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Nonanalytic magnetic response of Fermi and non-Fermi liquids

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We study the nonanalytic behavior of the static spin susceptibility of two-dimensional fermions as a function of temperature and magnetic field. For a generic Fermi liquid, $\chi_s(T,H)=\text{const}+c_1\max\{T,\mu_B|H|\}$, where c_1 is shown to be expressed via complicated combinations of the Landau parameters, rather than via the backscattering amplitude, contrary to the case of the specific heat. Near a ferromagnetic quantum critical point, the field dependence acquires a universal form $\chi_s^{-1}(H)=\text{const}-c_2|H|^{3/2}$, with $c_2>0$. This behavior implies a first-order transition into a ferromagnetic state. We establish a criterion for such a transition to win over the transition into an incommensurate phase.

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The nonanalytic behavior of thermodynamic quantities of a Fermi Liquid (FL) has attracted a substantial interest over the last few years. The Landau Fermi-liquid theory states that the specific heat coefficient $\gamma(T)=C(T)/T$ and uniform spin susceptibility $\chi_s(T,H)$ of an interacting fermionic system approach finite values at $T,H=0$, as in a Fermi gas. However, the temperature and magnetic field dependences of $\gamma(T,H)$ and $\chi_s(T,H)$ turn out to be nonanalytic. In two dimensions (2D), both γ and χ_s are linear rather than quadratic in T and $|H|$.[1] In addition, the nonuniform spin susceptibility $\chi_s(q)$ depends on the momentum as $|q|$ for $q\to 0$.[2,3]

Nonanalytic terms in γ and χ_s arise due to a long-range interaction between quasiparticles mediated by virtual particle-hole pairs. A long-range interaction is present in a Fermi liquid due to Landau damping at small momentum transfers and dynamic Kohn anomaly at momentum transfers near $2k_F$ [the corresponding effective interactions in 2D are $|\Omega|/r$ and $|\Omega|\cos(2k_Fr)/r^{1/2}$, respectively]. The range of this interaction is determined by the characteristic size of the pair, L_{ph}, which is large at small energy scales. To second order in the bare interaction, the contribution to the free energy density from the interaction of two quasiparticles via a single particle-hole pair can be estimated in 2D as $\delta F\sim u^2T/L_{\text{ph}}^2$, where u is the dimensionless coupling constant. As $L_{\text{ph}}\sim v_F/T$ by the uncertainty principle, $\delta F\propto T^3$ and $\gamma(T)\propto T$. Likewise, at $T=0$ but in a finite field a characteristic energy scale is the Zeeman splitting $\mu_B|H|$ and $L_{\text{ph}}\sim v_F/\mu_B|H|$. Hence $\delta F\propto |H|^3$ and $\chi_s(H)\propto|H|$.

A second-order calculation indeed shows[3–5] that γ and χ_s do depend linearly on T and $|H|$. Moreover, the prefactors are expressed only via two Fourier components of the bare interaction [$U(0)$ and $U(2k_F)$] which, to this order, determine the charge and spin components of the backscattering amplitude $f_{c,s}(\theta=\pi)$, where θ is the angle between the incoming momenta (the notations are as in Ref. 4). Specifically,

$$\frac{\delta\gamma(T,H)}{\gamma(0,0)}=-\frac{9\zeta(3)}{4\pi^2}\left[f_c^2(\pi)+3f_s^2(\pi)S_\gamma\left(\frac{\mu_B|H|}{T}\right)\right]\frac{T}{E_F},$$

$$\frac{\delta\chi_s(T,H)}{\chi_s(0,0)}=f_s^2(\pi)\frac{T}{2E_F}S_\chi\left(\frac{\mu_B|H|}{T}\right),$$

