# Vortex duality: Observing the dual nature using order propagators

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In condensed matter physics, Kramers-Wannier duality implies that the state disordered by quantum fluctuations or temperature actually corresponds to an ordered state formed from the topological excitations of the "original" ordered state. At first sight it might appear to be impossible to observe this dual order using means associated with the original order. Although true for Ising models, we demonstrate in this paper that this is not a general statement by considering the well-known vortex duality, in particular in the quantum interpretation in 2+1D where it is associated with the quantum phase transition from a superfluid to a Bose Mott insulator. Here, the disordered Mott insulating state is at the same time a dual superconductor corresponding to a Bose condensate of vortices. We present a simple formalism making it possible to compute the velocity propagator associated with the superfluid in terms of the degrees of freedom of the dual theory. The Mott insulator is characterized by a doublet of massive modes, and we demonstrate that one of these modes is nothing else than the longitudinal photon (gauged second sound) of the dual superconductor. For increasing momenta, the system rediscovers the original order, and the effect on the velocity correlator is that the longitudinal photon loses its pole strength. The quantum critical regime as probed by the velocity correlator is most interesting. We demonstrate that at infinite wavelength the continua of critical modes associated with second sound and the dual longitudinal photon are indistinguishable. However, at finite momenta they behave differently, tracking the weight reshuffling found in the quasiparticle spectrum of the disorder state closely.

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## I. INTRODUCTION

The notion of Kramers-Wannier duality[1] has been around for a long time in statistical physics and field theory but one can still wonder if its scope is fully appreciated. Especially in condensed matter physics it appears to be much more than a mathematical convenience. One can view it instead as a physical "relativity" principle associated with order. Order and disorder have no objective meaning but just depend on the viewpoint of the observer. For instance, according to the classic two-dimensional (2D) Ising model duality,[2] an observer equipped with machinery measuring two point correlators of the order degrees of freedom (the Ising spins) will be convinced that the low temperature state shows long range order while the high temperature state is just an entropy dominated featureless entity. On the other hand, the Kramers-Wannier duality demonstrates that the high temperature state in fact corresponds to an ordered state, a condensate, formed from the topological excitations (domain walls) associated with the low temperature order. An experimentalist probing the system with a machine sensitive to the domain wall order would be of the opinion that the *high* temperature state is ordered, while the low temperature state is entropy dominated. When duality is in charge, disorder is order in disguise and one might think that this camouflage act is perfect. The dual order is carried by the topological excitations of the direct order. In the continuum limit it takes an infinity of operations involving the order degrees of freedom to probe the topological excitations and this is beyond the capacity of any machine builder. Henceforth, it appears that it is fundamentally impossible for an observer which can only employ order degrees of freedom to directly measure the order associated with the dual side with the consequence that the "order-experimentalist" can only perceive dual order as disorder.

In this paper we demonstrate that the above is too strong a statement. This "duality censorship" seems absolute for the special case of the Ising model in 2D. However, in a way its scalar order parameter structure is too simple. Here we will focus on a more representative example: the 3D *XY* model which might be alternatively interpreted as the Bose-Hubbard model in 2+1D at zero chemical potential,[3] or either as the Abelian-Higgs model in 2+1D of high energy physics at $T=0$. It is characterized by the well-known Abelian-Higgs[4] or vortex duality which maps the global *XY* model on U(1)/U(1) gauge theory. In the Bose-Hubbard interpretation, the quantum disordered neutral superfluid corresponds to a dual Meissner phase characterized by Bose condensed vortex particles. This incompressible state corresponds physically to the Bose Mott-insulator.[3]

The excitations of the "dual side" are the Higgs (amplitude) mode and massive photons of the dual superconductor. Surely, the Higgs mode is subjected to dual censorship for the same reasons as found in the Ising model, but the photons are a different story. As is well-known, the Goldstone mode (second sound) of the superfluid turns into a doublet of massive excitations in the Mott-insulator corresponding to the propagating unoccupied and doubly occupied states in the Mott state. As we will show here, these actually correspond to linear combinations of the degenerate transversal and longitudinal photons of the dual superconductor. The latter is of course the usual extra gauge mode associated with the presence of the dual phase order, and one can say that the dual censorship does not prohibit the dual phase order to manifest itself on the order side.

Our workhorse is a simple expression relating the second sound propagators to the dual photons which just follows from the Legendre transformation (Sec. III). In an earlier work, connections between correlators of the ordered and of the dual side were studied, but these related the propagators

 

