r)^a=-i\varepsilon_{abc}b^{\dagger}_{b,r}b_{c,r}$ act only on occupied sites. That is, triplet projection operators $P_{\text{triplet}}(r)$ on both sides of $\mathbf{T}_r$ are implied. From these, we can conclude that our ladder Hamiltonian is equivalent to an $S=1$ bosonic t-J-like model *on a chain*;[54] the deviation from the self-dual models $B-D\neq 0$ introduces pair creation or annihilation processes. A special case of it ($C=-A<0$), where only the biquadratic interaction exists and the system has an enlarged SU(3) symmetry, was investigated by Albertini.[55] As has been mentioned in Sec. I, this connection might be quite useful in realizing the unconventional phases in the system of ultracold atomic gases.[39]

III. CONTINUUM LIMIT

A. Field-theory description of SU(4) point

In this section, we develop a low-energy approach to our problem starting from the SU(4)-symmetric point. Thanks to the exact Bethe-ansatz solution,[47] we know that the SU(4) point ($A=B=C=D>0$, $E=F=G=0$) is gapless and the (conformally invariant) field theory describing this massless fixed point was obtained by several authors.[56–58] It is given by the level-1 SU(4) Wess-Zumino-Witten (WZW) conformal field theory with central charge $c=3$ (for a review of WZW and related models; see, for instance, Ref. 35). For our purpose, it is more convenient to use an equivalent free-field description in terms of six real (Majorana) fermions.[57] The derivation is sketched briefly in Appendix C and the effective action corresponding to the fixed point is given by[57]

$$\mathcal{H}_{\mathrm{SO(6)}}=-\frac{iv}{2}\sum_{a=1}^{3}(\xi^a_{\mathrm{R}}\partial_x\xi^a_{\mathrm{R}}-\xi^a_{\mathrm{L}}\partial_x\xi^a_{\mathrm{L}}+\chi^a_{\mathrm{R}}\partial_x\chi^a_{\mathrm{R}}-\chi^a_{\mathrm{L}}\partial_x\chi^a_{\mathrm{L}}). \quad (31)$$

The above action describes six free massless Majorana fermions $(\xi^a_{\mathrm{L,R}},\chi^a_{\mathrm{L,R}})$ $(a=1,2,3)$ which are equivalent to six copies of critical two-dimensional (2D) Ising models. In general, SO(6)-invariant marginally irrelevant interactions should be added to $\mathcal{H}_{\mathrm{SO(6)}}$ in order to describe the low-energy physics of the SU(4) model (see Appendix C). In what follows, shorthand notations $\vec{\xi}_{\mathrm{R,L}}=(\xi^1_{\mathrm{R,L}},\xi^2_{\mathrm{R,L}},\xi^3_{\mathrm{R,L}})$, etc., will be used.

Using the quantum equivalence between the level-1 SU(4) WZW model and $\mathcal{H}_{\mathrm{SO(6)}}$ [or, the level-1 SO(6) WZW model], we can express all the 15 generators of SU(4) in terms of the above six Majorana fermions. In general, we may expect that local operators have the following expansions:

$$\mathcal{O}_{r,\text{lattice}}\sim\mathcal{O}+e^{i\pi r/2}\mathcal{N}_{\mathcal{O}}+e^{-i\pi r/2}\mathcal{N}^{*}_{\mathcal{O}}+(-1)^r n_{\mathcal{O}}. \quad (32)$$

For the SU(4) generators, they read

$$X^A_r\sim X^A_{\mathrm{R}}+X^A_{\mathrm{L}}+e^{i\pi r/2}\mathcal{N}^A+e^{-i\pi r/2}\mathcal{N}^{A,*}+(-1)^r n^A. \quad (33)$$

One can write down $X^A_{\mathrm{R,L}}$ in terms of six Majorana fermions by using the 6×6 representation of the SU(4) generators [see Appendix A and Eq. (1) for the definition of X^A]:

$$\mathbf{S}_{1,\mathrm{L/R}}=-\frac{i}{2}\vec{\xi}_{\mathrm{L/R}}\times\vec{\xi}_{\mathrm{L/R}},\quad \mathbf{S}_{2,\mathrm{L/R}}=-\frac{i}{2}\vec{\chi}_{\mathrm{L/R}}\times\vec{\chi}_{\mathrm{L/R}},$$

$$G_{ab,\mathrm{L/R}}=-i\xi^a_{\mathrm{L/R}}\chi^b_{\mathrm{L/R}} \quad (a,b=1,2,3). \quad (34a)$$

The staggered part (n^A) is given similarly as

$$\mathbf{S}_1=iB\vec{\xi}_{\mathrm{R}}\times\vec{\xi}_{\mathrm{L}},\quad \mathbf{S}_2=iB\vec{\chi}_{\mathrm{R}}\times\vec{\chi}_{\mathrm{L}},$$

$$G_{ab}=iB(\xi^a_{\mathrm{R}}\chi^b_{\mathrm{L}}-\chi^b_{\mathrm{R}}\xi^a_{\mathrm{L}}), \quad (34b)$$

where B is a regularization-dependent constant. Therefore both the uniform and the staggered correlations of X^A are written as products of two free fermion propagators and behave like x^{-2}. The second part, whose correlation decays as $x^{-3/2}$, is more complicated and given by a product of six order or disorder operators of the underlying Ising models.[57]

B. Symmetry operations

Aside from the internal $\mathrm{SO(6)_L}\times\mathrm{SO(6)_R}$ symmetry, various symmetry operations keep the fixed-point Hamiltonian (31) invariant. Among them, the following will play important roles:

(i) Time reversal ($\mathcal{T}$):

$$\xi^a_{\mathrm{R,L}} \overset{\mathcal{T}}{\to} -\xi^a_{\mathrm{L,R}}, \quad \chi^a_{\mathrm{R,L}} \overset{\mathcal{T}}{\to} \chi^a_{\mathrm{L,R}}. \tag{35}$$

A remark is in order here about $\mathcal{T}$. As is well known, the time-reversal operation is antiunitary and the complex conjugation must be taken after the transformation (35) is applied.

(ii) Translation by one-site ($T_{1\text{-site}}$):

$$\xi^a_{\mathrm{R}} \overset{T_{1\text{-site}}}{\to} -\xi^a_{\mathrm{R}}, \quad \xi^a_{\mathrm{L}} \overset{T_{1\text{-site}}}{\to} \xi^a_{\mathrm{L}},$$

$$\chi^a_{\mathrm{R}} \overset{T_{1\text{-site}}}{\to} -\chi^a_{\mathrm{R}}, \quad \chi^a_{\mathrm{L}} \overset{T_{1\text{-site}}}{\to} \chi^a_{\mathrm{L}}. \tag{36}$$

In the field-theory language, $T_{1\text{-site}}$ is nothing but chiral symmetry generated by γ^5.

(iii) Interchange of upper and lower chains ($\mathcal{P}_{12} \in \mathbb{Z}_2$):

$$\xi^a_{\mathrm{R,L}} \overset{\mathcal{P}_{12}}{\to} -\chi^a_{\mathrm{R,L}}, \quad \chi^a_{\mathrm{R,L}} \overset{\mathcal{P}_{12}}{\to} \xi^a_{\mathrm{R,L}}. \tag{37}$$

(iv) Site parity (P_{S}):

$$\xi^a_{\mathrm{R,L}}(x) \overset{P_{\mathrm{S}}}{\to} \xi^a_{\mathrm{L,R}}(-x), \quad \chi^a_{\mathrm{R,L}}(x) \overset{P_{\mathrm{S}}}{\to} \chi^a_{\mathrm{L,R}}(-x). \tag{38}$$

(v) Link parity (P_{L}):

$$\xi^a_{\mathrm{R}}(x) \overset{P_{\mathrm{L}}}{\to} \xi^a_{\mathrm{L}}(-x), \quad \xi^a_{\mathrm{L}}(x) \overset{P_{\mathrm{L}}}{\to} -\xi^a_{\mathrm{R}}(-x),$$

$$\chi^a_{\mathrm{R}}(x) \overset{P_{\mathrm{L}}}{\to} \chi^a_{\mathrm{L}}(-x), \quad \chi^a_{\mathrm{L}}(x) \overset{P_{\mathrm{L}}}{\to} -\chi^a_{\mathrm{R}}(-x). \tag{39}$$

(vi) Ising (or Kramers-Wannier) duality (s_1):

$$\xi^a_{\mathrm{R,L}} \overset{s_1}{\to} \xi^a_{\mathrm{R,L}},$$

$$\chi^a_{\mathrm{R}} \overset{s_1}{\to} -\chi^a_{\mathrm{R}}, \quad \chi^a_{\mathrm{L}} \overset{s_1}{\to} \chi^a_{\mathrm{L}}. \tag{40}$$

C. Spin-chirality transformation in the continuum limit

In this section, we look for the duality transformation for the Majorana fermions. To derive an expression of the "duality" transformation valid in the low-energy limit, it is convenient to use a generalized version (10):

$$\mathcal{U}(\theta) = \prod_{r\in\text{rung}} \exp\left[i\theta\left(\frac{3}{4} + \mathbf{S}_{1,r}\cdot\mathbf{S}_{2,r}\right)\right].$$

In terms of spin operators, $\mathcal{U}(\theta)$ is realized in a nonlinear manner as we have seen in Eqs. (9a) and (9b). Interestingly, it is realized in the continuum limit as

$$\mathcal{U}(\theta) = \mathcal{R}_{\mathrm{R}}(\theta)\mathcal{R}_{\mathrm{L}}(\theta) \tag{41}$$

by using the following *spin-independent* SO(2) transformation $\mathcal{R}_{\mathrm{L,R}}(\theta_{\mathrm{L,R}})$ for (ξ^a,χ^a):[38]

$$\tilde{\xi}^a_{\mathrm{L,R}} = \xi^a_{\mathrm{L,R}} \cos\frac{\theta_{\mathrm{L,R}}}{2} - \chi^a_{\mathrm{L,R}} \sin\frac{\theta_{\mathrm{L,R}}}{2},$$

$$\tilde{\chi}^a_{\mathrm{L,R}} = \xi^a_{\mathrm{L,R}} \sin\frac{\theta_{\mathrm{L,R}}}{2} + \chi^a_{\mathrm{L,R}} \cos\frac{\theta_{\mathrm{L,R}}}{2}. \tag{42}$$

Strictly speaking, different notations should be used to denote the original $\mathcal{U}(\theta)$ defined by Eq. (10) and its continuum version (41) and (42). However, the meaning is obvious from the context and we shall use the same notation for the two transformations.

In fact, the fixed-point Hamiltonian (31) is invariant under an even larger *chiral* (i.e., left-right independent) version $\mathrm{SO(2)_R}\times\mathrm{SO(2)_L}$ of the above SO(2) and this fact gives us a hint to find another set of order parameters. For readers who want to know more about the derivation of Eqs. (41) and (42), we give it in the Appendix D.

The following special cases are worth mentioning:

(i) $\theta=\frac{\pi}{2}$: $\mathcal{U}(\pi/2)\equiv\mathcal{D}$ reduces to the original spin-chirality duality transformation,[30] where the two fermions mix with equal weights. From Eqs. (12a)–(12g), it is clear that models with $E=0$ (no $\mathcal{T}$-odd term $\mathcal{H}_5$) form a closed subset of the full Hamiltonian.

(ii) $\theta=\pi$: This corresponds to the exchange of $\mathbf{S}_1$ and $\mathbf{S}_2$ (discrete $\mathbb{Z}_2$-symmetry $\mathcal{P}_{12}$). In the case of Majorana fermions, it accompanies a sign change:

$$\tilde{\xi}^a_{\mathrm{L/R}} = -\chi^a_{\mathrm{L/R}}, \quad \tilde{\chi}^a_{\mathrm{L/R}} = \xi^a_{\mathrm{L/R}}.$$

(iii) $\theta=2\pi$: Although this is nothing but an identity operation *in the original spin*-1/2 *language*, Majorana fermions change their signs,

$$\tilde{\xi}^a_{\mathrm{L/R}} = -\xi^a_{\mathrm{L/R}}, \quad \tilde{\chi}^a_{\mathrm{L/R}} = -\chi^a_{\mathrm{L/R}}.$$

This is already anticipated from what we have for $\theta=\pi$. That is, the $\mathbb{Z}_2$ exchange between $\mathbf{S}_1$ and $\mathbf{S}_2$ is *not* realized as a simple $\mathbb{Z}_2$ symmetry in terms of Majorana fermions.[59] Note that this sign inversion does not affect the physical operators since they are always written as fermion bilinears.

D. Order parameters as duality doublets

In Sec. II, we have pointed out that two quantities

$$\mathbf{S}_1 - \mathbf{S}_2 \text{ and } 2(\mathbf{S}_1\times\mathbf{S}_2)$$

form a doublet under the spin-chirality rotation $\mathcal{U}(\theta)$. However, as our system is one-dimensional, the expectation values of these vector order parameters are identically zero: $\langle\mathbf{S}_1-\mathbf{S}_2\rangle=\langle 2(\mathbf{S}_1\times\mathbf{S}_2)\rangle=\mathbf{0}$. Instead, we adopt the following two rotationally invariant order parameters:

$$\mathcal{O}^{\text{lattice}}_{\mathrm{SD}} = \mathbf{S}_{1,r}\cdot\mathbf{S}_{1,r+1} - \mathbf{S}_{2,r}\cdot\mathbf{S}_{2,r+1}, \tag{43a}$$

$$\mathcal{O}^{\text{lattice}}_{\mathrm{SC}} = (\mathbf{S}_{1,r}+\mathbf{S}_{2,r})\cdot(\mathbf{S}_{1,r+1}\times\mathbf{S}_{2,r+1}) + (\mathbf{S}_{1,r}\times\mathbf{S}_{2,r})\cdot(\mathbf{S}_{1,r+1}+\mathbf{S}_{2,r+1}). \tag{43b}$$

In numerical studies,[29] it was shown that phases which are characterized by nonvanishing $\mathcal{O}^{\text{lattice}}_{\mathrm{SD}}$ or $\mathcal{O}^{\text{lattice}}_{\mathrm{SC}}$ do exist in the phase diagram of the two-leg spin ladder with a ring-exchange interaction. The appearance of the latter describes an exotic phase since a nonzero value of $\mathcal{O}_{\mathrm{SC}}$ implies a nonmagnetic order with $\mathcal{T}$ breaking.

From Eqs. (11) and (43b), and

$$\mathcal{O}_{\rm SD}^{\rm lattice} = \frac{1}{2}(\mathbf{S}_1+\mathbf{S}_2)_r \cdot (\mathbf{S}_1-\mathbf{S}_2)_{r+1} + (r \leftrightarrow r+1),$$

it follows that these two order parameters transform as an SO(2) doublet:

$$\begin{pmatrix} \mathcal{O}_{\rm SD}^{\rm lattice} \\ \mathcal{O}_{\rm SC}^{\rm lattice} \end{pmatrix} \mapsto \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \mathcal{O}_{\rm SD}^{\rm lattice} \\ \mathcal{O}_{\rm SC}^{\rm lattice} \end{pmatrix}. \tag{44}$$

That is, as far as the spin-chirality SO(2) is concerned, $\mathcal{O}_{\rm SD}$ and $\mathcal{O}_{\rm SC}$ behave like $(\mathbf{S}_1-\mathbf{S}_2)$ and $2(\mathbf{S}_1\times\mathbf{S}_2)$, respectively. In particular, by the duality $\mathcal{D}$, two order parameters $\mathcal{O}_{\rm SD}^{\rm lattice}$ and $\mathcal{O}_{\rm SC}^{\rm lattice}$ are interchanged.[30]

Now let us find the continuum expressions for the above order parameters. As has been mentioned in Sec. III A, any local operator on a lattice has an expansion of the following form:

$$\mathcal{O}_{\rm lattice} \sim \mathcal{O}(x) + e^{i\pi x/2}\mathcal{N}(x) + e^{-i\pi x/2}\mathcal{N}^*(x) + (-1)^x\mathcal{O}^\pi(x). \tag{45}$$

By taking operator-product expansions (OPEs), we obtain the expressions of the staggered parts ($\mathcal{O}^\pi$) of $\mathcal{O}_{\rm SD}^{\rm lattice}$ and $\mathcal{O}_{\rm SC}^{\rm lattice}$:

$$\mathcal{O}_{\rm SD}^\pi = i(\vec{\xi}_{\rm R}\cdot\vec{\xi}_{\rm L} - \vec{\chi}_{\rm R}\cdot\vec{\chi}_{\rm L}) \quad \text{and } \mathcal{O}_{\rm SC}^\pi = i(\vec{\xi}_{\rm R}\cdot\vec{\chi}_{\rm L} + \vec{\chi}_{\rm R}\cdot\vec{\xi}_{\rm L}). \tag{46}$$

By using Eq. (41), it is straightforward to verify

$$\begin{pmatrix} \mathcal{O}_{\rm SD}^\pi \\ \mathcal{O}_{\rm SC}^\pi \end{pmatrix} \mapsto \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \mathcal{O}_{\rm SD}^\pi \\ \mathcal{O}_{\rm SC}^\pi \end{pmatrix}.$$

As has been mentioned before, the relations $B=D$ and $E=0$ define self-dual models, which are invariant under the *continuous* rotation $\mathcal{U}(\theta)\in \mathrm{SO}(2)$. From the fact that the above two order parameters transform as an SO(2) doublet, it readily follows that

$$\langle \mathcal{O}_{\rm SD}^{\rm lattice}\rangle = \langle \mathcal{O}_{\rm SC}^{\rm lattice}\rangle = 0 \tag{47}$$

for generic models on the self-dual manifold.[60]

E. Second set of order parameters

It would be interesting to look for the possibility of other order parameters. Let us restrict ourselves to those which are (i) spin singlet (that is, spin indices are contracted), (ii) Lorentz-invariant (i.e., Lorentz spin=0), and (iii) have a scaling dimension unity. Apparently, we have four such operators $\vec{\xi}_{\rm R}\cdot\vec{\xi}_{\rm L}$, $\vec{\chi}_{\rm R}\cdot\vec{\chi}_{\rm L}$, $\vec{\xi}_{\rm R}\cdot\vec{\chi}_{\rm L}$, and $\vec{\chi}_{\rm R}\cdot\vec{\xi}_{\rm L}$ made up of Majorana bilinears. We may recombine them into two scalars and two vectors under $\mathcal{U}(\theta)$:

$$\text{scalar:} \quad \begin{cases} \vec{\xi}_{\rm R}\cdot\vec{\xi}_{\rm L} + \vec{\chi}_{\rm R}\cdot\vec{\chi}_{\rm L} \\ \vec{\xi}_{\rm R}\cdot\vec{\chi}_{\rm L} - \vec{\chi}_{\rm R}\cdot\vec{\xi}_{\rm L}, \end{cases} \tag{48}$$

$$\text{vector:} \quad \begin{cases} \vec{\xi}_{\rm R}\cdot\vec{\xi}_{\rm L} - \vec{\chi}_{\rm R}\cdot\vec{\chi}_{\rm L} + i(\vec{\xi}_{\rm R}\cdot\vec{\chi}_{\rm L} + \vec{\chi}_{\rm R}\cdot\vec{\xi}_{\rm L}) \cdots e^{i\theta} \\ \vec{\xi}_{\rm R}\cdot\vec{\xi}_{\rm L} - \vec{\chi}_{\rm R}\cdot\vec{\chi}_{\rm L} - i(\vec{\xi}_{\rm R}\cdot\vec{\chi}_{\rm L} + \vec{\chi}_{\rm R}\cdot\vec{\xi}_{\rm L}) \cdots e^{-i\theta}. \end{cases} \tag{49}$$

While the latter two have already appeared, the former are new. Therefore it is suggested that we should add two more order parameters, which are scalars under the spin-chirality rotation $\mathcal{U}(\theta)$, to complete our analysis. Below, we shall use the following set of four order parameters:[38]

$$\mathcal{O}_{\rm SD}^\pi = i(\vec{\xi}_{\rm R}\cdot\vec{\xi}_{\rm L} - \vec{\chi}_{\rm R}\cdot\vec{\chi}_{\rm L}),$$
$$\mathcal{O}_{\rm SC}^\pi = i(\vec{\xi}_{\rm R}\cdot\vec{\chi}_{\rm L} + \vec{\chi}_{\rm R}\cdot\vec{\xi}_{\rm L}),$$
$$\mathcal{O}_{\rm Q}^\pi = i(\vec{\xi}_{\rm R}\cdot\vec{\xi}_{\rm L} + \vec{\chi}_{\rm R}\cdot\vec{\chi}_{\rm L}),$$
$$\mathcal{O}_{\rm RQ}^\pi = i(\vec{\xi}_{\rm R}\cdot\vec{\chi}_{\rm L} - \vec{\chi}_{\rm R}\cdot\vec{\xi}_{\rm L}). \tag{50}$$

As we already know, the spin-chirality SO(2) transforms the first pair $(\mathcal{O}_{\rm SD},\mathcal{O}_{\rm SC})$ as a doublet, while keeping the second $(\mathcal{O}_{\rm Q},\mathcal{O}_{\rm RQ})$ invariant:

$$\mathcal{U}(\theta): \quad \begin{pmatrix} \mathcal{O}_{\rm SD}^\pi \\ \mathcal{O}_{\rm SC}^\pi \end{pmatrix} \mapsto \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \mathcal{O}_{\rm SD}^\pi \\ \mathcal{O}_{\rm SC}^\pi \end{pmatrix},$$
$$\begin{pmatrix} \mathcal{O}_{\rm Q}^\pi \\ \mathcal{O}_{\rm RQ}^\pi \end{pmatrix} \mapsto \begin{pmatrix} \mathcal{O}_{\rm Q}^\pi \\ \mathcal{O}_{\rm RQ}^\pi \end{pmatrix}. \tag{51}$$

A similar property holds for the second pair as well. To see this, let us introduce the following product:[38]

$$\widetilde{\mathcal{U}}(\widetilde{\theta}) \equiv s_1\mathcal{U}(\widetilde{\theta})s_1 = \mathcal{R}_{\rm R}(\widetilde{\theta})\mathcal{R}_{\rm L}(-\widetilde{\theta}), \tag{52}$$

which is chiral (i.e., left-right asymmetric) and probably *nonlocal* in terms of the original lattice spins. Then, it is straightforward to show

$$\widetilde{\mathcal{U}}(\widetilde{\theta}): \quad \begin{pmatrix} \mathcal{O}_{\rm SD}^\pi \\ \mathcal{O}_{\rm SC}^\pi \end{pmatrix} \mapsto \begin{pmatrix} \mathcal{O}_{\rm SD}^\pi \\ \mathcal{O}_{\rm SC}^\pi \end{pmatrix},$$
$$\begin{pmatrix} \mathcal{O}_{\rm Q}^\pi \\ \mathcal{O}_{\rm RQ}^\pi \end{pmatrix} \mapsto \begin{pmatrix} \cos\widetilde{\theta} & -\sin\widetilde{\theta} \\ \sin\widetilde{\theta} & \cos\widetilde{\theta} \end{pmatrix} \begin{pmatrix} \mathcal{O}_{\rm Q}^\pi \\ \mathcal{O}_{\rm RQ}^\pi \end{pmatrix}. \tag{53}$$

As in the case of $\mathcal{U}(\theta)$, we will use the notation $\widetilde{\mathcal{D}}\equiv\widetilde{\mathcal{U}}(\widetilde{\theta}=\pi/2)$ to denote the second duality. The existence of the above two dualities (one is nonchiral and the other is chiral) will be useful in understanding the low-energy physics and the global phase structure. The transformation properties of these order parameters under the symmetry operations in Sec. III B are summarized in Table I.

