Perturbation theory for localized solutions of the sine-Gordon equation: Decay of a breather and pinning by a microresistor

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We develop a perturbation theory that describes bound states of solitons localized in a confined area. External forces and the influence of inhomogeneities are taken into account as perturbations to exact solutions of the sine-Gordon equation. We have investigated two special cases: a fluxon trapped by a microresistor and decay of a breather under dissipation. We have also carried out numerical simulations with the dissipative sine-Gordon equation and made a comparison with the McLaughlin-Scott theory. A significant distinction between the McLaughlin-Scott calculation for a breather decay and our numerical result indicates that the history dependence of the breather evolution cannot be neglected even for a small damping parameter.

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I. INTRODUCTION

Solitons or solitary waves are among the most interesting objects in nature. Observation of a solitary wave on water was first documented more than one and a half centuries ago. Besides, solitons occur naturally in many other substances like optical fibres,[1] nonlinear lattices,[2] and hot and cold plasma[3] and are even claimed to be responsible for Jupiter's red spots[4] and energy transfer in DNA.[5] Most intensively solitons have been studied in long Josephson contacts. The matter is that switching from the superconducting to the resistive state of the Josephson junction is related to the appearance and motion of solitons in these contacts which are known also as Josephson vortices or fluxons. Such solitons or fluxons are well described by the sine-Gordon equation.

In the ideal case, when the Josephson junction is infinitely long and narrow, Josephson solitons can be described analytically by well-known exact solutions of the sine-Gordon equation. However, there is always dissipation associated with quasiparticle current through the Josephson junction and inhomogeneities associated with its width and thickness. Moreover, real physical systems are always subjected to the influence of external forces. All these factors may have a significant impact on soliton behavior.

Although the strictly one-dimensional sine-Gordon equation is integrable,[6,7] perturbations to this equation associated with external forces and inhomogeneities spoil its integrability and the equation cannot be solved exactly. Nevertheless, if their influence is small, the solution can be found perturbatively. The perturbation theory for solitons was described in detail by Keener and McLaughlin,[8] while their interaction by McLaughlin and Scott.[9] Later, in application to the dynamics of vortices in Josephson contacts, perturbation analysis of the sine-Gordon equation was developed by McLaughlin and Scott.[10]

Many applications deal with localized oscillatory solutions of the sine-Gordon equation: for instance, when a Josephson vortex is pinned by an inhomogeneity or there is a bound state of a vortex and antivortex known as a breather. Breathers may appear as a result of collision of a fluxon with an antifluxon or even in the process of measurements of switching current characteristics.[11] The role of breathers is ambiguous. Depending on our expectations, they can be parasitic excitations or, vice versa, a good generator of THz waves. Recently we proposed a device that may deliberately generate and trap breathers.[12]

There have been many theoretical and numerical studies dedicated to continuous sine-Gordon breathers.[13–17] In particular, the decay of a breather into a fluxon and antifluxon induced by an external current has been studied by many authors.[13] Moreover, it was shown[14] that a breather can be stabilized by an ac drive even in the presence of energy losses. Also, the influence of the boundaries on breather dynamics[15] has been investigated and quantization of its energy spectrum[16] has been predicted.

Nevertheless, despite numerous theoretical studies, the dynamics of a breather under dissipation has not been fully understood. McLaughlin-Scott theory gets overcomplicated when applied to nontrivial solutions such as breathers, whereas its simplifications fail to predict the correct dynamics. We have performed numerical simulations of breather dynamics and found that there is a significant discrepancy with the McLaughlin-Scott calculation. In particular, it manifests itself in the dependence of the breather energy on time, Fig. 1. The thin line is the dependence following from the McLaughlin-Scott calculation [formula (5.5) in Ref. 10], and the solid line represents our numerical simulations. This discrepancy stimulated us to look into this problem once again and develop a perturbation theory that is designed especially for localized solutions of the sine-Gordon equation. We have found that at the construction of such theory it is very important to take into account the history dependence of the breather evolution. Also, we have carried out direct numerical simulations with the dissipative sine-Gordon equation. The numerical results appear in perfect agreement with our theory.

