Unconventional superconducting pairing symmetry induced by phonons

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The possibility of non-*s*-wave superconductivity induced by phonons is investigated using a simple model that is inspired by Sr_2RuO_4. The model assumes a two-dimensional electronic structure, a two-dimensional spin-fluctuation spectrum, and three-dimensional electron-phonon coupling. Taken separately, each interaction favors formation of spin-singlet pairs (of *s* symmetry for the phonon interaction and $d_{x^2-y^2}$ symmetry for the spin interaction), but in combination, a variety of more unusual singlet and triplet states are found, depending on the interaction parameters. These results suggest that phonons could play a key role in establishing the order-parameter symmetry in Sr_2RuO_4, and possibly in other unconventional superconductors.

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I. INTRODUCTION

The existence of superconducting states that break symmetries of the normal state beyond gauge symmetry has generated considerable interest. Such unconventional pairing was first discovered in ^{3}He, in which the three superfluid phases all have spin-triplet ($S=1$) atomic Cooper pairs with relative orbital angular momentum $L=1$ (p wave).[1] Strong evidence for unconventional superconductivity was subsequently found in heavy-fermion materials.[2] In the high-T_c cuprates, phase-sensitive experiments have provided convincing evidence for $L=2$ (d-wave) spin-singlet pairing.[3] The pairing symmetry in Sr_2RuO_4, a noncuprate perovskite oxide superconductor, has been unambiguously shown to be lower than the symmetry of the underlying crystal lattice, i.e., $L>0$.[4] It is generally believed that Sr_2RuO_4 is a triplet p-wave superconductor,[4] though it has been noted that a d-wave singlet state is also possible.[5] It has been suggested that some compounds could be superconductors with $L>2$.[6,7] More exotic types of pairing, such as order parameters with odd-frequency dependence and mixed singlet-triplet Cooper pairs, have also been considered for real materials.[7–9]

By itself, the electron-phonon interaction is always attractive, and hence it maximizes the pairing energy for an order parameter that is positive everywhere on the Fermi surface. Conventional wisdom thus suggests that $L>0$ pairing symmetries must be induced by nonphonon mechanisms, such as magnetic or Coulomb interactions. A number of authors have pointed out, however, that while phonons are always more pairing in the *s* channel, they may also be pairing in other channels as well.[10] If, for example, strong on-site Coulomb repulsion suppresses *s*-wave pairing because of the large overlap of the wave functions of the paired particles, or if there is an additional pairing contribution in an $L>0$ channel from other (nonphonon) mechanisms, unconventional pairing that results mainly from electron-phonon coupling is possible. Less well appreciated is the idea that a combination of electron-phonon coupling and another pairing interaction (e.g., spin-fluctuation exchange) may result in a superconducting state that is not the ground state of either of the two interactions taken separately.

In this paper we consider a simple model, loosely based on Sr_2RuO_4, that illustrates this effect. The model assumes a highly two-dimensional electronic structure, and it includes a two-dimensional spin-fluctuation-induced interaction that favors d-wave pairing of x^2-y^2 (or $\cos 2\phi$) symmetry,[11] and a three-dimensional electron-phonon interaction that favors an *s*-wave state. The spin interaction is based on the experimentally observed spin-fluctuation spectrum of Sr_2RuO_4, which has negligible dispersion along z.[12,13] While less is known about the electron-phonon coupling in Sr_2RuO_4, the three-dimensional nature of the crystal structure suggests that the electron-phonon interaction in the material may be much more three dimensional than the electronic structure itself. When the coupling strengths of the two interactions are comparable, we find that the symmetry of the emerging ground state can be changed by tuning the widths of the interactions in momentum space. Among the states that are found to be stable in certain regions of phase space are two-dimensional (z-independent) p-wave states and three-dimensional (z-dependent) singlet and triplet states, all of which arise from the interplay between the two interactions.

