

**High-temperature series for the bond-diluted Ising model in 3, 4, and 5 dimensions**Meik Hellmund<sup>1,3,\*</sup> and Wolfhard Janke<sup>2,3,†</sup><sup>1</sup>*Mathematisches Institut, Universität Leipzig, Augustusplatz 10/11, D-04109 Leipzig, Germany*<sup>2</sup>*Institut für Theoretische Physik, Universität Leipzig, Augustusplatz 10/11, D-04109 Leipzig, Germany*<sup>3</sup>*Centre for Theoretical Sciences (NTZ) of the Centre for Advanced Study (ZHS), Universität Leipzig, Emil-Fuchs-Straße 1, D-04105 Leipzig, Germany*

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In order to study the influence of quenched disorder on second-order phase transitions, high-temperature series expansions of the susceptibility and the free energy are obtained for the quenched bond-diluted Ising model in  $d=3-5$  dimensions. They are analyzed using different extrapolation methods tailored to the expected singularity behaviors. In  $d=4$  and 5 dimensions we confirm that the critical behavior is governed by the pure fixed point up to dilutions near the geometric bond percolation threshold. The existence and form of logarithmic corrections for the pure Ising model in  $d=4$  are confirmed, and our results for the critical behavior of the diluted system are in agreement with the type of singularity predicted by renormalization group considerations. In three dimensions we find large crossover effects between the pure Ising, percolation, and random fixed points. We estimate the critical exponent of the susceptibility to be  $\gamma=1.305(5)$  at the random fixed point.

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**I. INTRODUCTION**

For many years random Ising models have served as paradigmatic systems in which the influence of quenched disorder may be studied through different techniques. In the present work we report results obtained by high-temperature series expansions. Systematic series expansions<sup>1</sup> for statistical physics models defined on a lattice provide a useful complement to field-theoretical renormalization group studies and large-scale numerical Monte Carlo simulations. This is in particular true when studying phase transitions and critical phenomena of quenched, disordered systems.

Series expansion techniques treat the quenched disorder average exactly, and the infinite-volume limit is implicitly implied. Therefore one can obtain exact results up to a certain order in the inverse temperature for many quantities. Moreover, one can keep the disorder strength  $p$  as well as the dimension  $d$  as symbolic parameters and therefore analyze large regions of the parameter space of disordered systems. The critical part of the series expansion approach lies in the extrapolation techniques which are used in order to obtain information on the phase transition behavior from the finite number of known coefficients. While for pure systems this usually works quite well, one can question the use of these extrapolation techniques in disordered systems, where the singularity structure of the free energy or susceptibility may be very complicated, involving Griffiths-type singularities or logarithmic corrections.<sup>2</sup> Yet our work indicates that at least for the model we consider here, the extrapolation techniques are of comparable quality as for the case of the pure (no disorder) Ising model.

We consider in this paper the Ising model on a hypercubic  $d$ -dimensional lattice  $\mathbb{Z}^d$  with bond dilution as a realization of quenched and uncorrelated disorder. The pure model has a second-order phase transition for  $d \geq 2$ , and the upper critical dimension, where mean-field behavior sets in, is  $d_u=4$ . The influence of quenched disorder can be estimated by the Harris criterion:<sup>3</sup> For  $d > d_u=4$ , the disorder is expected to

change only nonuniversal quantities such as the transition temperature  $T_c$ . At the upper critical dimension  $d=4$ , logarithmic or even more subtle correction terms should appear, and in  $d=3$ , the phase transition of the disordered system should be governed by a new “random” fixed point since for the pure model the critical exponent  $\alpha$  of the specific heat is positive and disorder should hence be a relevant perturbation. Our high-temperature series analyses presented in this paper affirm this picture.

The rest of the paper is organized as follows: In Sec. II we first briefly recall the model and some of its properties. Section III is devoted to a description of the methods used by us for generating the series expansions, and Sec. IV first starts out with some remarks on the analysis techniques used. The main results are presented in the following subsections, where we discuss our results for the random-bond Ising model in five (Sec. IV A), four (Sec. IV B), and three (Sec. IV C) dimensions. Finally, Sec. V contains our conclusions.

**II. THE MODEL**

The ferromagnetic disordered Ising model on hypercubic lattices  $\mathbb{Z}^d$  is defined by the partition function

$$Z(\{J_{ij}\}) = \sum_{\{s_i\}} \exp\left(\beta \sum_{\langle ij \rangle} J_{ij} s_i s_j\right), \quad (1)$$

where  $\beta=1/k_B T$  is the inverse temperature,  $J_{ij}$  are quenched (non-negative) nearest-neighbor coupling constants, and the spins  $s_i$  can take on the two different values  $\pm 1$ . In our series expansion the combination  $v_{ij}=\tanh(\beta J_{ij})$  will be the relevant expansion parameter. Quenched disorder averages  $[\dots]_{P(J)}$  such as the free energy

$$- \beta F = [\ln Z]_{P(J)} \quad (2)$$

are taken over an uncorrelated bimodal distribution of the form

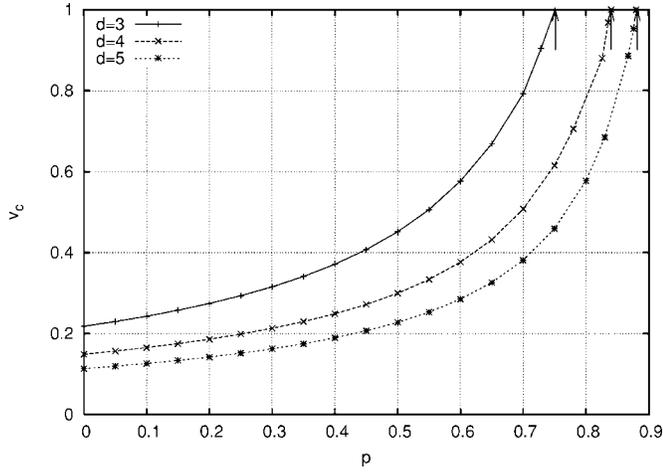


FIG. 1. Critical coupling  $v_c$  as function of the dilution  $p$  for  $d = 3, 4$ , and  $5$  dimensions. The arrows on the top axis indicate the percolation thresholds.