II. PERTURBATION THEORY FOR LOCALIZED SOLUTIONS

Consider the $(1+1)$-dimensional sine-Gordon equation

$$\ddot{\varphi} - \varphi_{xx} + \sin \varphi = 0. \tag{1}$$

The equation possesses solutions in the form of solitons (antisolitons):

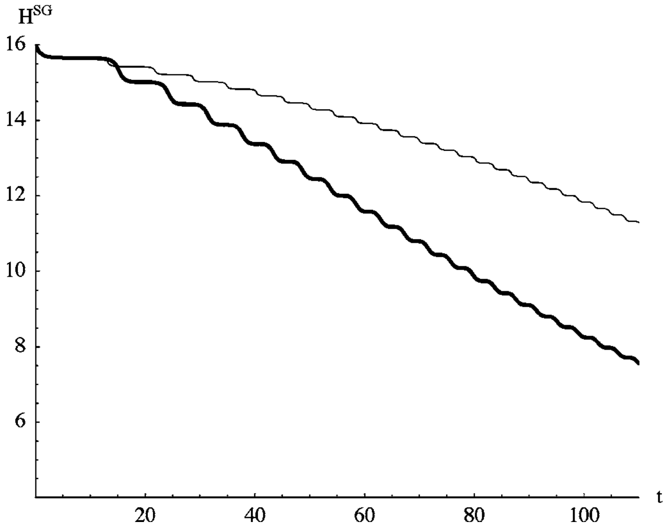

FIG. 1. Dissipative dynamics of a sine-Gordon breather: Dependence of the energy H^{SG} of a breather on time t calculated according to the McLaughlin-Scott formula (5.5) from Ref. 10 is presented by the thin solid line. Dependence of the energy H^{SG} of a breather on time t calculated by direct numerical simulations of sine-Gordon equation with damping is shown by the thick solid line. The damping constant is $\alpha=0.01$.

$$\varphi(x,t) = 4 \arctan \exp\left(\pm \frac{x-ut}{\sqrt{1-u^2}} \right). \tag{2}$$

The peculiar feature of solitons is that they keep their shapes while moving and even restore their shapes after collision. The attractive forces[9] between a soliton and antisoliton allow their bound state—a sine-Gordon breather[6,10] (sometimes called doublet or bion),

$$\varphi(x,t) = 4 \arctan \left(\frac{\sin \dfrac{ut}{\sqrt{1+u^2}}}{u \cosh \dfrac{x}{\sqrt{1+u^2}}} \right), \tag{3}$$

which oscillates periodically in time with frequency $\omega = u/\sqrt{1+u^2}$.

The reason why the McLaughlin-Scott formula for breather decay [formula (5.5) in Ref. 10] fails to predict correctly the dissipative dynamics of a breather is the following. The breather is an oscillatory solution that is characterized by some "phase" that depends on the history of the evolution. In their general formulation McLaughlin and Scott treat this difficulty by introducing the history-dependent term $\int_{t_0}^{t} u(t')dt'$ and allowing additional time-dependent modulation of the free parameters (such as initial positions of fluxons or phases of breathers) in the nonperturbed solution.[8,10] The modulation of the free parameters is governed by additional differential equations. Obviously, this leads to additional complications because of the coupled differential equations for the modulated parameters. Moreover, with such a modulation the original solution no longer satisfies the nonperturbed sine-Gordon equation exactly so that additional perturbation terms appear.[10] Here we describe a method that does not involve the modulation of free parameters, but correctly deals with the time-dependent dynamics due to an appropriately chosen ansatz of the nonperturbed sine-Gordon solution.