II. MODEL

The spin dependence of the pairing in a superconductor can be taken into account using a 2×2 matrix $\hat{\Delta}(\mathbf{k})$ to represent the gap function. Within this spin matrix formalism, the BCS gap equation has the form

$$\Delta_{s_1s_2}(\mathbf{k}) = \sum_{\mathbf{k}'s_3s_4} V_{s_2s_1s_3s_4}(\mathbf{k},\mathbf{k}')\Delta_{s_3s_4}(\mathbf{k}')F(\mathbf{k}',T), \qquad (1)$$

where

$$V_{s_1s_2s_3s_4}(\mathbf{k},\mathbf{k}') = \langle -\mathbf{k}s_1;\mathbf{k}s_2|\hat{V}|-\mathbf{k}'s_4;\mathbf{k}'s_3\rangle \qquad (2)$$

is the matrix element of the effective electron-electron pairing interaction $\hat{V}$, for which we adopt a convention in which positive V corresponds to an attractive interaction, and F is a function of the temperature T and the quasiparticle energy $E_{\mathbf{k}'}$.[14] In a crystal with inversion symmetry, the pairing potential can be separated into spin-triplet (+) and spin-singlet (−) channels,

$$V_{s_1s_2s_3s_4}(\mathbf{k},\mathbf{k}') = V^{(+)}(\mathbf{k},\mathbf{k}')S^{(+)}_{s_1s_2s_3s_4} + V^{(-)}(\mathbf{k},\mathbf{k}')S^{(-)}_{s_1s_2s_3s_4}, \tag{3}$$

where

$$S^{(\pm)}_{s_1s_2s_3s_4} = (\delta_{s_1s_4}\delta_{s_2s_3} \pm \delta_{s_1s_3}\delta_{s_2s_4})/2. \tag{4}$$

The gap equation decouples into separate equations for the singlet and triplet channels. For singlet pairing, the gap matrix has the form $\Delta_0(\mathbf{k})i\hat{\sigma}_y$, while for triplet pairing, it has the form $[\hat{\sigma}\cdot\mathbf{d}(\mathbf{k})]i\hat{\sigma}_y$, where $\hat{\sigma}$ are Pauli spin matrices. Since the total wave function must be antisymmetric under interchange of particles, the scalar order parameter $\Delta_0(\mathbf{k})$ for singlet pairing must be even in $\mathbf{k}$ and the vector order parameter $\mathbf{d}(\mathbf{k})$ must have odd parity.

Near $T=T_c$, the gap function is small and the gap equation can be linearized and reformulated as an eigenvalue problem:

$$f(T_c)\Delta_{s_1s_2}(\mathbf{k}) = \sum_{\mathbf{k}'s_3s_4} V_{s_2s_1s_3s_4}(\mathbf{k},\mathbf{k}')\Delta_{s_3s_4}(\mathbf{k}'). \tag{5}$$

In the singlet channel, this becomes an eigenvalue problem for $V^{(-)}(\mathbf{k},\mathbf{k}')$, with eigenfunctions given by the scalar order parameter $\Delta_0(\mathbf{k})$. For triplet pairing, the eigenvectors of $V^{(+)}(\mathbf{k},\mathbf{k}')$ correspond to components of the vector order parameter $\mathbf{d}(\mathbf{k})$. Among eigenfunctions that have parity compatible with the spin pairing, the one with the largest eigenvalue $f(T_c)$ produces the first superconducting instability that manifests as temperature is lowered, and hence defines the physical T_c.

In our model, the pairing potential has two contributions: an electron-phonon interaction V_{ph} that is the same (and always attractive) regardless of the symmetry of the spin pairing, and an interaction V_{sp} that is mediated by antiferromagnetic spin fluctuations. The latter is attractive in the triplet spin channel and repulsive in the singlet channel. In general, the momentum dependence of V_{sp} is also different in the triplet and singlet channels (see, for instance, Ref. 15 and references therein). In the simplest case of an isotropic contact magnetic coupling, the effective spin interaction in the singlet channel is equal to that in the triplet channel multiplied by a constant factor of -1 for 2D spins and -3 for 3D (rotationally invariant) spins. Even though the latter approximation is supposed to be reasonable for ^{3}He, practical applications have found that it overestimates the total strength of the interaction by at least a factor of 2.[15,16] Lacking detailed knowledge about the nature of the spin-fluctuation-induced pairing interaction in Sr_2RuO_4, we take the simplest choice, namely that the spin-induced interaction is the same in the triplet and singlet channels, but with opposite sign. (This approximation was discussed and used in an earlier study of ferromagnetic spin-fluctuation-induced p-wave pairing in Sr_2RuO_4.[17]) The total pairing interaction then has the form $V=(V_{ph}+V_{sp})S^{(+)}+(V_{ph}-V_{sp})S^{(-)}$.