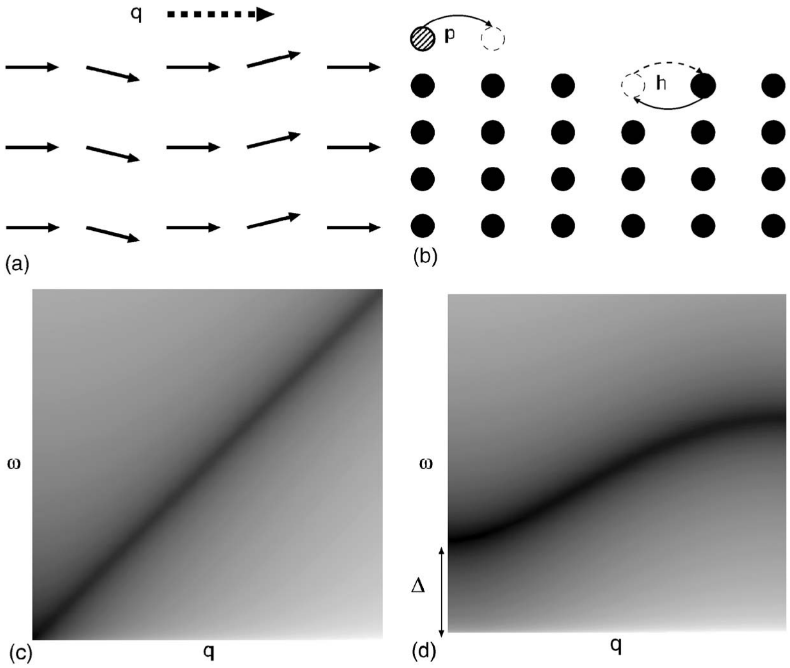


FIG. 1. The excitations in the weak/strong coupling limits of the Bose Hubbard model at zero chemical potential: The Goldstone boson (second sound) with linear dispersion (c) associated with the superfluid (phase ordered) state (a) at weak coupling. In the strong coupling limit (b) a doublet of massive "excitons" are found with gap $\Delta$ (d) corresponding to propagating unoccupied and doubly occupied sites, which can be alternatively understood at $q\to 0$ as the $\pm 1$ angular momentum eigenstates of a $O(2)$ quantum rotor.

at different coupling constants.[5] In combination with the essentially complete understanding of the physics resting on the dual order, we are able to describe with little effort some features of the order propagators in the disordered state which are to the best of our knowledge not recognized. On the Gaussian level (Secs. IV and V) we obtain the outcome sketched in Fig. 2(b): the single Goldstone of the ordered state [Fig. 2(a)] turns into a massive doublet in the disordered state. At zero momentum the two poles of the velocity propagator are indistinguishable but one of them loses its strength when momentum is exceeding the inverse London length of the dual superconductor. This makes sense because at shorter lengths order is reemerging and the simple Goldstone spectrum of the ordered state should be recovered.

The Abelian Higgs model in 2+1D is below its upper critical dimension and its critical state is strongly interacting. Resting on the complete description[6–8] of this critical state in the dual language, we will derive the critical velocity-velocity propagators (Sec. VI) with the surprising outcome that its transversal and longitudinal components appear to be quite different although governed by the same anomalous dimension, again reflecting the rather different status of the "order" (transversal) and "disorder" (longitudinal) photons when measured through velocity correlations. Before getting deeper into this, let us start with some simple considerations regarding the mode counting.

## II. A SIMPLE COUNTING ARGUMENT

The system of interest is the well-known Bose-Hubbard model in 2+1D at vanishing chemical potential, written in phase-number representation as[3]

$$H = \frac{1}{C}\sum_i n_i^2 - J\sum_{\langle ij\rangle} \cos(\phi_i - \phi_j) \tag{1}$$

defined on a 2D bipartite lattice; $n_i$ and $\phi_i$ are the number and phase operators on site $i$, satisfying the commutation relation $[n_i,\phi_j]=i\delta_{ij}$. The first and second term in Eq. (1) represents the charging and Josephson energy, respectively. When the coupling constant $\tilde{g}=2/(JC)$ is small the Josephson energy will dominate and the phase is ordered at zero temperature, while the excitation spectrum consists of a single Goldstone mode (phase mode or second sound) shown in Figs. 1(a) and 1(c). On the other hand, when $\tilde{g}$ is large the phase is quantum disordered and number condenses such that $n_i=0$ modulo local fluctuations, signaling the Mott-insulator. Of central interest is the excitation spectrum of the Mott-insulator. In the rotor language,[9] the ground state is the angular momentum singlet while the lowest lying excitations consist of a doublet of propagating $M=\pm 1$ modes characterized by a zero-momentum mass gap [Fig. 1(d)]. In the Bose-Hubbard interpretation these have a simple interpretation in the strong coupling limit ($\tilde{g}\to\infty$) as bosons added ($M=+1$) or removed ($M=-1$) from the charge-commensurate state [Fig. 1(b)], while their delocalization in the lattice produces an identical dispersion due to the charge conjugation symmetry of the model Eq. (1).

The above is, of course, very well understood. The addition of our work presented in this paper is that we map out the correspondence between the Mott-insulating phase and the dual (vortex) superconductor on the dynamical level, associating the excitation spectra of the two sides in full detail. After all, the doublet of massive modes are fully protected elementary excitations and they should arise regardless the way one wants to describe the system.

In the scaling limit, sufficiently close to the quantum phase transition where a continuum field-theoretic description applies, phase dynamics might as well be described in terms of the dual "disorder field theory." This turns out to be just the Ginzburg-Landau-Wilson theory of a relativistic 2+1D U(1) superconductor,[4]

$$\mathcal{L}_{EM,full} = \frac{\tilde{g}}{4} F_{\mu\nu}F_{\mu\nu} + \frac{1}{2}|(\partial_\mu - iA_\mu)\Psi_V|^2 + \frac{1}{2}m^2|\Psi_V|^2 + \omega|\Psi_V|^4. \tag{2}$$

The Higgs field $\Psi_V$ describes the Bose condensate of vortices, i.e., the tangle of vortex worldlines. In 2+1D vortex "particles" are in a precise sense indistinguishable from electromagnetically charged particles. The long range interactions between vortices mediated by the phase condensate can be described in terms of noncompact U(1) gauge fields $A_\mu$, U(1) appearing in the connections for the matter field and in the field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Of crucial importance for what comes, Eq. (2) is *fully relativistic*. The phase-dynamics problem is characterized by a single (spin-wave) velocity which we take to be one. Although in dual representation we are dealing with two separate fields $\Psi_V$ and $A_\mu$, both matter and gauge fields are governed by the same velocity.