$$\frac{\delta\chi_s(q)}{\chi_s(0)}=f_s^2(\pi)\frac{|q|}{3\pi k_F},\tag{1}$$

where $\delta\gamma(T,H)=\gamma(T,H)-\gamma(0,0)$, $\delta\chi_s(T,H)=\chi_s(T,H)-\chi_s(0,0)$, $\delta\chi_s(q)=\chi_s(q)-\chi_s(0)$, $f_c(\pi)=(m/\pi)[U(0)-U(2k_F)/2]$, $f_s(\pi)=-(m/2\pi)U(2k_F)$, $\gamma(0)=m\pi/3$, $\chi_s(0)=\mu_B^2m/\pi$, $E_F=mv_F^2/2$, μ_B is the Bohr magneton, and the limiting forms of the scaling functions are $S_\gamma(0)=S_\chi(0)=1$ and $S_\gamma(x\gg 1)=1/3$, $S_\chi(x\gg 1)=2x$. [Regular renormalizations of the effective mass and g factor are absorbed into $\gamma(0)$ and $\chi_s(0)$.] The second-order susceptibility increases with H and q, indicating a tendency either to a metamagnetic–first-order ferromagnetic transition or to a transition into a spiral state. These tendencies signal a possible breakdown of the Hertz-Millis scenario of the ferromagnetic quantum critical point (QCP).[6]

Experimentally, a linear T dependence of the specific heat coefficient has been observed in thin films of ^{3}He.[7] A linear increase of χ_s with magnetization (and thus H) has been observed in a 2D GaAs heterostructure.[8] Since none of these experiments correspond to the weak-coupling limit, there is obviously a need for a nonperturbative treatment of nonanalytic terms.

It has recently been shown[4,9] that the second-order result for $\gamma(T,0)$ in Eq. (1) becomes exact once the weak-coupling backscattering amplitudes $f_{c,s}(\pi)$ are replaced by the exact ones. This implies that the $O(T)$ term in γ is determined exclusively by 1D scattering events embedded into the 2D space. In these events, fermions with almost opposite momenta experience either a small or almost $2k_F$ momentum transfer. It has been conjectured in Ref. 4 that the nonanalytic part of the spin susceptibility can be generalized in the same way—i.e., by replacing weak-coupling f_s in Eq. (1) by its exact value. The same result was obtained within the supersymmetric theory of the Fermi liquid[10] and in the analysis of the scattering amplitude in the Cooper channel, $f_s^C(\theta)$.[11] Since an extension of the second-order result for $\delta\chi_s$ hints at far-reaching consequences for the ferromagnetic QCP, it is important to establish a general form of $\chi_s(T,H)$.

In this Rapid Communication, we present a general analysis of the nonanalytic behavior of the spin susceptibility in

2D. We show that, in contrast to the specific heat case, higher orders in the interaction are *not* absorbed into the renormalization of $f_s(\pi)$ [equal to $f_s^C(\pi)$], but give rise to extra $O(T,H)$ terms, whose prefactors are given by infinite series of the scattering amplitudes $f_s(\theta)$ and $f_s^C(\theta)$ at all angles. That higher-order terms in $f_s^C(\theta)$ are relevant was noticed in Ref. 11. We show that higher-order terms in $f_s(\theta)$ are also present. These terms are more important than higher powers of $f_s^C(\theta)$ as the latter are logarithmically reduced at small T and H. When the interaction is neither weak nor peaked in some particular channel, the total prefactor of the $O(T,H)$ term may, generally speaking, be of either sign. However, the universality is restored near a ferromagnetic QCP, where the $n=0$ partial component of $f_s(\theta)$ diverges. We show that near the QCP the inverse susceptibility $\chi_s^{-1}(T=0,H)$ behaves as $\xi^{-2}-A|H|^{3/2}$, where ξ is the correlation length. This dependence is dual to the $\xi^{-2}-B|q|^{3/2}$ behavior of the nonuniform susceptibility.[12] The signs of A and B are positive, so that the nonanalytic terms destroy a continuous transition towards a uniform ferromagnetic state, and depending on parameters, the system undergoes either a second-order transition into a spiral phase or a first-order transition into a ferromagnetic state.