We assume a simple tetragonal (not body-centered), three-dimensional crystal structure with D_{4h} symmetry and lattice constants a and c. The electronic structure is taken to be two dimensional, with a cylindrical Fermi surface of radius $k_F^{\parallel}=0.9\pi/a$, similar to the γ sheet of the Fermi surface in Sr_2RuO_4. The electron-phonon interaction is assumed to be three dimensional in nature while the antiferromagnetic-spin-fluctuation-induced interaction is two dimensional. Both interactions depend only on $\mathbf{q}=\mathbf{k}-\mathbf{k}'$. The spin-fluctuation-induced interaction $V_{sp}(\mathbf{q})$ is taken to be an isotropic Gaussian in q_x and q_y, centered on $\mathbf{q}=(2\pi/3a,2\pi/3a,q_z)$ (and all lines in reciprocal space that are equivalent by the rotational and translational symmetries of the tetragonal lattice), in accordance with both theory[18] and experiment.[12] The electron-phonon interaction $V_{ph}(\mathbf{q})$ is taken to be a superposition of three-dimensional Gaussian functions centered at $(\pm\pi/a,\pm\pi/a,\pm\pi/c)$, i.e., the corners of the Brillouin zone, and equivalent points in reciprocal space. Since details about the momentum-space structure of the electron-phonon interaction in Sr_2RuO_4 are not known, there is no compelling reason for this choice, beyond the observation that some materials similar to Sr_2RuO_4, such as Sr_2RhO_4 and Sr_2IrO_4, show an instability corresponding to a $(\pi/a,\pi/a,\pi/c)$ soft mode.[19–22] To enrich the model further, we allow the electron-phonon interaction to be anisotropic, with a different width in the z direction than in the plane. The anisotropy of the electron-phonon interaction is characterized by the parameter $\alpha=aw_{ph}^{\parallel}/cw_{ph}^{z}$, where the widths are defined in terms of the Gaussian form $\exp[-(q/w)^2]$.

It should be emphasized that although this model is inspired by Sr_2RuO_4, it is only a toy model chosen to demonstrate the possibility of unconventional pairing symmetries resulting from multiple pairing mechanisms acting in combination. Since we do not have detailed information about the form of the magnetic coupling, the spin anisotropy, the momentum-space dependence of the electron-phonon coupling interaction, etc., and since we do not take into account the full Fermiology, we do not expect the model to describe the real material Sr_2RuO_4.

Because the Fermi surface in the model is perfectly cylindrical and the interaction potentials are separable in Cartesian coordinates, the eigenvalue problem separates into z and planar parts, with product eigenfunctions. The in-plane eigenfunctions are of the form $\cos(m\phi)$ and $\sin(m\phi)$, while the z eigenfunctions are of the form $\cos(nk_zc)$ and $\sin(nk_zc)$, where m and n are integers and $\tan\phi=k_y/k_x$. For $n\neq 0$, the two-dimensional spin interaction cancels out since $\Sigma_{\mathbf{k}'}V_{sp}(\mathbf{k},\mathbf{k}')\cos(nk_z'c)=\Sigma_{\mathbf{k}'}V_{sp}(\mathbf{k},\mathbf{k}')\sin(nk_z'c)=0$ if V_{sp} has no z dependence. This means there are degenerate spin-triplet and spin-singlet states that have the same in-plane eigenfunction. The symmetry of the in-plane eigenfunction determines which one has the cosine solution and which one has the sine solution in k_z. In a real material, this degeneracy between triplet and singlet states would be broken by a variety of effects, including warping of the Fermi surface and the spin-orbit interaction.

Numerical calculations were carried out using a discrete grid of $\mathbf{k}$ points on the Fermi surface. The circumference of the Fermi surface was sampled with 40 equally spaced grid points, while 20 points were used along k_z. The grid was selected so that the spacing between grid points was small compared to the the widths of the interactions. We have also checked convergence by increasing the number of grid points to 60×30 and 80×40.