The ordered and disordered phases correspond to the Coulomb and Meissner phase, respectively, of the theory Eq. (2). Obviously, the observable consequences of the dual theory Eq. (2) and the original theory Eq. (1) have to be the same and for this to be the case, it is a necessary condition that the mode content of both theories is the same.

Let us first consider the superfluid phase/dual Coulomb phase. Since phase is condensed, the system should be characterized by a *single* Goldstone boson (spin-wave/second sound). How to count this on the dual side? The dual Lagrangian is just the 2+1D noncompact U(1) Maxwell Lagrangian $F_{\mu\nu}F^{\mu\nu}$. This is characterized by three vector potentials $A_\tau, A_x, A_y$ and one gauge constraint. Accordingly, the dual theory is characterized by *two* physical degrees of freedom. This may be confusing at first, but as we will discuss in some detail in the next section, it makes perfect sense. One photon is dynamical and transverse $\left(A_T = \frac{\nabla\times\mathbf{A}}{i\nabla}\right)$ and this is just the Goldstone mode in the dual language. The other physical photon is the temporal one $A_\tau$, describing the instantaneous ("Coulomb") interaction between static vortex sources: in the Coulomb gauge the longitudinal photon $\left(A_L = \frac{\nabla\cdot\mathbf{A}}{i\nabla}\right)$ drops out as we subject it to the Coulomb gauge fix $\nabla\cdot\mathbf{A} \equiv 0$. Hence we find the correct mode content: the gauge field description is just more efficient than the phase description because the instantaneous vortex-vortex interactions appear explicitly, while they have to be constructed by hand in terms of the phase degrees of freedom.

The full powers of the dual gauge theory unfold in the phase-disordered state. In terms of phase, all one has is a strong coupling expansion in the Hamiltonian language of Eq. (1) we already alluded to, leading to the conclusion that at least in the long wavelength limit one is dealing with a twofold degenerate massive "exciton." What is the dual gauge theory telling us? As we show later, the vortex amplitude "Higgs boson" $|\Psi_V|$ is the subject of dual censorship, and what remains are the original photons $A_\mu$ and the dual phase degree of freedom $\phi_V$ defined through $\Psi_V = |\Psi_V|e^{i\phi_V}$. If the system would be nonrelativistic [i.e., described by a time-independent Ginsburg-Landau-Wilson (GLW) action in Eq. (2)] such that $c_V/c \to \infty$, where $c_V$ and $c$ are the vortex condensate phase velocity and spin wave velocity, respectively, one would run into a problem. In this limit, the vortex phase drops out as a dynamical degree of freedom, and the photons are counted in the same way as in the Coulomb phase except that they are now massive: the interactions between static vortices are now screened, while the transversal "spin wave" photon acquires a mass. In this way one would find *one* propagating massive mode instead of two.

In the relativistic theory $(c_V \equiv c)$ it fits neatly: we encounter now four fields $(A_\tau, A_x, A_y, \phi_V)$ subjected to one gauge constraint. Of the remaining three physical fields, one takes care of static interactions, and we are left with two photons which are degenerate. Since only one of these photons is carried by the smooth phase field configurations, the other one reflects the phase rigidity of the vortex condensate: one of the two excitons lighting up in the phase world is just telling that the dual superconductor resists a twist in its phase. Stronger, we conclude that one needs to use the time-dependent GLW in the dual Lagrangian Eq. (2) in order to reproduce the spectrum of the Mott-insulator phase.

Having given away the bottom line, let us now substantiate these matters with some explicit computations.

## III. DISORDER FIELDS PROBED BY ORDER MEANS

It does not seem to be widely recognized that the propagators of order fields can be straightforwardly expressed in terms of the disorder fields. Such relations should not be confused with relations following directly from the Kramers-Wannier duality where the propagators of order fields are mapped onto propagators of the disorder fields at inverse coupling constant (or classically, temperature).[5] With the aim to compute order propagators using the disorder fields or vice versa, we proved that excitations expressed via either order or disorder language represent the same physical degrees of freedom. The relations we present here are true for all coupling constants as opposed to those used in an older work.[10] The correspondence of the propagators was discussed in an earlier paper dealing with field-theoretical elasticity,[11] and to save the reader the effort of learning this intricate affair, we rederive here these relations for the far simpler Abelian-Higgs case.