The temperature and magnetic field dependences of $\gamma(T,H)$ and $\chi_s(T,H)$ are most straightforwardly obtained by evaluating the thermodynamic potential at finite T and H, $\Xi(T,H)$, and then differentiating it twice with respect to T or H, respectively. To understand the difference between the spin susceptibility and the specific heat, consider for a moment the case of a contact interaction: $U(q)\equiv U$. To second order in U, the thermodynamic potential Ξ is expressed via the convolutions of the polarization bubbles Π (Ω_m,q,H) (with opposite spin projections for $H\neq 0$). The polarization bubble has a conventional form

$$\Pi_{\uparrow\downarrow}(\Omega_m,q) = -\frac{m}{2\pi}\left(1-\frac{|\Omega_m|}{\sqrt{(\Omega_m-2i\mu_B H)^2+v_F^2q^2}}\right) = -\frac{m}{2\pi}+\Pi_{\mathrm{dyn}}. \tag{2}$$

For large momenta $(v_Fq\gg|\Omega_m|\sim\mu_B|H|)$, the dynamic part Π_{dyn} behaves as $|\Omega_m|/q$. Consequently, the momentum integral $\int d^2q\Pi_{\mathrm{dyn}}^2$ diverges logarithmically and is cut at $q=\max\{|\Omega_m|,\mu_B|H|\}$. Because of the logarithm, the subsequent summation over frequencies yields a universal term $\Xi(T,H)\propto\max\{T^3,(\mu_B|H|)^3\}$. More precisely, $\Xi(T,H)\propto T^3p(\mu_BH/T)$, where $p(x\ll 1)=a+bx^2+\cdots$ and $p(x\gg 1)\sim|x|^3$. Accordingly, $\delta\gamma(T)/\gamma(0)\sim\delta\chi_s(T)/\chi_s(0)\propto T$ and $\delta\chi_s(H)\propto|H|$.

To second order, $\delta\gamma(T)$ and $\delta\chi_s(T,H)$ behave similarly. The difference between the two quantities shows up at higher orders in U. In the rest of the paper, we consider only scattering in the particle-hole channel. As we have already mentioned, there are higher-order contributions from the Cooper channel, but they are logarithmically small in a generic Fermi liquid and nonsingular near a ferromagnetic instability. A particle-hole contribution of order n contains integrals of $\Pi^n=\Pi_{\mathrm{dyn}}^n+c_{n-1}\Pi_{\mathrm{dyn}}^{n-1}+\cdots$, where c_n are the constants. The nonanalytic part of $\gamma(T)$ is solely related to the logarithmic divergence of the momentum integral $\int d^2q\Pi_{\mathrm{dyn}}^2\propto\ln|\Omega_m|$, because only the logarithmic singularity ensures the nonanalytic result of the subsequent Matsubara sum: $T\Sigma\Omega_m^2\ln|\Omega_m|=\mathrm{const}-O(T^3)$. The momentum integrals of Π_{dyn}^k with $k>2$ are not logarithmically divergent, and the subsequent frequency summation gives rise only to regular, $T^2,T^4,\ldots$ powers of T in $\gamma(T)$. As a result, higher-order diagrams for $\gamma(T)$ only renormalize the bare interaction U into a full backscattering amplitude. For $\chi_s(H)$, the situation is different—higher powers of Π_{dyn} do contribute additional $|H|^3$ terms to Ξ and, hence, additional $|H|$ terms to the susceptibility. Indeed, evaluating $\Xi(0,H)$ to third order in U and retaining only the contribution $\delta\Xi^{(3,3)}(0,H)$ from Π_{dyn}^3, we obtain

$$\delta\Xi^{(3,3)}(0,H) = \frac{u^3}{6\pi^2}\,\mathrm{Re}\int_0^{E_F}d\Omega_m\int_0^{\infty}dqq \times\frac{\Omega_m^3}{[(\Omega_m-2i\mu_BH)^2+v_F^2q^2]^{3/2}} = \frac{2}{3\pi}\frac{u^3}{v_F^2}\mu_B^3|H|^3, \tag{3}$$

where $u=mU/(2\pi)$. The momentum and frequency integrals in $\delta\Xi^{(3,3)}$ come from the region $|\Omega_m|\sim v_Fq\sim\mu_B|H|$. This implies that the $U^3|H|^3$ term in Eq. (3) appears by purely dimensional reasons and does not require the q integral in Ξ to be logarithmically divergent.