The effective strength of each interaction in coupling electrons on the Fermi surface is measured by

$$\lambda_{sp(ph)} = \sum_{\mathbf{k},\mathbf{k}' \in \mathbf{FS}} V_{sp(ph)}(\mathbf{k},\mathbf{k}'). \quad (6)$$

For fixed ratios of $R=\lambda_{ph}/\lambda_{sp}$, phase diagrams were mapped out as a function of interaction widths. Because of the discrete sampling of $\mathbf{k}$ points, results are presented only for interactions with Gaussian width greater than $0.1\pi/a$ in the plane. To construct the phase diagrams, we used a recursive algorithm that takes advantage of the fact that most phases occupy large regions in phase space. The gap equation is solved for all points along the border of the rectangular phase diagram. If the ground state is the same at all points on the border, all points in the interior are assumed to have the same ground state. Otherwise, the region is divided into two rectangles and the process is repeated recursively. For each phase diagram, phase space was sampled with a range of resolutions, from 5×5 to 400×400 points, to mitigate the chance of missing phases that might be stable in only small interior regions of the diagram.

III. RESULTS

We consider first the case when one of the interactions dominates. When the spin coupling is much larger than the phonon coupling ($R\ll1$), only one phase appears regardless of the widths of the spin and phonon interactions. This is the singlet $\cos 2\phi$ state (commonly denoted $k_x^2-k_y^2$). Because the spin interaction is repulsive in the singlet channel, to induce pair formation, it must scatter electrons between points $\mathbf{k}$ and $\mathbf{k}'$ on the Fermi surface that have order parameter of opposite sign [see Eq. (5)]. For $\cos 2\phi$ symmetry, three of the four lines in the Brillouin zone on which the spin interaction is maximal do this; the fourth couples nodal regions. Hence this d-wave state is stabilized by the antiferromagnetic spin fluctuations. When the phonon interaction dominates ($R\gtrsim5$), only an s-wave spin singlet appears in the phase diagram. This is expected since the attractive electron-phonon interaction is pair forming when the order parameter has the same sign at points $\mathbf{k}$ and $\mathbf{k}'$ on the Fermi surface.

Figure 1 shows the phase diagram for equal spin and phonon coupling strengths ($R=1$). In the diagram, the width of the spin-fluctuation-induced interaction increases from left to right, and the width of the electron-phonon interaction increases from top to bottom. The anisotropy parameter for the width of the electron-phonon interaction is $\alpha=1$. In the regions labeled B (green), the s-wave singlet favored by the electron-phonon interaction is stable, while in region C (blue), the $\cos 2\phi$ spin-singlet state favored by the spin-fluctuation-induced interaction is stable. New phases also appear. There is a manifold of degenerate triplet states with in-plane p-wave pairing (A, orange), and degenerate singlet and triplet states that have k_z-dependent order parameters (D, purple). The character of these phases is summarized in Table I. We have mapped out phase diagrams for coupling-strength ratios ranging from $R=0$ to $R=5$. With the electron-phonon anisotropy fixed at $\alpha=1$, no states other than the ones that are shown in the $R=1$ diagram are found to be stable.

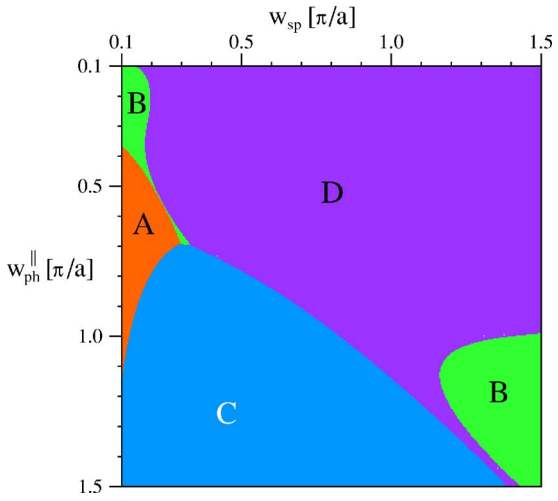

FIG. 1. (Color online) Phase diagram for coupling-strength ratio $R=\lambda_{ph}/\lambda_{sp}=1$ and electron-phonon anisotropy $\alpha=1$. See Table I for the symmetry of states that appear in this phase diagram.