$$P(J_{ij}) = (1-p)\delta(J_{ij}-J_0) + p\delta(J_{ij}), \quad (3)$$

which corresponds to bond dilution: With probability  $p$ , bonds are effectively absent from the lattice, so that  $p=0$  represents the pure system. The expansion parameter for averaged quantities is usually taken as  $v = \tanh(\beta J_0)$ .

### A. Thermal phase transitions

The pure ( $p=0$ ) model has for  $d \geq 2$  a second-order phase transition from the disordered high-temperature phase to a ferromagnetic low-temperature phase. Bond dilution decreases  $T_c$ . Approaching the geometric bond percolation threshold  $p=p_c$  the critical temperature decreases to zero ( $v_c \rightarrow 1$ ) since above this value only finite clusters of bonds are present and therefore no global ferromagnetic order is possible. Figure 1 gives an overview of the phase diagrams as calculated in this work from high-temperature series of the susceptibility. The influence of quenched disorder on the thermal fixed point is different above and below the upper critical dimension  $d_u=4$ . For  $d \geq 4$  the pure fixed point is, according to the Harris criterion,<sup>3</sup> stable against the influence of disorder and one expects that the renormalization group (RG) flow goes along the critical line from the unstable percolation fixed point straight to the stable pure fixed point. The critical behavior along this line of phase transitions is governed by the pure fixed point. For  $d=3$  the pure fixed point is unstable and one expects a new random fixed point governing the critical behavior in the region  $0 < p < p_c$ , attracting RG flow from both sides (pure and percolation fixed points). More specifically, for  $d=4$  random disorder changes the form of the logarithmic corrections to the Gaussian fixed point,<sup>4</sup> whereas for  $d=5$  Gaussian behavior is expected for  $p=0$  and  $p > 0$  as well.

### B. Crossover to the percolation point

The crossover behavior of the diluted system near the percolation threshold of the underlying lattice is well

understood<sup>5-7</sup> and independent of the dimension of the lattice. The percolation critical point  $p=p_c$ ,  $T_c=0$  ( $v_c=1$ ) is always unstable against thermal fluctuations<sup>6</sup> and plays for our system the role of the strong disorder fixed point.<sup>8</sup>

The crossover exponent  $\phi$  is exactly 1, and critical exponents approach their percolation values with a scaling behavior of the form

$$\chi \sim (p_c - p)^{-\gamma} f_\chi \left( \frac{e^{-2J_0/T}}{p_c - p} \right) \quad (4)$$

$$\sim (p_c - p)^{-\gamma} f_\chi \left( \frac{1}{p_c - p} \frac{1-v}{1+v} \right), \quad (5)$$

where  $f_\chi$  is a crossover-scaling function and the critical line near  $v=1$ ,  $p=p_c$  has the form

$$\frac{1-v}{1+v} \propto (p_c - p). \quad (6)$$

## III. STAR-GRAPH EXPANSION

The star-graph expansion method has been reviewed many times in the literature<sup>9</sup> as well as in our previous papers.<sup>10,11</sup> We will only present some algorithmic details of our approach and make some remarks comparing different techniques for series generation.

“Star graph” is an old-fashioned name for one-vertex-irreducible graphs. For observables  $F$  which allow a star-graph expansion (like the free energy and various inverse susceptibilities) this technique delivers a representation of  $F$  on an infinite, e.g., hypercubic, lattice,

$$F(\mathbb{Z}^d) = \sum_G E(G; \mathbb{Z}^d) W_F(G), \quad (7)$$

as sum over star graphs (one-vertex irreducible graphs) only. Here,  $E(G; \mathbb{Z}^d)$  denotes the weak embedding number of the graph  $G$  in the given lattice structure<sup>12</sup> and  $W_F(G)$  can be recursively calculated from knowledge of  $F(G)$  and  $F(G')$  for all star subgraphs  $G' \subset G$ .

A property of Eq. (7) important for us is that the quenched disorder average can be calculated on the level of single graphs as long as the disorder is uncorrelated. This is related to the fact that the method considers and counts weak embeddings of the graph into the lattice—i.e., embeddings where different bonds of a graph are always mapped to different bonds of the lattice. This is in contrast to the free embeddings used in linked-cluster expansions. Here, different vertices and bonds of the graph may correspond to the same vertex or bond of the lattice. Therefore in order to calculate the disorder average one has to classify the free embeddings of a graph according to their correspondence to embeddings of a collapsed graph (where some edges of  $G$  are identified). This is essentially the algorithm proposed in Ref. 13 where the collapsed graphs are called “multiple-line graphs.” This method has not yet been used for actual calculations of disordered systems, presumably due to its combinatorial complexity.

TABLE I. Number of star graphs with  $N \geq 8$  links and nonvanishing embedding numbers on  $Z^d$ . For  $N=1, 4, 6$ , and  $7$  only a single star graph exists.