However, this is not the end of the story. For the phases denoted by Q and RQ, it will turn out that the $q=\pi$ (i.e., period-2) components are not sufficient for the *full* characterization of the phases. This point will be discussed in Sec. IV D in conjunction with the ground-state degeneracy.

TABLE I. Four order parameters and discrete symmetries. Note that under s_1, $\mathcal{D}$, and $\widetilde{\mathcal{D}}$ the four order parameters transform onto each other.

Order param.	Symmetry					Duality		
	$\mathcal{T}$	$T_{\text{1-site}}$	$\mathcal{P}_{12}$	P_{S}	P_{L}	s_1	$\mathcal{D}$	$\widetilde{\mathcal{D}}$
$\mathcal{O}^{\pi}_{\text{SD}}$	+1	−1	−1	−1	+1	$\mathcal{O}^{\pi}_{\text{Q}}$	$\mathcal{O}^{\pi}_{\text{SC}}$	inv.
$\mathcal{O}^{\pi}_{\text{SC}}$	−1	−1	−1	−1	+1	$\mathcal{O}^{\pi}_{\text{RQ}}$	$\mathcal{O}^{\pi}_{\text{SD}}$	inv.
$\mathcal{O}^{\pi}_{\text{Q}}$	+1	−1	+1	−1	+1	$\mathcal{O}^{\pi}_{\text{SD}}$	inv.	$\mathcal{O}^{\pi}_{\text{RQ}}$
$\mathcal{O}^{\pi}_{\text{RQ}}$	+1	−1	+1	+1	−1	$\mathcal{O}^{\pi}_{\text{SC}}$	inv.	$\mathcal{O}^{\pi}_{\text{Q}}$

F. Interactions in the continuum limit

Now that we have identified the order parameters, we proceed to constructing interactions in the low-energy effective action. Basically, we have three different contributions to the interactions. The first one comes from the uniform ($q=0$) terms in the continuum expressions of the spin operators [see Eq. (33)] while the other two from the contraction of the $\pm 2k_{\text{F}}$ terms or of two $4k_{\text{F}}$ terms ($k_{\text{F}}=\pi/4$).

By using the Majorana expressions (34a) and (34b) for the spin operators, we can rewrite the nonoscillatory part of the seven interactions (4a)–(4g):

$$\mathcal{V}^0_1=(\vec{\xi}_{\text{R}}\cdot\vec{\xi}_{\text{L}})^2+(\vec{\chi}_{\text{R}}\cdot\vec{\chi}_{\text{L}})^2+(\vec{\xi}_{\text{R}}\cdot\vec{\chi}_{\text{L}})^2+(\vec{\chi}_{\text{R}}\cdot\vec{\xi}_{\text{L}})^2, \tag{54a}$$

$$\mathcal{V}^0_2=(\vec{\xi}_{\text{R}}\cdot\vec{\xi}_{\text{L}})^2+(\vec{\chi}_{\text{R}}\cdot\vec{\chi}_{\text{L}})^2-(\vec{\xi}_{\text{R}}\cdot\vec{\chi}_{\text{L}})^2-(\vec{\chi}_{\text{R}}\cdot\vec{\xi}_{\text{L}})^2, \tag{54b}$$

$$\mathcal{V}^0_3=2[(\vec{\xi}_{\text{R}}\cdot\vec{\xi}_{\text{L}})(\vec{\chi}_{\text{R}}\cdot\vec{\chi}_{\text{L}})-(\vec{\xi}_{\text{R}}\cdot\vec{\chi}_{\text{L}})(\vec{\chi}_{\text{R}}\cdot\vec{\xi}_{\text{L}})], \tag{54c}$$

$$\mathcal{V}^0_4=2[(\vec{\xi}_{\text{R}}\cdot\vec{\xi}_{\text{L}})(\vec{\chi}_{\text{R}}\cdot\vec{\chi}_{\text{L}})+(\vec{\xi}_{\text{R}}\cdot\vec{\chi}_{\text{L}})(\vec{\chi}_{\text{R}}\cdot\vec{\xi}_{\text{L}})], \tag{54d}$$

$$\mathcal{V}^0_5=2(\vec{\xi}_{\text{R}}\cdot\vec{\xi}_{\text{L}}-\vec{\chi}_{\text{R}}\cdot\vec{\chi}_{\text{L}})(\vec{\xi}_{\text{R}}\cdot\vec{\chi}_{\text{L}}+\vec{\chi}_{\text{R}}\cdot\vec{\xi}_{\text{L}}), \tag{54e}$$

$$\mathcal{V}^0_6=-\frac{i}{2}(\vec{\xi}_{\text{R}}\cdot\vec{\chi}_{\text{R}}+\vec{\xi}_{\text{L}}\cdot\vec{\chi}_{\text{L}}), \tag{54f}$$

$$\mathcal{V}^0_7=-\frac{1}{2}(\vec{\xi}_{\text{R}}\cdot\vec{\chi}_{\text{R}})(\vec{\xi}_{\text{L}}\cdot\vec{\chi}_{\text{L}}). \tag{54g}$$

The superscripts "0" denote terms coming from the uniform ($q=0$) components of the (local) generators. Actually, products of two n^A components contribute the same (but with a permuted order) terms to the interactions $\mathcal{V}_1,\ldots,\mathcal{V}_7$, which can be absorbed by the redefinition of the bare couplings. For this reason, we will suppress the superscripts "0" and consider the interaction

$$\mathcal{V}=\sum_{i=1}^{7}g_i\mathcal{V}_i \tag{55}$$

in what follows. These expressions are consistent with the transformation properties (12a)–(12g) under $\mathcal{U}(\theta)$ and the discrete symmetries of the lattice model [see Eqs. (35)–(39)]. Therefore we may conclude that with an appropriate choice of bare couplings the above seven interactions will describe (weak) perturbations from the SU(4)-invariant model.

It is also very suggestive to rewrite the main part of the interaction as ($g_6=g_7=0$):

$$\begin{aligned}\mathcal{V}&=g_1\mathcal{V}_1+g_2\mathcal{V}_2+g_3\mathcal{V}_3+g_4\mathcal{V}_4+g_5\mathcal{V}_5\\&=\frac{1}{2}(g_1+g_2+g_3+g_4)(\vec{\xi}_{\text{R}}\cdot\vec{\xi}_{\text{L}}+\vec{\chi}_{\text{R}}\cdot\vec{\chi}_{\text{L}})^2\\&\quad+\frac{1}{2}(g_1-g_2+g_3-g_4)(\vec{\xi}_{\text{R}}\cdot\vec{\chi}_{\text{L}}-\vec{\chi}_{\text{R}}\cdot\vec{\xi}_{\text{L}})^2\\&\quad+\frac{1}{2}(g_1+g_2-g_3-g_4)(\vec{\xi}_{\text{R}}\cdot\vec{\xi}_{\text{L}}-\vec{\chi}_{\text{R}}\cdot\vec{\chi}_{\text{L}})^2\\&\quad+\frac{1}{2}(g_1-g_2-g_3+g_4)(\vec{\xi}_{\text{R}}\cdot\vec{\chi}_{\text{L}}+\vec{\chi}_{\text{R}}\cdot\vec{\xi}_{\text{L}})^2+g_5\mathcal{H}_5,\end{aligned} \tag{56a}$$

which can be rewritten into a more convenient form in terms of the order parameters:

$$\begin{aligned}\mathcal{V}&=-\frac{1}{2}(g_1+g_2+g_3+g_4)(\mathcal{O}^{\pi}_{\text{Q}})^2\\&\quad-\frac{1}{2}(g_1-g_2+g_3-g_4)(\mathcal{O}^{\pi}_{\text{RQ}})^2\\&\quad-\frac{1}{2}(g_1+g_2-g_3-g_4)(\mathcal{O}^{\pi}_{\text{SD}})^2\\&\quad-\frac{1}{2}(g_1-g_2-g_3+g_4)(\mathcal{O}^{\pi}_{\text{SC}})^2-g_5\mathcal{O}^{\pi}_{\text{SD}}\mathcal{O}^{\pi}_{\text{SC}}.\end{aligned} \tag{56b}$$

From this, we may expect that one of the four competing quantum phases is selected according to the values of the seven couplings $g_1,\ldots,g_7$ in the low-energy limit. In order to investigate the low-energy behavior of the seven couplings, we shall carry out a RG analysis in the next section.

IV. RG ANALYSIS AND FOUR COMPETING ORDERS

A. β functions

As has been shown in the last section, we have seven interactions around the SU(4) [or SO(6)] fixed point. Since we are interested in the *spontaneous* breaking of time-reversal symmetry, we will suppress the $\mathcal{H}_5$ interaction:[61] $g_5=0$. The sixth interaction $\mathcal{H}_6$ is a kind of *magnetic field* or *chemical potential* (see Sec. II D) and may be incorporated after physics for $g_6=0$ is understood. For these reasons, we first consider the model with $E=F=0$:

$$\mathcal{H}=\mathcal{H}_{\mathrm{SO}(6)}+\mathcal{V}. \tag{57}$$

At the one-loop level, the calculation of the RG β function reduces to that of OPE defined below:[62]

$$\mathcal{H}_i(z,\bar{z})\mathcal{H}_j(w,\bar{w}) \sim \frac{C_{ij}^k}{|z-w|^2}\mathcal{H}_k(w,\bar{w}) \quad (i,j=1,2,3,4,7). \tag{58}$$

The nonzero OPE coefficients listed in Appendix E enable us to write down the RG β function:[38]

$$\begin{aligned}
\dot{g}_1 &= g_1^2+g_2^2+5g_3^2+g_4^2,\\
\dot{g}_2 &= 2g_1g_2+6g_3g_4+g_4g_7,\\
\dot{g}_3 &= 6g_1g_3+2g_2g_4,\\
\dot{g}_4 &= 2g_1g_4+6g_2g_3+g_2g_7,\\
\dot{g}_7 &= -16(g_1g_3-g_2g_4),
\end{aligned} \tag{59}$$

where dots denote the derivative with respect to the RG time: $\dot{g}=dg_i/(d\ln L)$.

It is interesting to observe that this set of RG equations (RGEs) determines a gradient flow in a five-dimensional space of coupling constants. In fact, if we make a change of variables:

$$h_1=g_1,\quad h_2=g_2,\quad h_3=\sqrt{2}g_3+\frac{1}{\sqrt{128}}g_7,$$
$$h_4=g_4,\quad h_5=\sqrt{\frac{5}{128}}g_7, \tag{60}$$

the RGE can be derived as

$$\dot{h}_i=-\frac{\partial V}{\partial h_i}$$

from a *single* RG potential:

$$\begin{aligned}
&V(h_1,h_2,h_3,h_4,h_5)\\
&\quad=-\frac{1}{3}h_1^3-h_1h_2^2-\frac{5}{2}h_1h_3^2-h_1h_4^2-\frac{1}{2}h_1h_5^2\\
&\qquad+\sqrt{5}h_1h_3h_5-3\sqrt{2}h_2h_3h_4-\sqrt{10}h_2h_4h_5.
\end{aligned} \tag{61}$$

A similar property was pointed out in quasi-one-dimensional electron systems[63] to explain the simple structure of the phase diagrams.

B. Symmetries of β functions

1. Sign-change and permutation symmetries

The above set (59) of β functions is invariant under discrete symmetries (sign change and permutations). To be concrete, the RGE is invariant under the s_1 transformation [see Eq. (40)]:

$$(g_1,g_2,g_3,g_4,0,0,g_7) \xrightarrow{s_1} (g_1,g_2,-g_3,-g_4,0,0,-g_7). \tag{62}$$

Of course, the spin-chirality duality $\mathcal{D}$ maps a set of couplings as

$$(g_1,g_2,g_3,g_4,0,0,g_7) \overset{\mathcal{D}}{\rightarrow} (g_1,g_4,g_3,g_2,0,0,g_7) \tag{63}$$

and keeps the self-dual manifold $g_2=g_4$ invariant. Combining this with s_1, we obtain for the second duality:

$$(g_1,g_2,g_3,g_4,0,0,g_7) \overset{\tilde{\mathcal{D}}}{\rightarrow} (g_1,-g_4,g_3,-g_2,0,0,g_7). \tag{64}$$

For the second duality $\tilde{\mathcal{D}}$, the self-dual manifold is characterized by

$$g_2=-g_4. \tag{65}$$

2. Self-dual manifolds

The existence of the two duality transformations $\mathcal{D}$ and $\tilde{\mathcal{D}}$ manifests itself in the invariance of the β functions under the interchange: $g_2\leftrightarrow \pm g_4$ (note that the time-reversal breaking term g_5 is already suppressed). That is, the RG flow is symmetric with respect to the four-dimensional self-dual manifolds defined by $g_2=\pm g_4$. Since it follows from the β function (59) that

$$(\dot{g}_2 \mp \dot{g}_4)=(2g_1 \mp 6g_3 \mp g_7)(g_2 \mp g_4),$$

systems which are self-dual initially will remain so even after renormalization. In fact, the original lattice model with $g_2=g_4$ has a continuous symmetry $\mathcal{U}(\theta)$—the spin-chirality rotation by an arbitrary angle θ—and the Mermin-Wagner-Coleman theorem guarantees the condition $g_2=g_4$ is preserved for all orders of perturbation. On this self-dual manifold, the β function simplifies to

$$\begin{aligned}
\dot{g}_1 &= g_1^2+2g_2^2+5g_3^2,\\
\dot{g}_2 &= g_2(2g_1+6g_3+g_7),\\
\dot{g}_3 &= 6g_1g_3+2g_2^2,\\
\dot{g}_7 &= -16(g_1g_3-g_2^2).
\end{aligned} \tag{66}$$

This reduced set of β functions has a further invariant manifold characterized by SU(3) symmetry:

$$g_1=g_3,$$

where we have only three coupled equations:

$$\begin{aligned}
\dot{g}_1 &= 6g_1^2+2g_2^2,\\
\dot{g}_2 &= 8g_1g_2+g_2g_7,\\
\dot{g}_7 &= -16(g_1^2-g_2^2).
\end{aligned} \tag{67}$$

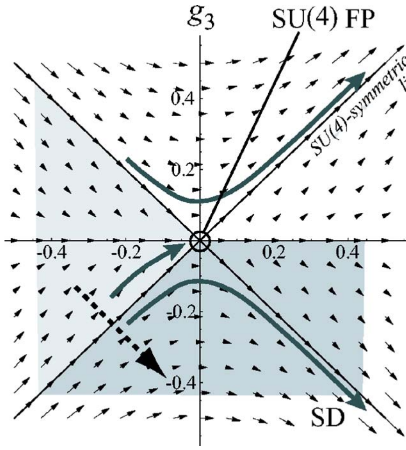

FIG. 3. (Color online) RG flow for the SO and dSO models. The origin [shown as SU(4) FP] corresponds to the level-1 SU(4) WZW model and a line $g_1 = g_3$ to self-dual models [SU(4) symmetric, in this case]. A dashed line roughly corresponds to a path across the SU(4) Sutherland model [extended massless → SU(4) → staggered dimer (SD)] discussed in literature (e.g., Ref. 57). An asymptotic trajectory in the lower-right portion (highlighted by a thick arrow) is responsible for the spontaneous breakdown of translational (for SO model) and time-reversal (for dSO model) symmetries.

3. Other invariant manifolds

Furthermore, by adding and subtracting equations in Eq. (59), we obtain several invariant manifolds:

Spin-orbital (SO) manifold.

$$g_1 = g_2, \quad g_3 = g_4, \quad g_7 = 0. \tag{68}$$

This means that as far as we consider the SO model [see Eq. (16)], the rung-rung four-body interaction will *never* be generated radiatively on the manifold. In fact, the underlying SU(2)×SU(2) [$SO(3)_\xi \times SO(3)_\chi$] symmetry guarantees that it persists even in *all* orders of perturbation expansion. On this manifold, only two couplings g_1 and g_3 suffice for describing the low-energy physics and the RG β functions substantially simplify to the following two coupled equations:

$$\dot{g}_1 = 2g_1^2 + 6g_3^2, \quad \dot{g}_3 = 8g_1 g_3. \tag{69}$$

When $g_3 < 0$ and $g_3 < g_1$ (which may be interpreted as the SO model with $J > K/4$. For a detailed discussion of the SO model, see Ref. 57.), the system is attracted in the infrared limit by an asymptotic trajectory:

$$g_1^*(=g_2^*) = -g_3^*(=-g_4^*)$$

(see a region shown by darker gray in Fig. 3) and breaks the translational symmetry spontaneously.[57] The exact ground state[46] at $K = 4J/3$ and the analysis for small $K(\ll J)$ (Ref. 25) are consistent with the above result. The region $g_1 < 0$, $g_3 < |g_1|$ (a portion of Fig. 3 marked by lighter gray) is the basin of the gapless $SU(4)_1$ fixed point.

Dual spin-orbital (dSO) manifold.

$$g_1 = g_4, \quad g_2 = g_3, \quad g_7 = 0. \tag{70}$$

This is the dual ($\mathcal{D}$) partner of the above invariant manifold obtained by applying $\mathcal{D}$ to the SO model and its existence is guaranteed by a hidden (dual) SU(2)×SU(2) symmetry of the model. Now two couplings g_1 and g_2 describe the system and, again, β functions reduce to the set of equations (69) with g_3 being replaced with g_2. For $g_2 < 0$ and $g_2 < g_1$, the system flows toward another asymptotic trajectory:

$$g_1^*(=g_4^*) = -g_2^*(=-g_3^*),$$

and, in this case, breaks time-reversal symmetry $\mathcal{T}$ (see the next subsection for the detail). The basin of $SU(4)_1$ fixed point of the SO manifold is now mapped onto $g_1 < 0$, $g_2 < |g_1|$. In terms of lattice models, this corresponds to the model Hamiltonian

$$\mathcal{H}_{\mathrm{dSO}} = \frac{1}{2} J \mathcal{H}_1 + \frac{1}{8} K \mathcal{H}_2 + \frac{1}{8} K \mathcal{H}_3 + \frac{1}{2} J \mathcal{H}_4$$

with $J < K$. Similarly, we have two more invariant manifolds,

$$g_1 = -g_2, \quad g_3 = -g_4, \quad g_7 = 0,$$

$$g_1 = -g_4, \quad g_2 = -g_3, \quad g_7 = 0, \tag{71}$$

on which the full RGE reduces to two coupled equations. Four RG-invariant lines (rays) defined by the intersections among these invariant manifolds (hyperplanes) will turn out to correspond to four dominant competing phases.

Away from these invariant manifolds, the rung-rung four-body interaction g_7 will be generated in the course of renormalization even though it is absent initially.

C. Asymptotic form of RG flow

The set of RG β functions (59) is complicated and complete analysis of it is not so easy. We numerically integrated the equations and found that the RG flow exhibited a striking feature in the low-energy limit. To see this more closely, we apply an ansatz proposed by Lin, Balents, and Fisher[15] in the study of a half filled two-leg Hubbard ladder.