It is interesting that the two-dimensional triplet manifold (A, orange) emerges despite the fact that the interactions individually stabilize singlet states. For this triplet state, the coupling matrix [Eq. (5)] has two degenerate eigenvalues, with eigenfunctions $\cos\phi$ and $\sin\phi$, for each component of the vector order parameter $\mathbf{d}$. This generates the set of allowed p-wave states on a cylindrical Fermi surface with tetragonal lattice symmetry, as derived, for example, in Ref. 14. Our model corresponds to the weak-coupling, nonrelativistic regime, where these states are all degenerate. A chiral member of this set of p-wave triplet states, with $\mathbf{d}\propto(k_x+ik_y)\hat{\mathbf{z}}$, is a leading contender for the order parameter in Sr_2RuO_4.[4] The two-dimensional triplet manifold is stable over a range of coupling strength ratios, but in all cases, it is limited to the region of the phase diagram corresponding to narrow spin interactions. This can be understood from the form of the spin interaction. In the triplet channel, the spin-fluctuation-mediated interaction is attractive, so to induce pairing it must scatter electrons between points on the Fermi surface that have order parameters of the same sign. Wave vectors along the line $\mathbf{q}=(4\pi/3a,4\pi/3a,q_z)$, for which the spin interaction is strong, have an in-plane component that is slightly larger than $2k_F^{\parallel}$, the diameter of the Fermi cylinder. Hence, if the spin interaction is narrowly peaked in the planar directions, these wave vectors contribute little to the total spin-coupling. With wider spin interactions, however, they have a strong pair-breaking effect if the order parameter has p-wave symmetry in the plane.

The singlet phase in region D (purple) has symmetry $\cos 2\phi\cos(k_zc)$ and can be thought of as an extended d-wave state, as illustrated in Fig. 2. It belongs to the same representation as the purely two-dimensional $\cos 2\phi$ state. For small w_{ph}, it is more stable than the $\cos 2\phi$ state. This is because in the two-dimensional $\cos 2\phi$ state favored by the spin interaction, the phonon interaction centered at $\mathbf{q}=(\pi/a,\pi/a,\pi/c)$ is always pair breaking, while the inclusion of a $\cos(k_zc)$ factor allows the phonon interaction to

TABLE I. Symmetry of gap functions of states that appear in Figs. 1–5. For orbital symmetry (L), the commonly used s, p, d wave, etc., nomenclature is adopted, with s^*, for example, denoting extended s-wave symmetry. For spin-singlet states ($S=0$), basis functions for the scalar order parameter Δ_0 are given, while for spin triplets ($S=1$), basis functions for the components of the vector order parameter d_i ($i=x,y,z$) are listed. In the weak-coupling, nonrelativistic approximation used in this work, all triplet states with the same **k** dependence are degenerate, independent of the direction of the **d** vector. For instance, all 11 p-wave states listed for tetragonal crystals in the Sigrist and Ueda review (Ref. 14) ($\hat{\mathbf{z}}k_x$, $\hat{\mathbf{x}}k_x \pm \hat{\mathbf{y}}k_y$, etc.) are degenerate in this approximation and correspond to the state labeled here as A. Actual gap functions include higher-order Fourier components that transform in the same way under the symmetry operations of the crystal (D_{4h}) as the basis functions shown in the table.

	S	L	$\Delta_0(\mathbf{k})$ or $d_i(\mathbf{k})$
A	1	p	$\sin\phi$, $\cos\phi$
B	0	s	1
C	0	d	$\cos 2\phi$
D	0	d^*	$\cos 2\phi \cos(k_z c)$
	1	f	$\cos 2\phi \sin(k_z c)$
E	0	s^*	$\cos(2k_z c)$
	1	p^*	$\sin(2k_z c)$
F	0	d	$\sin 2\phi$
G	0	g	$\sin 4\phi$
H	0	d	$\sin\phi \sin(k_z c)$, $\cos\phi \sin(k_z c)$
	1	p^*	$\sin\phi \cos(k_z c)$, $\cos\phi \cos(k_z c)$
I	0	d^*	$\cos 2\phi \cos(3k_z c)$
	1	f^*	$\cos 2\phi \sin(3k_z c)$
J	0	s^*	$\cos(4k_z c)$
	1	p^*	$\sin(4k_z c)$
K	0	d^*	$\sin\phi \sin(3k_z c)$, $\cos\phi \sin(3k_z c)$
	1	p^*	$\sin\phi \cos(3k_z c)$, $\cos\phi \cos(3k_z c)$
L	0	d^*	$\sin 2\phi \cos(2k_z c)$
	1	f^*	$\sin 2\phi \sin(2k_z c)$