Order $N$	8	9	10	11	12	13	14	15	16	17	18	19	20	21
No. in $d=3$	2	3	8	9	29	51	136	306	856	2237	6431	18487	55302	165730
No. in $d=4$ and $5$	2	3	8	9	29	51	142	330	951	2561	7622	22688		

For other series generation methods such as, e.g., the finite-lattice method which holds the world record for the pure three-dimensional (3D) Ising model<sup>14</sup> or methods using Schwinger-Dyson equations,<sup>15</sup> no practical generalization to disordered systems is known.

For the generation of graphs we employed the NAUTY package by McKay<sup>16</sup> which makes very fast isomorphism tests by calculating a canonical representation of the automorphism group of the graphs. By this means, we classified all star graphs up to order 21 that can be embedded in hypercubic lattices; see Table I. For each of these graphs we calculated their weak embedding numbers for  $d$ -dimensional hypercubic lattices (up to order 17 for arbitrary  $d$ , order 19 for dimensions  $d \leq 5$ , and order 21 for  $d \leq 3$ ). For this embedding count we implemented a refined version of the backtracing algorithm by Martin,<sup>12</sup> making use of a couple of simplifications for bipartite hypercubic lattices  $Z^d$ . After extensive tests to find the optimal algorithm for the innermost loop, the test for collisions in the embedding, we ended up using optimized hash tables.

In order to calculate the contribution  $W(G)$  of a graph  $G$  of order  $N$  to the susceptibility series, one needs the partition sum  $Z(\{J_{ij}\}|G)$  and the matrix of all spin-spin correlation functions  $M_{kl}(\{J_{ij}\}|G) = \text{Tr} \delta_{s_k, s_l} e^{-\beta H(J_{ij})}$  as polynomials in the  $N$  variables  $J_{ij}$ . This is achieved by using the cluster representation

$$Z \propto \sum_C 2^{c(C)} \prod_{\langle ij \rangle \in C} \frac{2v_{ij}}{1-v_{ij}}, \quad (8)$$

$$M_{kl} \propto \sum_{C_{kl}} 2^{c(C)} \prod_{\langle ij \rangle \in C} \frac{2v_{ij}}{1-v_{ij}}, \quad (9)$$

where the sum in Eq. (8) goes over all clusters  $C \subseteq G$  which have only vertices with an even number of bonds and in Eq. (9) the sum is restricted to all clusters  $C_{kl} \subseteq G$  in which the vertices  $k$  and  $l$  have an odd number of bonds and all other have an even number of bonds. The exponent  $c(C)$  counts the number of connected components of  $C$ . A graph with  $N$  bonds gives rise to  $2^N$  clusters: every bond may be present or absent. The clusters can therefore be numerated by  $N$ -bit integers. These  $2^N$  integers can be sorted in such a way that two consecutive numbers differ by exactly one bit (corresponding to the addition or deletion of one bond), an algorithm known as the ‘‘Gray code.’’<sup>17</sup> This gives a simple algorithm for the calculation of the cluster sums in Eqs. (8) and (9).

(i) Start with the full graph  $C=G$ , store the coordination number modulo 2 (‘‘coordination bit’’) for every vertex, and

count the total number  $r$  of odd vertices. Execute steps (b) and (c) from below for this cluster configuration.

(ii) Iterate over all other  $2^N - 1$  Gray codes  $C$ .

(a) Calculate the next Gray code (cluster configuration). Compared to the previous code, exactly one bond was added or deleted. Invert the coordination bit of the two involved vertices and calculate the change in  $r$ .

(b) If  $r=0$ , then add  $\prod_{n \in C} w_n$  to  $Z$ . The product has a factor  $w_n = 2v_{ij}/(1-v_{ij})$  for every nonzero bit  $n$  in  $C$ .

(c) If  $r=2$ , then add  $\prod_{n \in C} w_n$  to  $M_{kl}$  where  $k, l$  are the two odd vertices.

The prefactor  $2^{c(C)}$  is taken into account by monitoring the change in the number of connected components  $c(C)$  in each iteration step.

The further steps for the calculation of the susceptibility series are the following.

(i) Inversion of the  $Z$  polynomial as a series in the  $\{v_{ij}\}$  up to the desired order.

(ii) Averaging over quenched disorder,

$$N_{kl}(G) = [M_{kl}/Z]_{P(J)},$$

resulting in a matrix of polynomials in  $p$  and  $v$ .

(iii) Inversion of the matrix  $N_{kl}$  and subgraph subtraction,

$$W_\chi(G) = \sum_{k,l} (N^{-1})_{kl} - \sum_{G' \subset G} W_\chi(G').$$

(iv) Collecting the results from all graphs,

$$1/\chi = \sum_G E(G; Z^d) W_\chi(G).$$

All calculations are done in arbitrary-precision integer arithmetic using the open source library GMP. For the polynomial arithmetic we developed our own optimized C++ template library using a degree-sparse representation of polynomials as linked lists.<sup>18</sup> The calculations took around two CPU years on an Opteron Linux Cluster.

#### IV. SERIES ANALYSES

The estimate of critical parameters from a high-temperature series involves extrapolation from a finite number of exactly known coefficients to the asymptotic form of the function. Many such extrapolation techniques have been developed and tested for different series and are comprehensively reviewed in Ref. 19. These extrapolation techniques are not rigorous. They make some assumptions about the expected form of the singularity at the critical temperature. Field-theoretic techniques like the  $\epsilon$  expansion have similar problems.

In order to get a reliable picture, we will take into account several criteria, such as convergence of the analysis, scatter of different approximants, number of defective approximants, and agreement between different extrapolation methods.