These relations are associated with the first step of the duality, where the phase fields are turned into photons mediating interactions between the vortices. To set the stage, let us shortly review these matters.[4,8,12–20] After coarse graining, phase dynamics can be written in terms of the Lagrangian,

$$\mathcal{L}_{XY} = \frac{1}{2g}[(\partial_\tau\phi)^2 + c^2(\nabla\phi)^2] \to \frac{1}{2g}(\partial_\mu\phi)^2. \tag{3}$$

The coupling constant $g$ is proportional to the original coupling constant $\tilde{g}$. The spin-wave velocity is given by the ratio of the stiffness and compression moduli $c^2 = \rho_s/\kappa_s$ and set to 1 in the last step. It is left implicit that the phase field is compact, $\phi = \phi + 2\pi$. We take the superfluid "three" velocity

$$v_\mu(\mathbf{x}) = \partial_\mu\phi(\mathbf{x}) \tag{4}$$

as the natural, "primitive" observable of the orderly side. In the ordered phase, the momentum space velocity-velocity propagator is proportional to the phase-phase propagator,

$\langle\langle v_\mu | v_\nu \rangle\rangle_{q,\omega} = q_\nu^2 \delta_{\mu\nu} \langle\langle \phi | \phi \rangle\rangle_{q,\omega}$ and the latter suffices to calculate the order parameter propagator $\langle\langle e^{i\phi} | e^{i\phi} \rangle\rangle$ (e.g., Ref. 3). In the disordered phase $\phi$ itself becomes multivalued and meaningless, but $v_\mu$ continues to be single valued and meaningful.

In the phase-ordered state the theory Eq. (3) is Gaussian and the velocity propagator is easily computed by adding an external source term to the Lagrangian,

$$\mathcal{L}[\mathcal{J}_\mu] = \mathcal{L}_{XY} + \mathcal{J}_\mu \partial_\mu \phi, \tag{5}$$

followed by taking the functional derivative

$$\langle\langle v_\mu | v_\nu \rangle\rangle = \frac{1}{Z} \left. \frac{\partial^2 Z[\mathcal{J}_\mu]}{\partial \mathcal{J}_\mu \partial \mathcal{J}_\nu} \right|_{\mathcal{J}_\mu = 0}. \tag{6}$$

The nonrelativistic propagator measured in condensed matter experiments represents only the subset of components of the relativistic propagator Eq. (6) with spatial indices: $\langle\langle v_i | v_j \rangle\rangle$. In the phase ordered state of the $XY$ model one can integrate the Gaussian Goldstone fields in Eq. (5) with the result

$$Z[\mathcal{J}_\mu] = \prod_{p_\mu} \sqrt{\frac{2\pi g}{p^2}} e^{(g/2)\mathcal{J}_\mu (p_\mu p_\nu / p^2) \mathcal{J}_\nu} \tag{7}$$

and the propagators follows immediately from Eq. (5) identity. The relativistic and nonrelativistic versions are, respectively,

$$\langle\langle v_\mu | v_\nu \rangle\rangle = g \frac{p_\mu p_\nu}{p^2}, \tag{8}$$

$$\langle\langle v_i | v_j \rangle\rangle = g \frac{c^2 q^2}{\omega_n^2 + c^2 q^2} P_{ij}^L, \tag{9}$$

where the longitudinal and transversal (for later use) projection operators are

$$P_{ij}^L = \frac{q_i q_j}{q^2}, \quad P_{ij}^T = \delta_{ij} - \frac{q_i q_j}{q^2}. \tag{10}$$

Obviously, we find only a single mode: the single Goldstone boson of the scalar theory as shown in Fig. 2(a).

Of course, the above procedure no longer works in the absence of the phase condensate. At any finite disorder there are configurations present containing topological defects (vortices), which are ignored in the path integral Eq. (7) but these are easily handled in the language of the dual disorder field theory.

Let us turn to the duality itself. The first crucial step is a simple Legendre transformation. Introduce Hubbard-Stratanovich auxiliary fields $\xi_\mu$ such that the phase action including the external sources, Eq. (5), is written as

$$\mathcal{L}[\mathcal{J}_\mu] = -\frac{g}{2}\mathcal{J}_\mu \mathcal{J}_\mu + ig\mathcal{J}_\mu \xi_\mu + \frac{g}{2}\xi_\mu \xi_\mu + i\xi_\mu \partial_\mu \phi. \tag{11}$$

Let us ignore the external sources $\mathcal{J}_\mu$ for a moment to focus on the duality itself. The phase field is split into smooth and multivalued pieces: $\phi = \phi_{sm} + \phi_{MV}$. By reshuffling derivatives ($\xi_\mu \partial_\mu \phi_{sm} = -\phi_{sm} \partial_\mu \xi_\mu$) the Gaussian $\phi_{sm}$ turn into Lagrange multipliers imposing a conservation law on the auxiliary fields,

$$\partial_\mu \xi_\mu = 0. \tag{12}$$

In the phase dynamics interpretation, $\xi_\mu$ just represents the supercurrents and Eq. (12) is the hydrodynamical continuity equation governing superflow. In 2+1D continuity can be