become pair forming. Degenerate with this extended d-wave singlet is a set of triplet f states, one example of which is $\mathbf{d} \propto \cos 2\phi \sin(k_z c)\hat{\mathbf{z}}$. Similar f states have been discussed in connection with heavy-fermion superconductors, e.g., in UPt_3.[23] Note that because the two-dimensional spin interaction integrates out, the stabilization of the extended d-wave singlet and the degenerate f-wave triplets is due entirely to the electron-phonon interaction.

Figure 3 takes vertical and horizontal cuts through the phase diagram shown in Fig. 1 to illustrate how the eigenvalues change with the widths of the phonon and spin interactions. In Fig. 3(a), the width of the electron-phonon interaction is fixed at $w_{ph}^{\parallel}=0.625\pi/a$ with $\alpha=1$. In Fig. 3(b), the width of the spin-fluctuation-induced interaction is fixed at $w_{sp}=0.25\pi/a$. In addition to the four phases that appear in Fig. 1, corresponding to those states with the largest eigenvalues within the phase space of the diagram, phases with other symmetries appear in Fig. 3. Table I lists the symmetry

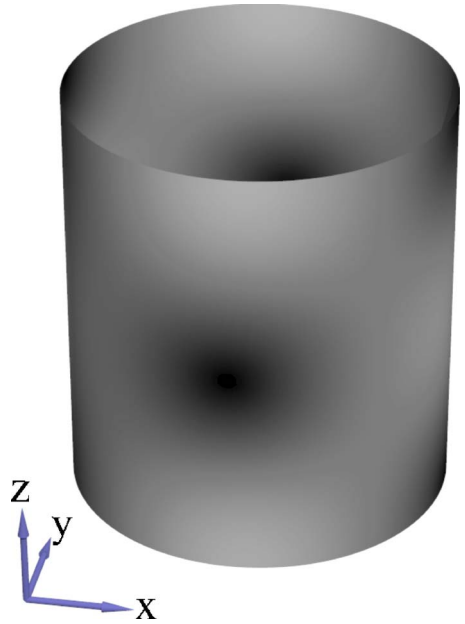

FIG. 2. (Color online) Order parameter of the singlet state stable in region D. White corresponds to the maximum positive value of Δ_0 and black corresponds to the minimum negative value. The Γ point is at the center of the cylinder. The full eigenfunction for $R=1$, $w_{sp}=0.5\pi/a$, $w_{ph}^{\parallel}=0.5\pi/a$, and $\alpha=1$ is plotted, but the lowest-order nonzero Fourier component, $\cos 2\phi \cos(k_z c)$, clearly dominates.

of these states. Curve H (gold), for example, corresponds to a manifold of three-dimensional singlets with Δ_0 given by linear combinations of $\sin\phi \sin(k_z c)$ and $\cos\phi \sin(k_z c)$, which includes the chiral d-wave state, $(k_x+ik_y)\sin(k_z c)$ that is compatible with existing experimental data on the symmetry of the order parameter in Sr_2RuO_4.[5] When both the spin and phonon interactions are very narrow, or when both are wide, this three-dimensional d-wave singlet (H) becomes more stable than the two-dimensional p-wave triplet (A) that is generally believed to be the most likely candidate for the pairing symmetry in Sr_2RuO_4.[4] The increased stability of the d-wave singlet as compared to the p-wave triplet in certain parameter regimes arises from the same mechanism that was discussed earlier in relation to the extended d-wave state (D, purple). Other symmetries, however, can make even better use of the structure of the pairing interaction in these parts of parameter space, so the three-dimensional d-wave singlet does not become the favored phase.

