Quantum spherical spin model on hypercubic lattices

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(Received 30 May 2006; revised manuscript received 23 August 2006; published 1 November 2006)

We present an alternative treatment to the quantum spherical spin model on $(d \geq 2)$-dimensional hypercubic lattices, focusing on the effects of quantum (g) and thermal (T) fluctuations, under a uniform magnetic field h, on the correlation function, correlation length, entropy, specific heat, and energy gap in the excitation spectrum. Explicit expressions for such quantities are provided close to the $d \geq 2$ quantum $(g=g_c, T=0)$ and $d \geq 3$ thermal $[T=T_c(g)]$ phase transitions in $h=0$, including the low-T quantum regimes near the quantum critical point. In particular, the calculation of the correlation function and correlation length generalizes the results on the $g=0$ classical spherical model. At $T=0$, the zero-field system is gapless at and below g_c; however, a gap opens in the quantum-disordered ground state, $g > g_c$. Conversely, the null gap for $T \leq T_c(g \neq 0)$ becomes finite as $T \to T_c(g \neq 0)^+$; thus, quantum fluctuations suppress the critical prefactors of observables near $T_c(g \neq 0)$, though they are irrelevant to the universality class shared with the gapless classical spherical model. The results on the entropy and specific heat in $g \neq 0$ circumvent the drawback in classical spherical models concerning the third law of thermodynamics, as $T \to 0$.

DOI: 10.1103/PhysRevB.74.184101 PACS number(s): 75.10.Jm

I. INTRODUCTION

Since the seminal work by Berlin and Kac,[1] it has been widely accepted that classical spherical models[2–4] have played an important role in statistical mechanics due to the opportunity they offer to rigorously study properties otherwise uncommonly probed through exact calculations, such as the critical behavior of observables close to thermal phase transitions and finite-size scaling hypotheses, just to name a few. The result by Stanley[5] that the spherical condition maps onto the limit of infinity spin dimensionality of the Heisenberg classical model has provided a way of contact between the spherical model and realistic spin systems. Consequently, over the last decades classical spherical models have been largely applied to study a rich variety of systems, including antiferromagnets[6] with competing interactions[7] and Lifshitz points,[8] critical phenomena involving long-range interactions[3,4,9] and topological considerations,[10] phase separation,[11] spin-charge effects in the context of the Hubbard model,[12] finite-size systems,[13] and disordered models,[14] such as electronic systems with localized states,[15] infinite-ranged spin glasses,[16] and random-field models.[17]

In spite of this, the early finding[1–4] of the anomaly present in classical spherical models regarding the third law of thermodynamics has provided a first indication that the suppression of quantum fluctuations can generally lead to unphysical behavior in very low temperatures T. Actually, the more recent emergence[18,19] of quantum phase transitions at $T=0$ has confirmed this scenario in regimes where the relevant fluctuations are of quantum rather than thermal nature, starting with the reformulation of the concept of hyperscale in $T=0$ transitions proposed by Hertz.[20] Since then, the interest in anomaly-free quantum spherical models[21] has grown in contexts as diverse as spin glasses,[22] thermodynamic properties,[23,24] and quantum phase transitions in d-dimensional hypercubic lattices,[25] including finite-size effects,[26] random field models,[27] quantum Lifshitz points,[28] systems with ferromagnetic coupling in transverse magnetic field,[29] and ferrimagnetic chains with AB_2 unit-cell topology.[30]

In this work we present an alternative treatment to the quantum spherical spin model on $(d \geq 2)$-dimensional hypercubic lattices. We focus on the effects of quantum (g) and thermal (T) fluctuations, under a uniform magnetic field h, on the correlation function, correlation length, entropy, and specific heat, described in detail close to both $d \geq 2$ quantum $(g=g_c, T=0)$ and $d \geq 3$ thermal $[T=T_c(g)]$ phase transitions in $h=0$, including the zero-field low-T quantum regimes near the quantum critical point. We consider a second-quantization Hamiltonian in which quantum fluctuations are introduced through a kinetic term on operators canonically conjugated to the spin degrees of freedom, with strength controlled by a tunable quantum parameter g, in a way similar to that of Refs. 25 and 26. In the latter, the authors have studied[26] finite-size scaling properties in $1 < d < 3$, with special attention to the low-T zero-field susceptibility and equation of state of the infinite system in $d=2$; in the former, the focus has been on the critical exponents near the quantum and thermal transitions.[25] Here we present explicit expressions for the correlation function, correlation length, entropy, and specific heat in a variety of possible critical paths around the $d \geq 2$ quantum and $d \geq 3$ thermal phase transitions. In particular, our results on the correlation function and correlation length generalize the calculation for the $g=0$ classical spherical model.[31] Moreover, a detailed study of the energy gap in the excitation spectrum in the mentioned regimes is provided on the basis of its relevance to the universality class of the transitions. Actually, in addition to the investigation on critical exponents, we are also interested in the influence of quantum fluctuations on the suppression of prefactors of critical observables. The $T \to 0$ behavior of the entropy and specific heat is also explicitly shown to obey the third law of thermodynamics in $g \neq 0$. In this sense, the low-T specific heat and entropy behaviors of quantum spherical models have been also investigated in Refs. 23 and 29. However, in the absence of canonically conjugated operators, the author in Ref. 23 has considered a model with double-spin field at each site and a couple of adjustable parameters to get rid of the classical anomaly in low T; and in Ref. 29, by taking into account such operators in the spherical constraint, a $z=2$ dynamical exponent has been obtained, in contrast with the value $z=1$ reported in Refs. 25 and 26 and in the present work.

The paper is organized as follows. In Sec. II we introduce the model Hamiltonian and diagonalize it in a second-quantization scheme to obtain the spectrum of eigenmodes. By studying the spherical constraint and the chemical potential calculation in the Appendix, we identify the critical and thermal phase transitions and confirm previous results on the magnetization, zero-field susceptibility, and equation of state. Sections III and IV are, respectively, devoted to the derivation of the correlation function, correlation length, and energy gap, and entropy and specific heat close to the quantum and thermal transitions, in the regimes devised in the Appendix. Finally, discussion and conclusions are presented in Sec. V.

II. QUANTUM SPHERICAL SPIN MODEL

We consider the quantum spherical spin model on a hypercubic lattice in $d \geq 2$:

$$H = \frac{g}{2}\sum_{\vec{R}} P_{\vec{R}}^2 - \frac{J}{2}\sum_{\langle \vec{R},\vec{R}'\rangle} S_{\vec{R}} S_{\vec{R}'} - h\sum_{\vec{R}} S_{\vec{R}} + \mu\sum_{\vec{R}} (S_{\vec{R}}^2 - 1/4), \tag{1}$$

where $J>0$ is the ferromagnetic coupling between $S=1/2$ spins at first-neighbor sites and h denotes a uniform magnetic field in energy units, $h \equiv \mu_{eff} H$, in which μ_{eff} is the effective Bohr magneton. The chemical potential μ ensures the mean spherical constraint[1,2]

$$\sum_{\vec{R}} \langle S_{\vec{R}}^2 \rangle = \frac{N}{4}, \tag{2}$$

where N is the total number of spins (or sites located at $\vec{R}$ lattice positions) and $\langle \cdots \rangle$ denotes the standard quantum thermal average. Notice that a continuous variation of the spin average value is allowed, with no upper and lower bounds, provided Eq. (2) is satisfied. We introduce quantum fluctuations by assigning[21] a canonically conjugated momentum $P_{\vec{R}}$ to each spin degree of freedom, so that the following commutation relations hold ($\hbar=1$): $[S_{\vec{R}}, S_{\vec{R}'}]=0$, $[P_{\vec{R}}, P_{\vec{R}'}]=0$, $[S_{\vec{R}}, P_{\vec{R}'}]=i\delta_{\vec{R},\vec{R}'}$, where $\delta_{\vec{R},\vec{R}'}$ is the Kronecker delta. The tunable quantum parameter $g>0$, in energy units, controls the strength of the quantum fluctuations responsible for the spin dynamics. Although this choice for the dynamics is not unique, it is by far the most usual one, as can be inferred from Refs. 21–29. The above features actually make the spin fields in quantum spherical models more like unit quantum rotors,[18,32–34] than standard SU(2) spin operators usually considered in Heisenberg models.