Consider a solution of the sine-Gordon equation (1) in the form $\varphi(g(u)x, g(u)ut, u)$ with $g(u)=1/\sqrt{1\pm u^2}$. Such a parametrization is natural for the sine-Gordon equation and obviously comprises the special cases of a soliton (2) and a breather (3). The sine-Gordon Hamiltonian is a functional of the field variable φ,

$$H^{SG}[\varphi] = \int_{-\infty}^{\infty} dx \left[\frac{\dot{\varphi}^2}{2} + \frac{\varphi_x^2}{2} + 1 - \cos \varphi \right]. \tag{4}$$

Substitution of $\varphi=\varphi(g(u)x, g(u)ut, u)$, gives the effective energy as a function of a single parameter u,

$$H_{eff}^{SG}(u) = H^{SG}[\varphi(g(u)x, g(u)ut, u)].$$

The second argument of $\varphi(g(u)x, g(u)ut, u)$, which we call here a phase $T(t)=g(u)ut$, can be written in different ways, such as $T(t)=g(u)\int_0^t u dt'$ or $T(t)=\int_0^t g(u)u dt'$. Obviously, in the case of u independent of time these cases are equivalent and the choice does not make any difference. However, this definition of the phase is very important when taking into account the influence of perturbations, as will be shown below.

In the presence of perturbations we assume that the dominant effect is to modulate the parameter $u=u(t)$. In other words, with an appropriate choice of $u(t)$ we may satisfy the perturbed sine-Gordon equation

$$\ddot{\varphi} - \varphi_{xx} + \sin \varphi = \epsilon f$$

by the function $\varphi=\varphi(g(u(t))x, T(t), u(t))$. Here, we take the perturbation ϵf in a general form

$$\epsilon f = -\sum_i \mu_i \delta(x-x_i) \sin \varphi - \gamma - \alpha \dot{\varphi},$$

which may include pointlike inhomogeneities μ_i that modify locally the critical current density of a Josephson junction, external bias γ, and damping characterized by parameter α. In contrast to the case of constant u, the choice of nonperturbative solution is not unique anymore. Indeed, depending on the choice of the phase $T(t)$, we come out with different functions of t. We will show that with the appropriate choice of the phase $T(t)$ we may correctly describe the time evolution of localized sine-Gordon solutions in the presence of perturbations. We describe the dynamics by a single modulated parameter $u=u(t)$ without introducing additional modulation of the free parameters. This gives considerable simplification and improvement because the other free parameters such as initial location of solitons or initial phases of breathers remain fixed and do not result in auxiliary differential equations like those introduced in Ref. 10.

Consider the ansatz

$$\varphi(g(u(t))x, T(t), u(t)), \tag{5}$$

$$T(t) = \int_0^t g(u(t'))u(t')dt',$$

where function φ is an exact solution of nonperturbative sine-Gordon equation (1). In further consideration we omit highlighting the explicit dependence of the functions $u=u(t)$ and $T=T(t)$ for typographical convenience. Obviously, the drawback of the time modulation of u affects the time derivative of φ,

$$\dot{\varphi} = \frac{d}{dt}\varphi(g(u)x,T,u) = \varphi_1 g_u \dot{u}x + \varphi_2 g u + \varphi_3 \dot{u},$$

where φ_1, φ_2, and φ_3 are derivatives of φ with respect to the first, second, and third arguments correspondingly. As we consider localized solutions confined in some area $|x|<C$, the term $\varphi_1 g'(u)\dot{u}x$ is of the order of $O(\epsilon)$. The third term also can be neglected as it does not contain explicit linear terms in x and t. Therefore, denoting $\varphi_T \equiv \varphi_2$, we get

$$\dot{\varphi} = \varphi_T g u + O(\epsilon), \tag{6}$$

which remains valid even in the limit of large times, $t\to\infty$. Obviously, another choice of $T(t)$ would spoil this equation with terms explicitly dependent on time t; e.g., for $T(t)=g(u(t))\int_0^t u(t')dt'$ we would have

$$\dot{\varphi}(g(u)x,T,u) = \varphi_1 g_u \dot{u}x + \varphi_2 g u + \varphi_2 g_u \dot{u}\int_0^t u(t')dt' + \varphi_3 \dot{u},$$

which contains a nonzero term $\int_0^t u(t')dt'$ proportional to t. Thus, in this case the dynamics would not be correctly described on large time scales $t\to\infty$. Mclaughlin and Scott overcome this problem introducing additional modulation of free parameters. Substituting Eq. (6) into Eq. (4) we obtain the effective energy as a function of $u(t)$,

$$H^{SG}[\varphi(g(u(t))x,T(t),u(t))] \simeq H^{SG}_{eff}(u(t)), \tag{7}$$

which is valid for any values of t. It is important to note that this expression coincides exactly with the effective energy of nonperturbed solution (4) and depends on time indirectly only via $u(t)$.