The basic methods we use are DLog-Padé approximation and inhomogeneous differential approximants (IDA's).<sup>19</sup> In order to analyze confluent nonanalytic and logarithmic corrections, these methods are applied to suitably transformed forms of the series. The parameters of these transformations are fine-tuned according to the criteria listed above, a technique pioneered in Ref. 20.

Usually, error estimates rely on the scatter of the results of extrapolations with different parameters (like  $[N/M]$  Padé approximants for different values of  $N$  and  $M$ ). In the cases where we have a quite large number of Padé or differential approximants, we quote the usual two standard deviations as statistical errors of our results. In the case of the  $M1$  and  $M2$  analyses, our error estimates come from estimating the stability and convergence of the analysis under parameter fine-tuning and we consider them to be comparable to the usual two standard deviation error indications. Nevertheless, this may seriously underestimate systematic errors coming from wrong assumptions about the structure of the singularity.

We obtain from the star-graph technique series in two variables  $v$  and  $p$ . Since the algorithm involves inversion of polynomials (and of matrices of polynomials), one has to find a consistent truncation criterion for the resulting series. The crucial observations are that (a) the coefficient matrix  $a_{mn}$  of the resulting series is triangular (cf. the Appendix), i.e., the series is of the form

$$\chi(p, v) = \sum_{n=0}^{\infty} \sum_{m=0}^n a_{mn} q^m v^n \quad \text{where } q = 1 - p, \quad (10)$$

and (b) the contribution of a graph of order  $N$  starts with terms of order  $q^N v^N$ . This allows different consistent truncation schemes. In scheme  $A$  we calculate series which are correct up to a given order  $N$  in  $v$ ; i.e., we calculate all nonvanishing coefficients  $a_{mn}$  with  $n \leq N$ . In scheme  $B$ , we calculate all nonvanishing coefficients  $a_{mn}$  which satisfy  $m+n \leq 2N$ .

Both schemes are useful in different regions of the parameter space. At small and medium disorder  $p$  we approach the critical point by extrapolating in  $v$  for constant values of  $p$ . Therefore we use the series calculated according to truncation scheme  $A$  in the form

$$\chi(v|p = \text{const}) \propto \sum_{n=0}^N \left( \sum_{m=0}^n a_{mn} q^m \right) v^n = \sum_{n=0}^N c_n v^n. \quad (11)$$

Near the percolation threshold  $p \lesssim p_c$  this gives less and less satisfactory results. As Eq. (6) and Fig. 1 show,  $p = \text{const}$  lines are reaching the critical line at a smaller and smaller angle. In this region, it is better to extrapolate along the locus  $w = q/v = \text{const}$ . This is achieved by taking the series coefficients calculated with scheme  $B$ :

$$\chi(v|w = \text{const}) \propto \sum_{m+n \leq 2N} a_{mn} w^m v^{n+m} = \sum_{n=0}^{2N} d_n v^n. \quad (12)$$

The analysis of the scheme  $B$  series brings much better results than scheme  $A$  in the highly diluted region  $p \lesssim p_c$ . But the quality of the analysis is still lower (e.g., the scatter of different Padé approximants is larger) than in the regions with small or medium dilution. This is not surprising since we cannot construct from our data a series consistently truncated in  $p$ . This would be necessary in order to extrapolate along the locus  $v = \text{const}$  and study the percolation transition directly. Nonetheless, a reliable estimate of the transition temperature  $v_c$  is possible up to  $p \approx p_c$ , giving the phase diagram in Fig. 1.

### A. Five dimensions

The analysis of the 19th-order series for the susceptibility of the 5D bond-diluted Ising model is a nice warmup in order to get an impression of the quality of different analysis methods. Unbiased DLog-Padé and IDA's along different  $p = \text{const}$  loci give values of  $\gamma$  between 1.02 and 1.04 for a wide range of dilutions  $p = 0 - 0.7$  with a slightly increasing tendency.

This effect—an overestimate of  $\gamma$  in a naive analysis not taking into account confluent corrections—is well known from the 3D pure Ising model. A good illustration is Fig. 2 in Ref. 21: whereas the results for many different theories in the Ising universality class ( $\phi^4$ , etc.) converge to  $\gamma \approx 1.237$ , even 25th-order series for the Ising model give  $\gamma \approx 1.244$ . The remedy is the use of analysis methods tailored for confluent nonanalytic corrections,

$$f(v) \sim A(v_c - v)^{-\gamma} [1 + A_1(v_c - v)^{\Delta_1} + \dots], \quad (13)$$

such as the methods from Ref. 20 called  $M1$  and  $M2$ . The method  $M1$  uses DLog-Padé approximants to

$$F(v) = (v_c - v) \frac{df}{dv} - \gamma f(v), \quad (14)$$

which has a pole at  $v_c$  with residue  $\Delta_1 - \gamma$ . For a given trial value of  $v_c$  the graphs of  $\Delta_1$  versus input  $\gamma$  are plotted for different Padé approximants and by adjusting  $v_c$  a point of optimal convergence is searched.

The  $M2$  method starts with a transformation of the series in  $v$  into a series in  $y = 1 - (1 - \frac{v}{v_c})^{\Delta_1}$  and then Padé approximants to

$$G(y) = \Delta_1(1 - y) \frac{d}{dy} \ln F(y) \quad (15)$$

are calculated which should converge to  $\gamma$  as  $y \rightarrow 1$ —i.e.,  $v \rightarrow v_c$ . These methods are especially useful when taken as biased approximants with a given value of  $\gamma$  or  $\Delta_1$  as input.