We see that the $|H|$ terms in $\delta\chi_s(H)$ coming from two and three or more dynamic bubbles correspond to physically distinct processes. The distinction becomes important for a generic Fermi liquid. The contribution to $\delta\chi_s(H)$ from two dynamic bubbles, which starts at order U^2, is generalized beyond the weak-coupling limit by replacing the bare interaction by a fully renormalized static vertex. Using the same computational procedure as in Ref. 9, we find that, similarly to the specific heat, the exact result for this contribution is expressed in terms of the spin part of the backscattering amplitude $f_s(\pi)$:

$$\delta\chi_s^{(2)}(H)\rightarrow\delta\chi_s^{BS}=\frac{2}{\pi v_F^2}f_s^2(\pi)\mu_B^3|H|. \tag{4}$$

The same result was obtained in Refs. 10 and 11. The logarithmic divergence of the momentum integral of Π_{dyn}^2 is the crucial element in the derivation of Eq. (4), as the higher-order corrections can be absorbed into static $f_s(\pi)$ only if typical v_Fq are much larger than typical $|\Omega_m|$, given by $\mu_B|H|$.

On the other hand, contributions to $\delta\chi_s(H)$ from three and more dynamic bubbles come from $v_Fq\sim|\Omega_m|$ and are expressed via the convolutions of the partial harmonics of the scattering amplitudes, which do not reduce to higher powers of the backscattering amplitude. As an illustration, we consider a generalization of the third-order contribution $\delta\Xi^{(3,3)}$, assuming that the spin component of the scattering amplitude has only two partial components: $n=0$, 1—i.e., $f_s(\theta)=f_{s,0}+\cos\theta f_{s,1}$. Replacing each interaction vertex by $f_s(\theta)$, we obtain

$$\delta\chi_s^{(3)}(H) \to \delta\chi_s^{\mathrm{any}} = \frac{4}{\pi v_F^2}\mu_B^3|H| \times [f_{s,0}^3 - a_1 f_{s,0}^2 f_{s,1} - a_2 f_{s,0} f_{s,1}^2 + a_3 f_{s,1}^3], \quad (5)$$

where $a_1=3(2\ln 2-1)$, $a_2=3(3\ln 2-2)$, and $a_3=(5/2-3\ln 2)$. This expression obviously does not reduce to the cube of the backscattering amplitude, which in this approximation would be equal to $f_s(\pi)=f_{s,0}-f_{s,1}$. Higher-order contributions are given by progressively more complicated combinations of $f_{s,0}$ and $f_{s,1}$.

The total result for $\delta\chi_s$ is a sum of backscattering and all-angle scattering contributions. Since $f_s(\pi)$ is equivalent to the spin component of the particle-particle scattering amplitude $f_s^C(\pi)$,[3,11] the repulsive interaction in the Cooper channel leads to a logarithmic reduction of $f_s(\pi)$.[10,11,13] For $T=0, H\to 0, f_s(\pi)\propto 1/\ln|H|$ and therefore $\delta\chi_s^{\mathrm{BS}}\propto |H|/\ln^2|H|$.[14] On the other hand, contributions to $\delta\chi_s$ from three and more dynamic bubbles contain angular averages of $f_s(\theta)$, which are not affected by the Cooper singularity. Therefore, the $|H|$ terms from these contributions do not acquire additional logarithms and win over the backscattering contribution for $T,H\to 0$ [and also over higher-order terms in $f_s^C(\theta)$, which vanish logarithmically at $T,H\to 0$]. Note that for $|f_{s,1}/f_{s,0}|<1$, the sign of χ_s^{any} in Eq. (5) is determined just by the sign of $f_{s,0}$: for negative $f_{s,0}$ (corresponding to enhanced ferromagnetic fluctuations), $\delta\chi_s$ decreases with H, whereas for positive $f_{s,0}$, $\delta\chi_s$ increases with H.