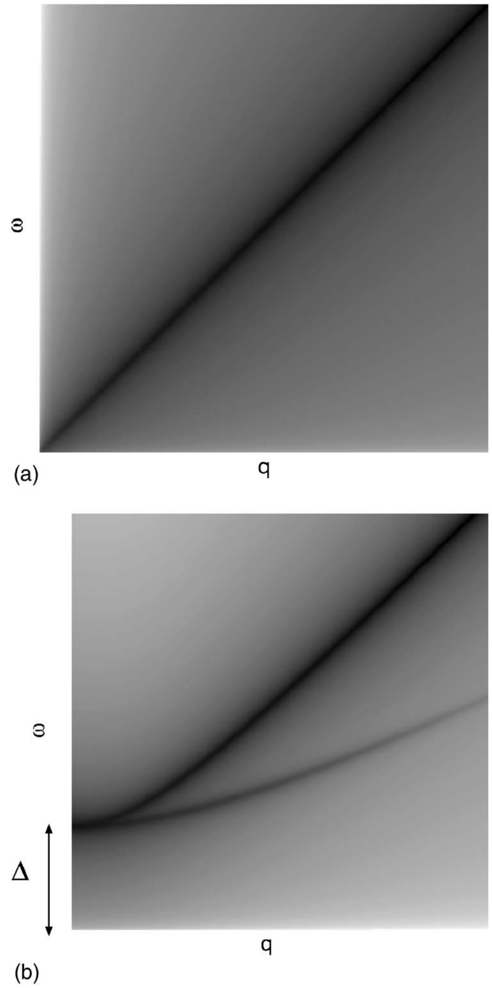


FIG. 2. Spectral functions associated with the superfluid velocity-propagator, computed on the Gaussian level using the dual theory. These results should be accurate deep inside the ordered and disordered phase. (a) The ordered (superfluid) phase: the second sound pole of Fig. 1(c) is recovered as it should. (b) The disordered (Mott insulating) phase: we used here a condensate velocity ($c_V$) which is half the sound velocity $c$ for the mere purpose to make visible the different behaviors of the strength of the second sound (higher branch) and dual (vortex) condensate (lower branch) poles. In reality these velocities are the same and the modes are degenerate. For $q \to 0$ the pole strengths of the two modes are the same, while they are governed by the same Higgs mass, and they can be combined in the ±1 helicity modes as expected from the strong coupling expansion in the Hamiltonian formalism [Fig. 1(d); see Sec. V]. However, for increasing momentum the condensate pole loses gradually strength while the second sound pole becomes more and more like the "orderly" result of (a), reflecting that at distances short compared to the dual London penetration depth the medium "rediscovers" the order.

imposed on the supercurrents by writing $\xi_\mu$ in terms of noncompact U(1) gauge fields $A_\mu$ as

$$\xi_\mu = \varepsilon_{\mu\nu\lambda}\partial_\nu A_\lambda, \tag{13}$$

and it follows that the remaining pieces of the Lagrangian Eq. (11) (ignoring the $\mathcal{J}$s) can be written as

$$\frac{g}{2}\xi_\mu\xi_\mu + i\xi_\mu\partial_\mu\phi_{MV} = \frac{g}{4}F_{\mu\nu}F^{\mu\nu} + iA_\mu J_\mu^V, \tag{14}$$

where $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$ is the field strength of the dual gauge sector while

$$J_\mu^V = \varepsilon_{\mu\nu\lambda}\partial_\nu\partial_\lambda\phi_{MV} \tag{15}$$

is the nonintegrability of the phase field, having the meaning of vortex current. This completes the key step of the duality: it shows that in 2+1D the rigidity of the phase medium can be exactly parametrized in "force carrying photons" (the $A_\mu$'s) while the vortices act as sources for these photons. The disorder field theory Eq. (2) follows immediately: vortices are indistinguishable from charged bosons interacting with electromagnetic fields and a system of such bosons is described by the Ginzburg-Landau-Wilson theory.[4]

This is all familiar territory but the following simple operation seems not to be commonly known. Let us include the external sources and use the identity Eq. (6) to calculate the phase propagator, but now using the action after the Hubbard-Stratanovich transformation,

$$\left.\frac{\partial^2 Z[\mathcal{J}_\mu]}{\partial\mathcal{J}_\mu\partial\mathcal{J}_\nu}\right|_{\mathcal{J}_\mu=0} = \int \mathcal{D}\xi_\mu\mathcal{D}\phi \left.\frac{\partial^2}{\partial\mathcal{J}_\mu\partial\mathcal{J}_\nu}e^{\int \mathrm{d}x_\nu((g/2)\mathcal{J}_\mu\mathcal{J}_\mu - ig\mathcal{J}_\mu\xi_\mu-(g/2)\xi_\mu\xi_\mu-i\xi_\mu\partial_\mu\phi)}\right|_{\mathcal{J}_\mu=0}$$
$$= \int \mathcal{D}\xi_\mu\mathcal{D}\phi(g\,\delta_{\mu\nu}-g^2\xi_\mu\xi_\mu)e^{\int \mathrm{d}x_\nu(-(g/2)\xi_\mu\xi_\mu-i\xi_\mu\partial_\mu\phi)} = Z(g\,\delta_{\mu\nu}-g^2\langle\langle\xi_\mu|\xi_\nu\rangle\rangle). \tag{16}$$

This implies an exact relationship between the velocity propagator and the propagator of the supercurrents, rooted in the Legendre transformation,

$$\langle\langle v_\mu|v_\nu\rangle\rangle = g\,\delta_{\mu\nu} - g^2\langle\langle\xi_\mu|\xi_\nu\rangle\rangle. \tag{17}$$

Because $\xi_\mu=\varepsilon_{\mu\nu\lambda}\partial_\nu A_\lambda$ this implies that in fact the phase velocity/spin wave propagator is proportional to a linear combination of the physical photon propagators of the dual gauge disorder-field theory. This implies that the poles of the magnon and photon propagator have to coincide and this has to be because both describe the same physics. However, the pole strengths might be quite different reflecting the "dual relativity principle:" pending the use of order or disorder "tools" one might get a very different view of the same underlying reality. The result Eq. (17), first derived in Ref. 11, shows that at least in the Abelian-Higgs case, the two observers should actually agree more on what they see than one could have expected *a priori*. The key is that although the vortex condensate falls prey to dual censorship, the orderly observer can still learn much about the dual world because he/she can probe the dual photons according to Eq. (17).