Using the  $M2$  method we find over a large range of the disorder strength,  $p = 0 - 0.7$ , excellent convergence of the Padé approximants with a nearly  $p$ -independent value of  $\Delta_1$ . As an example, Fig. 2 shows an  $M2$  plot at dilution  $p = 0.5$  from which we read off  $\gamma = 1.001(1)$ ,  $\Delta_1 = 0.51(2)$  and determine  $v_c = 0.227\,498$ . Our overall estimate for  $\Delta_1$  for all dis-

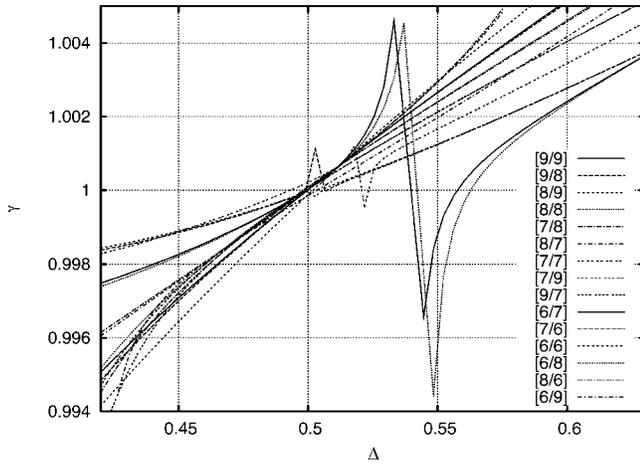


FIG. 2.  $M2$  analysis ( $\gamma$  as function of  $\Delta_1$ ) for the  $p=0.5$  diluted 5D Ising model at  $v_c=0.227\,498$  based on 15 different  $[N/M]$  Padé approximants.

order strengths  $p=0-0.7$  is  $\Delta_1=0.50(5)$ . The results of the  $M1$  method are compatible with this but have poorer convergence.

For the pure ( $p=0$ ) 5D Ising model our critical values obtained from the 19th-order series with the  $M2$  analysis biased with  $\gamma=1$  are  $v_c=0.113\,425(3)$ ,  $\Delta_1=0.50(2)$ . As usual, the errors are estimates of the scatter of different Padé approximants. This should be compared with the value from Monte Carlo (MC) simulations<sup>22</sup> of  $\beta_c=0.113\,915\,2(4)$ —i.e.,  $v_c=0.113\,425\,0(4)$ , which is in perfect agreement.

### B. Four dimensions

Four is the upper critical dimension of the Ising model. Without impurities the scaling behavior of the susceptibility and specific heat is thus expected to exhibit at the critical point logarithmic corrections. With  $t \equiv |1-T/T_c|$ , the precise form reads as<sup>23-26</sup>

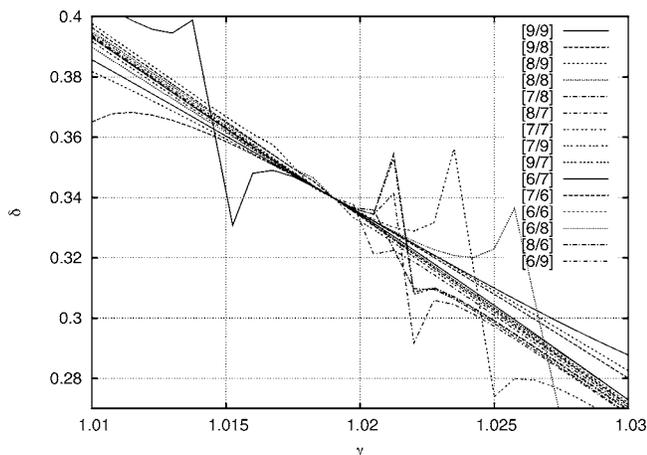


FIG. 3. Logarithmic exponent  $\delta$  as function of the leading exponent  $\gamma$  for the pure 4D Ising model via unbiased  $M3$  approximants at  $v_c=0.148\,607$ . A pronounced clustering of the different  $[N/M]$  Padé approximants around the point  $\gamma \approx 1.019$ ,  $\delta \approx 0.34$  is clearly observed.

$$\chi \sim t^{-1} |\ln t|^{1/3}, \quad (16)$$

$$C \sim |\ln t|^{1/3}, \quad (17)$$

where a nonsingular background term in  $C$  has been omitted. For the disordered case it was shown long ago<sup>4</sup> by a RG analysis that the critical behavior is modified to take the form

$$\chi \sim t^{-1} \exp[D |\ln t|^{1/2}], \quad (18)$$

$$C \sim -|\ln t|^{1/2} \exp[-2D |\ln t|^{1/2}], \quad (19)$$

where  $D = \sqrt{6/53} \approx 0.336$ . The crossover to mean-field behavior as well as higher-loop corrections<sup>27</sup> modify this unusual exponential term by factors of the form  $|\ln t|^{\Delta_1}$ . Notice that the expression in Eq. (19) approaches zero for  $t \rightarrow 0$  such that  $C$  is no longer logarithmic divergent but stays finite in the presence of disorder.

#### 1. Pure Ising model

Due to the logarithmic corrections in Eqs. (16) and (17), a special treatment of the series is needed. The analysis of logarithmic corrections of the general form

$$f(v) \sim (v_c - v)^{-\gamma} |\ln(v_c - v)|^\delta \quad (20)$$

is possible by<sup>28</sup> calculating approximants for

$$F(v) = (v_c - v) \ln(v_c - v) \left[ \frac{f'(v)}{f(v)} - \frac{\gamma}{v_c - v} \right], \quad (21)$$

where one expects for singularities of the form (20) that  $\lim_{v \rightarrow v_c} F(v) = \delta$ . In what follows we call this the  $M3$  method.