Next, we discuss the behavior of the spin susceptibility in the vicinity of a ferromagnetic QCP, where $f_{s,0}$ diverges, while other components of f_s remain finite. At any finite distance from the QCP, the backscattering amplitude still vanishes as $1/\ln\max\{\mu_B|H|,T\}$. However, at large $f_{s,0}$, this behavior is confined to an exponentially small range of H and T, which we will not consider below. Outside this range, the backscattering amplitude diverges as $f_{s,0}$, and the backscattering contribution $\delta\chi_s^{\mathrm{BS}}(T,H)$ diverges as $f_{s,0}^2$. All-angle contributions, however, diverge even more strongly, and one needs to sum up a full series of diagrams to obtain the behavior of $\delta\chi_s(T,H)$ near the QCP. To do this, we assume, as was done in Refs. 4 and 15, that the Eliashberg approximation is valid near the QCP because overdamped spin fluctuations are slow compared to fermions. In the Eliashberg theory, the field-dependent part of the thermodynamic potential is $\Xi=(1/2\pi)T\Sigma_{\Omega_m}\int dqq\ln\chi_0^{-1}(q,\Omega,H)$, where

$$\chi_0(q,\Omega_m,H) = \frac{m}{\pi}\frac{\mu_B^2}{\delta + a^2q^2 + (2\pi/m)\Pi_{\mathrm{dyn}}(\Omega_m,q)} \quad (6)$$

is the dynamic spin susceptibility without nonanalytic corrections, $\delta=|f_{s,0}|^{-1}$, and a is the radius of the exchange interaction, required to be large ($ak_F\gg 1$) for the Eliashberg theory to work.[16] Π_{dyn} differs from Eq. (2) in that (i) it is built on full Green's functions (containing self-energies) and (ii) μ_B in the denominator is replaced by $\mu_B^*=\mu_B/\delta$ (cf. Ref. 17). To illustrate once again the difference between the specific heat and spin susceptibility, we set $\xi=\infty$ in the denominator of Eq. (6) and neglect the self-energy renormalization for a moment. Evaluating the derivatives of $\Xi(T,H)$ with respect to T and H, we find that the prefactor of the T term in the specific heat coefficient diverges,[9] whereas the prefactor of the $|H|$ term in the spin susceptibility remains finite: $\delta\chi_s(H)/(\mu_B^*)^3=(2/\pi v_F^2)|H|$. This indicates dramatic cancellations between diverging terms in the perturbation theory for $\delta\chi_s(H)$.[18]

A complete result for the susceptibility near the QCP is obtained by including the self-energies when evaluating Π_{dyn} in Eq. (6) (the vertex corrections are small; see Ref. 15). The self-energy near the QCP interpolates between $\Sigma=\lambda\omega_m$ away from QCP and $\Sigma=\omega_0^{1/3}|\omega_m|^{2/3}$ near QCP, where $\lambda=3/(4k_Fa\sqrt{\delta})$ and $\omega_0=3\sqrt{3}E_F/[4(k_Fa)^4]$.[19,20] Using these expressions, we obtain for the inverse susceptibility

$$\chi_s^{-1}(H,T=0) \propto \delta - \frac{8}{3}\frac{\mu_B^*|H|}{v_F/a}\sqrt{\delta}K_H\left(\frac{\mu_B^*|H|ma^2}{\delta}\right), \quad (7)$$

where $K_H(0)=1$ and $K_H(x\gg 1)=1.25\sqrt{x}$. The limit of $x\to\infty$ describes the situation right at the QCP. Here, divergent ξ cancels out from the answer and the H dependence of χ_s^{-1} becomes $|H|^{3/2}$. We emphasize that the exponent of 3/2 is the consequence of non-Fermi-liquid behavior, manifested by the divergence of the "effective mass" $\partial\Sigma/\partial\omega_m\propto\omega_m^{-1/3}$.

The nonanalytic $|H|^{3/2}$ dependence exists only at $T\to 0$. At finite T, the field dependence of the spin susceptibility is analytic: $\delta\chi_s\propto H^2$. However, the prefactor scales as $1/(\lambda T)$ away from the QCP and as $T^{-1/6}$ at the QCP. At $H=0$, $\delta\chi_s(T)\propto T\ln T$.[15,21]

A complementary way to see the nonanalytic dependence of the susceptibility on the magnetic field is to analyze the thermodynamic potential itself. Viewed as a function of the magnetization $M=(m/\pi)\eta\mu_B$, where 2η is the difference of the Fermi energies for spin-up and spin-down fermions, the thermodynamic potential $\Xi(T=0,\eta)$ contains a nonanalytic $|\eta|^3$ term away from criticality: $\Xi_{na}(0,\eta)=-|\eta|^3/48\pi v_F^2\lambda$. Near the QCP, λ diverges and the $|\eta|^3$ dependence is replaced by $|\eta|^{7/2}$, in agreement with the $H^{3/2}$ field dependence of the spin susceptibility. At finite T, the $|\eta|^{7/2}$ term evolves into an analytic η^4 one with a singular prefactor $T^{-1/6}$.