Since only marginal interactions appear in the β functions, a natural ansatz valid for the infrared asymptotics may be[15]

$$g_i(t) = \frac{G_i}{t_0 - t} \quad (i = 1,2,3,4,7), \tag{72}$$

where the constant t_0 marks the crossover point where the weak-coupling perturbation breaks down. Plugging these into the RG equations and requiring consistency, we have a set of nonlinear equations which determines the IR asymptotics. We found various solutions and among them the followings are relevant for our analysis:

Self-dual.

$$(G_1, G_2, G_3, G_4, G_7) = \left(\frac{1}{6}, 0, -\frac{1}{6}, 0, \frac{4}{9}\right) \quad \text{(I)}, \tag{73a}$$

$$(G_1, G_2, G_3, G_4, G_7) = \left(\frac{1}{6}, 0, \frac{1}{6}, 0, -\frac{4}{9}\right) \quad \text{(II)}. \tag{73b}$$

The meaning of these rays will become clear if we plug Eqs. (73a) and (73b) into Eq. (56b):

$$\text{(I)}: \; -\frac{1}{6}g_{\mathrm{I}}^{*}[(\mathcal{O}_{\mathrm{SD}}^{\pi})^2+(\mathcal{O}_{\mathrm{SC}}^{\pi})^2]+\frac{4}{9}g_{\mathrm{I}}^{*}\mathcal{H}_7, \tag{74a}$$

$$\text{(II)}: \; -\frac{1}{6}g_{\mathrm{II}}^{*}[(\mathcal{O}_{\mathrm{Q}}^{\pi})^2+(\mathcal{O}_{\mathrm{RQ}}^{\pi})^2]-\frac{4}{9}g_{\mathrm{II}}^{*}\mathcal{H}_7. \tag{74b}$$

Therefore the above two rays (I) and (II) may be thought of as corresponding to SD-SC- and Q-RQ transitions, respectively.

From the argument in Sec. II, it is obvious that the latter case (II) has SU(3) symmetry, while the SU(3) symmetry of the former is hidden and appears only after the particle-hole transformation for the left movers is applied (see Sec. V A). Also it is worth noting that they are related to each other by the Ising duality s_1.

SO(6)-symmetric rays.

$$(G_1,G_2,G_3,G_4,G_7)=\left(\frac{1}{8},\frac{1}{8},-\frac{1}{8},-\frac{1}{8},0\right) \quad \text{(SD)}, \tag{75a}$$

$$(G_1,G_2,G_3,G_4,G_7)=\left(\frac{1}{8},-\frac{1}{8},-\frac{1}{8},\frac{1}{8},0\right) \quad \text{(SC)}, \tag{75b}$$

$$(G_1,G_2,G_3,G_4,G_7)=\left(\frac{1}{8},\frac{1}{8},\frac{1}{8},\frac{1}{8},0\right) \quad \text{(Q)}, \tag{75c}$$

$$(G_1,G_2,G_3,G_4,G_7)=\left(\frac{1}{8},-\frac{1}{8},\frac{1}{8},-\frac{1}{8},0\right) \quad \text{(RQ)}. \tag{75d}$$

Note that the latter two are invariant both under spin chirality SO(2) and under SU(3), while the former two are not. It is important to note that the low-energy effective Hamiltonians for these rays assume the following simple forms:

$$\mathcal{H}_{\mathrm{A}}=\mathcal{H}_{\mathrm{SO(6)}}-g_{A}^{*}(\mathcal{O}_{\mathrm{A}}^{\pi})^2 \quad (A=\mathrm{Q},\ \mathrm{RQ},\ \mathrm{SD}, \text{and SC}). \tag{76}$$

Semiclassical argument based on Eq. (56b) tells us that these symmetric rays correspond to the four competing phases where one of the order parameters has a nonzero expectation value: $\langle\mathcal{O}_{A}^{\pi}\rangle\neq 0$. In particular, from Table I, it follows that a spin-disordered ground state with broken $\mathcal{T}$ is realized along the ray "SC." Furthermore, we can show that all these rays correspond to IR-massive flows along which translational symmetry is broken and the SO(6) symmetry is restored asymptotically (see next section).

Transitions among dominant phases.

$$G_1=G_2=\frac{1}{2}, \quad G_3=G_4=0, \quad G_7=0\ (\mathrm{Q}\leftrightarrow\mathrm{SD}), \tag{77a}$$

$$G_1=-G_2=\frac{1}{2}, \quad G_3=G_4=0, \quad G_7=0\ (\mathrm{RQ}\leftrightarrow\mathrm{SC}), \tag{77b}$$

$$G_1=G_4=\frac{1}{2}, \quad G_2=G_3=0, \quad G_7=0\ (\mathrm{Q}\leftrightarrow\mathrm{SC}), \tag{77c}$$

$$G_1=-G_4=\frac{1}{2}, \quad G_2=G_3=0, \quad G_7=0\ (\mathrm{RQ}\leftrightarrow\mathrm{SD}). \tag{77d}$$

None of these restores the original SO(6) symmetry. Actually, these rays correspond to transitions among the above four dominant phases.

Case of four competing orders.

$$G_1=1, \quad G_2=G_3=G_4=G_7=0. \tag{78}$$

This ray describes nontrivial quantum criticality resulting from the competition among four different orders, which will be discussed in Sec. VI.

The point here is that in the low-energy limit our system is asymptotically characterized only by a *single* (diverging) coupling constant[15,63] $\sim 1/(t_0-t)$.

D. Four dominant phases

1. SO(6) restoration

To illustrate how SO(6) symmetry, which is explicitly broken in the bare action (57), is restored along the four rays (i.e., SD, SC, Q, and RQ), we take the two rays Q and SD. From the continuum expressions (56a) and (56b), it is obvious that the model along the ray Q is explicitly SO(6) invariant:

$$\mathcal{H}_{\mathrm{Q}}=\mathcal{H}_{\mathrm{SO(6)}}-g^{*}(\mathcal{O}_{\mathrm{Q}}^{\pi})^2=\mathcal{H}_{\mathrm{SO(6)}}+g^{*}(\vec{\xi}_{\mathrm{R}}\cdot\vec{\xi}_{\mathrm{L}}+\vec{\chi}_{\mathrm{R}}\cdot\vec{\chi}_{\mathrm{L}})^2. \tag{79}$$

This is nothing but the Hamiltonian of the SO(6) Gross-Neveu model.[64] This model is integrable. Its spectrum is known[65,66] to consist of the fundamental fermion with mass M together with a kink of mass $m=M/\sqrt{2}$. The $\mathbb{Z}_2$ symmetry $\mathcal{O}_{\mathrm{Q}}^{\pi}\leftrightarrow-\mathcal{O}_{\mathrm{Q}}^{\pi}$ is broken spontaneously and, as a result, we have a finite expectation value $\langle\mathcal{O}_{\mathrm{Q}}^{\pi}\rangle\neq 0$.

The application of the Ising duality s_1 [Eq. (40)] to this Hamiltonian $\mathcal{H}_{\mathrm{Q}}$ takes us to another ray SD and the associated Hamiltonian $\mathcal{H}_{\mathrm{SD}}$ since $\mathcal{H}_{\mathrm{SO(6)}}\mapsto\mathcal{H}_{\mathrm{SO(6)}}$ and $\mathcal{O}_{\mathrm{Q}}^{\pi}\mapsto\mathcal{O}_{\mathrm{SD}}^{\pi}$. A similar argument applies to the other pair ($\mathcal{H}_{\mathrm{RQ}}$ and $\mathcal{H}_{\mathrm{SC}}$) as well.

Moreover, the spin-chirality duality $\mathcal{D}$ ($\tilde{\mathcal{D}}$) maps $\mathcal{H}_{\mathrm{SD}}$ ($\mathcal{H}_{\mathrm{Q}}$) onto $\mathcal{H}_{\mathrm{SC}}$ ($\mathcal{H}_{\mathrm{RQ}}$). These facts imply that all these four low-energy Hamiltonians $\mathcal{H}_{\mathrm{Q}}$, $\mathcal{H}_{\mathrm{RQ}}$, $\mathcal{H}_{\mathrm{SD}}$, and $\mathcal{H}_{\mathrm{SC}}$ can be transformed back to the SO(6)-invariant one (79) by applying symmetry operations of $\mathrm{O(6)_L}\times\mathrm{O(6)_R}$. This is the reason why we named the asymptotic rays (75a)–(75d) "SO(6) symmetric." The relationship among these four phases is summarized in Fig. 4.

2. Ground-state degeneracy

Now we know that four competing (gapped) phases exist around the SU(4) point and that they are mapped onto each other by the two discrete transformations $\mathcal{D}$ and $\tilde{\mathcal{D}}$. From this, we may conclude that all these phases have the same ground-state degeneracy (from the exact solution[46] for the SD phase, it might seem natural that we should have the degeneracy 2 for the others as well). But there is a pitfall

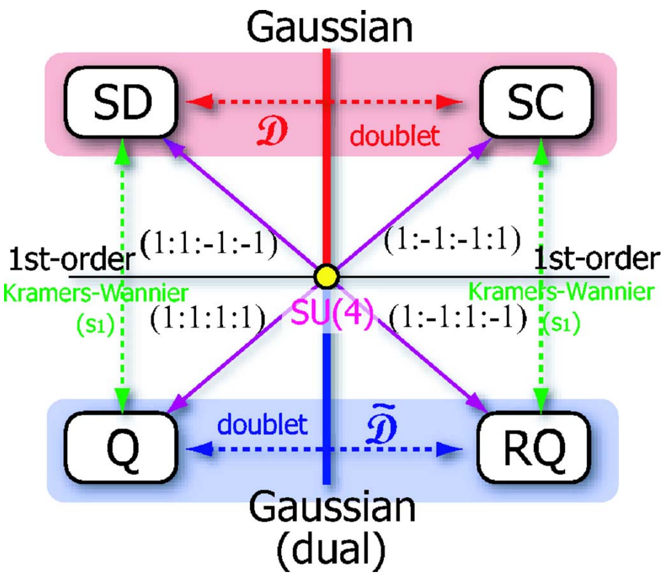

FIG. 4. (Color online) Relationship among the four dominant phases. Ratio of the couplings g_1, g_2, g_3, and g_4 along each symmetric ray is shown.

here; the discrete transformation $\tilde{\mathcal{D}}$ that relates SD to Q is *not* a local one and there is no reason to believe that $\tilde{\mathcal{D}}$ maps a phase onto another one with the same ground-state degeneracy. A typical example of this kind of *nonlocal* transformation is the so-called Kennedy-Tasaki transformation,[67] which maps a unique (infinite-volume) ground state onto fourfold degenerate ones.

To find the correct answer, we first note that Dirac fermions used to construct our SU(4) Hamiltonian (see Appendix C) are expressed as exponentials of boson operators and that the semiclassical bosonic ground states should be considered modulo the following gauge-equivalence:

$$\phi_{a,\sigma,\mathrm{L/R}} \sim \phi_{a,\sigma,\mathrm{L/R}} + \sqrt{\pi} N_{a,\sigma,\mathrm{L/R}} \quad (N_{a,\sigma,\mathrm{L/R}} \in \mathbb{Z}), \quad (80)$$

which keeps the bosonic expressions of the fermions (B2) unchanged.

We carefully counted the number of inequivalent ground states for all four cases Q, SD, SC, and RQ. Although the calculation was slightly more complicated than that in Ref. 15, the procedure was essentially the same (see Appendix F for details) and the results are summarized as (see also Table II)

TABLE II. Four dominant phases and the ground-state degeneracy. In all four phases, translation symmetry is broken.

	GS degeneracy and symmetry				
Phase	Degen.	$\mathcal{T}$	$\mathcal{P}_{12}$	P_{S}	P_{L}
SD	2	+1	−1	−1	+1
SC	2	−1	−1	−1	+1
Q	4	+1	+1	−1	+1
RQ	4	+1	+1	+1	−1

(i) *Q phase*: Four ground states; $2k_{\mathrm{F}}$ part of $\langle X_r^A X_{r+1}^A \rangle$ is non vanishing (a kind of quadrumerized phase) and period-4 ground states appear;

(ii) *SD phase*: Two ground states; $2k_{\mathrm{F}}$ part is vanishing (staggered phase) and the ground states have a period 2;

(iii) *SC phase*: Two ground states; $2k_{\mathrm{F}}$ part is vanishing;

(iv) *RQ phase*: Four ground states; $2k_{\mathrm{F}}$ part is nonvanishing (period-4 density wave of the pseudospin $\langle \mathbf{S}_{1,r} \cdot \mathbf{S}_{2,r} \rangle$).

For some reasons, the period-4 ($q = \pi/2$) contribution in the order-parameter correlations vanishes for the SD and SC phases, as is expected from the exact ground state[46] for which we have $\langle \mathcal{O}_r \mathcal{O}_{r+1} \rangle = (-1)^r 3/4$ (when the amplitude of the order parameter is maximal). On the other hand, we have the $2k_{\mathrm{F}}$ components in the Q and RQ phases. This implies that the $4k_{\mathrm{F}}$ component is not sufficient to fully characterize the latter two phases.

So far, the nature of the ground states appearing in the Q or the RQ phase is not so clear. This is mainly because the transformation s_1 relating $\mathcal{O}_{\mathrm{SD}}^{\pi}$ to $\mathcal{O}_{\mathrm{Q}}^{\pi}$ is nonlocal and we do not know the microscopic (or lattice) expressions for $\mathcal{O}_{\mathrm{Q,RQ}}^{\pi}$. By symmetry arguments, we can restrict the possible forms of lattice operators corresponding to $\mathcal{O}_{\mathrm{Q}}^{\pi}$. We have five building blocks for the lattice order parameters all of which have (at least for the leading contributions) the same $4k_{\mathrm{F}}$ (or $q = \pi$) term $\mathcal{O}_{\mathrm{Q}}^{\pi}$ in the continuum. Exactly on the SU(4)-invariant line, these five will condense with the same amplitude to describe a phase with SU(4) quadrumerization (note that we have four-fold degeneracy). Unfortunately, our continuum approach cannot tell anything about which of the five will become dominant when we move away from the SU(4) manifold.

Similarly, since the $4k_{\mathrm{F}}$ part of $\mathbf{S}_{1,r} \cdot \mathbf{S}_{2,r}$ is written as $\mathcal{O}_{\mathrm{RQ}}^{\pi}$, the RQ phase may be thought of as a density-wave phase of the pseudospin $\mathcal{S}^z$. By using the analogy in Sec. II D 2, this is nothing but the bosonic charge-density wave state with period 4.

V. PHYSICS OF SELF-DUAL MODELS AND PHASE TRANSITIONS

Now that we have identified four dominant phases in our SO(6) problem, the next question to ask would be the quantum phase transitions among them. In this section, we discuss the nature of the phase transition between SD and SC phases as well as the Q-RQ transition. As described in Sec. IV C, the transition should belong to the self-dual manifolds ($g_2 = \pm g_4$) and we first present a fully quantum description of the low-energy physics along these manifolds.

A. SU(3×U(1)) bosonization

According to the analysis presented in Sec. IV C, the one-loop RG flow on the self-dual manifold, which is invariant under the spin-chirality rotation $\mathcal{U}(\theta)$ or $\tilde{\mathcal{U}}(\theta)$, is attracted to one of the following two special rays: (I): $g_1 = -g_3 = g_{\mathrm{I}}^*/6$, $g_2 = g_4 = 0$, $g_7 = 4g_{\mathrm{I}}^*/9$, and (II): $g_1 = g_3 = g_{\mathrm{II}}^*/6$, $g_2 = g_4 = 0$, $g_7 = -4g_{\mathrm{II}}^*/9$ with $g_{\mathrm{I,II}}^* > 0$. Along these rays, the low-energy field theory reads as follows:

$$\mathcal{H}_{\rm IR}^{(\rm I)} = \mathcal{H}_{\rm SO(6)} + \frac{g_{\rm I}^*}{6}[(\vec{\xi}_{\rm R}\cdot\vec{\xi}_{\rm L} - \vec{\chi}_{\rm R}\cdot\vec{\chi}_{\rm L})^2 + (\vec{\xi}_{\rm R}\cdot\vec{\chi}_{\rm L} + \vec{\chi}_{\rm R}\cdot\vec{\xi}_{\rm L})^2] - \frac{2g_{\rm I}^*}{9}(\vec{\xi}_{\rm R}\cdot\vec{\chi}_{\rm R})(\vec{\xi}_{\rm L}\cdot\vec{\chi}_{\rm L}), \tag{81a}$$

$$\mathcal{H}_{\rm IR}^{(\rm II)} = \mathcal{H}_{\rm SO(6)} + \frac{g_{\rm II}^*}{6}[(\vec{\xi}_{\rm R}\cdot\vec{\xi}_{\rm L} + \vec{\chi}_{\rm R}\cdot\vec{\chi}_{\rm L})^2 + (\vec{\xi}_{\rm R}\cdot\vec{\chi}_{\rm L} - \vec{\chi}_{\rm R}\cdot\vec{\xi}_{\rm L})^2] + \frac{2g_{\rm II}^*}{9}(\vec{\xi}_{\rm R}\cdot\vec{\chi}_{\rm R})(\vec{\xi}_{\rm L}\cdot\vec{\chi}_{\rm L}). \tag{81b}$$

It is interesting to observe that the two low-energy field theories $\mathcal{H}_{\rm IR}^{(\rm I)}$ and $\mathcal{H}_{\rm IR}^{(\rm II)}$ interchange themselves under the Ising duality symmetry s_1 [Eq. (40)].

As has been shown in Sec. IV C, the low-energy effective Hamiltonians (81a) and (81b) describe the competition between two different orders [see Eqs. (73a) and (73b)]: the staggered dimerized $\mathcal{O}_{\rm SD}^{\pi}$ and the staggered scalar chirality $\mathcal{O}_{\rm SC}^{\pi}$ orders for the model $\mathcal{H}_{\rm IR}^{(\rm I)}$ [Eq. (81a)], and the two quadrumerized phases $\mathcal{O}_{\rm Q}^{\pi}$ and $\mathcal{O}_{\rm RQ}^{\pi}$ for the model $\mathcal{H}_{\rm IR}^{(\rm II)}$ [Eq. (81b)]. The main question concerns the nature of the quantum phase transitions that result from the competition between these orders ($\mathcal{O}_{\rm SD}^{\pi}\leftrightarrow\mathcal{O}_{\rm SC}^{\pi}$ for case I and $\mathcal{O}_{\rm Q}^{\pi}\leftrightarrow\mathcal{O}_{\rm RQ}^{\pi}$ for case II). In this section, we present a full quantum description to this issue.

The key observation, which will be crucial in the following analysis, is that the low-energy theories (81a) and (81b) display a hidden symmetry which is not $\mathrm{SU}(2)\times\mathrm{U}(1)$ but, in fact, a larger U(3) symmetry. In the following, we are going to discuss only the case of the first model (81a), i.e., the nature of the phase transition between the staggered dimerized and the scalar chirality phases. The physical properties of the second model (81b) can then be derived readily by applying the Ising duality symmetry s_1 (40).

In the following analysis, it would be convenient to combine three pairs of Majorana fields $\xi_{\rm L/R}^a$ and $\chi_{\rm L/R}^a$ to form three Dirac fermions $\Psi_{a,\rm L/R}$:

$$\Psi_{a,\rm R} = \frac{\xi_{\rm R}^a + i\chi_{\rm R}^a}{\sqrt{2}}, \quad \Psi_{a,\rm L} = \frac{\xi_{\rm L}^a - i\chi_{\rm L}^a}{\sqrt{2}}, \tag{82}$$

with $a=1,2,3$. The reason for this left-right *asymmetric* definition is that SU(3) symmetry is clear in this notation. The symmetric notation will be useful in describing the second ray (II) [Eq. (73b) or the model (81b)].[68]

From Eq. (82), it is straightforward to express the order parameters (50) in terms of these Dirac fermions:

$$\mathcal{O}_{\rm Q}^{\pi} = i\sum_{a=1}^{3}(\Psi_{a,\rm R}\Psi_{a,\rm L} + \Psi_{a,\rm R}^\dagger\Psi_{a,\rm L}^\dagger),$$

$$\mathcal{O}_{\rm RQ}^{\pi} = \sum_{a=1}^{3}(-\Psi_{a,\rm R}\Psi_{a,\rm L} + \Psi_{a,\rm R}^\dagger\Psi_{a,\rm L}^\dagger),$$

$$\mathcal{O}_{\rm SD}^{\pi} = i\sum_{a=1}^{3}(\Psi_{a,\rm R}\Psi_{a,\rm L}^\dagger + \Psi_{a,\rm R}^\dagger\Psi_{a,\rm L}),$$

$$\mathcal{O}_{\rm SC}^{\pi} = \sum_{a=1}^{3}(\Psi_{a,\rm R}\Psi_{a,\rm L}^\dagger - \Psi_{a,\rm R}^\dagger\Psi_{a,\rm L}). \tag{83}$$

In this notation, two order parameters $\mathcal{O}_{\rm Q}^{\pi}$ and $\mathcal{O}_{\rm RQ}^{\pi}$, which are invariant under $\mathcal{U}(\theta)$, transform like "Cooper pairs," while $\mathcal{O}_{\rm SD}^{\pi}$ and $\mathcal{O}_{\rm SC}^{\pi}$ look like the density operators. If we had adopted the left-right symmetric definition instead of Eq. (82), $\mathcal{O}_{\rm SD}^{\pi}$ and $\mathcal{O}_{\rm SC}^{\pi}$ would have behaved like the Cooper pairs. This analogy can also be revealed by mentioning the expression of the spin-chirality rotation $\mathcal{U}(\theta)$ [Eq. (41)] and the Ising duality symmetry s_1 [Eq. (40)] on these Dirac fermions:

$$\Psi_{a,\rm R} \xrightarrow{\mathcal{U}(\theta)} e^{+i(\theta/2)}\Psi_{a,\rm R}, \quad \Psi_{a,\rm L} \xrightarrow{\mathcal{U}(\theta)} e^{-i(\theta/2)}\Psi_{a,\rm L}, \tag{84a}$$

$$\Psi_{a,\rm R} \xrightarrow{\tilde{\mathcal{U}}(\tilde{\theta})} e^{-i(\tilde{\theta}/2)}\Psi_{a,\rm R}, \quad \Psi_{a,\rm L} \xrightarrow{\tilde{\mathcal{U}}(\tilde{\theta})} e^{-i(\tilde{\theta}/2)}\Psi_{a,\rm L}, \tag{84b}$$

$$\Psi_{a,\rm R} \xrightarrow{s_1} \Psi_{a,\rm R}^\dagger, \quad \Psi_{a,\rm L} \xrightarrow{s_1} \Psi_{a,\rm L}. \tag{84c}$$

The appearance of the left-right asymmetric expressions for $\mathcal{U}(\theta)$ is just an artifact of our definition (82); if the symmetric definition had been used, Eqs. (84a) and (84b) would have been interchanged. One thus observes that the spin-chirality rotation acts as a pseudocharge U(1) symmetry on these fermions.