In the presence of external forces, we may write the full Hamiltonian,

$$H = H^{SG} + H^P$$

and take into account the dissipative perturbations affecting the energy dissipation rate:[10]

$$\frac{dH}{dt} = -\int_{-\infty}^{\infty} \alpha\dot{\varphi}^2 dx. \tag{8}$$

The Hamiltonian H^P serves to describe nondissipative perturbations induced by external potential forces. This could be microshorts, microresistors, or applied driving current:

$$H^P = \int_{-\infty}^{\infty}\left(\sum_i \mu_i \delta(x-x_i)(1-\cos\varphi) + \gamma\varphi\right)dx.$$

Thus, from Eq. (8),

$$\frac{dH^{SG}}{dt} = -\int_{-\infty}^{\infty}\left(\sum_i \mu_i \delta(x-x_i)\dot{\varphi}\sin\varphi + \gamma\dot{\varphi} + \alpha\dot{\varphi}^2\right)dx.$$

Substituting (7), we obtain the equation for parameter u,

$$\dot{u}\frac{dH^{SG}_{eff}}{du} = -\int_{-\infty}^{\infty}\left(\sum_i \mu_i \delta(x-x_i)\dot{\varphi}\sin\varphi + \gamma\dot{\varphi} + \alpha\dot{\varphi}^2\right)dx, \tag{9}$$

where $\varphi=\varphi(g(u)x,T)$ and $\dot{\varphi}\simeq\varphi_T g(u)u$ should be substituted. Equation (9) is coupled to the equation for T,

$$\dot{T} = g(u)u. \tag{10}$$

In some cases it can be convenient to rewrite this system of differential equations for independent variable T,

$$\frac{du}{dT} = -\left(\frac{dH^{SG}_{eff}}{du}g(u)u\right)^{-1}\int_{-\infty}^{\infty}\left(\sum_i \mu_i \delta(x-x_i)\dot{\varphi}\sin\varphi + \gamma\dot{\varphi} + \alpha\dot{\varphi}^2\right)dx,$$

$$\frac{dt}{dT} = [g(u)u]^{-1}. \tag{11}$$

The dynamics is described by $u(T(t))$, where $T(t)$ is an inverse function of $t(T)$.

III. PINNING BY A MICRORESISTOR

Let us consider the following ansatz for a single soliton solution:

$$\varphi(g(u)x,T) = 4\arctan\exp[g(u)x - T], \quad g(u) = 1/\sqrt{1-u^2},$$

subjected to the attractive potential of a microresistor representing an inhomogeneity with locally decreased critical current density of a Josephson junction. Then the appropriate term, $\sin\varphi$, of the sine-Gordon equation (1) is modified with the perturbation

$$\epsilon f = -\mu\delta(x)\sin\varphi, \quad \mu<0,$$

which represents the local change of the critical current density. Here the parameter μ describes the relative change of the critical current density on the microresistor. The energy of a soliton is equal to

$$H^{SG}_{eff}(u) \simeq \frac{8}{\sqrt{1-u^2}}.$$

From Eqs. (9) and (10) we obtain the next system of coupled differential equations,

$$\dot{u} = \frac{1}{2}\mu(1-u^2)\mathrm{sech}^2[T(t)]\tanh[T(t)],$$

$$\dot{T} = \frac{u}{\sqrt{1-u^2}}. \tag{12}$$

We have found that after some simplifications, the McLaughlin-Scott formula (4.3) from Ref. 10 can be reduced