We now study the consequences of the nonanalytic behavior of $\chi_s(T,H,q)$. First, we see from Eq. (7) that the spin susceptibility diverges at some finite value of H, which implies that a second-order ferromagnetic QCP is preempted by the first-order one. This possibility was discussed in detail in Ref. 22—our analysis differs from this work in that we include the fermionic self-energy and nonanalytic T dependence of the susceptibility. Assuming that the first-order transition occurs near the QCP, where the nonanalytic term in $\Xi(T=0,\eta)$ is $\eta^{7/2}$, we have

$$\frac{\pi}{m}\Xi(T=0,\eta) = \frac{\delta}{2}\eta^2 - \frac{|\eta|^{7/2}}{E^{3/2}} + b^2\eta^4, \quad (8)$$

with $E=3.82E_F/(k_Fa)^{4/3}$. Because of the nonanalytic term, Ξ has a minimum at finite η. The first-order transition occurs at $\delta_H=(k_Fa/3.32)^8/(bE_F)^6$ when $\Xi=0$ at this minimum. By an

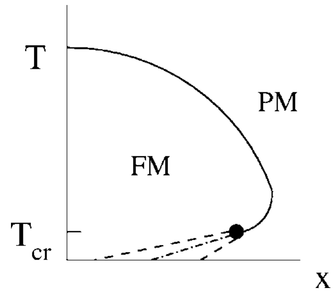

FIG. 1. Schematic phase diagram near a ferromagnetic QCP for $\rho<1$ (see text). x is a control parameter—e.g., pressure. The second-order phase transition becomes first-order below a tricritical point because of the nonanalytic $\eta^{7/2}$ term in Eq. (8). At small T, the transition line has an S-shaped form because of the negative T dependence of $\chi_s^{-1}(T)$.

order of magnitude, $b\sim 1/E_F$. The first-order transition occurs in the critical region ($\delta_H<1$) for $k_F a<3.32(bE_F)^{3/4}$.

The susceptibility at $H=0$ but finite q is given by

$$\chi_s^{-1}(q)=\delta+a^2(q^2-cq^{3/2}k_F^{1/2}), \qquad (9)$$

where $c\approx 0.25$.[15] $\chi_s^{-1}(q)$ diverges at $q=q_0=0.035k_F$ for $\delta_q=0.42\times 10^{-3}(ak_F)^2$. This signals a transition into an incommensurate phase.

Which of the two instabilities occurs first depends on the nonuniversal parameter $\rho=\delta_q/\delta_H=(1.35bE_F/ak_F)$.[6] For $\rho>1$, the first instability is into the incommensurate phase; for $\rho<1$, the first-order transition occurs first (see Fig. 1). Although formally ak_F should be large, both situations are actually possible, particularly if bE_F is a large number.

At finite T, the transition line has an S-shaped form (see Fig. 1) because of the negative T dependence of $\chi^{-1}(T)$.[23] The tricritical point separating the second- and first-order transitions results from the balance between the $b^2\eta^4$ term in Eq. (8) and the $\eta^4/T^{1/6}$ term which replaces the $|\eta|^{7/2}$ term at finite T.

To summarize, in this paper we considered the temperature and magnetic field behavior of the spin susceptibility of a 2D Fermi liquid, both away and near a ferromagnetic QCP. We found that in a Fermi-liquid phase, $\delta\chi_s(T,H)\propto\max\{T,|H|\}$, but the prefactor is not expressed solely in terms of the backscattering amplitude, in contrast to the specific heat. At a ferromagnetic QCP, the magnetic field dependence of $\chi^{-1}(T=0,H)$ becomes $H^{3/2}$, with a universal, negative prefactor. This behavior favors a first-order transition to ferromagnetism and competes with the $q^{3/2}$ behavior of $\chi^{-1}(T=H=0,q)$ which favors an incommensurate spin ordering.

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