## IV. MAGNONS AS PHOTONS

Let us exercise the notions of the previous section in the simple case of the phase ordered state. We know the answer [the Goldstone mode, Eq. (8)], we know the dual side (Maxwell theory, $F_{\mu\nu}F^{\mu\nu}$), and we know how these relate [the "Zaanen-Mukhin" relation, Eq. (17)]. It is indeed a straightforward exercise.

Although the dynamics is fully relativistic the questions of relevance to condensed matter experimentalists are not relativistic: only the spatial components of the propagator are measurable [Eq. (9)]. This makes it convenient to use the Coulomb gauge fix. The Maxwell action in momentum-Matsubara frequency space becomes, including the external sources $J_\mu^{ext.}$,

$$\mathcal{L}_{EM} = \frac{g}{2}(A_\tau^\dagger,\mathbf{A}^\dagger)\begin{pmatrix} q^2 & -i\omega_n\langle\mathbf{q}| \\ i\omega_n|\mathbf{q}\rangle & \omega_n^2\hat{1}+c^2q^2\hat{P}^T\end{pmatrix}\begin{pmatrix}A_\tau\\ \mathbf{A}\end{pmatrix} + iJ_\tau^{ext}A_\tau + i\mathbf{J}^{ext}\cdot\mathbf{A}^\dagger, \tag{18}$$

where we have explicitly indicated the time ($X_\tau$) and space ($\mathbf{X}$) components of the gauge fields and currents. The bra and ket in the gauge field propagator represent rows and columns $q_i$, respectively. Notice that from now on we keep the Goldstone/spin-wave velocity $c$ explicit, for purposes which will become clear later.

Provided that we choose a gauge fix $\mathcal{F}$ that does not act on the temporal component $A_\tau$, the temporal component can be integrated out. This yields the usual Lagrangian with Coulomb interactions between static sources,

$$\mathcal{L}_{EM} = \frac{1}{2g}\frac{J_\tau^\dagger J_\tau}{q^2} + \frac{g}{2}(\omega_n^2+c^2q^2)\mathbf{A}^\dagger\hat{P}^T\mathbf{A} + i\left(J_L - \frac{i\omega}{q}J_\tau\right)A_L^\dagger + i\mathbf{J}\hat{P}^T\mathbf{A}^\dagger. \tag{19}$$

The longitudinal component $A_L$ is unphysical (its source is $i\omega_nJ_\tau - qJ_L \to \partial_\tau J_\tau + \partial_i J_i = 0$) and it should be removed by the Coulomb gauge fix

$$0 = \partial_i A_i = -qA_L. \tag{20}$$

We end up with two propagators for the gauge fields, as it should in 2+1D. We find one dynamical photon,

$$\langle\langle A_i^\dagger | A_j \rangle\rangle = \frac{P_{ij}^T}{g(\omega_n^2 + c^2 q^2)} \tag{21}$$

and a propagator taking care of the Coulomb interactions between the static sources,

$$\langle\langle A_\tau^\dagger | A_\tau \rangle\rangle = \frac{1}{gq^2}. \tag{22}$$

This is, of course, textbook electromagnetism, but be aware of the twist in the interpretation. The photons now keep track of the capacity of the phase condensate to respond to external influences. The outcome is: it is carrying a "Goldstone photon" [$A_T$, Eq. (21)] and it can mediate as well interactions between static vortices [$A_\tau$, Eq. (22)].

We are now in the position to evaluate the "Zaanen-Mukhin" relation Eq. (17). For this purpose, we are only interested in the spatial components of the supercurrents $\xi_\mu$. The supercurrent propagator is easily found by using the definition Eq. (13), and the results for the gauge field propagators Eqs. (21) and (22), and we find for its spatial components

$$\langle\langle \xi_i^\dagger | \xi_j \rangle\rangle = \frac{1}{g}\left[\frac{\omega_n^2}{\omega_n^2 + c^2 q^2} P_{ij}^L + P_{ij}^T\right]. \tag{23}$$

Using now the Zaanen-Mukhin relation Eq. (17),

$$\begin{aligned}\langle\langle v_i | v_j \rangle\rangle &= g\delta_{ij} - g^2 \langle\langle \xi_i | \xi_j \rangle\rangle = g[P_{ij}^L + P_{ij}^T] \\ &\quad - g\left[\frac{\omega_n^2}{\omega_n^2 + c^2 q^2} P_{ij}^L + P_{ij}^T\right] = g\frac{c^2 q^2}{\omega_n^2 + c^2 q^2} P_{ij}^L.\end{aligned} \tag{24}$$

After this long detour, we indeed have managed to recover the spin wave propagator Eq. (9).