In order to diagonalize Eq. (1) in a second-quantization scheme, we first introduce creation ($a_{\vec{R}}^\dagger$) and annihilation ($a_{\vec{R}}$) bosonic operators through $S_{\vec{R}}=(g/8\mu)^{1/4}(a_{\vec{R}}+a_{\vec{R}}^\dagger)$ and $P_{\vec{R}}=-i(\mu/2g)^{1/4}(a_{\vec{R}}-a_{\vec{R}}^\dagger)$. By Fourier transforming $a_{\vec{R}}^\dagger$ and $a_{\vec{R}}$, Eq. (1) becomes

$$H = \sqrt{2g\mu}\sum_{\vec{k}}\left(1-\frac{J_{\vec{k}}}{2\mu}\right)a_{\vec{k}}^\dagger a_{\vec{k}} - \sqrt{2g\mu}\sum_{\vec{k}}\frac{J_{\vec{k}}}{4\mu}(a_{-\vec{k}}a_{\vec{k}} + a_{-\vec{k}}^\dagger a_{\vec{k}}^\dagger) - \sqrt{\frac{N}{2}}\left(\frac{g}{2\mu}\right)^{1/4}(a_0 + a_0^\dagger)h + \sqrt{2g\mu}\frac{N}{2} - \frac{\mu N}{4}, \tag{3}$$

where $J_{\vec{k}}=J\sum_{i=1}^d \cos(k_i)$, with $\vec{k}$ in the first Brillouin zone, and $a_0=a_{\vec{k}=0}$. Now, defining

$$\alpha_{\vec{k}\pm} = \frac{1}{2\sqrt{2}}\left(1-\frac{J_{\vec{k}}}{2\mu}\right)^{1/4}\left[\left(1+\frac{1}{\sqrt{1-J_{\vec{k}}/(2\mu)}}\right)(a_{\vec{k}} \pm a_{-\vec{k}}) \pm \left(1-\frac{1}{\sqrt{1-J_{\vec{k}}/(2\mu)}}\right)(a_{\vec{k}}^\dagger \pm a_{-\vec{k}}^\dagger)\right], \tag{4}$$

$$\alpha_0 = \frac{1}{2}\left(1-\frac{J_0}{2\mu}\right)^{1/4}\left[\left(1+\frac{1}{\sqrt{1-J_0/(2\mu)}}\right)a_0 + \left(1-\frac{1}{\sqrt{1-J_0/(2\mu)}}\right)a_0^\dagger\right] - \sqrt{\frac{N}{2}}\left[\frac{g}{2(\mu-dJ)}\right]^{1/4}\frac{h}{\sqrt{2g(\mu-dJ)}}, \tag{5}$$

Eq. (1) is diagonalized as follows:

$$H = \sum_{\vec{k}>0}\omega_{\vec{k}}(\alpha_{\vec{k}+}^\dagger\alpha_{\vec{k}+} + 1/2) + \sum_{\vec{k}>0}\omega_{\vec{k}}(\alpha_{\vec{k}-}^\dagger\alpha_{\vec{k}-} + 1/2) + \omega_0(\alpha_0^\dagger\alpha_0 + 1/2) - \frac{Nh^2}{4(\mu-dJ)} - \frac{\mu N}{4}, \tag{6}$$

where

$$\omega_{\vec{k}} = \sqrt{2g(\mu - J_{\vec{k}})} \tag{7}$$

are the model eigenfrequencies. Since $J_{\vec{k}}$ assumes its maximum value $J_0=dJ$ at $\vec{k}=0$, we notice that values $\mu>dJ$ implies in $\omega_{\vec{k}}>0$ for all $\vec{k}$ modes. Conversely, values $\mu<dJ$ are not allowed and the special choice $\mu=dJ$ causes the $\vec{k}=0$ eigenfrequency ω_0 to vanish.

From Eqs. (6) and (7) the Helmholtz free energy is calculated ($k_B=1$):

$$F = T\sum_{\vec{k}}\ln[2\sinh(\beta\omega_{\vec{k}}/2)] - \frac{Nh^2}{4(\mu-dJ)} - \frac{\mu N}{4}. \tag{8}$$

The spherical constraint, Eq. (2), is derived through $\partial F/\partial\mu=0$:

$$\sum_{\vec{k}}\frac{g}{2\omega_{\vec{k}}}\coth(\beta\omega_{\vec{k}}/2) + \frac{Nh^2}{4(\mu-dJ)^2} = \frac{N}{4}. \tag{9}$$

In the Appendix we consider Eq. (9) in the continuous $N \gg 1$ limit and obtain the solutions for $\mu(g,T,h)$ near the $d\geq 2$ quantum ($g=g_c$, $T=0$) and $d\geq 3$ thermal $[T=T_c(g)]$ $h=0$ transitions. We mention that in Ref. 26 the authors have considered a distinct approach to Eq. (9) in $N \gg 1$ and $d=2$, near the quantum critical point as $T \to 0$; when appropriate, comparison between the two approaches is provided.

III. CORRELATION FUNCTION, CORRELATION LENGTH, AND GAP DISCUSSION

In this section the correlation function and correlation length of the quantum spherical model in $d \geq 2$ are provided, generalizing the results to the classical spherical model.[31] Although in Refs. 25 and 26 similar quantum spherical models have been investigated, no explicit expressions for the correlation function, correlation length, and energy gap have been reported in the variety of $d \geq 2$ quantum and $d \geq 3$ thermal critical paths considered below. Indeed, whereas in Ref. 26 the authors have presented a detailed study of finite-size scaling properties in $1 < d < 3$, with special attention to the low-T zero-field susceptibility and the equation of state of the infinite system, in Ref. 25 the focus has been on the critical exponents of the quantum and thermal transitions through scaling arguments.

With the aid of the following two-operator quantum thermal averages,

$$\langle \Delta_r^\dagger \Delta_s \rangle = \frac{\delta_{r,s}}{e^{\beta\omega_r} - 1}, \quad \langle \Delta_r \Delta_s \rangle = 0,$$

$$\langle \Delta_r^\dagger \Delta_s^\dagger \rangle = 0, \quad \langle \Delta_r \Delta_s^\dagger \rangle = \frac{\delta_{r,s} e^{\beta\omega_r}}{e^{\beta\omega_r} - 1}, \tag{10}$$

where $\{\Delta_r, \Delta_s\} = \{\alpha_{\vec{k}+}, \alpha_{\vec{k}-}, \alpha_0\}$, $\{r,s\} = \{\vec{k}+, \vec{k}-, 0\}$ and $\omega_r = \omega_{\vec{k}}$ [see Eqs. (4) and (5)], we obtain the spin-spin correlation function as follows:

$$G(\vec{R},\vec{R}') = \langle S_{\vec{R}} S_{\vec{R}'} \rangle - \langle S_{\vec{R}} \rangle \langle S_{\vec{R}'} \rangle = \frac{1}{N} \sum_{\vec{k}} \frac{g}{2\omega_{\vec{k}}} \coth(\beta\omega_{\vec{k}/2}) \cos[\vec{k} \cdot (\vec{R} - \vec{R}')]. \tag{11}$$

By expanding $\coth(\beta\omega_{\vec{k}}/2)$ in partial fractions, using the identity $y^{-1} = \int_0^\infty dx \exp(-yx)$, $y>0$, and the Euler-Maclaurin formula for $N \gg 1$, we find

$$G(\vec{R},\vec{R}') = \frac{T}{2} \int_0^\infty dx\, \vartheta_3(e^{-(2\pi^2 T^2 x)/g}) e^{-\mu x} \prod_{i=1}^{d} I_{n_i}(Jx), \tag{12}$$

where $\vec{R} - \vec{R}' = \sum_{i=1}^d n_i \hat{e}_i$, $\hat{e}_i$ is the unit vector along direction i, and $I_{n_i}(x)$ denotes the n_ith-order modified Bessel function of the first kind.

We now analyze the correlation function and correlation length in the regimes devised in the Appendix.