The next step is to express the low-energy field theory corresponding to model (81a) in such a way that U(3) symmetry of our problem is manifest. To this end, it is useful to introduce the chiral $\mathrm{SU}(3)_1$ currents built from the three Dirac fermions:

$$\mathcal{J}_{\rm R/L}^A = \sum_{a,b}\Psi_{a,\rm R/L}^\dagger T_{ab}^A\Psi_{b,\rm R/L}, \tag{85}$$

where the 3×3 matrices T^A, $A=1,\ldots,8$ are the generators of SU(3) in the fundamental representation **3**, which are normalized so that $\mathrm{Tr}(T^AT^B)=\delta^{AB}/2$. Using the identity

$$\sum_A T_{ab}^A T_{cd}^A = \frac{1}{2}\left(\delta_{ad}\delta_{bc} - \frac{1}{3}\delta_{ab}\delta_{cd}\right) \tag{86}$$

and the definition (82), one obtains the following identity:

$$\sum_A \mathcal{J}_{\rm R}^A\mathcal{J}_{\rm L}^A = \frac{1}{8}[(\xi_{\rm R}\cdot\xi_{\rm L} - \chi_{\rm R}\cdot\chi_{\rm L})^2 + (\xi_{\rm R}\cdot\chi_{\rm L} + \chi_{\rm R}\cdot\xi_{\rm L})^2] - \frac{1}{6}(\xi_{\rm R}\cdot\chi_{\rm R})(\xi_{\rm L}\cdot\chi_{\rm L}). \tag{87}$$

We thus deduce that the low-energy field theory (81a) associated to the ray (I) exhibits an exact U(3) symmetry:

$$\mathcal{H}_{\rm IR}^{(\rm I)} = -iv\sum_{a=1}^{3}(\Psi_{a,\rm R}^\dagger\partial_x\Psi_{a,\rm R} - \Psi_{a,\rm L}^\dagger\partial_x\Psi_{a,\rm L}) + \frac{4g^*}{3}\sum_A \mathcal{J}_{\rm R}^A\mathcal{J}_{\rm L}^A. \tag{88}$$

Noting the well-known fact[35] that the three massless Dirac fermions (the first term) can be bosonized in terms of a single scalar field and the above $SU(3)_1$ currents, we can further recast the above Hamiltonian as

$$\mathcal{H}_{\mathrm{IR}}^{(\mathrm{I})} = \frac{v}{2}[(\partial_x \varphi)^2 + (\partial_x \vartheta)^2] + \frac{\pi v}{2} \sum_{A=1}^{8} (\mathcal{J}_{\mathrm{R}}^A \mathcal{J}_{\mathrm{R}}^A + \mathcal{J}_{\mathrm{L}}^A \mathcal{J}_{\mathrm{L}}^A) + \frac{4g^*}{3} \sum_{A=1}^{8} \mathcal{J}_{\mathrm{R}}^A \mathcal{J}_{\mathrm{L}}^A, \tag{89}$$

where φ and ϑ are the U(1) free bosonic field and its dual, respectively. There is a "*spin-charge separation*" between the "charge" sector described by the Tomonaga-Luttinger (TL) model [first term in Eq. (89)] and the SU(3) non-Abelian "spin" sector described by the second and third term—the SU(3) Gross-Neveu (GN) model. The appearance of SU(3) is not so surprising; as has been mentioned in Sec. II, we can embed SU(3) symmetry into the triplet (spin-1) sector of the original ladder models. Therefore we may conclude that the SU(3) sector describes the dynamics of the (real) spin degrees of freedom. The physical meaning of the remaining "charge" sector will be clarified below by using a slightly different bosonization scheme.

As is known from the exact solution,[69] for $g^* > 0$ a spectral gap (spin gap) is formed in the spin [SU(3)] sector and the low-energy physics is dictated by massive SU(3) spinons and antispinons which transform like the fundamental representations **3** and $\bar{\mathbf{3}}$, respectively. The low-energy field theory (89) displays, nevertheless, a $c=1$ quantum criticality due to the decoupled charge degrees of freedom which are described by a massless free boson field (TL model) at the free-fermion point. Therefore the quantum phase transition between the staggered dimerized phase and the scalar chirality phase generically continuous and belongs to the $c=1$ universality class.

To clarify the role of the charge U(1) (φ) and the spin SU(3) sector, we apply the Abelian bosonization to the Hamiltonian (88). As in Appendix B, three bosonic fields are introduced to bosonize the three Dirac fermions as follows:

$$\Psi_{a,\mathrm{R}} = \frac{\tilde{\kappa}_a}{\sqrt{2\pi a_0}} \exp(i\sqrt{4\pi}\phi_{a,\mathrm{R}}),$$

$$\Psi_{a,\mathrm{L}} = \frac{\tilde{\kappa}_a}{\sqrt{2\pi a_0}} \exp(-i\sqrt{4\pi}\phi_{a,\mathrm{L}}). \tag{90}$$

Since we have adopted the left-right asymmetric definition in the above equations to bosonize our three Dirac fermions, the spin-chirality rotation simply reads [see Eq. (84a)]

$$\phi_{a,\mathrm{L/R}} \mapsto \phi_{a,\mathrm{L/R}} + \frac{\theta}{4\sqrt{\pi}} \quad (a=1,2,3). \tag{91}$$

The next step is to switch to a new basis where the U(1) and the SU(3) degrees of freedom are separated from each other:

$$\varphi = \frac{1}{\sqrt{3}}(\phi_1 + \phi_2 + \phi_3),$$

$$\varphi_{\mathrm{s}} = \frac{1}{\sqrt{2}}(\phi_1 - \phi_2),$$

$$\varphi_{\mathrm{f}} = \frac{1}{\sqrt{6}}(\phi_1 + \phi_2 - 2\phi_3). \tag{92}$$

The two bosonic fields φ_{s} and φ_{f} are compactified fields with special radii $R_{\mathrm{s,f}}$ so as to capture the underlying SU(3) symmetry of the problem:

$$\varphi_{\mathrm{s,f}} \sim \varphi_{\mathrm{s,f}} + 2\pi R_{\mathrm{s,f}}, \tag{93}$$

with $R_{\mathrm{s}}=1/\sqrt{2\pi}$ and $R_{\mathrm{f}}=\sqrt{3/2\pi}$. The spin-chirality transformation $\mathcal{U}(\theta)$ affects only the charge sector:

$$\varphi \mapsto \varphi + \sqrt{\frac{3}{4\pi}}\theta, \quad \vartheta \mapsto \vartheta, \tag{94a}$$

while the dual spin chirality $\tilde{\mathcal{U}}(\tilde{\theta})$ changes only the dual field ϑ:

$$\varphi \mapsto \varphi, \quad \vartheta \mapsto \vartheta + \sqrt{\frac{3}{4\pi}}\tilde{\theta}. \tag{94b}$$

Now the physical meaning of the charge fields φ and ϑ is clear; the field φ (respectively ϑ) describes the angular (or, phase) fluctuation of the doublet $(\mathcal{O}_{\mathrm{SD}}^{\pi}, \mathcal{O}_{\mathrm{SC}}^{\pi})$ [respectively $(\mathcal{O}_{\mathrm{Q}}^{\pi}, \mathcal{O}_{\mathrm{RQ}}^{\pi})$]. In the bosonization approach, two fields φ and ϑ are conjugate (or, dual) to each other. Therefore two doublets $(\mathcal{O}_{\mathrm{SD}}^{\pi}, \mathcal{O}_{\mathrm{SC}}^{\pi})$ and $(\mathcal{O}_{\mathrm{Q}}^{\pi}, \mathcal{O}_{\mathrm{RQ}}^{\pi})$ are mutually dual objects which are analogous to the Cooper pairs and the density-wave operators in strongly correlated electron systems.

Using this Abelian bosonization, it is then possible to write down the SU(3) current-current interaction in Eq. (88) in terms of the bosonic fields:

$$\begin{aligned}\sum_A \mathcal{J}_R^A \mathcal{J}_L^A &= \frac{1}{2\pi}(\partial_x \varphi_{\mathrm{s,R}} \partial_x \varphi_{\mathrm{s,L}} + \partial_x \varphi_{\mathrm{f,R}} \partial_x \varphi_{\mathrm{f,L}}) \\ &\quad - \frac{1}{2\pi^2 a_0^2} \cos(\sqrt{2\pi}\varphi_{\mathrm{s}}) \cos(\sqrt{6\pi}\varphi_{\mathrm{f}}) \\ &\quad - \frac{1}{4\pi^2 a_0^2} \cos(\sqrt{8\pi}\varphi_{\mathrm{s}}) \\ &= \frac{1}{2\pi}(\partial_x \varphi_{\mathrm{s,R}} \partial_x \varphi_{\mathrm{s,L}} + \partial_x \varphi_{\mathrm{f,R}} \partial_x \varphi_{\mathrm{f,L}}) \\ &\quad - \frac{1}{8}[(\mathcal{O}_{\mathrm{SD}}^{\pi})^2 + (\mathcal{O}_{\mathrm{SC}}^{\pi})^2].\end{aligned} \tag{95}$$

A straightforward semiclassical analysis of the potential part of Eq. (95) together with the identification (93) show that the ground state in the SU(3) spin sector is threefold degenerate with expectation values

$$\langle\varphi_s\rangle = 0, \quad \langle\varphi_f\rangle = 0,$$

$$\langle\varphi_s\rangle = \sqrt{\frac{\pi}{2}}, \quad \langle\varphi_f\rangle = \sqrt{\frac{\pi}{6}},$$

$$\langle\varphi_s\rangle = 0, \quad \langle\varphi_f\rangle = 2\sqrt{\frac{\pi}{6}}. \tag{96}$$

Now the role of the current-current interaction $\Sigma\mathcal{J}_R^A\mathcal{J}_L^A$ is clear. For the semiclassical vacuum configurations $\langle\varphi_s\rangle$ and $\langle\varphi_f\rangle$, the modulus of the order-parameter doublet $(\mathcal{O}_{SD}^\pi, \mathcal{O}_{SC}^\pi)$ is given by

$$\langle(\mathcal{O}_{SD}^\pi)^2 + (\mathcal{O}_{SC}^\pi)^2\rangle = \frac{6}{\pi^2 a_0^2}.$$

That is, fluctuations of the doublet in the *radial* direction is suppressed by the marginally relevant interaction $\Sigma_A\mathcal{J}_R^A\mathcal{J}_L^A$. The only remaining massless fluctuations in the azimuthal direction are described by the TL (or Gaussian) model [the first term in Eq. (89)].

It is then interesting to express the order parameters of Eq. (83) in terms of these bosonic fields:

$$\mathcal{O}_Q^\pi = -\frac{1}{\pi a_0}[2\cos(\sqrt{2\pi}\vartheta_s)\cos(\sqrt{4\pi/3}\vartheta + \sqrt{2\pi/3}\vartheta_f) + \cos(\sqrt{4\pi/3}\vartheta - \sqrt{8\pi/3}\vartheta_f)], \tag{97a}$$

$$\mathcal{O}_{SD}^\pi = \frac{1}{\pi a_0}[2\cos(\sqrt{2\pi}\varphi_s)\cos(\sqrt{4\pi/3}\varphi + \sqrt{2\pi/3}\varphi_f) + \cos(\sqrt{4\pi/3}\varphi - \sqrt{8\pi/3}\varphi_f)], \tag{97b}$$

$$\mathcal{O}_{SC}^\pi = \frac{1}{\pi a_0}[2\cos(\sqrt{2\pi}\varphi_s)\sin(\sqrt{4\pi/3}\varphi + \sqrt{2\pi/3}\varphi_f) + \sin(\sqrt{4\pi/3}\varphi - \sqrt{8\pi/3}\varphi_f)], \tag{97c}$$

$$\mathcal{O}_{RQ}^\pi = -\frac{1}{\pi a_0}[2\cos(\sqrt{2\pi}\vartheta_s)\sin(\sqrt{4\pi/3}\vartheta + \sqrt{2\pi/3}\vartheta_f) + \sin(\sqrt{4\pi/3}\vartheta - \sqrt{8\pi/3}\vartheta_f)]. \tag{97d}$$

It is straightforward to observe that for the type-I self-dual model, i.e., for the model described by the field theory (81a), one has $\langle\mathcal{O}_D^\pi\rangle = \langle\mathcal{O}_{SD}^\pi\rangle = \langle\mathcal{O}_{SC}^\pi\rangle = \langle\mathcal{O}_{RD}^\pi\rangle = 0$ due to the quantum criticality of the charge degrees field in Eq. (89). From Eqs. (97a)–(97d), we also deduce that the first doublet $\mathcal{O}_{SD}^\pi$ and $\mathcal{O}_{SC}^\pi$ has correlation functions decaying as $x^{-2/3}$, i.e., has quasi-long-range coherence, whereas the second one $\mathcal{O}_Q^\pi$ and $\mathcal{O}_{RQ}^\pi$ is exponentially decaying due to strong quantum fluctuations. The situation is completely reversed for the second (type-II) self-dual model.

B. Small deviation from self-dual models

Now let us discuss the effect of small deviation from the self-dual model (81a). The deviation may be incorporated by adding the following symmetry-breaking perturbation to the Hamiltonian (89):

$$\mathcal{H}_{SB} = \epsilon(\mathcal{H}_2 - \mathcal{H}_4) = \epsilon[(\vec{\xi}_R\cdot\vec{\xi}_L - \vec{\chi}_R\cdot\vec{\chi}_L)^2 - (\vec{\xi}_R\cdot\vec{\chi}_L + \vec{\chi}_R\cdot\vec{\xi}_L)^2]. \tag{98}$$

We can also express this symmetry-breaking perturbation in terms of the three Dirac fermions using Eq. (82):

$$\mathcal{H}_{SB} = 2\epsilon\sum_{a,b}[\Psi_{a,R}\Psi_{a,L}^\dagger\Psi_{b,R}\Psi_{b,L}^\dagger + \Psi_{a,R}^\dagger\Psi_{a,L}\Psi_{b,R}^\dagger\Psi_{b,L}]. \tag{99}$$

The effect of this term can be elucidated by means of the Abelian bosonization (90) of the Dirac fermions. Moving to the basis (92), we obtain the bosonized form of the symmetry-breaking term (98):

$$\mathcal{H}_{SB} = \frac{-2\epsilon}{\pi^2 a_0^2}[\cos(\sqrt{2\pi}\varphi_s)\cos(2\sqrt{4\pi/3}\varphi - \sqrt{2\pi/3}\varphi_f) + \cos(2\sqrt{4\pi/3}\varphi + \sqrt{8\pi/3}\varphi_f)]. \tag{100}$$

The stability of the critical line described by the model (89) with respect to the small perturbation (100) with $|\epsilon| \ll 1$ can be investigated by a naive semiclassical analysis: for $|\epsilon| \ll 1$, the bosonic fields $\varphi_{s,f}$ are still frozen to one of the ground-state configurations $(\langle\varphi_s\rangle, \langle\varphi_f\rangle)$ and the non-abelian spin degrees of freedom are still gapfull.

The only difference from the previous case is that here we have couplings between φ and $\varphi_{s,f}$ and they may shift the (semiclassical) values $(\langle\varphi_s\rangle, \langle\varphi_f\rangle)$ from those obtained for $B = D$ [Eq. (96)]. However, as far as the value of ε is small enough, we may expect that $\varphi_{s,f}$ still are pinned at the same values. Assuming that $\varphi_{s,f}$ are locked to, for instance, the first set in Eq. (96), we deduce that the U(1) charge degrees of freedom acquire now an extra term:

$$\mathcal{H}_c \simeq \frac{v}{2}[(\partial_x\varphi)^2 + (\partial_x\vartheta)^2] - \frac{4\epsilon}{\pi^2 a_0^2}\cos(2\sqrt{4\pi/3}\varphi). \tag{101}$$

The low-energy field theory for the charge degrees of freedom takes thus the form of a quantum sine-Gordon model. The interaction has scaling dimension $\Delta = 4/3 < 2$ so that the perturbation is relevant and the charge degrees of freedom acquire a gap. For $\epsilon > 0$ i.e., $g_2 > g_4$ ($\epsilon < 0$, i.e., $g_2 \le g_4$), the U(1) bosonic field is locked on $\langle\varphi\rangle = 0$ or $\langle\varphi\rangle = \sqrt{3\pi}/2$ ($\langle\varphi\rangle = \sqrt{3\pi}/4$ or $\langle\varphi\rangle = 3\sqrt{3\pi}/4$). Using the identifications (97a)–(97d), we then deduce $\langle\mathcal{O}_{SD}^\pi\rangle \neq 0$, and $\langle\mathcal{O}_Q^\pi\rangle = \langle\mathcal{O}_{SC}^\pi\rangle = \langle\mathcal{O}_{RQ}^\pi\rangle = 0$ for $\epsilon > 0$, i.e., for $g_2 > g_4$, so that one enters the staggered dimerized phase SD. In contrast, when $\epsilon < 0$, i.e., $g_2 < g_4$, we have $\langle\mathcal{O}_{SC}^\pi\rangle \neq 0$, and $\langle\mathcal{O}_Q^\pi\rangle = \langle\mathcal{O}_{SD}^\pi\rangle = \langle\mathcal{O}_{RQ}^\pi\rangle = 0$ and the scalar chirality (SC) phase is stabilized by the small symmetry-breaking term. A similar result can be obtained by considering the second ground state $\langle\varphi_s\rangle = \sqrt{\pi/2}$, $\langle\varphi_f\rangle = \sqrt{\pi/6}$. In Fig. 5, we illustrate how the fluctuation of the order parameter doublet is frozen as the symmetry is lowered.

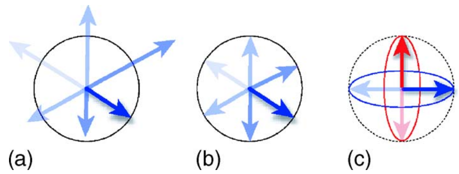

FIG. 5. (Color online) Schematic picture illustrating how the doublet of order parameters is fixed: (a) At the SU(4) point, the doublet fluctuate both in the radial and angular directions. (b) For the self-dual models, the modulus of it is fixed by the interactions in the SU(3) sector, whereas fluctuations in the azimuthal direction are still gapless. (c) The remaining angular fluctuation gets locked by a small deviation from the self-dual models.

Finally, we remark that a similar approach can be applicable also to the competition between the quadrumerized phase (Q) and the RQ phase described by the low-energy field theory (81b). Repeating the same steps as before, we see that the theory obtained by making the replacement $\varphi \leftrightarrow \vartheta$ in Eq. (101) dictates the competition.

C. On the possibility of umklapp

In the previous subsection, we have seen that on the self-dual manifold the enlarged U(1) symmetry forbids the cosine interactions with explicit φ dependence from appearing in the low-energy Hamiltonian. However, we are on a lattice and any interactions compatible with both the U(1) symmetry and the *discrete* lattice symmetries are allowed in principle. Now let us discuss briefly about the possibility of a gap generation by such umklapp interactions.

To this end, it is necessary to know how the discrete one-site translation is realized in terms of our bosons. Apparently, the one-site translation concerns only the *charge* boson φ and its dual ϑ. In order for the expression of the translation [see Eq. (36)]

$$(\vec{\xi}_{\rm L},\vec{\chi}_{\rm L},\vec{\xi}_{\rm R},\vec{\chi}_{\rm R}) \mapsto (\vec{\xi}_{\rm L},\vec{\chi}_{\rm L},-\vec{\xi}_{\rm R},-\vec{\chi}_{\rm R})$$

written in terms of the Majorana sextet to be translated correctly to a bosonic expression, we should take

$$\varphi \mapsto \varphi + \sqrt{\frac{3\pi}{4}}, \quad \vartheta \mapsto \vartheta - \sqrt{\frac{3\pi}{4}}. \tag{102}$$

From Eqs. (97a)–(97d), it is easy to check that it indeed changes the sign of four order parameters $\mathcal{O}_A^{\pi}$ (note our order parameters are staggered ones). Using this, we can obtain a selection rule for the umklapp.