The simple lesson following from this simple exercise is that the dual photon language is in a way more complete than the description in terms of phase fields, in the sense that the gauge fields keep track in an explicit way of both the capacity of the medium to propagate Goldstone bosons and the fact that it mediates interactions between its topological excitations. The Zaanen-Mukhin relation filters out the Goldstone sector from the "omnipotent" dual gauge sector, keeping its topological side (the Coulomb propagator, requiring vortex sources) completely hidden from the eye from an "orderly" observer.

## V. WATCHING THE DISORDERED STATE WITH ORDERLY MEANS

Surely, the dual route of the previous section is a rather inefficient way to derive the propagator of a Goldstone mode. This changes drastically in the phase disordered state. Resting on the fact that the dual gauge theory is now governed by order, precise information on the second sound propagator can be extracted with barely any extra investments. The only other option is the strong coupling expansion in the Hamiltonian language and this becomes very tedious at intermediate couplings.

The phase disordered state corresponds to the Higgs phase of the gauge theory Eq. (2), corresponding to the state where vortex loops have blown out and the vortices have Bose-condensed. As a consequence, the bosonic disorder field $\Psi = |\Psi_0| e^{i\phi_V}$ acquires a finite expectation value. This theory is fully relativistic, as we explained, and this vortex condensate is literally like the U(1) Higgs phase of high energy physics.[21] It will turn out to be quite convenient for the interpretation of the results to consider a nonrelativistic extension of the theory characterized by a condensate velocity $c_V$, which is different from the spin-wave velocity $c$, entering the time components of the covariant derivatives $\sim \frac{1}{c_V}(\partial_\tau - iA_\tau)$. Of course, the real theory is characterized by $c_V = c$ as implied by the Lorentz-invariance of the action Eq. (3).

Let us employ the usual unitary gauge, corresponding to fixing the condensate phase $\phi_V = 0$. The finite expectation value of the disorder field results in the familiar Higgs term in the action

$$\mathcal{L}_{Higgs} = \frac{1}{2}|\Psi_0|^2\left[\frac{1}{c_V^2} A_\tau A_\tau + A_i A_i\right]. \tag{25}$$

The only specialty is the velocity $c_V$. In high energy physics this is the light velocity while in the nonrelativistic condensates of condensed matter physics $c_V$ is the sound velocity (in BCS theory $\sim v_F$[22,23]), which is vanishingly small compared to the light velocity with the consequence that one can get away with a time independent Ginzburg-Landau theory. In our final result we have to set $c_V = c$.

We now follow the same route as in the previous section. Adding the Higgs term the Lagrangian becomes

$$\begin{aligned}\mathcal{L}_{full} &= \frac{g}{2}(A_\tau^\dagger, \mathbf{A}^\dagger)\begin{pmatrix} q^2 + \frac{\Omega^2}{c_V^2} & -\omega_n \langle \mathbf{q}| \\ -\omega_n |\mathbf{q}\rangle & (\omega_n^2 + \Omega^2)\hat{\mathbf{1}} + c^2 q^2 \hat{P}^T \end{pmatrix}\begin{pmatrix} A_\tau \\ \mathbf{A} \end{pmatrix} \\ &\quad + iJ_\tau^{ext} A_\tau^\dagger + i\mathbf{J}^{ext} \cdot \mathbf{A}^\dagger.\end{aligned} \tag{26}$$

The currents $J_\mu^{ext}$ represent external currents, artificially inserted from outside (by an observer) and these do not include the vortex current contribution from the (disordering) vortex tangle. Its contribution has already been accounted by the Higgs term, Eq. (25). Introducing a Higgs mass $\Omega$, defined by

$$\Omega^2 = \frac{|\Psi_0|^2}{g}. \tag{27}$$

Since the gauge has already been fixed, the temporal components $A_\tau$ can be safely integrated out,

$$\begin{aligned}\mathcal{L}_{full} &= \frac{g}{2}\mathbf{A}^\dagger\left[\frac{\Omega^2(\omega^2 + c_V^2 q^2 + \Omega^2)}{c_V^2 q^2 + \Omega^2}\hat{P}^L + (\omega_n^2 + c^2 q^2 + \Omega^2)\hat{P}^T\right]\mathbf{A} \\ &\quad + i\mathbf{J}^{ext}\left[\hat{\mathbf{1}} - \frac{c_V^2 q^2}{c_V^2 q^2 + \Omega^2}\hat{P}^L\right]\mathbf{A}^\dagger + \frac{1}{2g}\frac{J_\tau^{ext\dagger} J_\tau^{ext}}{q^2 + \frac{\Omega^2}{c_V^2}}.\end{aligned} \tag{28}$$

The last term corresponds to the interactions between the static vortices which are now short ranged. The interest is in the dynamics of the gauge fields itself. As before, we find a transversal photon $A_T$ characterized by second sound propagator which has acquired a Higgs mass. In addition, we find an extra longitudinal photon (the first term) which is now physical. This is also characterized by the same Higgs mass but it is propagating at the condensate velocity showing that it represents the phase rigidity of the dual superconducting matter sector.