A. $T=0$ and $T \to 0$, $g=g_c$, and $g \to g_c$, $\mu \to dJ$

The ground state of the $d \geq 2$ system evolves from the fully saturated long-range-ordered ferromagnetism, with magnetization $M=1/2$ (see the Appendix) if quantum fluctuations are absent, to a $M=0$ critical state at the quantum critical point, $g=g_c$, due to strong quantum fluctuations. Indeed, as $\tau_g = g/g_c - 1 \to 0^-$ quantum critical fluctuations gradually dominate over the low-lying excitation modes, suppressing the order towards a quantum disordered paramagnetic state for $g > g_c$. To evaluate G just above g_c at $T=0$, we use Eq. (A4), $I_{n_i}(x) \sim I_0(x) \exp(-n_i^2/2x)$, and the asymptotic expression[31,36] for $I_0(x)$ given by Eq. (A5):

$$G(\vec{R},\vec{R}') = \frac{\sqrt{g/J}}{2(2\pi)^{(d+1)/2}} \left(\frac{1}{r\xi}\right)^{(d-1)/2} K_{(d-1)/2}(r/\xi), \quad d \geq 2, \tag{13}$$

where $r^2 = \sum_{i=1}^d n_i^2$, $K_\nu(x)$ represents the νth-order modified Bessel function of the second kind, Eq. (A9), and $\xi = [2(\mu - dJ)/J]^{-1/2}$ is the correlation length in $h=0$. In fact, by considering the asymptotic form for $K_\nu(x)$, Eq. (A9), we obtain as $\tau_g \to 0^+$, $T=0$,

$$G(\vec{R},\vec{R}') \sim \frac{\sqrt{g/J}}{2\xi^{(d-2)/2}(2\pi r)^{d/2}} e^{-r/\xi}; \tag{14}$$

conversely, at $g=g_c$, $T=h=0$, and using[36] $K_\nu(x) \sim (2/x)^\nu [\Gamma(\nu)/2]$, $x \to 0$, $\nu > 0$:

$$G(\vec{R},\vec{R}') \sim \frac{\sqrt{g_c/J}\,\Gamma((d-1)/2)}{4\pi^{(d+1)/2}} \frac{1}{r^{d-1}}. \tag{15}$$

At $T=0$, Eq. (A13) implies in infinite ξ in the quantum-ordered regime below g_c in $d \geq 2$. On the other hand, as $\tau_g \to 0^+$, $T=0$, we consider Eqs. (A19), (A21), and (A22), so that

$$\xi \approx \left(\frac{g_c}{\pi^3 e^2 J}\right)^{1/2} \tau_g^{-1}, \quad d=2, \tag{16}$$

$$\xi \approx \frac{1}{\sqrt{6}} \left\{ \frac{\ln\{[(\pi^2 e)/3]\sqrt{J/g_c}\,\tau_g\}}{[(\pi^2 e)/3]\sqrt{J/g_c}\,\tau_g} \right\}^{1/2}, \quad d=3, \tag{17}$$

and

$$\xi \approx \sqrt{-2J\sqrt{g_c} R_g'(dJ)}\, \tau_g^{-1/2}, \quad d \geq 4, \tag{18}$$

are finite in the quantum disordered regime—i.e., $g > g_c$. The critical behavior $\xi \sim \tau_g^{-\nu}$, $\tau_g \to 0^+$, $T=0$, provides $\nu=1$ in $d=2$ and $\nu=1/2$ in $d \geq 3$. In $d=3$ the log dependence on τ_g implies in log corrections to the leading power-law singularity in ξ (log corrections also apply to other exponents of the quantum transition at the associated upper critical dimension, $d_c=3$).

At finite T and for any g the system does not display long-range order in $d=2$. In particular, close to g_c as $T \to 0$ and $h=0$, Eqs. (A14), (A16), and (A29) imply, in the renormalized classical, quantum-critical, and quantum-disordered regimes,[18,19,25,26]

$$\xi \approx \frac{\sqrt{Jg}}{T} \exp\left[\frac{\pi J\sqrt{g_c}}{2T(\sqrt{g_c} + \sqrt{g})} |\tau_g|\right], \quad d=2, \tag{19}$$

$$\xi \approx \frac{\sqrt{Jg_c}}{T}, \quad d=2, \tag{20}$$

and

$$\xi \approx \frac{\sqrt{g}(\sqrt{g}+\sqrt{g_c})}{\pi\sqrt{Jg_c}}\tau_g^{-1}, \quad d=2, \tag{21}$$

as $\tau_g \to 0^-$, $\tau_g=0$, and $\tau_g \to 0^+$, respectively. As $T\to 0$, notice the exponential divergence in Eq. (19), the quantum-critical behavior $\xi \sim T^{-\nu}$, with $\nu=1$ in Eq. (20), and the independence of ξ on T in Eq. (21), up to exponentially small corrections. These regimes are separated by crossover lines[18,19,33] $T \propto |\tau_g|/(\sqrt{g/g_c}+1)$, with crossover exponent $\phi = z\nu = 1$, where $T\sim|\tau_g|^{\phi}$ (see Sec. III B below for the identification of the dynamical exponent $z=1$).On the other hand, in $d>2$, by writing[19] $\xi \sim |\tau_g|^{-\nu} f_{\xi}(T/|\tau_g|^{\phi})$ at nonzero T as $\tau_g \to 0^+$, where f_ξ is a scaling function, the low-T expansion, $f_\xi \sim (T/|\tau_g|^{\phi})^x$, cancels out the dependence on $|\tau_g|$ in the quantum-critical regime, $\tau_g=0$, if $x=-\nu/\phi$, implying in the quantum-critical behavior $\xi \sim T^{-1/z} = T^{-1}$, $T\to 0$, for $d\geq 2$.

It is also worth noticing that the appearance of a quantum-critical regime in $d=2$ and a quantum-disordered paramagnetic phase in $d\geq 2$ are not allowed in quantized ferromagnets.[18,37] Indeed, in the present case the influence of strong quantum fluctuations under a spherical constraint on quantum-rotor-like spin fields and a dynamics driven by quadratic conjugated momenta, are relevant ingredients to induce these low-T regimes.

B. Finite $T=T_c(g)$ and $T\to T_c(g)$, $\mu \to dJ$, and gap discussion

In this regime, the same procedure leads to G as in Eqs. (14) and (15), with the only replacements $\sqrt{g/J} \to T/J$ and $d+z \to d$. In particular,[19] $G \sim r^{-(d+z+\eta-2)}$ at $g=g_c$, $T=0$, is replaced by $G\sim r^{-(d+\eta-2)}$ at $T=T_c(g)$, thus fixing[20] $z=1$ and $\eta=0$ [compare with Eq. (15)] at the quantum-critical transition and $\eta=0$ at the thermal-critical line. Conversely, by using Eq. (7) for $k\to 0$, the infrared states[33] with $\omega_{\vec{k}} \approx \sqrt{gJ}[\xi^{-2z}+k^2]^{1/2}$ are such that the scaling[20,33] of the low-lying quantum fluctuation modes, with $k\sim \xi^{-1}$ and $\omega \sim \xi^{-z}$—i.e., $k\sim w^{1/z}$—alternatively provides $z=1$.

Regarding the correlation length, Eq. (A32) implies in infinite ξ in the ordered phase below $T_c(g)$ in $d\geq 3$. Furthermore, Eqs. (A33)–(A35) as $\tau_T = T/T_c(g)-1 \to 0^+$ lead to

$$\xi \approx \frac{T_c(g)}{\pi^{3/2} eJ}\tau_T^{-1}, \quad d=3, \tag{22}$$

$$\xi \approx \frac{1}{2\sqrt{2}}\left\{\frac{\ln(\{(\pi^4 eJ)/[8T_c(g)]\}\tau_T)}{\{(\pi^4 eJ)/[8T_c(g)]\}\tau_T}\right\}^{1/2}, \quad d=4, \tag{23}$$

and

$$\xi \approx \sqrt{-JT_c(g)R'(dJ)}\,\tau_T^{-1/2}, \quad d\geq 5, \tag{24}$$

from which $\xi \sim \tau_T^{-\nu}$, $\tau_T\to 0^+$, and $\nu=1$ in $d=3$ and $\nu=1/2$ in $d\geq 4$ (log corrections to the exponents are present at the upper critical dimension of the thermal transition, $d_c=4$). We note in Eqs. (22)–(24) the same critical behavior of the $g=0$ classical spherical model,[31] but with the prefactors suppressed by quantum fluctuations controlled by g. Interestingly, regarding the $g=0$ limit of Eqs. (22)–(24) the critical amplitudes can be made identical to those of the classical spherical model by performing the integral (A20) over $[[\kappa(\mu-dJ)]^{-1},\infty)$, with the dimensionless factor $\kappa=4.77$. This indicates that, although not important to the overall critical behavior close to the transition, the intermediate-x regime in Eq. (A20) may actually contribute, to some extent, to the prefactor of critical observables.