The estimates for the pure 4D Ising model based on our 19th-order series are (a) unbiased estimate, Fig. 3,  $\gamma = 1.019(1)$ ,  $\delta = 0.34(1)$ ,  $v_c = 0.148\,607(10)$  and (b) biased to  $\gamma = 1$ , Fig. 4,  $\delta = 0.429$ ,  $v_c = 0.148\,583(3)$ . This should be

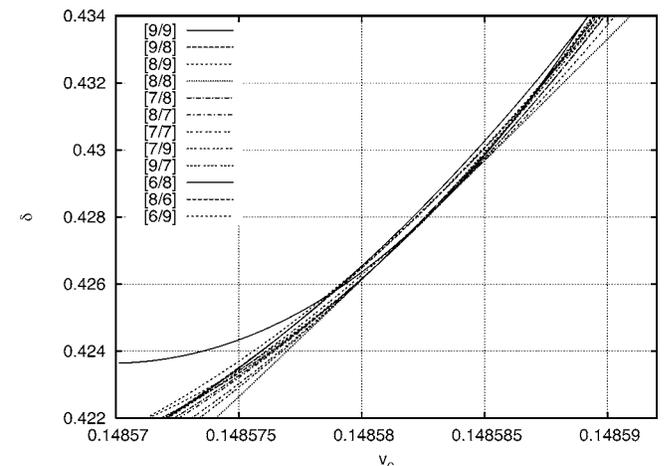


FIG. 4. Logarithmic exponent  $\delta$  as function of  $v_c$  for the pure 4D Ising model via  $M3$  approximants biased to  $\gamma=1$ . From the clustering of the  $[N/M]$  Padé approximants one reads off  $v_c \approx 0.148\,583$ ,  $\delta \approx 0.429$ . When assessing the clustering property notice that the  $y$  scale is 10 times finer than in Fig. 3.

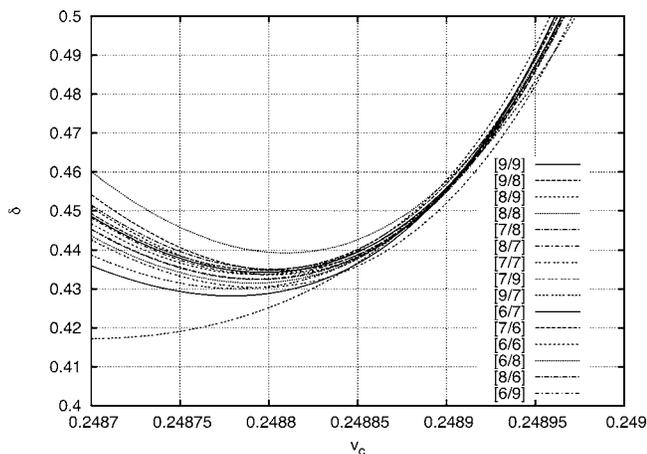


FIG. 5. Logarithmic exponent  $\delta$  as function of  $v_c$  for the  $p=0.4$  diluted 4D model, results of an  $M3$  analysis.

compared to the analysis in Ref. 29 of a 17th-order series<sup>30</sup> which gave  $v_c=0.148\,588$  and to MC simulations<sup>31</sup> which found  $\beta_c=0.149\,697(2)$ —i.e.,  $v_c=0.148\,589(2)$ . While the agreement of the estimates for  $v_c$  is excellent, this analysis also shows how difficult it is to obtain reliable estimates for the critical exponent  $\delta$  of the logarithmic correction term.

## 2. Bond dilution

An extensive MC study of the *site*-diluted 4D model was done in Ref. 23. The authors found data compatible with the theoretical prediction of a Gaussian fixed point with logarithmic corrections, but a precise fit of the logarithmic corrections was not possible.

A modification of the  $M3$  analysis method which we refer to as  $M3a$  is able to take into account the specific form of Eq. (18): For

$$f(v) \sim (v_c - v)^{-\gamma} \exp[D \ln|v_c - v|^{1/2}], \quad (22)$$

we calculate approximants to

$$F(v) \sim (v_c - v) [-\ln(v_c - v)]^{1/2} \left[ \frac{f'(v)}{f(v)} - \frac{\gamma}{v_c - v} \right], \quad (23)$$

which should behave as  $\lim_{v \rightarrow v_c} F(v) = D/2$ . We made an extensive analysis of our 19th-order susceptibility series assuming the three forms (13), (20), and (22). It turns out that, while form (13) is not acceptable, both forms of logarithmic corrections allow a fit of the series data, both with a disorder-dependent exponent  $\delta$  or  $D$ , respectively. The quality of the  $M3a$  fits is, however, better than for the  $M3$  fits. This is demonstrated in Figs. 5 and 6 where equal ranges of  $v_c$  ( $\Delta v_c = 0.0003$ ) and of the exponent ( $\Delta\delta = \Delta D = 0.1$ ) are shown.

We interpret this as an indication for the validity of the RG prediction, Eq. (18). Figure 7 shows the dilution dependence of our estimates for the critical parameter  $D$ . It illustrates the difficulty in differentiating between a behavior of the form (20) and the form (22) by giving a nonzero result for  $D$  at  $p=0$ . It is nevertheless quite impressive that we can see at all such weak correction terms to the leading singularity and even estimate the parameters.