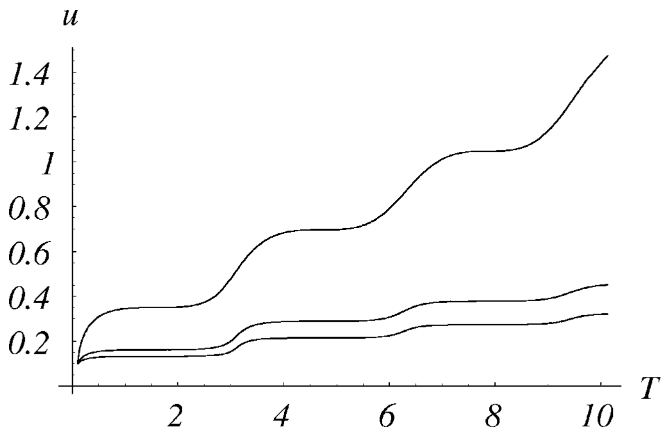

FIG. 2. The dependence of the speed $u(T)$ on the phase T calculated using perturbation theory for localized sine-Gordon solutions at different damping rates. The top curve corresponds to the damping constant $\alpha=0.05$, the middle curve to $\alpha=0.01$, and the lower curve to $\alpha=0.005$.

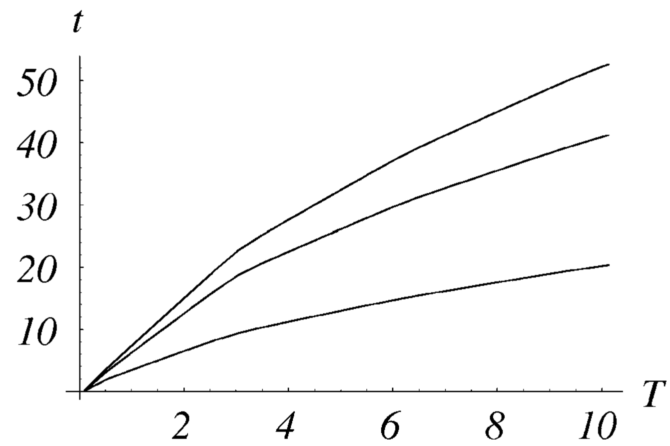

FIG. 3. The dependence of the time t on the phase T calculated using perturbation theory for localized sine-Gordon solutions at different damping rates. The top curve corresponds to the damping constant $\alpha=0.005$, the middle curve to $\alpha=0.01$, and the lower curve to the highest damping $\alpha=0.05$.

to exactly the same system of differential equations. Although both approaches lead to exactly the same result, McLaughlin and Scott's formulation is, obviously, more cumbersome.

IV. DECAY OF A BREATHER

Consider a breather solution

$$\varphi(g(u)x,T,u) = 4\arctan\left(\frac{\sin T}{u\cosh[g(u)x]}\right),$$

with $g(u)=1/\sqrt{1+u^2}$. As a perturbation we consider a dissipative term in the form

$$\epsilon f = -\alpha\dot{\varphi}.$$

Such a term may describe a normal current in the Josephson junction and, therefore, it is responsible for Ohmic losses. The effective energy is

$$H_{eff}^{SG}(u(t)) \simeq \frac{16}{\sqrt{1+u(t)^2}}.$$

From Eq. (11) we obtain the next system of coupled differential equations,

$$\frac{du}{dT} = \alpha\frac{(1+u^2)^{3/2}\cos^2 T}{\sin^2 T + u^2}\left[1 + \frac{u^2 \operatorname{arctanh}\left(\frac{\sin T}{\sqrt{\sin^2 T + u^2}}\right)}{\sin T\sqrt{\sin^2 T + u^2}}\right],$$

$$\frac{dt}{dT} = \frac{\sqrt{1+u^2}}{u}, \qquad (13)$$

where $u=u(T(t))$. This is a result that may not be obtained from the McLaughlin-Scott theory by straightforward manipulation. The system can be solved numerically. For an illustration we present the solution of these equations in Figs 2 and 3. There the initial conditions are taken as $u(0.1)=0.1$ and $t(0.1)=0.1$. One may notice the steplike character of the dependence $u(T)$; see Fig. 2. The size of the steps increases with damping, indicating the importance of the introduction of the phase $T(t)$. This phase has also nontrivial dependence on time t. Its inverse function $t(T)$ is presented in Fig. 3. One may notice that at some values of T there is a fast change of the slope. Obviously this is related to the steplike character of the dependence $u(T)$. The dissipative dynamics of a breather is also well reflected by the time dependence of its energy, Fig. 4. The results are in agreement with our numerical simulations using the complete sine-Gordon equation with dissipative term.