From Eq. (7) with $k\to 0$, we note that the ground state of the $d\geq 2$ system is separated from the low-lying excitation modes by an energy gap $\Delta \approx [2g(\mu-dJ)]^{1/2} \sim \xi^{-z}$. Using Eqs. (A13), (A19), (A21), and (A22), we find that the system at $T=h=0$ is gapless in the ordered phase, for $g<g_c$, and at the quantum-critical point, $g=g_c$ [see Fig. 1(a) in $d=2$]; however, the gap opens in the quantum-disordered ground state, $g>g_c$, so that $\Delta \propto \tau_g^{z\nu}$, with $\nu=1$ in $d=2$ and $\nu=1/2$ in $d\geq 4$, and $\Delta \propto [\tau_g/\ln(|\text{const}\times\tau_g|)]^{1/2}$ in $d=3$. On the other

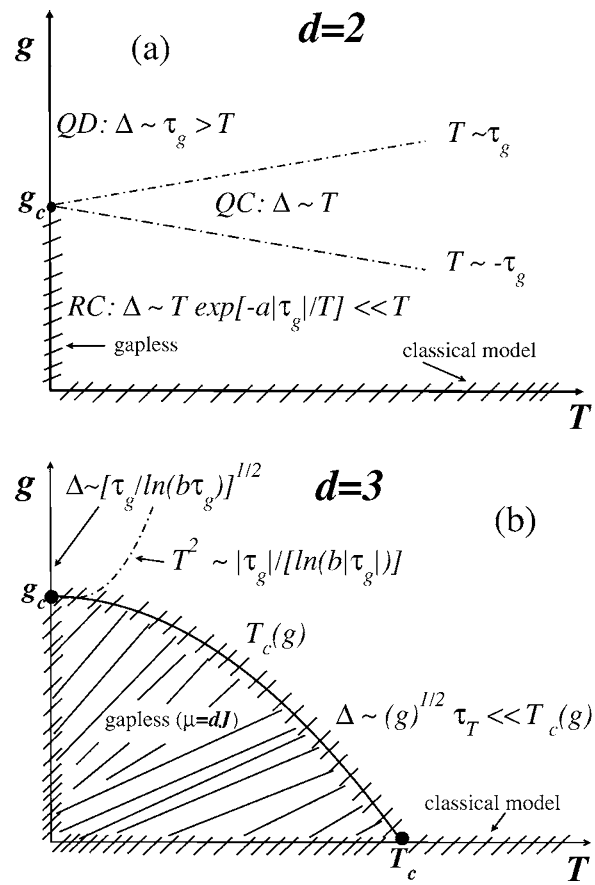

FIG. 1. Phase diagram of the quantum spherical model in $h=0$, displaying the quantum parameter g vs temperature T, in energy units. (a) $d=2$: an ordered gapless ground state (vertical dashed line) is found for $g<g_c$, where g_c denotes the $T=0$ quantum critical point; for $g>g_c$ strong quantum fluctuations lead to a quantum disordered paramagnetic phase at $T=0$, with gap $\Delta\sim\tau_g = g/g_c-1$. Low-$T$ regimes and their respective gaps are also indicated: renormalized classical (RC), quantum-critical (QC), and quantum-disordered (QD), with boundaries at the dot-dashed crossover lines, $T\sim|\tau_g|$. (b) $d=3$: the critical line $T_c(g)$, sketched in the figure, separates the gapless ordered (dashed region) and gapped disordered phases; the dot-dashed crossover line marks the boundary between the quantum-critical and quantum-disordered regimes. In (a) and (b) the gapless classical spherical model maps onto the axis $g=0$. (a and b are positive constants.)

hand, in the absence of long-range order as $T\to 0$ in $d=2$, Eqs. (A14), (A16), and (A29) lead to $\Delta\approx T\exp(-|\text{const}\times\tau_g|/T)\ll T$, $\Delta\approx T$, and $\Delta\propto\tau_g\gtrsim T$ in the renormalized classical, quantum-critical, and quantum-disordered regimes, respectively; we note in the latter two regions near the quantum-critical point that the gap is not negligible compared with thermal fluctuations, in contrast with the renormalized classical regime.

On the other hand, near the thermal transition at $T_c(g)$, Eqs. (A32)–(A35) lead to a gapless spectrum in the ordered phase, $T<T_c(g)$, and at $T=T_c(g)$, but to a gap in the disordered regime [see Fig. 1(b) in $d=3$]: $\Delta\propto(\sqrt{g}\tau_T)^\nu$ as $\tau_T\to 0^+$ and $g\neq 0$, with $\nu=1$ in $d=3$ and $\nu=1/2$ in $d\geq 5$, and $\Delta\propto[\sqrt{g}\tau_T/\ln(\text{const}\times\tau_T)]^{1/2}$ in $d=4$. In any case, the result $\Delta\ll T$ just above $T_c(g)$ indicates that, compared with the thermal energy at finite $T\to T_c(g\neq 0)^+$, the presence of a nonzero gap is not relevant to the critical behavior in this region. Indeed, by setting $g=0$ in Eq. (7) we note that $\Delta=0$ in the classical case, though the $g\neq 0$ and $g=0$ models display the same thermal critical behavior. Actually, a gap $\Delta\sim T_c(g\neq 0)$ can be found as $\mu\to dJ$ only by approaching the quantum-critical point, $T_c(g\neq 0)\to 0$, in agreement with the result in the quantum-critical regime.

IV. ENTROPY AND SPECIFIC HEAT

Entropy and specific heat are obtained using Eq. (8), $S=-\partial F/\partial T$ and $C=-T\partial^2 F/\partial T^2$.

We first consider the very-low-temperature regime $T\to 0$, in which only low-lying modes with $k\to 0$ contribute significantly to the system properties. In such a case,

$$\omega_{\vec{k}}\approx\sqrt{2g(\mu-dJ)}[1+J(\mu-dJ)^{-1}k^2/4], \tag{25}$$

provided $\mu\neq dJ$. Such condition is fulfilled as $T\to 0$, e.g., by applying $h\neq 0$, in the absence of long-range order. In the continuous limit the use of the Euler-Maclaurin formula and Eq. (25) allows us to write S and C as Gaussian integrals, which as $\mu\to dJ$ give rise to

$$\frac{S}{N}\sim D^d\left(\frac{\Theta}{T}\right)^{1-d/2}\exp\left(-\frac{\Theta}{T}\right) \tag{26}$$

and

$$\frac{C}{N}\sim D^d\left(\frac{\Theta}{T}\right)^{2-d/2}\exp\left(-\frac{\Theta}{T}\right), \tag{27}$$

where $D=[(\mu_0-dJ)/(\pi J)]^{1/2}$, $\Theta=[2g(\mu_0-dJ)]^{1/2}$, and $\mu_0\equiv\mu(g,T=0,h\neq 0)>dJ$. In Eqs. (26) and (27) the T dependence is a combination of the typical result of Einstein's model, $T^{-2}\exp(-\Theta/T)$, and the spin-wave-like contribution $T^{d/2}$ which emerge from the low-$\vec{k}$ behavior of the eigenfrequencies, Eq. (25), with a finite energy gap $\Delta=\Theta$ due to $h\neq 0$ and a magnon-like ferromagnetic dependence on k^2. Such kind of behavior has been also observed in the quantum spherical model in ferrimagnetic AB_2 chains,[30] as well as in anisotropic quantum ferromagnetic and antiferromagnetic Heisenberg models in $d=1$.[35]

On the other hand, the zero-field system presents a vanishing gap as $T\to 0$ and $g\leq g_c$ (see Sec. III). Therefore, Eq. (25) cannot be applied to calculate S and C in this region via Gaussian integrals. Instead, by substituting Eqs. (A14) and (A16) in Eq. (7) and performing the integrals with a suitable cutoff in the momentum space, the specific heat in the renormalized classical and quantum-critical regimes in $h=0$ reads, respectively,

$$\frac{C}{N}\approx\frac{\lambda_{d=2}T^2}{Jg}\exp\left[\frac{2\pi J\sqrt{g_c}}{T(\sqrt{g}+\sqrt{g_c})}\tau_g\right],\quad d=2, \tag{28}$$

where $\tau_g\to 0^-$ and $\lambda_{d=2}$ is some finite integral with a T-independent dominant term, and

$$\frac{C}{N}\approx\frac{\lambda^0_{d=2}T^2}{Jg_c},\quad d=2, \tag{29}$$

with $\lambda^0_{d=2}\equiv\lambda_{d=2}(\tau_g=0)$. In the quantum-critical region, $C/T\sim T^{-\alpha}$ provides $\alpha=-1$, as $T\to 0$, $g=g_c$. Indeed, the quantum-critical behavior is derived through the scaling of the Helmholtz free energy,[19] $F\sim|\tau_g|^{2-\alpha}f_F(T/|\tau_g|^\phi)$, at nonzero T as $\tau_g\to 0^+$, with the use of the hyperscaling relation $\nu(d+z)=2-\alpha$, so that[18,19,33] $\alpha=1-d/z$.

Note from Eqs. (26)–(29) that the presence of quantum fluctuations controlled by g actually fixes the well-known drawback (diverging S and finite C as $T\to 0$) of classical spherical models[1,3,4] concerning the third law of thermodynamics. Indeed, in the $g=0$ classical case all eigenmodes are null in $h=0$ [see Eq. (7)] and anomalies arise. In this sense, the low-T entropy and specific heat of quantum spherical models have been also investigated in Refs. 23 and 29. In spite of this, in the absence of canonically conjugated operators, the author in Ref. 23 has considered a model with double spin field at each site and a couple of adjustable parameters so to get rid of the anomaly; in Ref. 29 the use of a spherical constraint different from Eq. (2), involving both $S_{\vec{R}}$ and $P_{\vec{R}}$, has led to a dynamical exponent $z=2$ and its consequences to the critical behavior of observables (see, e.g., last paragraph).