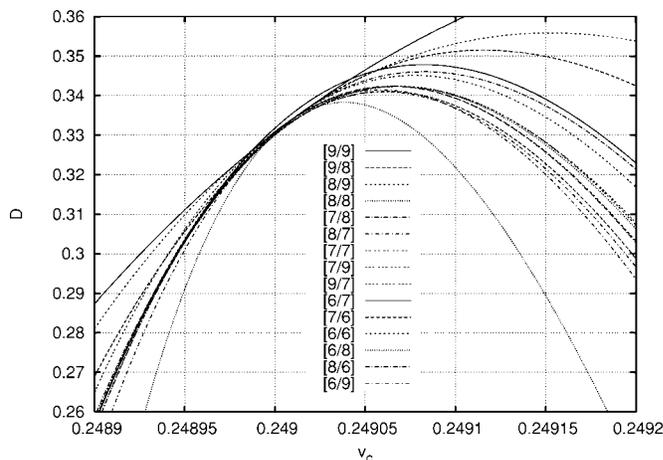


FIG. 6. Parameter  $D$  as function of  $v_c$  for the  $p=0.4$  diluted 4D model, results of an  $M3a$  analysis.

For the specific heat, one expects the weak logarithmic singularity in the pure case [Eq. (17)] to be washed out by the disorder. An analysis of the free energy series  $F(v) = \sum a_i(p)v^i$  is usually more difficult and gives less satisfactory results compared to the susceptibility series. One loses two orders in  $v$  by calculating  $C = -\beta^2 \frac{d^2 F}{d\beta^2}$  and, more importantly, on bipartite lattices the series for  $F(v)$  includes only even powers of  $v$ . So it has to be considered as a much shorter series in  $v^2$ . In four dimensions we calculated  $F(v, p)$  up to order 18, which gives a series for  $C(v^2)$  of order 8 in  $v^2$ .

Another difficulty is that the nonsingular background terms are more influential in this case. Therefore DLog-Padé approximants show very poor convergence and one has to use IDA's which are able to take background contributions into account. By this method, polynomials  $P_N(v)$ ,  $Q_M(v)$ , and  $R_L(v)$  (of order  $N$ ,  $M$ , and  $L$ , respectively) are determined such that

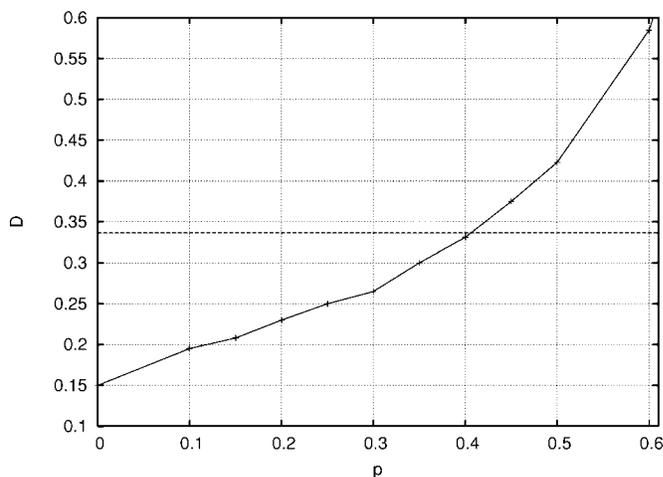


FIG. 7. Parameter  $D$  as function of  $p$  for the diluted 4D model, results of an  $M3a$  analysis. The horizontal line at  $D=0.336\dots$  shows the predicted value  $\sqrt{6/53}$ .

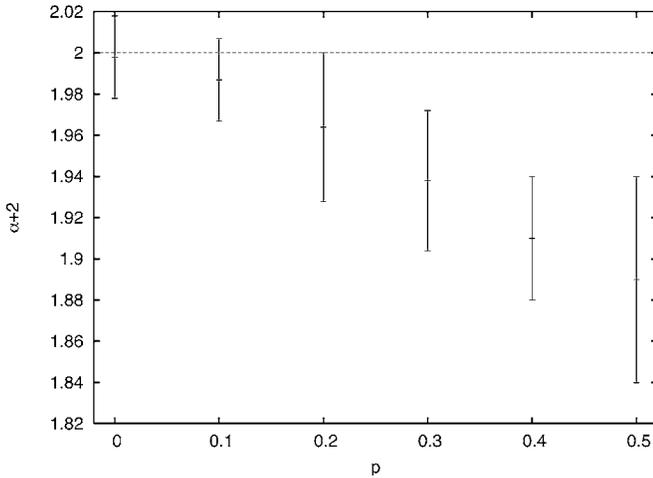


FIG. 8. Critical exponent of  $\partial^4 f(v)/\partial v^4$  as function of  $p$  for the diluted 4D model.

$$P_N(v)\hat{D}f(v) + Q_M(v)f(v) + R_L(v) = o(v^{(N+M+L)}), \quad (24)$$

where many different triples  $(N, M, L)$  with  $N+M+L \ll (\text{order of } f)$  and two different variants of the differential operator  $\hat{D}$ , either  $\hat{D}_1 = vd/dv$  or  $\hat{D}_2 = d/dv$ , are used. The critical point  $v_c$  is then given by the smallest positive real root of  $P_N(v)$  and the critical exponent by  $Q_M(v_c)/P'_N(v_c)$ .

We applied IDA's tailored to powerlike singularities  $\sim t^{-\alpha-2}$  to the fourth derivative  $\partial^4 f(v)/\partial v^4$ . The results are shown in Fig. 8. In the pure case we find a result consistent with  $\partial^4 f(v)/\partial v^4 \sim (v_c - v)^{-2}$ , indicating a logarithmic singularity in  $C \sim \partial^2 f(v)/\partial v^2$ . At nonzero values of the dilution parameter  $p$  the singularity of the fourth derivative of  $f$  is clearly weaker than  $(v_c - v)^{-2}$ , indicating the absence of a divergence in the specific heat in the disordered case.