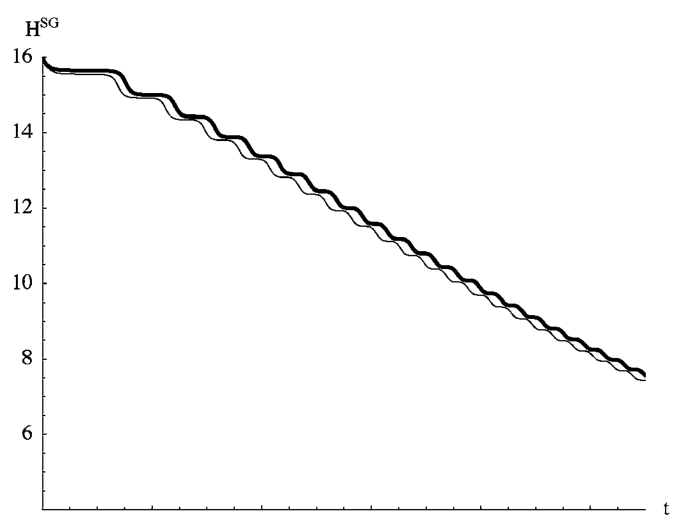

FIG. 4. Dissipative dynamics of a sine-Gordon breather: Dependence of the energy H^{SG} of a breather on time t calculated using perturbation theory for localized sine-Gordon solutions (thin line). Dependence of the energy H^{SG} of a breather on time t calculated by direct numerical simulations of the sine-Gordon equation with damping is shown by the thick solid line. The damping constant is $\alpha=0.01$.

In the case of Josephson junctions, which are usually described by the sine-Gordon equation, the time parameter t is normalized to the plasma frequency ω_p. Our results indicate that the breather lifetime is about $1/(\alpha\omega_p)$. Typically, $\omega_p = 10^{12}\ \mathrm{s}^{-1}$ for Josephson junctions implemented with Nb and $\omega_p = 10^{11}\ \mathrm{s}^{-1}$ for high-temperature superconductors such as BSCCO (bismuth strontium calcium copper oxide). Thus, the lifetime of a breather excitation in Josephson junctions made from Nb or BSCCO characterized by the level of dissipation constant $\alpha = 0.01$ would be of the order of nanoseconds.

V. CONCLUSION

In summary, we have shown that our perturbation theory describes well the dynamics of localized excitations subjected to the influence of external forces such as various inhomogeneities and damping associated with quasiparticle current. In particular, we have described a fluxon trapped in a potential well which could be related to a microresistor in the Josephson junction. Here the equations derived with the use of our method coincide identically with equations derived by McLaughlin and Scott.[10] However, the derivation of these equations obtained by our method is significantly simpler. Second, we have described the decay of the breather under dissipation. In this case, the equations describing such a decay are different from McLaughlin and Scott's.[10] Moreover, according to our calculation, the one-dimensional sine-Gordon breather decays significantly faster, see Fig. 1. In order to resolve this difference we have performed numerical simulations with the dissipative sine-Gordon equation. The results of these numerical simulations are in close agreement with our theoretical results. The comparison of our perturbation theory with the McLaughlin-Scott calculation[10] indicates that the history dependence of the breather evolution has a strong influence on its dynamics even at low damping.

To conclude, we have developed a perturbation theory which perfectly describes the localized-in-space solutions of the sine-Gordon equation. Such a study can be important for new devices such as fluxon colliders or other devices[12] based on the recently discovered flux cloning effect[18] where the dissipative dynamics and the breather excitations may play a key role. The theory may allow generalization to higher dimensions. This can be of use to study localized pulsating solutions of the sine-Gordon equation in two spatial dimensions.[19]

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