Regarding the behavior of C near $T_c(g)$, we observe that μ is T independent in $h=0$ and $T\lesssim T_c(g)$ [see Eq. (A32)]. Thus, the relevant term in $C[T\to T_c(g)^+]-C[T\to T_c(g)^-]$ is proportional to $\partial\mu/\partial T$ evaluated as $T\to T_c(g)^+$, $\mu\to dJ$. From Eqs. (A33)–(A35),

$$C[T\to T_c(g)^+]-C[T\to T_c(g)^-]\approx-\frac{A\pi^3e^2J^3}{[T_c(g)]^3}\tau_T,\quad d=3, \tag{30}$$

$$C[T\to T_c(g)^+]-C[T\to T_c(g)^-]\approx-\frac{4AJ}{T_c(g)}\frac{(\pi^4eJ)/[8T_c(g)]}{\ln\{(\pi^4eJ)/[8T_c(g)]\tau_T\}},\quad d=4, \tag{31}$$

and

$$C[T \to T_c^+(g)] - C[T \to T_c(g)^-] \approx \frac{A}{2[T_c(g)]^2 R'(dJ)}, \quad d \geq 5, \tag{32}$$

with the proportionality constant $A = \Sigma_{\vec{k}}[g\beta^2 \exp(\beta\omega_{\vec{k}})]/[\exp(\beta\omega_{\vec{k}})-1]^2$, $T \to T_c(g)^+$. As $\tau_T \to 0$ a cusp in C can be possibly found, so that, by writing $C \sim \tau_T^{-\alpha}$, $\tau_T \to 0$, we identify $\alpha = -1$ in $d=3$; the log dependence on τ_T in $d=4$ also results in a continuous maximum in C at $T_c(g)$, whereas $\alpha = 0$ in $d \geq 5$ indicates discontinuity at the transition. We note that the classical spherical model in $h=0$ presents the same critical behavior of Eqs. (30)–(32); in particular, a cusp at T_c also occurs in $d=3$, where[1,3] $C(T \to T_c^+) - C(T \to T_c^-) = 2NK^2 dz_s/dK \propto -d\mu/dT \to 0$, with $K = J/8T$ in the $S = \pm 1/2$ case and $z_s = \mu/J$.

V. DISCUSSION AND CONCLUSIONS

We have presented an alternative second-quantization treatment to the quantum ferromagnetic spherical spin model on $d \geq 2$ hypercubic lattices, with focus on the effects of quantum (g) and thermal (T) fluctuations, under a uniform field h, on the correlation function, correlation length, entropy, and specific heat, as well as on the energy gap in the excitation spectrum. Here, explicit expressions for these quantities have been provided in the variety of possible critical paths devised in the Appendix.

We have described the properties of the system near both $d \geq 2$ quantum ($g = g_c$, $T=0$) and $d \geq 3$ thermal $[T = T_c(g)]$ phase transitions in $h=0$. In particular, the calculation of the correlation function and correlation length generalizes the result on the $g=0$ classical spherical model[31] and leads to exponents $z=1$ (dynamical) and $\eta=0$ in $d \geq 2$, confirming the mapping[20,25,26] of the d-dimensional quantum onto the $(d+z)$-thermal critical behaviors in $d \leq d_c$, with the upper critical dimension $d_c = 3$ ($d_c = 4$) in the quantum (thermal) transition. It follows that the critical indexes obtained independently were found to satisfy the scaling relations near the quantum and thermal transitions, with $\nu d = 2 - \alpha$, valid for the thermal exponents, replaced by $\nu(d+z) = 2-\alpha$ in the quantum case.[20] Above d_c, as expected, the critical indexes are Gaussian. Moreover, the relation $\chi \sim \xi^2$ has been also verified in any regime considered.

At $T=0$ we have also noticed that below g_c the ordered system in $h=0$ is gapless, as well as at $g = g_c$; however, a gap in the excitation spectrum opens in the quantum-disordered ground state, $g > g_c$. As $T \to 0$, the gap is relevant regarding quantum and thermal fluctuations in the quantum-disordered and quantum-critical regimes. Conversely, as $\tau_T \to 0^+$, $g \neq 0$, a nonzero gap is present, although much smaller than the thermal energy, but nullifies for $T \leq T_c(g \neq 0)$. In this sense, the gap present above $T_c(g \neq 0)$ is irrelevant to the universality class, since the gapless classical spherical model shares the same exponents of the quantum model as $\tau_T \to 0^+$. In spite of this, quantum fluctuations do suppress the critical prefactors of observables near $T_c(g \neq 0)$, as compared with the $g=0$ ones.

At last, our calculation of the entropy and specific heat in $g \neq 0$ circumvents the drawback in classical spherical models concerning the third law of thermodynamics, as $T \to 0$.

ACKNOWLEDGMENTS

We thank Alba Theumann and Sílvio R. A. Salinas for very useful discussions. This work was partially supported by CNPq, CAPES, FINEP, and FACEPE (Brazilian agencies).

APPENDIX: CHEMICAL POTENTIAL CALCULATION

The continuous limit of the spherical constraint, Eq. (9), is obtained by expanding $\coth(\beta\omega_{\vec{k}}/2)$ and using the Euler-Maclaurin formula $\Sigma_{\vec{k}} \to N\Pi_{i=1}^d \int_{-\pi}^{\pi} dk_i/(2\pi)$, $N \gg 1$:

$$\frac{T}{2}R(\mu) + \frac{h^2}{4(\mu - dJ)^2} = \frac{1}{4}, \tag{A1}$$

with

$$R(\mu) = \int_0^\infty dx e^{-\mu x}\left[\sum_{n=-\infty}^{\infty} q^{n^2}\right][I_0(Jx)]^d, \tag{A2}$$

where we identify the third Jacobi theta function[36] $\vartheta_3(q,z) = \Sigma_{n=-\infty}^{\infty} q^{n^2} e^{2niz}$, with $z=0$, $q = \exp(-2\pi^2 T^2 x/g)$, and the zeroth-order modified Bessel function of the first kind, $I_0(x)$. The convergence analysis of Eqs. (A1) and (A2) needs the following asymptotic limits:[36]

$$I_0(x) \sim 1 + \frac{1}{4}x^2, \quad x \to 0; \tag{A3}$$

$$\vartheta_3(q) \equiv \vartheta_3(q, z=0) \sim \sqrt{-\frac{\pi}{\ln q}}, \quad q \to 1^-, \quad x \to 0; \tag{A4}$$

$$I_0(x) \sim \frac{e^x}{\sqrt{2\pi x}}, \quad x \to \infty; \tag{A5}$$

$$\vartheta_3(q) \sim 1 + 2q, \quad q \to 0, \quad x \to \infty. \tag{A6}$$

We first consider the $T=0$ case. By substituting Eqs. (A3)–(A6) into Eq. (A2), no singularity appears in the $x \to 0$ limit, but convergence as $x \to \infty$ and $\mu = dJ$ exists only in $d \geq 2$. Since the choice $\mu = dJ$ implies in a singular behavior of F due to $\omega_{\vec{k}=0} = 0$ [Eqs. (7) and (8)], it is possible to define a critical quantum parameter g_c at $T = h = 0$ and $\mu = dJ$:

$$g_c^{-1/2} = 2\int_0^\infty dx \frac{[e^{-Jx} I_0(Jx)]^d}{\sqrt{2\pi x}}, \quad d \geq 2. \tag{A7}$$

Similarly, the analysis at finite T shows convergence in the $x \to \infty$, $\mu = dJ$ limit only if $d \geq 3$, providing the critical line $[T_c(g), h=0]$, with

TABLE I. Numerical estimates for the critical parameters $g_c(T=0)$ and $T_c(g=0)$, in $h=0$, as calculated from Eqs. (A7) and (A8).

d	$g_c(T=0)$	$T_c(g=0)$
2	$0.6049J$	
3	$1.2058J$	$0.9892J$
4	$1.7471J$	$1.6136J$
5	$2.2667J$	$2.1621J$

$$[T_c(g)]^{-1} = 2\int_0^\infty dx\,\vartheta_3(e^{-2\pi^2 T_c^2 x/g})[e^{-Jx}I_0(Jx)]^d, \quad d \geq 3. \tag{A8}$$

Some of these critical values are shown in Table I. In particular, the value $g_c(T=0)\approx 0.6049J$ in $d=2$ compares quite well with $g_c(T=0)\approx 0.6051J$ obtained by Chamati *et al.*[26] for the $d=2$ quantum spherical model, and $T_c(g=0)\approx 0.9892J$ in $d=3$ coincides with the value for the $d=3$ classical spherical model.[3] This analysis also shows that no long-range order can be found for $T>0$ in $d\leq 2$ and $T>T_c(g)$ in $d\geq 3$.