### C. Three dimensions

Let us finally turn to the physically most important case of three dimensions. The quest for a determination of the prop-

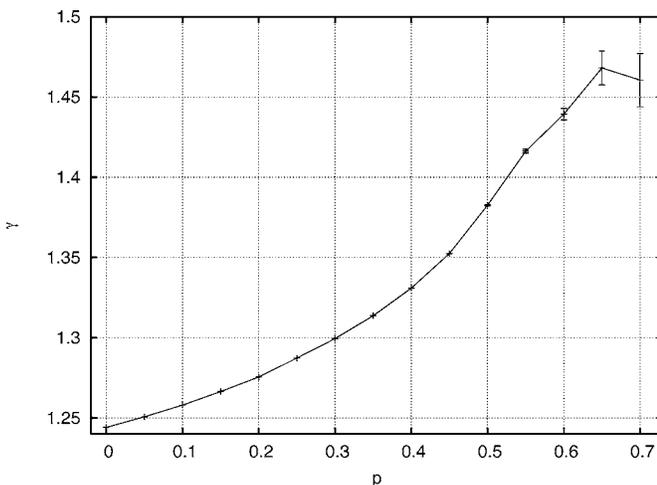


FIG. 9. Critical exponent  $\gamma$  as function of  $p$  for the diluted 3D model obtained from about 300 DLog-Padé and inhomogeneous differential approximants for each dilution value.

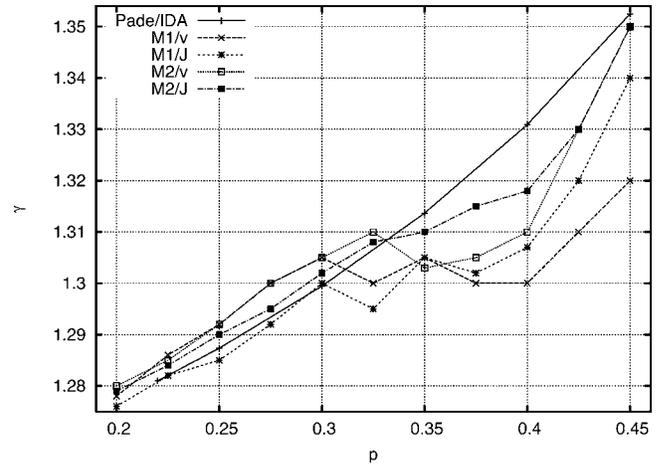


FIG. 10. Critical exponent  $\gamma$  as function of  $p$  for the diluted 3D model obtained from series analyses with methods  $M1$  and  $M2$  (see text for details). The data of Fig. 9 (Pade/IDA) are also shown for comparison.

erties of the expected random fixed point in the 3D disordered Ising model is already rather long. A comprehensive compilation of results can be found in Refs. 32–34, showing a wide scatter in the critical exponents of different groups, presumably due to large crossover effects. Recent MC simulations<sup>35–38</sup> provide evidence for the random fixed point but also show large crossover effects due to the interference with the pure fixed point for  $p \rightarrow 0$  and with the percolation fixed point for  $p \rightarrow p_c$  (recall the discussion in Sec. II B).

Early attempts using series expansions<sup>39,40</sup> already indicated that the series are slower converging and more difficult to analyze than in the pure case. A review of earlier work can be found in Ref. 41. Series analysis in crossover situations is, in fact, extremely difficult. If the parameter  $p$  interpolates between regions governed by different fixed points, the exponent obtained from a finite number of terms of a series expansion must cross somehow between the two universal values and does this usually quite slowly.<sup>42</sup> The mere existence of a plateau in  $\gamma(p)$ , however, is an indication that here a truly different critical behavior is seen.

Our results are obtained from a large number of DLog-Padé and inhomogeneous differential approximants (around 300 for every dilution value) applied to the 21st-

TABLE II. Summary of results. In three dimensions, longer series for the pure Ising model exist; therefore, we do not quote here our results for  $v_c$  or  $\gamma$ .

Model	$v_c$	$\gamma$	Confluent correction
5D Pure	0.113 425(3)	1.001(1)	Powerlike, $\Delta_1=0.50(2)$
5D Disordered		1.001(1)	Powerlike, $\Delta_1=0.50(5)$
4D Pure	0.148 607(3)	1.019(20)	Logarithmic, $\delta=0.34(2)$
4D Disordered			Form of Eq. (18) confirmed
3D Disordered		1.305(5)	



model disappears in the disordered case due to logarithmic corrections of a special kind. We confirm the presence of these special corrections also in the case of the susceptibility. We furthermore determine the exponent of the logarithmic correction in the pure case with satisfying precision.

The case of three dimensions is, in a sense, the most difficult to analyze. We clearly see that the pure fixed point is unstable and does not describe the random system. Extrapolation methods not taking into account confluent corrections give a value for  $\gamma$  which changes monotonously with  $p$ . By using methods tailored to the consideration of confluent corrections we identify a plateau in the curve  $\gamma(p)$  with critical exponent  $\gamma=1.305(5)$  which we see as evidence for the random critical point.

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## APPENDIX: COEFFICIENTS OF THE 3D SUSCEPTIBILITY

Table III gives complete information for calculating the susceptibility of the 3D bond-diluted Ising model up to the order  $v^{21}$  for any dilution  $p$ .

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