In what follows, we shall look for terms which (i) does not contain a φ field and are (ii) translationally invariant. To this end, we note that the charge bosons φ and ϑ enter into the expressions of (chiral) Dirac fermions $\Psi_{a,{\rm L/R}}$ like

$$\Psi_{a,{\rm R}} \sim e^{i\sqrt{\pi}(\varphi_a-\vartheta_a)} \sim e^{i\sqrt{\pi/3}(\varphi-\vartheta)+\cdots},$$

$$\Psi_{a,{\rm L}} \sim e^{-i\sqrt{\pi}(\varphi_a+\vartheta_a)} \sim e^{-i\sqrt{\pi/3}(\varphi+\vartheta)+\cdots},$$

where the ellipses denote the contributions from $\varphi_{\rm s}$ ($\vartheta_{\rm s}$) and $\varphi_{\rm f}$ ($\vartheta_{\rm f}$). In particular, Cooper pairs are U(1) invariant and will take the following form:

$$\Psi_{a,{\rm R}}\Psi_{b,{\rm L}} \sim e^{-i\sqrt{4\pi/3}\vartheta+\cdots}.$$

Since an operator

$$(\Psi_{a,{\rm R}}\Psi_{b,{\rm L}})^N \sim e^{-i\sqrt{4\pi/3}N\vartheta+\cdots}$$

acquires a phase $N\pi$ after the one-site translation, the integer N should be even. This gives us the first constraint.

However, this is not the whole story. In general, interactions constructed this way will contain not only $\varphi_{\rm s}$ and $\varphi_{\rm f}$ but also their duals $\vartheta_{\rm s,f}$ which are not pinned by the interaction $\Sigma \mathcal{J}_{\rm R}^A \mathcal{J}_{\rm L}^A$; if these interactions include the dual fields they will be suppressed by strong quantum fluctuations (as was pointed out by Schulz[50] in the context of spin chains). For this reason, we have to look for combinations with (i) N =even and (ii) *no* explicit $\vartheta_{\rm s}$ ($\vartheta_{\rm f}$) dependence (these interactions will be generated by perturbations even if they do not exist in the bare Hamiltonian). By a direct enumeration, we checked that no such interactions appear up to $N=4$. For $N=6$, we have several combinations, e.g.,

$$(\Psi_{1,{\rm R}}\Psi_{1,{\rm L}})(\Psi_{2,{\rm R}}\Psi_{2,{\rm L}})(\Psi_{2,{\rm R}}\Psi_{1,{\rm L}})$$

$$(\Psi_{2,{\rm R}}\Psi_{3,{\rm L}})(\Psi_{3,{\rm R}}\Psi_{2,{\rm L}})(\Psi_{3,{\rm R}}\Psi_{3,{\rm L}}) \tag{103}$$

(note that due to the Fermi statistics these expressions should be understood as short-distance expansions). In general, terms containing both $\Psi_{a,{\rm R}}\Psi_{b,{\rm L}}$ and $\Psi_{c,{\rm R}}^{\dagger}\Psi_{d,{\rm L}}^{\dagger}$ are also allowed. However, up to $N=6$ we did not find such combinations satisfying our requirements. (In principle, we can consider interactions made up of $N/2$ pieces of $\Psi_{a,{\rm R}}\Psi_{b,{\rm L}}$ and the same number of $\Psi_{a,{\rm R}}^{\dagger}\Psi_{b,{\rm L}}^{\dagger}$. But they do not contain a ϑ field and are not umklapp.)

Therefore possible umklapp interactions with the lowest dimensions will be of the following form:

$$\cos\left(6\sqrt{\frac{4\pi}{3}}\vartheta + \text{const}\right). \tag{104}$$

Around the SU(4) point, it has scaling dimensions larger than 2 and is thus irrelevant; strong renormalization of the Luttinger K is needed for the umklapp (104) to stabilize a gapped phase on the self-dual manifold.

VI. OTHER QUANTUM CRITICALITIES

Now let us discuss the properties of the low-energy effective Hamiltonian on the asymptotic rays (77a)–(77d) and (78) which correspond to the transitions among the four dominant phases. Combination of the Abelian- and non-Abelian bosonization enables us to obtain nonperturbative solutions to the problem. Although it looks more complicated to treat the last one (78), technically it is slightly simpler and we will begin with the case of four competing orders (78).

A. Case of four competing orders

On the ray (78), the effective action reads as follows:

$$\mathcal{H}_{\text{Q-SD-SC-RQ}} = \mathcal{H}_{\text{SO(6)}} - \frac{g^*}{2}[(\mathcal{O}^{\pi}_{\text{SD}})^2 + (\mathcal{O}^{\pi}_{\text{SC}})^2 + (\mathcal{O}^{\pi}_{\text{Q}})^2 + (\mathcal{O}^{\pi}_{\text{RQ}})^2] = -\frac{iv}{2}\sum_{a=1}^{3}(\xi^a_{\text{R}}\partial_x\xi^a_{\text{R}} - \xi^a_{\text{L}}\partial_x\xi^a_{\text{L}} + \chi^a_{\text{R}}\partial_x\chi^a_{\text{R}} - \chi^a_{\text{L}}\partial_x\chi^a_{\text{L}}) + g^*[(\vec{\xi}_{\text{R}}\cdot\vec{\xi}_{\text{L}})^2 + (\vec{\chi}_{\text{R}}\cdot\vec{\chi}_{\text{L}})^2 + (\vec{\xi}_{\text{R}}\cdot\vec{\chi}_{\text{L}})^2 + (\vec{\chi}_{\text{R}}\cdot\vec{\xi}_{\text{L}})^2], \tag{105}$$

with $g^* > 0$.

As has been mentioned in Sec. IV C, the model (105) describes the competition between the four different orders of our problem: the staggered dimerized (SD), scalar chiral (SC), and the two quadrumerized (i.e., period 4) orders (Q and RQ).

We apply the SU(3)×U(1) bosonization scheme to our Hamiltonian (105). After some algebra, we obtain the following simple Hamiltonian:

$$\mathcal{H}_{\text{Q-SD-SC-RQ}} = \frac{v}{2}[(\partial_x\varphi)^2 + (\partial_x\vartheta)^2] + \frac{v}{2}[(\partial_x\vec{\phi})^2 + (\partial_x\vec{\theta})^2] - \frac{g^*}{\pi^2}\sum_{i=1}^{3}[\cos(\sqrt{8\pi}\vec{\alpha}_i\cdot\vec{\phi}) + \cos(\sqrt{8\pi}\vec{\alpha}_i\cdot\vec{\theta})], \tag{106}$$

where $\vec{\phi}=(\varphi_{\text{s}},\varphi_{\text{f}})$, $\vec{\theta}=(\vartheta_{\text{s}},\vartheta_{\text{f}})$, and $\vec{\alpha}_i$ are the positive roots of the SU(3) algebra: $\vec{\alpha}_1=(1/2,\sqrt{3}/2)$, $\vec{\alpha}_2=(1/2,-\sqrt{3}/2)$, and $\vec{\alpha}_3=(1,0)$. Equation (106) describes a Lie-algebraic generalization[70,71] of the self-dual sine-Gordon models considered in Ref. 72.

From Eq. (106), one immediately observes that Hamiltonian (105) displays U(1) quantum critical behavior due to the noninteracting bosonic field φ. The interacting part of Eq. (105) takes the form of an SU(3) self-dual sine-Gordon model with a marginal interaction. From the self-duality symmetry $\vec{\phi}\leftrightarrow\vec{\theta}$ in the SU(3) part, we may naively expect additional critical degrees of freedom resulting from this symmetry. However, this is not the case. To see this, we first note that the interaction part of Eq. (105) is written compactly as

$$\mathcal{H}_{\text{int}} = -\frac{g^*}{2}(\vec{\xi}_{\text{R}}\times\vec{\xi}_{\text{R}} + \vec{\chi}_{\text{R}}\times\vec{\chi}_{\text{R}})\cdot(\vec{\xi}_{\text{L}}\times\vec{\xi}_{\text{L}} + \vec{\chi}_{\text{L}}\times\vec{\chi}_{\text{L}}) = 2g^*\vec{\mathcal{I}}_{\text{R}}\cdot\vec{\mathcal{I}}_{\text{L}}, \tag{107}$$

where we have introduced the level-2 SO(3) currents $\vec{\mathcal{I}}_{R,L}$ defined by

$$\vec{\mathcal{I}}_{\text{R,L}} \equiv -\frac{i}{2}(\vec{\xi}_{\text{R,L}}\times\vec{\xi}_{\text{R,L}} + \vec{\chi}_{\text{R,L}}\times\vec{\chi}_{\text{R,L}}). \tag{108}$$

It is interesting to note that $\vec{\mathcal{I}}_{\text{L,R}}$ are simply written as

$$\mathcal{I}^a_{\text{R,L}} = \vec{\Psi}^{\dagger}_{\text{R,L}}(\mathcal{S}^a)\vec{\Psi}_{\text{R,L}}, \tag{109}$$

where $(\mathcal{S}^a)_{bc}=-i\varepsilon_{abc}$ $(a=x,y,z)$ is a 3×3 representation of spin operators which form an SO(3) subgroup of SU(3).

By using these SO(3) currents, we can then rewrite the initial Hamiltonian (105), which governs the competition of the four orders, in a current-current form:

$$\mathcal{H}_{\text{Q-SD-SC-RQ}} = \frac{v}{2}[(\partial_x\varphi)^2 + (\partial_x\vartheta)^2] + \frac{\pi v}{3}(\vec{\mathcal{I}}^2_{\text{R}} + \vec{\mathcal{I}}^2_{\text{L}}) + 2g^*\vec{\mathcal{I}}_{\text{R}}\cdot\vec{\mathcal{I}}_{\text{L}}. \tag{110}$$

The second part of this Hamiltonian is that of the level-2 SO(3) WZW model perturbed by an SO(3)-invariant current-current interaction and is integrable.[74] Now the relation to the previous model [Eq. (89)] is clear; if one adds current interactions coming from the remaining five SU(3) currents, the $\vec{\theta}$-dependent part of Eq. (106) is canceled and the form (95) is recovered.

For $g^*>0$, which is relevant to our problem, the second part has a spectral gap and a nontrivial structure of massive spinon. We thus deduce that the initial Hamiltonian (105) exhibits only a U(1) quantum criticality due to the free boson φ. The transition that results from the competition between the four orders is thus of a U(1) Gaussian type.

B. Transitions among dominant phases

The transitions between two dominant phases within the same (spin-chirality) doublet (i.e., SD-SC and Q-RQ) have been already discussed in the previous section in conjunction with the low-energy physics of the self-dual models. Here we concentrate on the transitions between different doublets (e.g., Q-SD).

Let us consider, for instance, the Q-SD transition [Eq. (77a)] and begin with rewriting the interaction term $(\mathcal{O}^{\pi}_{\text{Q}})^2 + (\mathcal{O}^{\pi}_{\text{SD}})^2$. Plugging the bosonized expressions (97a) and (97b) into the above, we obtain

$$(\mathcal{O}^{\pi}_{\text{Q}})^2 + (\mathcal{O}^{\pi}_{\text{SD}})^2 = -\frac{1}{\pi}[(\partial_x\vec{\varphi})^2 + (\partial_x\vec{\vartheta})^2] + \frac{1}{\pi^2 a_0^2}\times\sum_{r=1}^{6}[\cos(\sqrt{8\pi}\vec{\beta}_r\cdot\vec{\varphi}) + \cos(\sqrt{8\pi}\vec{\beta}_r\cdot\vec{\vartheta})], \tag{111}$$

where $\vec{\varphi}=(\varphi,\varphi_{\text{s}},\varphi_f)$, $\vec{\vartheta}=(\vartheta,\vartheta_{\text{s}},\vartheta_f)$, and the last summation here is taken over all six positive roots of SU(4): $\vec{\beta}_r = (1/2,\pm\sqrt{3}/2,0)$, $(1/2,\pm 1/(2\sqrt{3}),\mp\sqrt{2/3})$, $(0,1/\sqrt{3},\sqrt{2/3})$, and (1, 0, 0) normalized so that $|\beta_r|=1$. Therefore the effective Hamiltonian corresponding to the ray (77a) reads

$$\mathcal{H}_{\text{Q-SD}} = \frac{v'}{2}[(\partial_x\vec{\varphi})^2 + (\partial_x\vec{\vartheta})^2] - \frac{g^*}{\pi^2 a_0^2}\sum_{r=1}^{6}[\cos(\sqrt{8\pi}\vec{\beta}_r\cdot\vec{\varphi}) + \cos(\sqrt{8\pi}\vec{\beta}_r\cdot\vec{\vartheta})], \tag{112}$$

where we have rescaled the velocity v so that $(\partial_x\vec{\varphi})^2$ terms coming from interactions may be absorbed. If we replace SU(3) ($\vec{\alpha}$) by SU(4) ($\vec{\beta}$) and $\vec{\phi}$ by $\vec{\varphi}$ in the previous section, this is nothing but the second part of Eq. (106). Therefore the argument here goes similarly to that in the previous section; the effective Hamiltonian (112) can be written here in terms of SO(4) currents $\mathcal{J}^{ij}_{\mathrm{R,L}}$ $(1 \leq i < j \leq 4)$ as[73]

$$\mathcal{H}_{\text{Q-SD}} = \frac{\pi v'}{4} \sum_{i<j} [(\mathcal{J}^{ij}_{\mathrm{R}})^2 + (\mathcal{J}^{ij}_{\mathrm{L}})^2] + 2g^* \sum_{i<j} \mathcal{J}^{ij}_{\mathrm{R}} \mathcal{J}^{ij}_{\mathrm{L}}. \tag{113}$$

That is, we have obtained the level-2 SO(4) WZW model (with central charge $c=3$) perturbed by an SO(4)-invariant marginal current-current interaction which is an integrable field theory.[74] In contrast to the previous case, we have no critical degrees of freedom here and we expect that the transition between the Q phase and the SD phase is of first order. A similar result holds for the RQ-SC transition as well.

VII. GLOBAL STRUCTURE OF THE PHASE DIAGRAM

In this section, we shall investigate the main effects of the interactions ($\mathcal{H}_{5,6}$) in Eq. (6) that we have so far neglected in our field-theoretical approach. In addition, we shall also use variational and strong-coupling analyses to figure out the global phase diagram of model (6).

A. Effects of $\mathcal{H}_5$

So far, we have neglected the $\mathcal{T}$-breaking three-spin interaction $\mathcal{H}_5$ of Eq. (4e). In this section, we shall give a hand-waving argument for its main effect. In Sec. V, it has been argued that the $\mathcal{T}$-breaking and the appearance of the SD and SC phases can be understood as a pinning of the vector doublet $(\mathcal{O}^{\pi}_{\mathrm{SD}}, \mathcal{O}^{\pi}_{\mathrm{SC}})$ in the spin-chirality plane (see Fig. 5). Now the role of $\mathcal{H}_5$ can be discussed in a similar manner by means of a semiclassical argument applied to the bosonized Hamiltonian.

According to Eq. (56b), the $\mathcal{T}$-breaking interaction $\mathcal{H}_5$ changes the form of the effective potential as

$$-\frac{g^*}{6}[(\mathcal{O}^{\pi}_{\mathrm{SD}})^2 + (\mathcal{O}^{\pi}_{\mathrm{SC}})^2] \to -\frac{g^*}{6}[(\mathcal{O}^{\pi}_{\mathrm{SD}})^2 + (\mathcal{O}^{\pi}_{\mathrm{SC}})^2] - g_5 \mathcal{O}^{\pi}_{\mathrm{SD}} \mathcal{O}^{\pi}_{\mathrm{SC}}. \tag{114}$$

In the presence of $\mathcal{H}_5$, the spin-chirality plane is no longer isotropic and the principal axes of the ellipsoid (see Fig. 5) are tilted by $\pi/4$ (the sign of g_5 determines the direction of the longer axis.). Then, we may expect that the two order parameters $\mathcal{O}^{\pi}_{\mathrm{SD}}$ and $\mathcal{O}^{\pi}_{\mathrm{SC}}$ simultaneously take nonzero values.

To make the above argument more quantitative, we add the g_5 term to the interaction (95). The interaction takes its minima when $\langle \varphi_{\mathrm{s}} \rangle$ and $\langle \varphi_{\mathrm{f}} \rangle$ are given by Eq. (96). Then the expectation values of the φ field are determined by finding the minima of trigonometric functions. For $\langle \varphi_{\mathrm{s}} \rangle = \langle \varphi_{\mathrm{f}} \rangle = 0$, $\langle \varphi \rangle$ is determined by minimizing $-9g_5/2 \sin(\sqrt{16\pi/3}\varphi)$. The calculation goes similarly for the other cases as well.

This analysis tells us that the doublet $(\mathcal{O}^{\pi}_{\mathrm{SD}}, \mathcal{O}^{\pi}_{\mathrm{SC}})$ takes one of the following values when $\mathcal{T}$-breaking interaction g_5 is present:

$$(\mathcal{O}^{\pi}_{\mathrm{SD}}, \mathcal{O}^{\pi}_{\mathrm{SC}}) = \begin{cases} \left(\pm \frac{3}{\sqrt{2}\pi a_0}, \pm \frac{3}{\sqrt{2}\pi a_0} \right) & \text{for } g_5 > 0 \\ \left(\pm \frac{3}{\sqrt{2}\pi a_0}, \mp \frac{3}{\sqrt{2}\pi a_0} \right) & \text{for } g_5 < 0. \end{cases} \tag{115}$$

The above argument can be easily generalized to the case of non-self-dual models.

B. Effects of $\mathcal{H}_6$

Now let us discuss the effect of the $\mathcal{H}_6$ interaction [Eq. (4f)] which has been neglected in the preceding analysis. As has been mentioned in Sec. II, our model (spin) Hamiltonian can be rewritten in terms of spin-1 (hardcore) boson $b_{r,a}$ ($a = x, y, z$) as

$$\begin{aligned} \mathcal{H} &= A\mathcal{H}_1 + B\mathcal{H}_2 + C\mathcal{H}_3 + D\mathcal{H}_4 + E\mathcal{H}_5 + F\mathcal{H}_6 + G\mathcal{H}_7 \\ &= (B+D)\sum_{r,a} (b^{\dagger}_{r,a} b_{r+1,a} + b^{\dagger}_{r+1,a} b_{r,a}) + (B-D) \\ &\quad \times \sum_{r,a} (b^{\dagger}_{r,a} b^{\dagger}_{r+1,a} + b_{r+1,a} b_{r,a}) \\ &\quad + E\sum_{r} \varepsilon_{abc} [b^{\dagger}_{r,a} b^{\dagger}_{r+1,b} b_{r,c} + b^{\dagger}_{r,c} b_{r,a} b_{r+1,b} + (r \leftrightarrow r+1)] \\ &\quad + \sum_{r} [(A+C)\mathbf{T}_r \cdot \mathbf{T}_{r+1} + 2C(\mathbf{T}_r \cdot \mathbf{T}_{r+1})^2] \\ &\quad + G\sum_{r} n^{\mathrm{B}}_r n^{\mathrm{B}}_{r+1} + \left(-4C + F - \frac{3}{2}G \right) \sum_{r} n^{\mathrm{B}}_r, \end{aligned} \tag{116}$$

where the projection onto occupied states is implied for the fourth term (the triplet-triplet interaction). From this, it can be easily read off that the coupling F of $\mathcal{H}_6$ affects the chemical potential of the hardcore boson. Therefore for sufficiently large values of $|F|$, the system becomes either a carrierless insulator (i.e., $n^{\mathrm{B}}_r = 0$ for all sites r) or a fully occupied state (i.e., $n^{\mathrm{B}}_r = 1$ for all r). Apparently, the ground state of the carrierless insulator is trivial and is, in terms of the original spins, given by a tensor product of local (rung) singlets. The fate of the spin wave function in the latter case depends strongly on the two couplings A and C which dictate the spin-spin interaction between hardcore bosons. Here, we will mainly focus on the case $A + C > 2|C|$ which corresponds to the Haldane phase.[75]

When $|F|$ is decreased, we have quantum phase transitions to a conducting state. Let us first consider a transition from the carrierless insulator. Then, the transition is equivalent to the superfluid (SF)-onset transition of spin-1 bosons. For the moment, let us neglect the interactions among different species of particles (including the hardcore repulsion). As is well known,[76] the bosonic two-body interactions are relevant at the $z=2$ SF-onset transition and the system flows toward the strong-coupling fixed point. In the presence of U(1)-breaking anisotropy ($B \neq D$), the particle number is no longer conserved and the fixed point is replaced by that of the $z=1$ critical 2D Ising model.[77,78] Since we have three copies of such Ising models, we may expect that the fixed point here

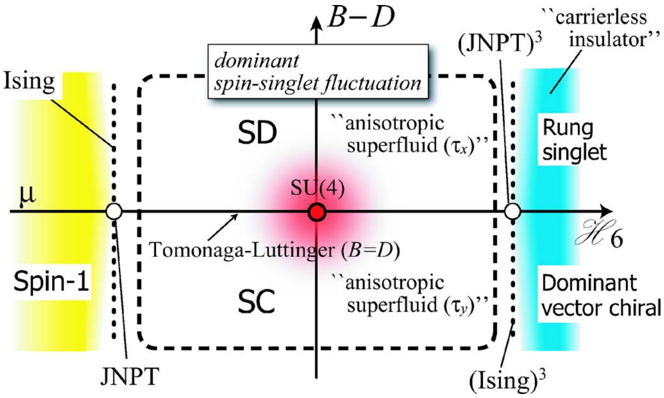

FIG. 6. (Color online) A schematic picture illustrating the effects of $\mathcal{H}_6$ and the anisotropy $\varepsilon = B - D$ in the spin-chirality plane on the global structure of the phase diagram of our model. According to the sign of the anisotropy, we have two different orderings; an anisotropic SF in the x direction ($\mathcal{T}$ even) for $\varepsilon > 0$ and one in the y direction ($\mathcal{T}$ odd) for $\varepsilon < 0$. The duality symmetry $\mathcal{D}$ interchanges the region with $\varepsilon > 0$ and that with $\varepsilon < 0$. On both sides of the region enclosed by a dashed line (*dominant singlet fluctuations*), we find conventional phases (spin-1 or rung singlet), which can be interpreted as insulating ones.

is described by the level-2 SU(2) WZW model which is equivalent to three massless Majorana fermions with $c = 3/2$. Now let us estimate the impact of the neglected boson-boson interactions. Since they are given by O(3)-invariant (i.e., spin-symmetric) products of four Fermi fields, the only possible one should be the marginal interaction of the form $\mathbf{J}_R \cdot \mathbf{J}_L$ [$\mathbf{J}_{L,R}$ are the level-2 SU(2) currents of the WZW model]. Therefore we may conclude that the SF-onset transition from the rung-singlet phase (*carrierless insulator*) to the staggered dimer-(SD) or the scalar-chirality (SC) phase belongs to the level-2 SU(2) WZW criticality; the SD or SC order appears as a consequence of the SF ordering in the presence of the U(1)-breaking anisotropy. This is consistent with the results of weak-coupling approach[25,26,28] and numerical analysis.[27] Since the number of the bosonic particles is conserved, the transition at $B = D$ (self-dual) is exceptional and belongs essentially to the Japaridze-Nersesyan-Pokrovsky-Talapov (JNPT) universality class[79] (note that we have three copies of the JNPT criticality here).