As $T\to 0$ in the vicinity of the quantum critical point, we approach Eqs. (A1) and (A2) through a distinct strategy. First, we apply the Jacobi identity $\sum_{n=-\infty}^{\infty}\exp(-un^2) = (\pi/u)^{1/2}\sum_{n=-\infty}^{\infty}\exp(-\pi^2 n^2/u)$ to Eq. (A2); next, we introduce $I_0(Jx)$, in the limit given by Eq. (A5), and the modified Bessel function[36] of the second kind and order $\nu=1/2$,

$$K_\nu(\sqrt{4uv}) = \frac{1}{2}\left(\frac{u}{v}\right)^{\nu/2}\int_{-\infty}^{\infty}\frac{dx}{x^{\nu+1}}e^{-vx-u/x} \sim \left(\frac{\pi}{2\sqrt{4uv}}\right)^{1/2} e^{-\sqrt{4uv}}, \quad uv \to \infty; \tag{A9}$$

after summing the whole series term by term in n, we obtain in $h=0$, as $T\to 0$, $\mu\to dJ$,

$$\left(\frac{T}{2\pi J}\right)^{d/2}\left[\frac{2(\mu-dJ)}{g}\right]^{(d-2)/4}[\delta + \mathrm{Li}_{d/2}(\delta)] = \frac{1}{4}\left(1-\sqrt{\frac{g}{g_c}}\right), \quad d\geq 2, \tag{A10}$$

where $\delta\equiv\exp[-\sqrt{2g(\mu-dJ)}/T]$ and $\mathrm{Li}_s(z)=\sum_{m=1}^{\infty}z^m/m^s$ is the polylogarithm, or Jonquiére's function.[36] In $d=2$, considering $\mathrm{Li}_1(z)=-\ln(1-z)$ in Eq. (A10), we find[26]

$$\sinh\left[\frac{\sqrt{g(\mu-dJ)/2}}{T}\right] \approx \frac{1}{2}\exp\left[\frac{\pi J\sqrt{g_c}}{2T(\sqrt{g_c}+\sqrt{g})}\tau_g\right], \quad d=2, \tag{A11}$$

where $\tau_g=g/g_c-1\to 0$, $T\to 0$, $\mu\to dJ$. In $d>2$ the low-T, $\tau_g=0$ quantum-critical behavior is studied through a suitable scaling analysis (see Sec. III and below).

In what follows we obtain solutions for $\mu(g,T,h)$ in the vicinity of the quantum ($g=g_c$, $T=h=0$) and thermal [$T=T_c(g)>0$, $h=0$] phase transitions in $d\geq 2$ and $d\geq 3$, respectively. We also calculate the magnetization $M=\langle S_{\vec{R}}\rangle = h/[2(\mu-dJ)]$ and zero-field susceptibility $\chi=(\partial M/\partial h)_{h=0}$ in order to check with previously reported results.[25,26]

1. $T=0$ and $T\to 0$, $g=g_c$, and $g\to g_c$, $\mu\to dJ$

Just below g_c at $T=0$, the integral $TR(\mu)/2$ in Eq. (A1), with the use of Eq. (A4), is dominated by its value at $\mu=dJ$, so that we can write Eq. (A1) as

$$\frac{1}{4}\sqrt{\frac{g}{g_c}} + \frac{h^2}{4(\mu-dJ)^2} \approx \frac{1}{4}, \tag{A12}$$

from which it follows, as $\tau_g\to 0^-$, that

$$\mu - dJ \approx \sqrt{2}h(-\tau_g)^{-1/2}, \quad d\geq 2. \tag{A13}$$

The ground-state magnetization for $g\leq g_c$ in $d\geq 2$ thus reads $M\approx(1-g/g_c)^{1/2}/\{2[1+(g/g_c)^{1/2}]^{1/2}\}$, where $g_c\equiv g_c(T=0)$, from which $M\sim(-\tau_g)^\beta$, $\tau_g\to 0^-$, with $\beta=1/2$ in any $d\geq 2$. Moreover, Eq. (A13) implies in an infinite $T=h=0$ susceptibility for $g\leq g_c$.

On the other hand, in the regime $T\to 0$, $\tau_g\to 0^-$, $h=0$, Eq. (A11) provides

$$\mu - dJ \approx \frac{T^2}{2g}\exp\left[\frac{\pi J\sqrt{g_c}}{T(\sqrt{g_c}+\sqrt{g})}\tau_g\right], \quad d=2, \tag{A14}$$

and

$$\chi \approx \frac{g}{T^2}\exp\left[-\frac{\pi J\sqrt{g_c}}{T(\sqrt{g_c}+\sqrt{g})}\tau_g\right], \quad d=2. \tag{A15}$$

In such a case, precisely at $g=g_c$ we obtain, as $T\to 0$,

$$\mu - dJ \approx \frac{T^2}{2g_c}, \quad d=2, \tag{A16}$$

and

$$\chi \approx \frac{g_c}{T^2}, \quad d=2. \tag{A17}$$

Equations (A15) and (A17), with T dependence typical of the $d=2$ renormalized classical and quantum-critical regimes,[26,33] have been also found by Chamati *et al.*,[26] with the differences in the numerical factors only due to the $S=\pm 1$ spin variables used in that work. In $d>2$, by writing[19] $\chi\sim|\tau_g|^{-\gamma}f_\chi(T/|\tau_g|^\phi)$ at nonzero T as $\tau_g\to 0^+$, where f_χ is the corresponding scaling function, we find, from the exponents identified in Eqs. (17), (18), (A27), and (A28), that the $\chi\sim T^{-2}$ quantum critical behavior also arises in $d>2$.

Just above g_c at $T=0$, we follow the arguments by Thompson[4] to study the behavior of $R_g(\mu)=TR(\mu)/\sqrt{g}$, using Eq. (A4), and its derivative $R_g'(\mu)=\partial R_g/\partial\mu$, as $\mu\to dJ$. First, we note that the same arguments above that guaranteed the convergence of $R(\mu)$ at $\mu=dJ$ in $d\geq 2$ can be applied to assure that $R_g'(\mu)$ converges at $\mu=dJ$ in $d>3$; the borderline cases $d=2$ and $d=3$ should be considered carefully. In $d>3$ we find, in $h=0$,

$$R_g(\mu) - R_g(dJ) \approx R_g'(dJ)(\mu - dJ), \tag{A18}$$

which combined with the expression $R_g(\mu)-R_g(dJ) \approx -\tau_g/(4\sqrt{g_c})$, obtained from Eqs. (A1) and (A2) at $T=h=0$, as $\tau_g \to 0^+$, $\mu \to dJ$, leads to

$$\mu - dJ \approx -\frac{1}{4\sqrt{g_c}R_g'(dJ)}\tau_g, \quad d \geq 4, \tag{A19}$$

with $R_g'(dJ)<0$. In the cases $d=2$ and $d=3$ we first calculate $R_g'(\mu)$ from Eq. (A2) at $T=h=0$, so to obtain, with the aid of Eq. (A4),

$$R_g'(\mu) = -\int_0^\infty \frac{dx}{\sqrt{2\pi}} x^{1/2} e^{-\mu x}[I_0(Jx)]^d. \tag{A20}$$

From the expansions of $I_0(Jx)$ and the incomplete γ function we note that the integrand makes a nonsingular contribution as $x \to 0$, $\mu \to dJ$. On the other hand, the singular contribution as $x \to \infty$, $\mu \to dJ$ can be evaluated by integrating by parts the variable x over $[(\mu-dJ)^{-1}, \infty)$ and making use of Eq. (A5). The $T=0$, $\tau_g \to 0^+$ results read

$$\mu - dJ \approx \frac{\pi^3 e^2 J^2}{2g_c}\tau_g^2, \quad d=2, \tag{A21}$$

and

$$\mu - dJ \approx 3J\frac{[(\pi^2 e)/3]\sqrt{J/g_c}\tau_g}{\ln\{[(\pi^2 e)/3]\sqrt{J/g_c}\tau_g\}}, \quad d=3. \tag{A22}$$

The $T=0$ equation of state as $\tau_g \to 0^+$ is obtained by expressing $R_g(\mu)-R_g(dJ)$ as $\mu \to dJ$ from Eqs. (A1), (A2), and (A4), along with Eq. (A18) in $d \geq 4$ and the singular contribution of the integration by parts of $R_g'(\mu)=\partial R_g/\partial\mu$, Eq. (A20), in $d=2$ and $d=3$:

$$h \approx \frac{64\pi^3 e^2 J^2}{g_c}M(M^2+\tau_g/8)^2, \quad d=2, \tag{A23}$$

$$h \approx 6J\frac{[(8\pi^2 e)/3]\sqrt{J/g_c}M(M^2+\tau_g/8)}{\ln\{[(8\pi^2 e)/3]\sqrt{J/g_c}(M^2+\tau_g/8)\}}, \quad d=3, \tag{A24}$$

and

$$h \approx -\frac{4}{\sqrt{g_c}R_g'(dJ)}M(M^2+\tau_g/8), \quad d \geq 4. \tag{A25}$$