By narrowing the width of the electron-phonon interaction in the z direction, the stabilizing mechanism for states with k_z-dependent order parameters can be enhanced. Figure 4 illustrates how the eigenvalues change with anisotropy in the phonon interaction. The eigenvalues depend on α only if the order parameter varies with k_z. As expected, the three-dimensional d-wave singlet (H, gold) is further stabilized as α is increased, but within our model and within the parameter space explored, it never emerges as the most stable phase. Nevertheless, one can imagine that with a different structure of electron-phonon coupling in reciprocal space, this solution could become stable in some parameter regime. When w_{ph}^{z} is very small, an extended s-wave singlet state (E,

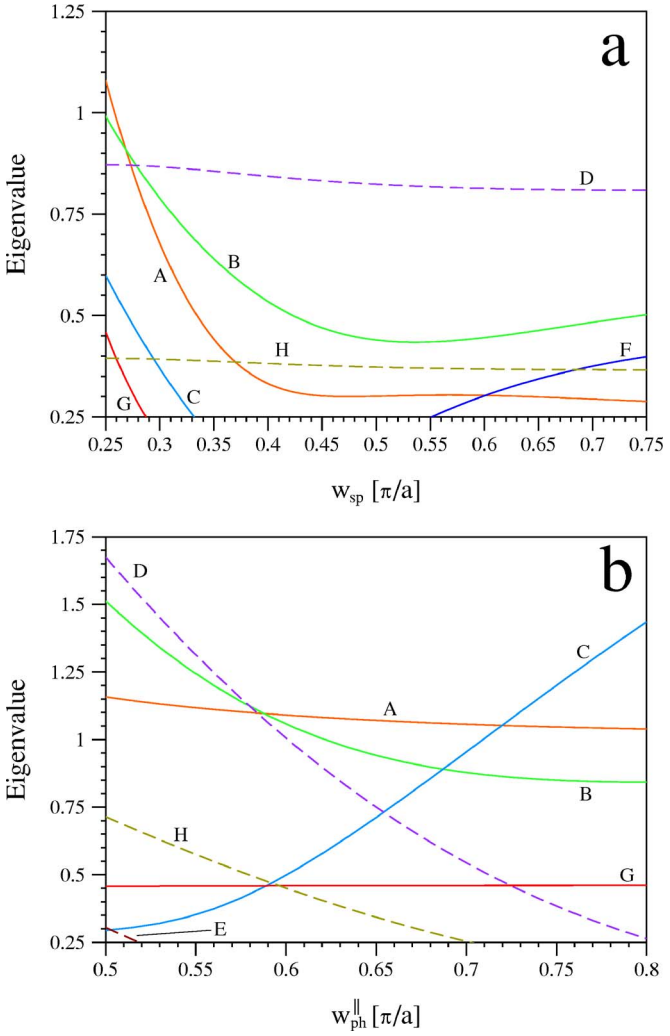

FIG. 3. (Color online) Dependence of eigenvalues on (a) width of spin-fluctuation-induced interaction, w_{sp}, and (b) in-plane width of electron-phonon interaction, $w_{ph}^{\parallel}$, for coupling-strength ratio $R=1$ and electron-phonon anisotropy $\alpha=1$. In (a), $w_{ph}^{\parallel}=0.625\pi/a$, and in (b), $w_{sp}=0.25\pi/a$. Dashed lines indicate states with k_z-dependent eigenfunctions. See Table I for the full symmetry of the eigenfunctions.

dark red) with symmetry $\cos(2k_z c)$ becomes stable and occupies a large area of the phase diagram (Fig. 5). As discussed earlier, for states with k_z-dependent order parameters, the two-dimensional spin fluctuations integrate out and play no role (i.e., they are neither pairing nor pair breaking). This is reflected in Fig. 5 by the fact that the phase boundary between region D (purple) and region E (dark red) is perfectly horizontal.[24] Each of those phases represents one singlet (extended d or extended s, for D and E, respectively) and several triplet (f or extended p for D and E, respectively) states.

IV. CONCLUSIONS

The combination of spin- and phonon-mediated pairing interactions with different structure in momentum space can lead to a rich variety of unconventional pairing states, many with symmetries and parities that could never be stabilized by either interaction alone. Within the toy model considered here, with a cylindrical Fermi surface, two-dimensional spin-fluctuation-mediated pairing, and three-dimensional phonon-mediated pairing, there are two-dimensional pairing states with no dispersion in z and three-dimensional k_z-dependent states. For the two-dimensional states, the interplay of the phonon and spin interactions, each of which favors a two-dimensional singlet state, can result in a triplet state. For the three-dimensional states, the symmetry of the model suppresses the effect of the spin interaction, and phonon-induced non-s-wave states become possible.