A similar argument applies to the transition from the fully occupied state as well. The elementary particle here is a rung singlet moving in the sea of rung triplets (note that we have assumed that we are in the spin-gapped phase and the magnon excitations occur at the energy scale of the "Haldane" gap). Then the dynamics of these excited singlets may be modeled by the interacting spinless fermion. The sign of the fermion-fermion interaction will be positive (negative) when G is much larger (smaller) than A and C. When A and C are not so large and the anisotropy is absent ($B = D$), the F-driven transition is of the JNPT universality class again. The xy anisotropy ($B \neq D$) alters the transition to that of the Ising type. We show a schematic phase diagram illustrating the effects of $\mathcal{H}_5$ and $\mathcal{H}_6$ in Fig. 6.

C. *XYZ* analogy and pseudospin description

In order to demonstrate the relevance of the pseudospin picture (see Sec. II D 1), we present two approaches which are complementary to that developed in the previous sections. In both approaches, the key is how to identify the pseudospin degrees of freedom on the lattice.

1. Variational approach

Let us begin with the self-dual models. In the above, we have seen how the U(1) Gaussian model emerges as the low-energy effective-field theory after the spin sector is gapped. The spin and the (pseudo)charge decouple from each other at low energies and the physics of the spin sector may be described by the following bilinear-biquadratic interaction (see Sec. II and Appendix A 2 for more details)

$$A\mathcal{H}_1 + C\mathcal{H}_3 \sim \sum [(A+C)\mathbf{T}_r \cdot \mathbf{T}_{r+1} + 2C(\mathbf{T}_r \cdot \mathbf{T}_{r+1})^2].$$

For this reason, we may expect that the Haldane-gap physics *a la* Affleck, Kennedy, Lieb, and Tasaki[75] dominates in a reasonably large region of the phase diagram. Of course, we have the additional pseudocharge degrees of freedom (i.e., motion of triplet rungs in the rung-singlet background) here and the stability of the Haldane state is not so obvious. However, from a simple argument[80] we know that the valence-bond-solid state proposed in Ref. 75 is stable against the motion of singlet rungs [or, holes in the spin-1 t-J model given by Eqs. (24)–(30) or Eq. (116)]. In fact, numerical simulations carried out for the spin-1 bosonic t-J model[81] (the set of parameters $B = D = t/2$, $A = J$, $C = 0$ was used there) show the existence of a finite spin gap and the Luttinger-liquid behavior in the pseudocharge sector, which is consistent with our conclusion in Sec. V A.

Now that we know that the spin sector is *generically* gapped and decoupled from the pseudocharge sector, the next question would be what kind of degrees of freedom determines the global structure of the phase diagram. From the bosonization analysis presented above, a natural guess would be the pseudospin $\mathcal{S} = 1/2$ (or charge in the t-J language). As was described in Sec. II D, $\mathcal{H}_{2,4}$ correspond to the pseudospin-flipping processes and $\mathcal{H}_6$ and $\mathcal{H}_7$ to magnetic field and the $\mathcal{S}^z\mathcal{S}^z$ interaction, respectively. The remaining parts concern dynamics in the (true) spin sector, which is separated from the low-energy sector by a finite spin gap.

To develop a variational theory for this kind of spin liquid, we first construct a *coherent state* of the pseudospin $\mathcal{S}$ by combining a local singlet $|s\rangle_r$ and a triplet $|\mathbf{t}\rangle_r$:

$$|\mathbf{\Omega}\rangle' = \bigotimes_{r \in \text{rung}} \left[e^{-i[\phi(r)/2]} \cos\frac{\theta(r)}{2} \left(e^{i\phi(r)} \tan\frac{\theta(r)}{2} |\mathrm{s}\rangle_r + |\mathbf{t}\rangle_r \right) \right],$$

where $\phi(r)$ and $\theta(r)$ are, respectively, azimuthal and polar angles of the spin vector $\mathbf{\Omega}_r$ at rung r. Apparently, this is unsatisfactory because we still have local spin degrees of freedom represented by $|\mathbf{t}\rangle_r$. To kill the spin degrees of freedom and construct a spin-gapped wave function out of the above coherent states, the most natural way would be to replace triplet states $|\mathbf{t}\rangle_r$ on rth rung by the following 2×2 matrix:[82,83]

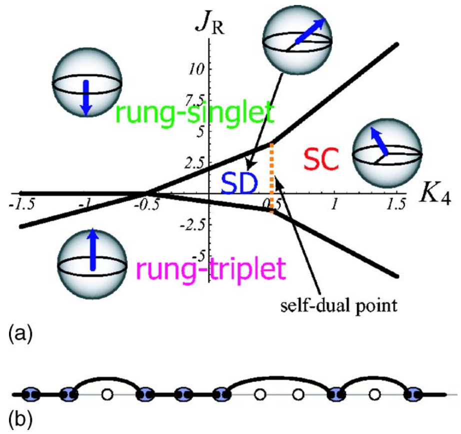

FIG. 7. (Color online) (a) Variational phase diagram obtained by the ansatz (117). Shown is the pseudospin direction (ϕ, θ) for each phase: conventional rung-singlet and rung-triplet (Haldane or VBS) phases are *spin-polarized states* while the SD and SC phases may be viewed as *canted states* with different XY projections. (b) Typical spin-singlet configuration contained in our variational wave function (117). Ovals and open circles respectively denote triplet (occupied sites) and singlet (unoccupied sites) rungs. The spin-1 bosons totally form a spin-singlet valence-bond state. Due to motion and pair creation or annihilation of triplet bosons, the wave function (117) describes a strongly fluctuating state.

$$\mathbf{g}_r \equiv \frac{1}{\sqrt{3}} \begin{pmatrix} |t_0\rangle_r & -\sqrt{2}|t_1\rangle_r \\ +\sqrt{2}|t_{-1}\rangle_r & -|t_0\rangle_r \end{pmatrix}$$

($|t_a\rangle$ is the rung-triplet state with $T^z = a$) and use the modified ansatz

$$|\mathbf{\Omega}\rangle = \underset{r\in \text{rung}}{\otimes} \left(\sqrt{3} e^{i\phi(r)} \tan \frac{\theta(r)}{2} |\text{s}\rangle_r \mathbf{1} + \mathbf{g}_r \right). \qquad (117)$$

Note that this is essentially the same as the building block of the matrix-product ground state adopted in Ref. 46. Taking the trace of the above matrix product, we can obtain the desired (unnormalized) spin-singlet wave function with a finite correlation length. This wave function may be thought of as the valence-bond-solid (VBS) state[75] randomly "diluted" by vacancies [see Fig. 7(b)].

To illustrate the usefulness of our pseudospin picture, we map out the variational phase diagram of the two-leg spin ladder with four-spin cyclic exchange $\mathcal{H}_{\text{ladder+4-spin}}$ [see Eq. (20)]. As a variational ansatz, we choose

$$(\phi(r), \theta(r)) = \begin{cases} (\phi_1, \theta_1) & \text{for } r = \text{even} \\ (\phi_2, \theta_2) & \text{for } r = \text{odd}. \end{cases} \qquad (118)$$

Thanks to the special form of $|\Omega\rangle$, we can reduce the computation of the ground-state expectation values $\langle \Omega | \mathcal{H}_{\text{ladder+4-spin}} | \Omega \rangle$ to that of 4×4 matrices[82,83] and we obtain the variational energy $E_{\text{var}}(\phi_1, \theta_1, \phi_2, \theta_2)$ to minimize. Since the explicit form of $E_{\text{var}}(\phi_1, \theta_1, \phi_2, \theta_2)$ is unimportant, we only show the resulting phase diagram in Fig. 7. It should be compared with the phase diagram of the same model obtained by large-scale numerical simulations[29] (density-matrix renormalization group). The path searched in Ref. 29 corresponds to the line $J_R = 1$ in Fig. 7. Three dominant phases (rung-singlet, SD, and SC) found there appear in our variational phase diagram as well, while the phase denoted by "dominant vector chirality,"[29,30] whose nature is not clear currently, is missing in ours.

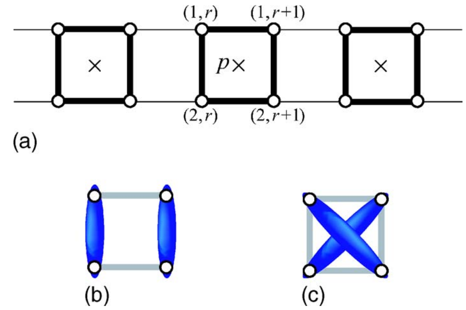

FIG. 8. (Color online) (a) A two-leg ladder as a weakly coupled plaquettes. Strongly coupled plaquettes, on which pseudospin 1/2's and coarse-grained spin-1's are defined, are shown by thick lines. All interactions (both two spin and four spin) connecting these plaquettes are multiplied by $\lambda(\ll 1)$. (b), (c) Two triplet states which constitute a (nearly) degenerate ground-state manifold in the vicinity of the SU(4) point. Two spin-1/2's contained in each oval form a spin triplet. In both (b) and (c), two triplets (ovals) are antisymmetrized to form a total spin triplet on a plaquette.

Similarly, the ferromagnetic phase is beyond the scope of our simple variational calculation since our variational wave function (117) contains only spin-singlet phases. In our picture, both rung-triplet- and rung-singlet phases correspond to *(pseudo)spin-polarized states* as is shown in Fig. 7. The spin-chirality transformation $\mathcal{U}(\theta)$ rotates the pseudospin vector along the z axis. When the external magnetic field $(\mathcal{H}_6)$ is not very strong, *spin-canted states* (SD and SC) are favored by the xy coupling $\mathcal{H}_{2,4}$; in the SD (SC) phase, the xy projection of the pseudospin lies in the $x(y)$ direction.

2. Strong-coupling approach

Another way to see the role of the pseudospin degrees of freedom would be a strong-coupling expansion. A natural starting point might be isolated rungs (as in the usual ladder systems). Unfortunately, however, all the phases that we have found in the field-theory analysis break the periodicity of the original Hamiltonian and the limit of isolated rungs is not quite helpful (the phases accessible from the limit would be more or less trivial). Instead, we divide the whole lattice into plaquettes [see Fig. 8(a)] and introduce a new coupling λ which controls the coupling between neighboring plaquettes. On these plaquettes, a kind of *coarse-grained* degrees of freedom will be defined by which the low-energy sector of our ladder system may be described.

To explore the vicinity of the SU(4) point, we adopt the following Hamiltonian:

$$\mathcal{H} = \sum_{r=\text{rung}} J_r \{ h_1(r) + (1+\delta_2) h_2(r) + (1+\delta_3) h_3(r) + (1+\delta_4) h_4(r) \} + \delta_6 \sum_r \mathbf{S}_{1,r} \cdot \mathbf{S}_{2,r}, \qquad (119)$$

where the couplings J_r alternate like

$$J_r = 1 \text{ (for } r \text{ even)}, \quad J_r = \lambda(\ll 1) \text{ (for } r \text{ odd)} \tag{120}$$

and $h_i(r)$ denote local Hamiltonians obtained by restricting $\mathcal{H}_i$ to two rungs r and $r+1$.

Let us begin with the case of $\lambda=0$, i.e., the limit of isolated plaquettes. On a single plaquette, we have 16 states which can be decomposed into the following multiplets:

$$(\mathbf{0})^2 \oplus (\mathbf{1})^3 \oplus (\mathbf{2}). \tag{121}$$

Around the SU(4) point, two of the three triplets [see Figs. 8(b) and 8(c)] form nearly degenerate ground states [exactly at the SU(4) point, they are completely degenerate]. The explicit forms of these states are given as

$$|\boxtimes\rangle = \frac{1}{\sqrt{2}}(|\downarrow\ \Vert\rangle - |\Vert\ \downarrow\rangle) = \frac{1}{\sqrt{2}}(|\overset{\rightarrow}{=}\rangle - |\underset{\rightarrow}{=}\rangle) \tag{122a}$$

$$|[\,]\rangle = \frac{1}{\sqrt{2}}(|\Vert\ \Vert\rangle - |\Vert\ \Vert\rangle) = \frac{1}{\sqrt{2}}(|\overset{\rightarrow}{=}\rangle + |\underset{\rightarrow}{=}\rangle), \tag{122b}$$

where arrows and double lines in the above respectively denote spin singlets and triplets (ovals in Fig. 8).

It should be remarked that we shall construct an effective theory in terms of new degrees of freedom (spin-1 and species of triplets) defined on these plaquettes. In this plaquette picture, our ladder is already dimerized and the two staggered phases (SD and SC) should be understood as *uniform* ones. Correspondingly, we shall introduce order parameters defined on each plaquette:

$$\mathcal{O}_{\rm SD}(p) = \mathbf{S}_{1,r}\cdot\mathbf{S}_{1,r+1} - \mathbf{S}_{2,r}\cdot\mathbf{S}_{2,r+1},$$

$$\mathcal{O}_{\rm SC}(p) = (\mathbf{S}_{1,r}\times\mathbf{S}_{2,r})\cdot(\mathbf{S}_{1,r+1}+\mathbf{S}_{2,r+1}) + (r\leftrightarrow r+1), \tag{123}$$

where p labels the plaquette formed by four points $(1,r)$, $(2,r)$, $(2,r+1)$, and $(1,r+1)$ (see Fig. 8).

The point here is that the eigenstates of $\mathcal{O}_{\rm SD}$ and $\mathcal{O}_{\rm SC}$ can be constructed out of the two triplets shown in Fig. 8:

$$\begin{aligned} |\pm 1\rangle_{\rm SD} &= \frac{1}{\sqrt{2}}(|[\,]\rangle \mp |\boxtimes\rangle) \\ |\pm 1\rangle_{\rm SC} &= \frac{1}{\sqrt{2}}(|[\,]\rangle \mp i|\boxtimes\rangle), \end{aligned} \tag{124}$$

where ± 1 denote the eigenvalues of $\mathcal{O}_{\rm SD,SC}(p)$.

The next step is to identify the states having the pseudospin ($\boldsymbol{\tau}$) up and down. To this end, we consider the coherent state of a spin-1/2:

$$|\boldsymbol{\Omega}\rangle = \cos\frac{\theta}{2}|\uparrow\rangle + e^{+i\phi}\sin\frac{\theta}{2}|\downarrow\rangle.$$

If we identify the following eigenstates of τ^x and τ^y

$$|\tau^x = \pm 1/2\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle \pm |\downarrow\rangle),$$

$$|\tau^y = \pm 1/2\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle \pm i|\downarrow\rangle) \tag{125}$$

with the eigenstates (124) of the order parameters ($\mathcal{O}_{\rm SD}$ and $\mathcal{O}_{\rm SC}$), it is suggested that we should take

$$|\uparrow\rangle \leftrightarrow |[\,]\rangle, \quad |\downarrow\rangle \leftrightarrow -|\boxtimes\rangle \tag{126a}$$

and

$$\mathcal{O}_{\rm SD}(p) = 2\tau_p^x, \quad \mathcal{O}_{\rm SC}(p) = 2\tau_p^y$$

$$\tau_p^z = \mathbf{S}_{1,r}\cdot\mathbf{S}_{2,r} + \mathbf{S}_{1,r+1}\cdot\mathbf{S}_{2,r+1}. \tag{126b}$$

Obviously, τ_p^z is a generalization of $\mathcal{S}^z$ discussed in Sec. II D to a plaquette system.

Now that we have identified the pseudospin degrees of freedom, it is straightforward to carry out a perturbation expansion for (nearly) degenerate ground-state manifold made up of $|\boxtimes\rangle$ and $|[\,]\rangle$. Within this manifold, operators for spin, p-nematic, and n-nematic read

$$\mathbf{S}_{1,r}+\mathbf{S}_{2,r} = \mathbf{S}_{1,r+1}+\mathbf{S}_{2,r+1} = \frac{1}{2}\mathbf{1}\otimes\mathbf{T}_p, \tag{127a}$$

$$\mathbf{S}_{1,r}-\mathbf{S}_{2,r} = \mathbf{S}_{1,r+1}-\mathbf{S}_{2,r+1} = \tau^x\otimes\mathbf{T}_p, \tag{127b}$$

$$2(\mathbf{S}_{1,r}\times\mathbf{S}_{2,r}) = 2(\mathbf{S}_{1,r+1}\times\mathbf{S}_{2,r+1}) = \tau^y\otimes\mathbf{T}_p, \tag{127c}$$

$$\begin{aligned} Q_r^{\alpha\beta} &\equiv (S_{1,r}^{\alpha}S_{2,r}^{\beta} + S_{1,r}^{\beta}S_{2,r}^{\alpha}) \\ &= -\tau^z\otimes\left\{\frac{1}{2}(\mathrm{T}_p^{\alpha}\mathrm{T}_p^{\beta} + \mathrm{T}_p^{\beta}\mathrm{T}_p^{\alpha}) - \delta^{\alpha\beta}\right\} = Q_{r+1}^{\alpha\beta}, \end{aligned} \tag{127d}$$

where $\mathbf{T}_p$ denotes a spin-1 operator *on the plaquette* p. Plugging these equations into the original Hamiltonian, we obtain the (first-order) effective Hamiltonian which describes the physics around the SU(4) point:

$$\begin{aligned} \mathcal{H}_{\rm eff} &= \lambda\sum_{p=\rm plaq}\left\{(1+\delta_2)\tau_p^x\tau_{p+1}^x + (1+\delta_4)\tau_p^y\tau_{p+1}^y + \frac{1}{4}\right\}\mathbf{T}_p\cdot\mathbf{T}_{p+1} \\ &\quad + \lambda(1+\delta_3)\sum_p(\tau_p^z\tau_{p+1}^z)\{\mathbf{T}_p\cdot\mathbf{T}_{p+1} + 2(\mathbf{T}_p\cdot\mathbf{T}_{p+1})^2 - 2\} \\ &\quad + \sum_p(\delta_2-\delta_3+\delta_4+\delta_6)\tau_p^z = \lambda\sum_{p=\rm plaq}\left\{(1+\delta_2)\tau_p^x\tau_{p+1}^x\right. \\ &\quad \left. + (1+\delta_4)\tau_p^y\tau_{p+1}^y + \frac{1}{4}\right\}\mathbf{T}_p\cdot\mathbf{T}_{p+1} + 2\lambda(1+\delta_3)\sum_p(\tau_p^z\tau_{p+1}^z) \\ &\quad \times\left\{\sum_{\alpha,\beta}\tilde{Q}_p^{\alpha\beta}\tilde{Q}_{p+1}^{\alpha\beta} - 1\right\} + \sum_p(\delta_2-\delta_3+\delta_4+\delta_6)\tau_p^z, \end{aligned} \tag{128}$$

where we have introduced the coarse-grained n-type nematic operators $\tilde{Q}_p^{\alpha\beta} \equiv (T_p^{\alpha}T_p^{\beta} + T_p^{\beta}T_p^{\alpha})/2$ on plaquettes. This is similar to the well-known Kugel-Khomskii effective Hamiltonian[43] for the orbital-degenerate systems; here the role of two degenerate orbitals is played by two types of triplets |⊠⟩ and |[ ]⟩. It should be noted that the xy-components of the pseudospin (AF τ^x and p-nematic τ^y) couple to the dipolar part $(\mathbf{T}_p \cdot \mathbf{T}_{p+1})$ of the magnetic Hamiltonian while the z component is associated with the (n-type) nematic part $(\widetilde{Q}^{\alpha\beta}\widetilde{Q}^{\alpha\beta})$.

In order to further simplify the effective Hamiltonian, we again assume that the spin sector is well described by the VBS wave function.[75] Then the dipolar and the nematic part of the effective Hamiltonian may be replaced by the following expectation values:

$$\langle \mathbf{T}_p \cdot \mathbf{T}_{p+1} \rangle_{\mathrm{VBS}} = -\frac{4}{3}, \quad \left\langle \sum_{\alpha,\beta} \widetilde{Q}_p^{\alpha\beta} \widetilde{Q}_{p+1}^{\alpha\beta} \right\rangle_{\mathrm{VBS}} = +\frac{4}{3} \tag{129}$$

and the above effective Hamiltonian (128) reduces to that for the pseudospins:

$$\mathcal{H}_{\mathrm{eff}} = -\frac{4}{3}\lambda \sum_{p=\mathrm{plaq}} \left\{ (1+\delta_2)\tau_p^x \tau_{p+1}^x + (1+\delta_4)\tau_p^y \tau_{p+1}^y + \frac{1}{4} \right\} + \frac{2}{3}\lambda(1+\delta_3) \sum_p \tau_p^z \tau_{p+1}^z + \sum_p (\delta_2 - \delta_3 + \delta_4 + \delta_6)\tau_p^z. \tag{130}$$

This is nothing but the $S=1/2$ XYZ Hamiltonian in a magnetic field.[84] For small δ_3 the system is xy-like (easy plane) and *ferromagnetic* ordering will occur mainly in the xy plane; when $\delta_2 > \delta_4$ ($\delta_2 < \delta_4$) the pseudospins align in the $x(y)$ direction and the SD (SC) phase will be stabilized [see Eq. (126b)]. For $\delta_2 = \delta_4$ (the self-dual models), on the other hand, the system reduces to the well-known XXZ Hamiltonian and the low-energy sector is generically described by the TL model (B6b). These are exactly what we have found by a field-theoretical analysis in Sec. V.