Thus, the $T=h=0$ magnetization is null in the quantum-disordered regime, $g>g_c$. Precisely at $g=g_c$, $M \sim h^{1/\delta}$, $h \to 0$, leads to $\delta=5$ in $d=2$ and $\delta=3$ in $d \geq 3$, with the log correction in $d=3$ related to the x^{-1} integrand of Eq. (A20) as $x \to \infty$, $\mu \to dJ$. Now, from Eqs. (A19), (A21), and (A22), we find at $T=0$, $\tau_g \to 0^+$,

$$\chi \approx \frac{g_c}{\pi^3 e^2 J^2}\tau_g^{-2}, \quad d=2, \tag{A26}$$

$$\chi \approx \frac{1}{6J}\frac{\ln\{[(\pi^2 e)/3]\sqrt{J/g_c}\tau_g\}}{[(\pi^2 e)/3]\sqrt{J/g_c}\tau_g}, \quad d=3, \tag{A27}$$

and

$$\chi \approx -2\sqrt{g_c}R_g'(dJ)\tau_g^{-1}, \quad d \geq 4, \tag{A28}$$

which are finite in the quantum-disordered phase. These results lead to $\chi \sim \tau_g^{-\gamma}$, $\tau_g \to 0^+$, with $\gamma=1$ in $d \geq 3$. In $d=2$, Eqs. (A23) and (A26) present the same form as obtained by Chamati *et al.*[26] near the quantum-critical point, but with distinct amplitudes. However, by integrating Eq. (A20) over $[[\kappa(\mu-dJ)]^{-1}, \infty)$, we find that the amplitudes match for $\kappa \approx 14.43$. This suggests that, similarly to the classical transition (see Sec. III), by keeping only the $x \to \infty$ regime while calculating $R'(\mu)$, the correct critical behavior in the quantum transition is achieved; the larger κ value indicates that the contribution of the intermediate x regime to the amplitude of critical observables in the quantum phase transition might be more important than in the classical one.

As $T \to 0$, $g \gtrsim g_c$, $h=0$, the analysis of Eq. (A11) leads to

$$\mu - dJ \approx \frac{\pi^2 J^2 g_c}{2g(\sqrt{g}+\sqrt{g_c})^2}\tau_g^2, \quad d=2, \tag{A29}$$

and the correspondent quantum-disordered susceptibility,[26,33]

$$\chi \approx \frac{g(\sqrt{g}+\sqrt{g_c})^2}{\pi^2 J^2 g_c}\tau_g^{-2}, \quad d=2. \tag{A30}$$

2. Finite $T=T_c(g)$ and $T \to T_c(g)$, $\mu \to dJ$

As above, for $T \lesssim T_c(g)$ the integral (A2) is dominated by its value at $\mu=dJ$, so that

$$\frac{T}{4T_c(g)} + \frac{h^2}{4(\mu-dJ)^2} \approx \frac{1}{4}, \tag{A31}$$

giving rise to

$$\mu - dJ \approx h(-\tau_T)^{-1/2}, \quad d \geq 3, \tag{A32}$$

near the thermal transition, $\tau_T=T/T_c(g)-1 \to 0^-$. In this regime, $M \approx [1-T/T_c(g)]^{1/2}/2$, $d \geq 3$, which implies in $M \sim (-\tau_T)^\beta$, $\tau_T \to 0^-$, with $\beta=1/2$ in any $d \geq 3$. Furthermore, Eq. (A32) leads to an infinite zero-field susceptibility for $T \lesssim T_c(g)$.

For $T \gtrsim T_c(g)$, similarly to case for $g \gtrsim g_c$, $T=0$, convergence of $R'(\mu)$ at $\mu=dJ$ is assured in $d>4$. By using Eq. (A18) with the expression $R(\mu)-R(dJ) \approx -\tau_T/[2T_c(g)]$, obtained from Eq. (A1) in $h=0$, as $\tau_T \to 0^+$, $\mu \to dJ$, we find

$$\mu - dJ \approx -\frac{1}{2T_c(g)R'(dJ)}\tau_T, \quad d \geq 5, \tag{A33}$$

with $R'(dJ)<0$; also, the singular contribution of $R'(\mu)$ as $x \to \infty$ gives, in $h=0$, $\tau_T \to 0^+$,

$$\mu - dJ \approx \frac{\pi^3 e^2 J^3}{2[T_c(g)]^2}\tau_T^2, \quad d=3, \tag{A34}$$

and

$$\mu - dJ \approx 4J\frac{\{(J\pi^4 e)/[8T_c(g)]\}\tau_T}{\ln(\{(Je\pi^4 e)/[8T_c(g)]\}\tau_T)}, \quad d=4. \tag{A35}$$

To derive the equation of state as $\tau_T \to 0^+$, we express $R(\mu)-R(dJ)$ from Eq. (A1) and use Eq. (A18) in $d \geq 4$ and the singular contribution of $R'(\mu)$ in $d=3$ and $d=4$:

$$h \approx \frac{16\pi^3 e^2 J^3}{[T_c(g)]^2} M(M^2 + \tau_T/4)^2, \quad d=3, \tag{A36}$$

$$h \approx 8J\frac{\{(128eJ)/[\pi^4 T_c(g)]\}M(M^2+\tau_T/4)}{\ln(\{(128eJ)/[\pi^4 T_c(g)]\}(M^2+\tau_T/4))}, \quad d=4, \tag{A37}$$

and

$$h \approx -\frac{4}{T_c(g)R'(dJ)} M(M^2 + \tau_T/4), \quad d \geq 5, \tag{A38}$$

so that the $h=0$ magnetization is null above $T_c(g)$. At $T=T_c(g)$ we define $M \sim h^{1/\delta}$, $h \to 0$, with $\delta=5$ in $d=3$ and $\delta=3$ in $d \geq 4$.

The zero-field susceptibility is calculated from Eqs. (A33)–(A35):

$$\chi \approx \frac{[T_c(g)]^2}{\pi^3 e^2 J^3}\tau_T^{-2}, \quad d=3, \tag{A39}$$

$$\chi \approx \frac{1}{8J}\frac{\ln\{\{(\pi^4 eJ)/[8T_c(g)]\}\tau_T\}}{\{(\pi^4 eJ)/[8T_c(g)]\}\tau_T}, \quad d=4, \tag{A40}$$

and

$$\chi \approx -\frac{T_c(g)}{R'(dJ)}\tau_T^{-1}, \quad d \geq 5, \tag{A41}$$

implying in $\chi \sim \tau_T^{-\gamma}$, $\tau_T \to 0^+$, with $\gamma=2$ in $d=3$ and $\gamma=1$ in $d \geq 4$. The explicit presence of $T_c(g)$ in Eqs. (A36)–(A41) indicates that, under the influence of quantum fluctuations, $g \neq 0$, the critical prefactors are suppressed, as compared with the $g=0$ ones. However, the thermal-critical behaviors of the d-dimensional quantum and d-dimensional classical[3] spherical models do belong to the same universality class, since in this case quantum fluctuations should play only a secondary role.

At last, it is worth noticing from the results of Sec. III and this Appendix that the relation $\chi \sim \xi^2$ holds in all regimes considered, thus extending to the quantum spherical case a general relation previously known in the context of $g=0$ classical spherical models.[31]