This toy model is only loosely connected with the unconventional superconductor Sr_2RuO_4, yet it suggests a resolu-

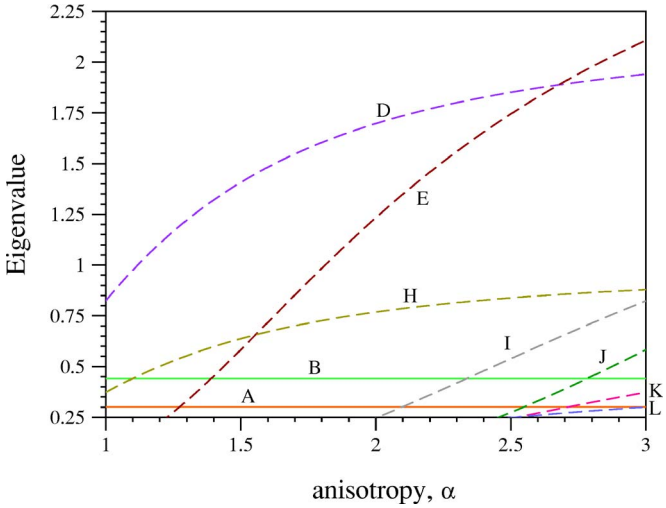

FIG. 4. (Color online) Dependence of eigenvalues on the anisotropy of the electron-phonon interaction, for coupling-strength ratio $R=1$, spin-interaction width $w_{sp}=0.5\pi/a$, and in-plane phonon-interaction width $w_{ph}^{\parallel}=0.625\pi/a$. Dashed lines indicate states with k_z-dependent eigenfunctions. See Table I for the full symmetry of the eigenfunctions.

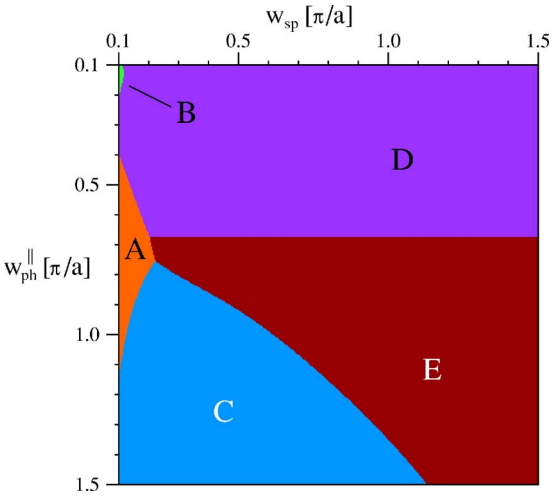

FIG. 5. (Color online) Phase diagram for coupling-strength ratio $R=1$ and electron-phonon anisotropy $\alpha=2.5$. See Table I for the symmetry of states that appear in this phase diagram.

tion to the seeming paradox regarding the symmetry of the superconducting state in this compound. While most researchers believe that superconductivity is induced by the strong spin fluctuations observed in this system, the momentum dependence of these fluctuations, well known from neutron measurements,[12,13] should stabilize the same $d_{x^2-y^2}$ pairing symmetry as in the cuprates. Yet this symmetry can be excluded based on a variety of experiments.[4,25] On the other hand, a large isotope effect on T_c indicates the importance of phonons,[26] yet the s or extended s symmetry favored by phonons can be excluded with even more confidence. Our results suggest the unconventional and rather unusual (in all likelihood, chiral) symmetry of the superconducting state in Sr_2RuO_4 may emerge as a result of interplay of electron coupling with both phonons and spin fluctuations, although neither of these agents separately can stabilize such a state. A quantitative application of these ideas to Sr_2RuO_4 requires a complete mapping of the electron-phonon coupling in the Brillouin zone to complement the existing map of spin fluctuations, as well as consideration of the full, multisheeted Fermi surface. It should also address possible quantitative differences between the spin-fluctuation-induced interactions in the triplet and singlet channels.

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