For appropriate choices of δ_2, δ_3, and δ_4, the zz part will dominate and staggered ordering along the z axis will occur (note that the sign of $\tau^z\tau^z$ is antiferromagnetic). Since this ground state has a period two in the "plaquette" picture, it has, when translated back to the original ladder, a period 4.

VIII. CONCLUDING REMARKS

In the present paper, we have developed a unifying approach to the problem of unconventional phases appearing in generalized two-leg spin ladders. Although the low-energy properties of the two-leg spin ladder with only two-spin (or, ordinary exchange) interactions are fairly well known, not much is known for the case with four-spin interactions.

To treat the problem from a wider viewpoint, we have constructed, out of $S=1/2$ spins and their bilinears, a general lattice model Hamiltonian with SU(2) (rotational) invariance and symmetry under the exchange of two constituent chains. We have adopted the so-called SU(4)-symmetric model as the starting point to elucidate the interplay between two-spin and four-spin interactions. In particular, the SU(4) symmetry is the maximal continuous symmetry that one can have in general two-leg $S=1/2$ spin ladders. The greatest advantage of this extended symmetry approach is that it can unify the unconventional phases (e.g., the SC phase) of the two-leg spin ladder with four-spin exchange interactions.

In the present paper, we have shown that four unconventional phases (Q, SD, SC, and RQ) of the generalized two-leg spin ladder are unified at the SU(4) multicritical point. These phases, which emerge in regions characterized by substantial strength of the four-spin interactions, is quite difficult to describe by means of a perturbative approach starting from the limit of two decoupled chains. Fortunately, however, we have a well-controlled starting point—the SU(4) point—in our problem. In the field-theory language, the low-energy properties of the SU(4) point are described by the level-1 SO(6) WZW model which is equivalent to six copies of the 2D Ising model at its critical point. The continuum expressions of the interactions can then be written in terms of this basis and we have revealed the existence of a duality symmetry $\mathcal{D}$, the so-called spin-chirality duality,[30] and a new hidden one $\widetilde{\mathcal{D}}$ which is an emergent symmetry. The one-loop RG flow exhibits quite a simple structure (four dominant phases and symmetric rays toward them) in spite of the complexity in the RG equations. The duality transformation $\mathcal{D}$ and its dual ($\widetilde{\mathcal{D}}$) map the dominant phases onto each other (see Fig. 4). The existence of these duality symmetries enables us to investigate the nature of the quantum phase transitions between these competing orders which occur along the self-dual manifolds. The behavior on the self-dual manifold $B=D$ is intriguing; the spin sector dies away by opening a gap and in turn the *pseudospin* degrees of freedom come into play in the low-energy physics. Their gapless spin-singlet fluctuations are described by the $c=1$ TL model. In fact, what controls the competition and quantum phase transitions between the unconventional spin-singlet phases (SD and SC) is this pseudospin degrees of freedom and we demonstrated this picture by a simple variational calculation in Sec. VII C. The fact that the pseudospin sector displays an U(1) quantum critical behavior helps us understand the global phase structure in the same manner as we do in the $S=1/2$ XXZ model. Note that the above U(1) criticality is *not* an emergent one, whereas the other one within the manifold $g_2=-g_4$ is emergent in the sense of Ref. 14.

Another interesting point concerning the physics on the self-dual manifold is the *emergent* SU(3) symmetry. Although this is not so obvious in the lattice analysis, our field-theory analysis tells us that instead of the original SO(3) an enlarged SU(3) symmetry appears in the low-energy limit. As a byproduct, we established an interesting description of our problem in terms of the spin-1 (hardcore) bosons. In this respect, we hope that our results obtained for spin systems will be of some help in understanding the phases of the spin 1 boson systems.[85]

ACKNOWLEDGMENTS

The authors would like also to thank P. Azaria, E. Boulat, A. Läuchli, H-H. Lin, T. Momoi, and A. A. Nersesyan for very useful discussions. K.T. was supported by Grant-in-Aid No. 16740222 from Ministry of Education, Culture, Sports, Science, and Technology of Japan. He is grateful to LPTM at Cergy-Pontoise University and the Condensed-Matter-

Theory Laboratory of RIKEN for hospitality extended to him during his stay. P.L. is grateful to the Abdus Salam ICTP for hospitality where this work was completed.

APPENDIX A: SUMMARY OF SU(4)

1. Generators

In Sec. II, we introduced 15 SU(4) generators X^i in the form of 4×4 matrices. Below, we give the explicit forms of them. Note that we use the singlet-triplet basis defined by

$$|\mathrm{s}\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle), \quad |\mathrm{t}_x\rangle = \frac{-1}{\sqrt{2}}(|\uparrow\uparrow\rangle - |\downarrow\downarrow\rangle),$$

$$|\mathrm{t}_y\rangle = \frac{i}{\sqrt{2}}(|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle), \quad |\mathrm{t}_z\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle), \quad \text{(A1)}$$

$$S_1^1 = X^1 = \begin{pmatrix} 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{-i}{2} \\ 0 & 0 & \frac{i}{2} & 0 \end{pmatrix},$$

$$S_1^2 = X^2 = \begin{pmatrix} 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{i}{2} \\ \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{-i}{2} & 0 & 0 \end{pmatrix},$$

$$S_1^3 = X^3 = \begin{pmatrix} 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & \frac{-i}{2} & 0 \\ 0 & \frac{i}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 \end{pmatrix}, \quad \text{(A2a)}$$

$$S_2^1 = X^4 = -\begin{pmatrix} 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{i}{2} \\ 0 & 0 & \frac{-i}{2} & 0 \end{pmatrix},$$

$$S_2^2 = X^5 = -\begin{pmatrix} 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{-i}{2} \\ \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{i}{2} & 0 & 0 \end{pmatrix},$$

$$S_2^3 = X^6 = -\begin{pmatrix} 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & \frac{i}{2} & 0 \\ 0 & \frac{-i}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 \end{pmatrix}. \quad \text{(A2b)}$$

The remaining nine generators are essentially given by the tensor product of two Pauli matrices:

$$G_{ab} = \frac{1}{2}\sigma^a \otimes \sigma^b = 2S_1^a S_2^b \quad (a,b = 1,2,3),$$

$$G_{11} = X^7 = \begin{pmatrix} -\frac{1}{2} & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix},$$

$$G_{22} = X^8 = \begin{pmatrix} -\frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix},$$

$$G_{33}=X^9=\begin{pmatrix} -\frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & -\frac{1}{2} \end{pmatrix}, \tag{A2c}$$

$$G_{12}=X^{10}=\begin{pmatrix} 0 & 0 & 0 & \frac{i}{2} \\ 0 & 0 & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & 0 & 0 \\ \frac{-i}{2} & 0 & 0 & 0 \end{pmatrix},$$

$$G_{13}=X^{11}=\begin{pmatrix} 0 & 0 & \frac{-i}{2} & 0 \\ 0 & 0 & 0 & -\frac{1}{2} \\ \frac{i}{2} & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & 0 \end{pmatrix},$$

$$G_{21}=X^{12}=\begin{pmatrix} 0 & 0 & 0 & \frac{-i}{2} \\ 0 & 0 & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & 0 & 0 \\ \frac{i}{2} & 0 & 0 & 0 \end{pmatrix}, \tag{A2d}$$

$$G_{23}=X^{13}=\begin{pmatrix} 0 & \frac{i}{2} & 0 & 0 \\ \frac{-i}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & 0 \end{pmatrix},$$

$$G_{31}=X^{14}=\begin{pmatrix} 0 & 0 & \frac{i}{2} & 0 \\ 0 & 0 & 0 & -\frac{1}{2} \\ \frac{-i}{2} & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & 0 \end{pmatrix},$$

$$G_{32}=X^{15}=\begin{pmatrix} 0 & \frac{-i}{2} & 0 & 0 \\ \frac{i}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & 0 \end{pmatrix}. \tag{A2e}$$

2. SU(3)×U(1)

The spin-chirality transformation introduces a natural categorization of the above generators in terms of a subgroup SU(3)×U(1). The easiest way of understanding the appearance of SU(3) would be to decompose the whole Hilbert space into a singlet and a triplet [of course, the singlet corresponds to U(1) factor]; the SU(3) part acts only to the triplet subspace.

With this identification, the SU(3) Gell-Mann matrices are given as follows:

$$G^1=-(G_{12}+G_{21})=\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$G^2=S_1^3+S_2^3=\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$G^3=-G_{11}+G_{22}=\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$G^4=-(G_{13}+G_{31})=\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

$$G^5 = -(S_1^2 + S_2^2) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix},$$

$$G^6 = -(G_{23} + G_{32}) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

$$G^7 = S_1^1 + S_2^1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix},$$

$$G^8 = -\frac{1}{\sqrt{3}}(G_{11} + G_{22} - 2G_{33}) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{3}} & 0 \\ 0 & 0 & 0 & \frac{-2}{\sqrt{3}} \end{pmatrix}.$$

They are supplemented by the generator of (spin chirality) U(1):

$$G_{\mathrm{U}(1)} = -\sqrt{\frac{2}{3}}(G_{11} + G_{22} + G_{33}),$$

which satisfies

$$[G_{\mathrm{U}(1)}, G^a] = 0 \quad (a = 1, \ldots, 8).$$

These nine operators generate a subgroup SU(3)×U(1).

It is a straightforward task to rewrite our building blocks Eqs. (4a)–(4g) in terms of the above generators. In particular, one has

$$\begin{aligned} A\mathcal{H}_1 + C\mathcal{H}_3 = \sum_r \{ & A[G^2(r)G^2(r+1) + G^5(r)G^5(r+1) \\ & + G^7(r)G^7(r+1)] + C[G^1(r)G^1(r+1) \\ & + G^3(r)G^3(r+1) + G^4(r)G^4(r+1) \\ & + G^6(r)G^6(r+1) + G^8(r)G^8(r+1) \\ & + G_{\mathrm{U}(1)}(r)G_{\mathrm{U}(1)}(r+1)]\}. \end{aligned} \tag{A3}$$

From which we observe that the model with $A=C$ has an SU(3) symmetry.

Interestingly, the remaining six SU(4) generators

$$\mathbf{S}_1 - \mathbf{S}_2, \quad 2(\mathbf{S}_1 \times \mathbf{S}_2)$$

are order parameters of the above SU(3)×U(1). In fact,

$$(\mathbf{S}_1 - \mathbf{S}_2) + i2(\mathbf{S}_1 \times \mathbf{S}_2) \text{ and } (\mathbf{S}_1 - \mathbf{S}_2) - i2(\mathbf{S}_1 \times \mathbf{S}_2)$$

respectively transform like **3** and $\bar{\mathbf{3}}$ under SU(3). The spin-chirality U(1) acts like

$$(\mathbf{S}_1 - \mathbf{S}_2) \pm i2(\mathbf{S}_1 \times \mathbf{S}_2) \mapsto e^{\pm i\theta}[(\mathbf{S}_1 - \mathbf{S}_2) \pm i2(\mathbf{S}_1 \times \mathbf{S}_2)].$$

APPENDIX B: BOSONIZATION DICTIONARY

To establish the notations, we briefly summarize the main formulas used in our bosonization analysis. Consider N free Dirac fermions defined by the following Hamiltonian:

$$\mathcal{H}_{\mathrm{Dirac}} = -iv \int dx \sum_{a=1}^{N} (\Psi_{a,\mathrm{R}}^{\dagger}\partial_x \Psi_{a,\mathrm{R}} - \Psi_{a,\mathrm{L}}^{\dagger}\partial_x \Psi_{a,\mathrm{L}}). \tag{B1}$$

Let us first introduce N bosonic fields to bosonize N Dirac fermions as follows:[35]

$$\Psi_{a,\mathrm{R}} = \frac{\tilde{\kappa}_a}{\sqrt{2\pi a_0}} \exp(i\sqrt{4\pi}\phi_{a,\mathrm{R}}),$$

$$\Psi_{a,\mathrm{L}} = \frac{\tilde{\kappa}_a}{\sqrt{2\pi a_0}} \exp(-i\sqrt{4\pi}\phi_{a,\mathrm{L}})$$

$$(a = 1, \ldots, N), \tag{B2}$$

a_0 being some ultraviolet cutoff (typically the lattice spacing) and we use the following normalization for the chiral bosonic fields:

$$\langle \phi_{a,\mathrm{R}}(\bar{z})\phi_{b,\mathrm{R}}(\bar{w})\rangle = -\frac{\delta_{ab}}{4\pi}\ln(\bar{z} - \bar{w}),$$

$$\langle \phi_{a,\mathrm{L}}(z)\phi_{b,\mathrm{L}}(w)\rangle = -\frac{\delta_{ab}}{4\pi}\ln(z - w), \tag{B3}$$

where the complex coordinates z and $\bar{z}$ are defined by $z = v\tau + ix$ and $\bar{z} = v\tau - ix$, respectively. The total bosonic fields ϕ_a and their dual fields θ_a are defined as $\Phi_a \equiv \phi_{a,\mathrm{R}} + \phi_{a,\mathrm{L}}$, $\Theta_a \equiv -\phi_{a,\mathrm{R}} + \phi_{a,\mathrm{L}}$. We impose the following commutation relation for the chiral bosonic fields:

$$[\phi_{a,\mathrm{R}}, \phi_{b,\mathrm{L}}] = \frac{i}{4}\delta_{ab}, \tag{B4}$$

so that the left and the right movers of the same species anticommute: $\{\Psi_{a,\mathrm{R}}(x), \Psi_{a,\mathrm{L}}(y)\} = 0$. The anticommutation between fermions of *different* species is guaranteed by the presence of the Klein factors (here Majorana fermions) $\tilde{\kappa}_a$ in the definition (B2), which obey the anticommutation rule

$$\{\tilde{\kappa}_a, \tilde{\kappa}_b\} = 2\delta_{ab}. \tag{B5}$$

Using these bosons, $\mathcal{H}_{\mathrm{Dirac}}$ can be written as a Hamiltonian of the so-called Tomonaga-Luttinger (TL) model:[35]

$$\mathcal{H}_{\mathrm{Dirac}} = \pi v \int dx \sum_{a=1}^{N} (J_{\mathrm{L},a}^2 + J_{\mathrm{R},a}^2) \tag{B6a}$$

$$=\frac{v}{2}\int dx\sum_{a=1}^{N}\{(\partial_x\Phi_a)^2+(\partial_x\Theta_a)^2\}=\mathcal{H}_{\rm TL}, \quad \text{(B6b)}$$

where the operators of the form A^n should always be understood as normal-ordered: $:A^n:$.[35,36] The above equivalence is established by the following bosonization formulas for fermion bilinears:

$$J_{{\rm L},a}=\Psi^{\dagger}_{a,{\rm L}}\Psi_{a,{\rm L}}=\frac{1}{\sqrt{\pi}}\partial_x\phi_{a,{\rm L}}=\frac{1}{\sqrt{4\pi}}\partial_x(\Phi_a+\Theta_a), \quad \text{(B7a)}$$

$$J_{{\rm R},a}=\Psi^{\dagger}_{a,{\rm R}}\Psi_{a,{\rm R}}=\frac{1}{\sqrt{\pi}}\partial_x\phi_{a,{\rm R}}=\frac{1}{\sqrt{4\pi}}\partial_x(\Phi_a-\Theta_a), \quad \text{(B7b)}$$

$$J^2_{{\rm L},a}=+\frac{i}{\pi}\Psi^{\dagger}_{a,{\rm L}}\partial_x\Psi_{a,{\rm L}}, \quad J^2_{{\rm R},a}=-\frac{i}{\pi}\Psi^{\dagger}_{a,{\rm R}}\partial_x\Psi_{a,{\rm R}}, \quad \text{(B7c)}$$

$$\Psi^{\dagger}_{a,{\rm R}}\Psi_{a,{\rm L}}=-\frac{i}{2\pi a_0}e^{-i\sqrt{4\pi}\Phi_a}, \quad \text{(B7d)}$$

$$\Psi^{\dagger}_{a,{\rm L}}\Psi_{a,{\rm R}}=+\frac{i}{2\pi a_0}e^{+i\sqrt{4\pi}\Phi_a}.$$

APPENDIX C: SU(4) EFFECTIVE ACTION FROM HUBBARD MODEL

In this section, we sketch the derivation of the SU(4) fixed-point Hamiltonian from the two-band Hubbard model at quarter filling:

$$\mathcal{H}_{\rm Hubbard}=-t\sum_{i,a,\sigma}(c^{\dagger}_{a,\sigma,i+1}c_{a,\sigma,i}+c^{\dagger}_{a,\sigma,i}c_{a,\sigma,i+1})+\frac{U}{2}\sum_{\substack{i,a,b,\\ \sigma,\sigma'}}n_{a,\sigma,i}n_{b,\sigma',i}(1-\delta_{a,b}\delta_{\sigma,\sigma'}), \quad \text{(C1)}$$

where $c^{\dagger}_{i,a,\sigma}$ creates an electron with spin $\sigma(=\uparrow,\downarrow)$ in orbital $a(=1,2)$ of the site-i and $n_{a,\sigma,i}=c^{\dagger}_{a,\sigma,i}c_{a,\sigma,i}$. The key idea is the following. When the Hubbard interaction U is sufficiently large, the system becomes a Mott insulator and the charge degrees of freedom gets decoupled from the low-energy physics.[56,86] However, we still have four states ($a=1,2$, $\sigma=\uparrow,\downarrow$) on each site and we identify them with four states of the fundamental representation of **4** of SU(4). These states constitute the low-energy sector of the system. As in the usual single-band case, the effective Hamiltonian describing the low-energy physics is obtained by the second-order perturbation:

$$\mathcal{H}_{\rm eff}=\frac{2t^2}{U}P_{\rm G}\sum_{\substack{a,\sigma\\ b,\sigma'}}(c^{\dagger}_{a,\sigma,i+1}c_{a,\sigma,i}+c^{\dagger}_{a,\sigma,i}c_{a,\sigma,i+1})\times(c^{\dagger}_{b,\sigma',i+1}c_{b,\sigma',i}+c^{\dagger}_{b,\sigma',i}c_{b,\sigma',i+1})P_{\rm G}=\frac{4t^2}{U}\sum_i P_{i,i+1}=\frac{4t^2}{U}\sum_{i,a}X^a_iX^a_{i+1}+\text{const}, \quad \text{(C2)}$$

where the projection onto the subspace with one electron per site is enforced by $P_{\rm G}$. The operator $P_{i,i+1}$ appearing in the last line is the SU(4) permutation operator acting on $\mathbf{4}\otimes\mathbf{4}$ on neighboring sites. The Hamiltonian $\Sigma X^a_iX^a_{i+1}$ is nothing but the SU(4)-invariant model (17) in Sec. II.