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[1] T. H. Berlin and M. Kac, Phys. Rev. **86**, 821 (1952).
[2] H. W. Lewis and G. H. Wannier, Phys. Rev. **88**, 682 (1952).
[3] G. S. Joyce, in *Phase Transitions and Critical Phenomena*, edited by C. Domb and M. S. Green (Academic, New York, 1972), Vol. 2, and references therein. Notice that in this work the author has considered a classical spherical model with spin variables $S=\pm 1$, such that, e.g., the $d=3$ critical temperature for simple cubic lattices with short-range interactions should read $T_c=(1/4)(2dJ/K_c)\approx 0.9892J$ in the $S=\pm 1/2$ case, with $K_c \approx 1.5164$.
[4] C. J. Thompson, *Classical Equilibrium Statistical Mechanics* (Oxford University Press, New York, 1988), and references therein.
[5] H. E. Stanley, Phys. Rev. **176**, 718 (1968); see also H. E. Stanley, *Introduction to Phase Transitions and Critical Phenomena* (Oxford University Press, New York, 1971).
[6] B. S. Gourary and R. W. Hart, Phys. Rev. **95**, 676 (1954); R. M. Mazo, J. Chem. Phys. **39**, 2196 (1963); H. J. Knops, Phys. Rev. B **8**, 4209 (1973); W. K. Theumann, *ibid.* **15**, 3534 (1977).
[7] L. Kalok and L. C. de Menezes, Z. Phys. B **20**, 223 (1975); L. Kalok and G. Obermair, J. Phys. C **9**, 819 (1975).
[8] R. M. Hornereich, M. Luban, and S. Shtrikman, Physica A **86**, 465 (1977); C. S. O. Yokoi, M. D. Coutinho-Filho, and S. R. Salinas, Phys. Rev. B **24**, 4047 (1981).
[9] G. S. Joyce, Phys. Rev. **146**, 349 (1966); W. K. Theumann, Phys. Rev. B **2**, 1396 (1970).
[10] A. C. Ribeiro Teixeira and D. A. Stariolo, Phys. Rev. E **70**, 016113 (2004); S. Risau-Gusman, A. C. Ribeiro-Teixeira, and D. A. Stariolo, Phys. Rev. Lett. **95**, 145702 (2005).
[11] A. E. Patrick, J. Stat. Phys. **72**, 665 (1993); J. G. Brankov and N. S. Tonchev, Phys. Rev. B **50**, 2970 (1994); D. A. Garanin, J. Phys. A **29**, 2349 (1996).
[12] M. L. Lyra, M. D. Coutinho-Filho, and A. M. Nemirovsky, Phys. Rev. B **48**, 3755 (1993).
[13] M. E. Fisher and M. N. Barber, Phys. Rev. Lett. **28**, 1516 (1972); M. N. Barber and M. E. Fisher, Ann. Phys. (N.Y.) **77**, 1 (1973); S. Singh and R. K. Pathria, *ibid.* **55**, 347 (1985); Phys. Rev. B **31**, 4483 (1985); **36**, 3769 (1987); J. G. Brankov and N. S. Tonchev, J. Stat. Phys. **52**, 143 (1988); **59**, 1431 (1990); Physica A **189**, 583 (1992); J. G. Brankov and D. M. Danchev, J. Phys. A **26**, 4485 (1993); D. Danchev, Phys. Rev. E **53**, 2104 (1996).
[14] L. A. Pastur, J. Stat. Phys. **27**, 119 (1982). See also A. M. Khorunzhy, B. A. Khoruzhenko, L. A. Pastur, and M. V. Scherbina, in *Phase Transitions and Critical Phenomena*, edited by C. Domb and J. Lebowitz (Academic, New York, 1992), Vol. 15, p. 73, and references therein.
[15] T. Vojta and M. Schreiber, Phys. Rev. B **49**, 7861 (1994).
[16] J. M. Kosterlitz, D. J. Thouless, and R. C. Jones, Phys. Rev. Lett. **36**, 1217 (1976); T. M. Nieuwenhuizen, Phys. Rev. B **31**, 7487 (1985); A. Theumann, J. Phys. A **20**, 25 (1987); A. Jagannathan, S. Eva, and J. Rudnick, *ibid.* **24**, 2193 (1991); A. Crisanti and L. Leuzzi, Phys. Rev. Lett. **93**, 217203 (2004); Phys. Rev. B **73**, 014412 (2006).
[17] M. Schwartz, Phys. Lett. **76A**, 408 (1980); R. M. Hornreich and H. G. Schuster, Phys. Rev. B **26**, 3929 (1982); J. F. Perez, W. F. Wreszinski, and J. L. van Hemmen, J. Stat. Phys. **35**, 89 (1984);

W. K. Theumann and J. F. Fontanari, *ibid.* **45**, 99 (1986); Physica A **149**, 341 (1988); W. F. Wreszinski and S. R. A. Salinas, *Disorder and Competition in Soluble Lattice Models* (World Scientific, Singapore, 1993), p. 35, and references therein; T. Vojta, J. Phys. A **26**, 2883 (1993); T. Vojta and M. Schreiber, Phys. Rev. B **50**, 1272 (1994).

[18] See, e.g., S. Sachdev, *Quantum Phase Transitions* (Cambridge University Press, Cambridge, England, 1999), and references therein.

[19] See, e.g., M. A. Continentino, *Quantum Scaling in Many-Body Systems* (World Scientific, Singapore, 2001), and references therein.

[20] J. A. Hertz, Phys. Rev. B **14**, 1165 (1976).

[21] G. Obermair, in *Dynamical Aspects of Critical Phenomena*, edited by J. I. Budnick and M. P. Kawatra (Gordon and Breach, New York, 1972), p. 137. See also a related quantum-mechanical lattice-dynamical model in the context of critical hydrodynamics: L. Sásvari, F. Schwabl, and P. Szépfalusy, Physica A **81**, 108 (1975).

[22] P. Shukla and S. Singh, Phys. Lett. **81A**, 477 (1981); T. M. Nieuwenhuizen, Phys. Rev. Lett. **74**, 4289 (1995); T. K. Kopeć and K. D. Usadel, *ibid.* **78**, 1988 (1997); T. M. Nieuwenhuizen and F. Ritorti, Physica A **250**, 8 (1998).

[23] T. M. Nieuwenhuizen, Phys. Rev. Lett. **74**, 4293 (1995).

[24] Y. Tu and P. B. Weichman, Phys. Rev. Lett. **73**, 6 (1994).

[25] T. Vojta, Phys. Rev. B **53**, 710 (1996).

[26] H. Chamati, E. S. Pisanova, and N. S. Tonchev, Phys. Rev. B **57**, 5798 (1998). In this article the authors have used spin variables $S=\pm 1$. As a consequence, their $d=2$ critical value should read $g_c(T=0)/J=(\lambda_c/4)^2 \approx (3.1114/4)^2 \approx 0.6051$ in the $S=\pm 1/2$ case. See also H. Chamati, D. M. Danchev, and N. S. Tonchev, Eur. Phys. J. B **14**, 307 (2000); J. G. Brankov, D. M. Danchev, and N. S. Tonchev, *Theory of Critical Phenomena in Finite-Size Systems* (World Scientific, Singapore, 2000), Chap. 10, and references therein, where the low-T zero-field free energy is calculated in $d=4$, though, except for χ as $T \to 0$, $g \approx g_c$, no explicit expressions for the quantum- and thermal-critical behaviors of observables or the equation of state are provided.

[27] T. Vojta and M. Schreiber, Phys. Rev. B **53**, 8211 (1996).

[28] A. Dutta, B. K. Chakrabarti, and J. K. Bhattacharjee, Phys. Rev. B **55**, 5619 (1997).

[29] R. Serral Gracià and T. M. Nieuwenhuizen, Phys. Rev. E **69**, 056119 (2004). In this work the authors have also studied the low-T behavior of the entropy and specific heat by considering quantum spherical models with or without canonically conjugated momenta in transverse magnetic field, in a path integral formalism using coherent states. Differently from Eq. (2), however, they have worked out a spherical constraint involving both $S_{\vec{R}}$ and $P_{\vec{R}}$, and concluded that in the presence (absence) of momenta the model is analog to the large-n limit of SU(n) Heisenberg ferromagnets (antiferromagnets), with dynamical exponent $z=2$ ($z=1$). Such result is distinct from ours and those of Refs. 25 and 26, which have found $z=1$ in the presence of momenta, longitudinal field, and spherical constraint only dependent on $S_{\vec{R}}$, Eq. (2).

[30] M. H. Oliveira, M. D. Coutinho-Filho, and E. P. Raposo, Phys. Rev. B **72**, 214420 (2005).

[31] See, e.g., R. K. Pathria, *Statistical Mechanics* (Elsevier, Oxford, 2004), and references therein.

[32] S. Sachdev and T. Senthil, Ann. Phys. (N.Y.) **251**, 76 (1996).

[33] S. Chakravarty, B. I. Halperin, and D. R. Nelson, Phys. Rev. Lett. **60**, 1057 (1988); Phys. Rev. B **39**, 2344 (1989).

[34] E. P. Raposo and M. D. Coutinho-Filho, Phys. Rev. Lett. **78**, 4853 (1997); Phys. Rev. B **59**, 14384 (1999).

[35] See, e.g., J. Johnson, J. Appl. Phys. **52**, 1991 (1981); M. Takahashi, *Thermodynamics of One-Dimensional Solvable Models* (Cambridge University Press, Cambridge, England, 1999), and references therein.

[36] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products* (Academic Press, New York, 1965); A. P. Prudnikov, Y. A. Brychkov, and O. I. Marichev, *Integrals and Series* (Gordon and Breach, Amsterdam, 1990), Vol. 3.

[37] P. Kopietz and S. Chakravarty, Phys. Rev. B **40**, 4858 (1989).