If we repeat the same steps starting from the continuum expression of the two-band Hubbard model (C1), we may obtain the desired continuum field theory for the SU(4) model (17).[56–58]

Let us begin with the first term of $\mathcal{H}_{\rm Hubbard}$ Eq. (C1). As is well known in the standard bosonization treatment of one-dimensional electron systems, the spectrum of the four electrons ($c_1\equiv c_{1,\uparrow}$, $c_2\equiv c_{1,\downarrow}$, $c_3\equiv c_{2,\uparrow}$, $c_4\equiv c_{2,\downarrow}$) near the Fermi points $k=\pm k_{\rm F}=\pm\pi/4$ may be linearized to obtain the Hamiltonian of the Dirac fermions (B1) with $N=4$ and $v=2t\sin k_{\rm F}$. In order to obtain the continuum expression of the second term (Hubbard interaction $\mathcal{H}_U$), we need the expression of the density operator $n_{b,i}$. Since we have two species of Dirac fermions (L and R) for each fermion, the electron density is made up of four pieces:

$$\frac{1}{a_0}n_{i,b}\approx n_b(x)=\Psi^{\dagger}_{b,{\rm R}}\Psi_{b,{\rm R}}+\Psi^{\dagger}_{b,{\rm L}}\Psi_{b,{\rm L}}+e^{-2ik_{\rm F}x}\Psi^{\dagger}_{b,{\rm R}}\Psi_{b,{\rm L}}+e^{+2ik_{\rm F}x}\Psi^{\dagger}_{b,{\rm L}}\Psi_{b,{\rm R}}\ (b=1,2,3,4). \quad \text{(C3)}$$

With the help of Eqs. (B7a)–(B7d), the right-hand side can be readily bosonized as

$$n_b(x)=\frac{1}{\sqrt{\pi}}\partial_x\Phi_b-\frac{i}{2\pi a_0}e^{-2ik_{\rm F}x}e^{-i\sqrt{4\pi}\Phi_b}+\frac{i}{2\pi a_0}e^{+2ik_{\rm F}x}e^{+i\sqrt{4\pi}\Phi_b}. \quad \text{(C4)}$$

Using this bosonized expression, one finds that the effective Hamiltonian which describes the low-energy properties of the two-band Hubbard model at quarter filling consists of four terms:

$$\mathcal{H}_U=\mathcal{H}_{\rm c}+\mathcal{H}_{\rm SU(4)}+\mathcal{H}_{2k_{\rm F}}+\mathcal{H}_{4k_{\rm F}}. \quad \text{(C5)}$$

The two parts $\mathcal{H}_{2k_{\rm F}}$ and $\mathcal{H}_{4k_{\rm F}}$ contains terms oscillating with wave vectors $q=\pm 2k_{\rm F}$ and $q=4k_{\rm F}$, respectively, and may be neglected in the first approximation. Before presenting the explicit forms of $\mathcal{H}_{\rm c}$ and $\mathcal{H}_{\rm SU(4)}$, it is convenient to move to another set of boson basis:

$$\begin{pmatrix}\Phi_c\\ \Phi_s\\ \Phi_f\\ \Phi_{sf}\end{pmatrix}=\frac{1}{2}\begin{pmatrix}1&1&1&1\\ 1&-1&1&-1\\ 1&1&-1&-1\\ 1&-1&-1&1\end{pmatrix}\begin{pmatrix}\Phi_1\\ \Phi_2\\ \Phi_3\\ \Phi_4\end{pmatrix}. \tag{C6}$$

Since the above is an orthogonal transformation, the Dirac part is transformed to

$$\mathcal{H}_{\rm Dirac}=\frac{v}{2}\int dx\sum_{a={\rm c,s,f,sf}}\{(\partial_x\Phi_a)^2+(\partial_x\Theta_a)^2\}. \tag{C7}$$

The U-dependent part $(\mathcal{H}_c+\mathcal{H}_{SU(4)})$ can be recasted as

$$\begin{aligned}&\frac{U}{2\pi}\sum_{a,b=1}^{4}\partial_x\Phi_a(\mathbb{M}-\mathbf{1})_{a,b}\partial_x\Phi_b+\frac{2U}{(2\pi a_0)^2}\\ &\quad\times\sum_{a<b}^{4}\cos[\sqrt{4\pi}(\Phi_a-\Phi_b)]\\ &=\frac{U}{2\pi}\left\{3(\partial_x\Phi_c)^2-\sum_{a={\rm s,f,sf}}(\partial_x\Phi_a)^2\right\}\\ &\quad+\frac{U}{\pi^2a_0^2}\{\cos(\sqrt{4\pi}\Phi_s)\cos(\sqrt{4\pi}\Phi_f)\\ &\quad+\cos(\sqrt{4\pi}\Phi_f)\cos(\sqrt{4\pi}\Phi_{sf})\\ &\quad+\cos(\sqrt{4\pi}\Phi_{sf})\cos(\sqrt{4\pi}\Phi_s)\},\end{aligned} \tag{C8}$$

where $\mathbb{M}$ is a 4×4 matrix with all matrix elements equal to unity. From Eqs. (C7) and (C8), we can read off

$$\mathcal{H}_c=\int dx\left\{\frac{v}{2}[(\partial_x\Phi_c)^2+(\partial_x\Theta_c)^2]+\frac{3U}{2\pi}(\partial_x\Phi_c)^2\right\} \tag{C9a}$$

and

$$\begin{aligned}\mathcal{H}_{SU(4)}&=\int dx\sum_{a={\rm s,f,sf}}\left\{\frac{v}{2}[(\partial_x\Phi_a)^2+(\partial_x\Theta_a)^2]-\frac{U}{2\pi}(\partial_x\Phi_a)^2\right\}\\ &\quad+\frac{U}{\pi^2a_0^2}\int dx\{\cos(\sqrt{4\pi}\Phi_s)\cos(\sqrt{4\pi}\Phi_f)\\ &\quad+\cos(\sqrt{4\pi}\Phi_f)\cos(\sqrt{4\pi}\Phi_{sf})\\ &\quad+\cos(\sqrt{4\pi}\Phi_{sf})\cos(\sqrt{4\pi}\Phi_s)\},\end{aligned} \tag{C9b}$$

which is written only in terms of three bosons Φ_s, Φ_f, and Φ_{sf} and dictates the physics of the SU(4) sector. Comparing with the bosonization treatment of the single-band Hubbard model, one finds that the $(4k_F)$ umklapp scattering present in the single-band case is absent here while the SU(4) part ["SU(4)" should be replaced with spin in the single-band case] contains the marginally irrelevant backscattering as in the case of the single band. This may seem contradicting since we know the two-band Hubbard model will become insulating (i.e., charge-gapped) for sufficiently large U ($U\sim2.8t$).[86] This paradox is remedied by taking into account the second-order (U^2) perturbation coming from the OPE $\mathcal{H}_{4k_F}\mathcal{H}_{4k_F}$. In fact, the above OPE yields the contribution like

$$-\frac{3U^2}{8\pi^4a_0^2}\cos(\sqrt{16\pi}\Phi_c)$$

which will pin the charge Φ_c field for large enough U. Therefore we may fix the value of the Φ_c field and drop $\mathcal{H}_c$ in the following analysis.

The final step is to rewrite the SU(4) Hamiltonian $\mathcal{H}_{SU(4)}$ in terms of six Majorana fermions. The mapping is provided by the following correspondence:

$$\begin{aligned}\xi^2_{R/L}+i\xi^1_{R/L}&=\frac{\eta_1}{\sqrt{\pi a_0}}\exp(-ip\sqrt{4\pi}\varphi_{s,R/L}),\\ \chi^1_{R/L}+i\chi^2_{R/L}&=\frac{\eta_2}{\sqrt{\pi a_0}}\exp(+ip\sqrt{4\pi}\varphi_{f,R/L}),\\ \chi^3_{R/L}+i\xi^3_{R/L}&=\frac{\eta_3}{\sqrt{\pi a_0}}\exp(-ip\sqrt{4\pi}\varphi_{sf,R/L}),\end{aligned} \tag{C10}$$

where the integer p is defined by $p=+1(L)$ and $p=-1(R)$. The Majorana fermions $\eta_a(a=1,2,3)$ have been introduced to guarantee the anticommutation. The following formulas proved by taking OPEs are useful in rewriting the SU(4) Hamiltonian (C9b):

$$\begin{aligned}\partial_x\varphi_{s,R/L}&=i\sqrt{\pi}\xi^1_{R/L}\xi^2_{R/L},\\ \partial_x\varphi_{f,R/L}&=i\sqrt{\pi}\chi^1_{R/L}\chi^2_{R/L},\\ \partial_x\varphi_{sf,R/L}&=i\sqrt{\pi}\xi^3_{R/L}\chi^3_{R/L},\end{aligned} \tag{C11}$$

$$\begin{aligned}\cos(\sqrt{4\pi}\Phi_s)&=i\pi a_0(\kappa_1+\kappa_2),\\ \cos(\sqrt{4\pi}\Phi_f)&=i\pi a_0(\kappa_4+\kappa_5),\\ \cos(\sqrt{4\pi}\Phi_{sf})&=i\pi a_0(\kappa_3+\kappa_6).\end{aligned} \tag{C12}$$

The Majorana fermions are normalized so that

$$\begin{aligned}\langle\xi^a(z)\xi^b(w)\rangle=\langle\chi^a(z)\chi^b(w)\rangle&=\frac{\delta_{a,b}}{2\pi(z-w)},\\ \langle\xi^a(z)\chi^b(w)\rangle&=0,\end{aligned} \tag{C13}$$

and the Majorana bilinears κ_i are defined as

$$\kappa_a\equiv\xi^a_R\xi^a_L,\quad \kappa_{a+3}\equiv\chi^a_R\chi^a_L\ (a=1,2,3). \tag{C14}$$

By using Eqs. (C11) and (C12), the first term in Eq. (C9b) can be written as

$$\begin{aligned}&-\frac{i}{2}\tilde{v}\sum_{a=1}^{3}\int dx(\xi^a_R\partial_x\xi^a_R-\xi^a_L\partial_x\xi^a_L+\chi^a_R\partial_x\chi^a_R-\chi^a_L\partial_x\chi^a_L)\\ &\quad-U\int dx(\kappa_1\kappa_2+\kappa_4\kappa_5+\kappa_3\kappa_6),\end{aligned} \tag{C15}$$

where we have introduced the renormalized velocity $\tilde{v}=[v-U/(2\pi)]$. Similarly, Eq. (C12) enables us to recast the second term of $\mathcal{H}_{\mathrm{SU}(4)}$ as

$$-\frac{U}{2}\int dx\left(\sum_{a=1}^{6}\kappa_a\right)^2+U\int dx(\kappa_1\kappa_2+\kappa_4\kappa_5+\kappa_3\kappa_6). \tag{C16}$$

From these equations, we finally obtain the desired result:

$$\mathcal{H}_{\mathrm{SU}(4)}=-\frac{i}{2}\tilde{v}\sum_{a=1}^{3}\int dx(\xi_{\mathrm{R}}^a\partial_x\xi_{\mathrm{R}}^a-\xi_{\mathrm{L}}^a\partial_x\xi_{\mathrm{L}}^a+\chi_{\mathrm{R}}^a\partial_x\chi_{\mathrm{R}}^a-\chi_{\mathrm{L}}^a\partial_x\chi_{\mathrm{L}}^a)-\frac{U}{2}\int dx\left(\sum_{a=1}^{6}\kappa_a\right)^2. \tag{C17}$$

For repulsive interaction $U>0$, the interaction (the last term) is marginally irrelevant and known to yield logarithmic corrections[87,88] to physical quantities.

APPENDIX D: DERIVATION OF THE DUALITY TRANSFORMATION IN THE CONTINUUM LIMIT

If the duality transformation (10) works not only on a lattice but also in a low-energy field theory, the uniform part of the generators derived above should obey the *same* transformation rule (since the duality transformation does not change momentum of the system). Plugging Eqs. (42) and (34a) into Eq. (11), we obtain the following set of equations for the uniform part of $\mathbf{S}_1$:

$$\tilde{\xi}^a\tilde{\xi}^b=\cos^2\left(\frac{\theta}{2}\right)\xi^a\xi^b+\sin^2\left(\frac{\theta}{2}\right)\chi^a\chi^b-\frac{1}{2}\sin\theta(\xi^a\chi^b-\xi^b\chi^a), \tag{D1}$$

where we have dropped the indices L/R and $(a,b)=(2,3)$, (3,1), (1,2). Since all three equations are satisfied simultaneously for a transformation which is independent both of color $(a=1,2,3)$ and of the chirality (L/R), we try the following:

$$\begin{pmatrix}\tilde{\xi}_{\mathrm{L/R}}^a\\ \tilde{\chi}_{\mathrm{L/R}}^a\end{pmatrix}=\begin{pmatrix}\cos\phi & -\sin\phi\\ \sin\phi & \cos\phi\end{pmatrix}\begin{pmatrix}\xi_{\mathrm{L/R}}^a\\ \chi_{\mathrm{L/R}}^a\end{pmatrix}. \tag{D2}$$

After some algebra, we see that the above equations are satisfied if we choose the parameter ϕ as

$$\phi=\frac{\theta}{2}.$$

Moreover, it is not difficult to verify that the above choice works for the equations for $\mathbf{S}_2$ as well:

$$\tilde{\chi}^a\tilde{\chi}^b=\sin^2\left(\frac{\theta}{2}\right)\xi^a\xi^b+\cos^2\left(\frac{\theta}{2}\right)\chi^a\chi^b+\frac{1}{2}\sin\theta(\xi^a\chi^b-\xi^b\chi^a). \tag{D3}$$

Therefore we may conclude that the following color and chirality (L/R) independent SO(2) transformation $R(\theta)$ generates the "duality" in the continuum limit:

$$\tilde{\xi}_{\mathrm{L/R}}^a=\xi_{\mathrm{L/R}}^a\cos\frac{\theta}{2}-\chi_{\mathrm{L/R}}^a\sin\frac{\theta}{2},$$

$$\tilde{\chi}_{\mathrm{L/R}}^a=\xi_{\mathrm{L/R}}^a\sin\frac{\theta}{2}+\chi_{\mathrm{L/R}}^a\cos\frac{\theta}{2}. \tag{D4}$$

This is not the end of the story. Besides the uniform part discussed above, the continuum expression of the SU(4) generators contain two kinds of oscillating parts; one has momentum $\pi(=4k_{\mathrm{F}})$ and oscillates in a staggered manner, and the others exhibits $\pm\pi/2(=2k_{\mathrm{F}})$ oscillation. The argument goes similarly for the $4k_{\mathrm{F}}$ part and we can verify that the choice (D4) works.

The easiest way to write down the transformation of the $2k_{\mathrm{F}}$ part would be to use the Dirac fermions instead of Majorana fermions ξ and χ and look for the correct transformation rule for the Dirac quartet. Again, as in Eq. (D1), we require that the $2k_{\mathrm{F}}$ part of the *transformed* SU(4) generators

$$\hat{\tilde{G}}=\tilde{L}^{\dagger}X^a\tilde{R}(2k_{\mathrm{F}})\ \text{or}\ \tilde{R}^{\dagger}X^a\tilde{L}(-2k_{\mathrm{F}})$$

be correctly reproduced by linear combinations of the original generators as in Eqs. (9a) and (9b). The answer is quite simple and is given as follows:

$$\tilde{R}=U(\theta)R,\quad \tilde{L}=U(\theta)L, \tag{D5}$$

where

$$R=(\Psi_{\mathrm{R},1\uparrow},\Psi_{\mathrm{R},1\downarrow},\Psi_{\mathrm{R},2\uparrow},\Psi_{\mathrm{R},2\downarrow}),$$

$$L=(\Psi_{\mathrm{L},1\uparrow},\Psi_{\mathrm{L},1\downarrow},\Psi_{\mathrm{L},2\uparrow},\Psi_{\mathrm{L},2\downarrow}),$$

and $U(\theta)$ is given by Eq. (10).

APPENDIX E: NONZERO OPE COEFFICIENTS

The operator-product expansion of interaction operators is defined as

$$\mathcal{H}_i(z,\bar{z})\mathcal{H}_j(w,\bar{w})\sim\frac{C_{ij}^k}{|z-w|^2}\mathcal{H}_k(w,\bar{w})\quad(i,j=1,2,3,4,7).$$

Since $\mathcal{H}_i$ are written in terms of free fermions ξ^a and χ^a, it is straightforward to compute the right-hand side. The only nonzero OPE coefficients are listed below:

$$C_{11}^1=C_{22}^1=C_{44}^1=-\frac{1}{\pi^2},\quad C_{33}^1=-\frac{5}{\pi^2},$$

$$C_{12}^2=C_{21}^2=-\frac{1}{\pi^2},\quad C_{34}^2=C_{43}^2=-\frac{3}{\pi^2},$$

$$C_{47}^2=C_{74}^2=-\frac{1}{2\pi^2},\quad C_{13}^3=C_{31}^3=-\frac{3}{\pi^2},$$

$$C^3_{24}=C^3_{42}=-\frac{1}{\pi^2},\quad C^4_{14}=C^4_{41}=-\frac{1}{\pi^2},$$

$$C^4_{23}=C^4_{32}=-\frac{3}{\pi^2},\quad C^4_{27}=C^4_{72}=-\frac{1}{2\pi^2},$$

$$C^7_{13}=C^7_{31}=\frac{8}{\pi^2},\quad C^7_{24}=C^7_{42}=-\frac{8}{\pi^2}.$$

APPENDIX F: GROUND-STATE DEGENERACY

In this appendix, we describe how to obtain ground-state degeneracy by inspecting the classical ground states within the bosonization approach. As has been mentioned before, bosonic expressions of the physical operators have a kind of "gauge redundancy" and the ground states should be counted modulo the gauge redundancy.

To this end, we have first to identify physical operators for which we define "gauge transformations." One obvious choice may be U(3) Dirac fermions in Eq. (82). However, the expansion of physical (lattice) operators in terms of the continuum ones contains the $2k_F$ terms which cannot be expressed by these Dirac fermions. This obscures how to impose the gauge equivalence for three bosons φ, φ_s, and φ_f. Instead, we start from the *four* Dirac fermions used in Appendix C to obtain the effective Hamiltonian of the SU(4) model. Throughout this section, we shall use the same notations as in Appendix C. By using the SU(4) bosons Φ_s, Φ_f, and Φ_{sf} [see Eq. (C6)], the interactions $\mathcal{V}_A=-\lambda_A(\mathcal{O}^{\pi}_A)^2$ (A =Q, SD, SC, and RQ) read

$$\mathcal{V}_Q=-\frac{2\lambda_Q}{\pi^2a_0^2}\{\cos(\sqrt{4\pi}\Phi_s)\cos(\sqrt{4\pi}\Phi_f)+\cos(\sqrt{4\pi}\Phi_s)\cos(\sqrt{4\pi}\Phi_{sf})+\cos(\sqrt{4\pi}\Phi_f)\cos(\sqrt{4\pi}\Phi_{sf})\},\tag{F1a}$$

$$\mathcal{V}_{SD}=-\frac{2\lambda_{SD}}{\pi^2a_0^2}\{-\cos(\sqrt{4\pi}\Phi_s)\cos(\sqrt{4\pi}\Phi_f)+\cos(\sqrt{4\pi}\Phi_s)\cos(\sqrt{4\pi}\Theta_{sf})-\cos(\sqrt{4\pi}\Phi_f)\cos(\sqrt{4\pi}\Theta_{sf})\},\tag{F1b}$$

$$\mathcal{V}_{SC}=-\frac{2\lambda_{SC}}{\pi^2a_0^2}\{\sin[\sqrt{\pi}(\Phi_s+\Theta_s+\Phi_f-\Theta_f)]\times\sin[\sqrt{\pi}(\Phi_s-\Theta_s+\Phi_f+\Theta_f)]+\sigma^z\sin[\sqrt{\pi}(\Phi_s+\Theta_s+\Phi_f-\Theta_f)]\sin(\sqrt{4\pi}\Theta_{sf})+\sigma^z\sin([\sqrt{\pi}(\Phi_s-\Theta_s+\Phi_f+\Theta_f)]\sin(\sqrt{4\pi}\Theta_{sf})\},\tag{F1c}$$

$$\mathcal{V}_{RQ}=-\frac{2\lambda_{RQ}}{\pi^2a_0^2}\{-\sin[\sqrt{\pi}(\Phi_s+\Theta_s+\Phi_f-\Theta_f)]\times\sin[\sqrt{\pi}(\Phi_s-\Theta_s+\Phi_f+\Theta_f)]-\sigma^z\sin([\sqrt{\pi}(\Phi_s+\Theta_s+\Phi_f-\Theta_f)]\sin(\sqrt{4\pi}\Phi_{sf})+\sigma^z\sin[\sqrt{\pi}(\Phi_s-\Theta_s+\Phi_f+\Theta_f)]\sin(\sqrt{4\pi}\Phi_{sf})\}.\tag{F1d}$$

These expressions can be readily obtained from Eqs. (83). The 2×2 Hamiltonian is diagonal and we can freely choose one of the eigenvalues (say, +1) of the Pauli matrix σ^z throughout the calculation. In principle, our system may be described only by these three bosons. However, the relationship between three bosons $\Phi_s(\Theta_s)$, $\Phi_f(\Theta_f)$, $\Phi_{sf}(\Theta_{sf})$ and physical operators is not so obvious and it is convenient to go back to the original (two-band) Hubbard model (see Appendix C). For this reason, we recover the charge boson $\Phi_c(\Theta_c)$ and add the following umklapp term to find the semiclassical vacua of our problem:

$$-\lambda_{\text{umklapp}}\cos(\sqrt{16\pi}\Phi_c).\tag{F2}$$

The procedure is as follows. First of all, the gauge redundancy[15] of the original four Dirac fermions (see Appendix C for the definition of these fermions),

$$\Psi_{a,L}=\frac{\kappa_a}{\sqrt{2\pi a_0}}\exp(-i\sqrt{4\pi}\Phi_{a,L}),$$

$$\Psi_{a,R}=\frac{\kappa_a}{\sqrt{2\pi a_0}}\exp(+i\sqrt{4\pi}\Phi_{a,R})\quad(a=1,\ldots,4),$$

reads

$$\Phi_{a,L/R}\mapsto\Phi_{a,L/R}+\sqrt{\pi}N_{a,L/R}\quad(N_{a,L/R}\in\mathbb{Z}).\tag{F3}$$

That is, physical quantities should be unchanged even if we make the above shift. Therefore the semiclassical ground states which are "gauge-equivalent" by this shift should be treated as one and the true ground states are the equivalence classes of this gauge transformation.

For the two phases Q and SD, a straightforward semiclassical analysis (for either Φ or Θ) is applicable and we can proceed in essentially the same manner as in Ref. 15. For the cases of SC and RQ phases, however, the situation is slightly tricky. Since both Φ and Θ appear in a single sine cosine interaction, we should introduce new fields

$$\Phi'_s=\frac{1}{\sqrt{2}}(\Phi_s+\Phi_f),\quad\Phi'_f=\frac{1}{\sqrt{2}}(\Phi_s-\Phi_f),$$

$$\Theta'_s=\frac{1}{\sqrt{2}}(\Theta_s+\Theta_f),\quad\Theta'_f=\frac{1}{\sqrt{2}}(\Theta_s-\Theta_f)\tag{F4}$$

before applying a semiclassical argument. Then we can readily find the semiclassical ground states of $\mathcal{V}_{SC,RQ}$ to apply the method of Ref. 15.

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