The propagators for gauge fields are easily determined from the inverse of the full action Eq. (26). The superfluid current propagator is decomposed into longitudinal and transversal parts $\xi^{L,T}$ (parallel and perpendicular to the momentum $\mathbf{q}$, respectively) and the propagators are found to be

$$\langle\langle \xi_L | \xi_L \rangle\rangle = \frac{1}{g}\frac{\omega_n^2}{\omega_n^2 + c^2q^2 + \Omega^2}, \tag{29}$$

$$\langle\langle \xi_T | \xi_T \rangle\rangle = \frac{1}{g}\frac{\omega_n^2 + c_V^2q^2}{\omega_n^2 + c_V^2q^2 + \Omega^2}. \tag{30}$$

Using now the Zaanen-Mukhin relation Eq. (17) and the momentum propagators Eqs. (29) and (30), we obtain the result for the nonrelativistic propagator for the superfluid velocity in the disordered phase,

$$\langle\langle v_i | v_j \rangle\rangle = g\left[\frac{c^2q^2 + \Omega^2}{\omega_n^2 + c^2q^2 + \Omega^2}P_{ij}^L + \frac{\Omega^2}{\omega_n^2 + c_V^2q^2 + \Omega^2}P_{ij}^T\right]. \tag{31}$$

The spectral response from this propagator is plotted in Fig. 2(b).

The longitudinal (first) term represents, as before [Eq. (24)], the correlations associated with the smooth part of the phase field: this is literally second sound acquiring a mass associated with the disappearance of the superfluid rigidity at large lengths and times. We notice that in the static limit ($\omega_n \to 0$) the longitudinal part becomes a constant, signaling that even at the shortest distances superfluid correlations have disappeared. This makes sense: when vortices populate the whole system, then any long living correlation is destroyed even between two neighboring sites when one waits long enough.

The second, transversal term is the interesting one: we indeed find a second mode and although it has the same mass as the gapped second sound it propagates with the condensate velocity. It is of course the longitudinal photon reflecting the dynamics of the dual superconducting vortex matter. In order for the superfluid velocity correlator to acquire a nonzero transversal component it is actually a requirement that the phase field becomes nonintegrable. This becomes clear by inspecting the transversal part of the supercurrent Eq. (30),

$$\xi_T = -ie_i^T\xi^i = -e_i^T\frac{\partial_i(\phi_{sm} + \phi_{MV})}{g} = -\frac{1}{g}e_i^T(iqe_i^L\phi_{sm} + \partial_i\phi_{MV})$$
$$= -\frac{i}{g}e_i^T\partial_i\phi_{MV}, \tag{32}$$

where the smooth part has disappeared since it makes no sense to have derivatives in the transversal direction for the case of smooth fields.

Let us analyze the result Eq. (31) in more detail. The velocity $c_V$ has done its job in establishing that the transversal poles of Eq. (31) are indeed due to the superconducting vortex matter, and we can impose Lorentz invariance by setting $c_V = c$: the longitudinal and transversal modes become degenerate as they should because of the degeneracy associated with the $n = \pm 1$ excitons, trivially seen in the Hamiltonian formalism. A different issue is the pole strength of the second sound ($I_L$) and condensate poles ($I_T$) as measured by the velocity-velocity correlator, characterized by the ratio

$$\frac{I_L}{I_T} = 1 + \frac{c^2q^2}{\Omega^2}. \tag{33}$$

Giving this a minute of thought this makes perfect sense. First, at $q = 0$ both modes at real frequency $\omega = \Omega$ have the same weight. It follows from the Hamiltonian formalism that at wavelengths large compared to the London length the excitons correspond to the helicity ±1 eigenstates of the angular momentum of the O(2) quantum rotors. The supercurrents have the status of canonical momenta and should therefore be combined in currents with definite helicity $\xi_{\pm 1} \sim \xi_L \pm i\xi_T$. The implications are obvious: at $q \to 0$ the longitudinal and transversal poles of the velocity propagator should have the same strength because all that exist in this limit are the helicity eigenstates.

What is changing at smaller distances? The characteristic momentum scale is of course the inverse dual London penetration depth $q_L \simeq 1/\lambda_L = \Omega/c$ and from Eq. (33) it follows that at larger momenta the strength of the dual condensate pole decreases quadratically in momentum relative to that of the second sound pole. Within the confines of this Gaussian treatment this makes sense again. At these short times and distances one enters a regime where one is probing mainly the phase ordered matter forming the background in which the vortices move. This matter is the same stuff as the fully phase ordered matter and accordingly it should carry the same Goldstone excitation. Equation (31) tells how to interpolate between the disorder physics at $q \to 0$ with its number eigenstates and the phase ordered regime at large momenta: the dual condensate pole loses its weight gradually, in fact in the same way as the Higgs mass loses its influence on the dispersion.

## VI. DUAL VIEW ON THE CRITICAL REGIME

We are not done yet. We have implicitly assumed up to this point that the fields are noninteracting. Modulo perturbative corrections this would have been fine in dimensions above the upper critical dimension but the Abelian-Higgs model in 2+1D is below its upper critical dimension $d_{uc} = 3+1$D. One has now to be cautious with considerations like the one in the previous section. Upon exceeding the scale $\Omega$ one does not reenter the ordered phase but instead one enters the quantum critical regime which has no longer to do with order or disorder but has acquired its own identity due to the strongly interacting nature of the critical point. Away from the critical coupling, the order (second sound) and dual order

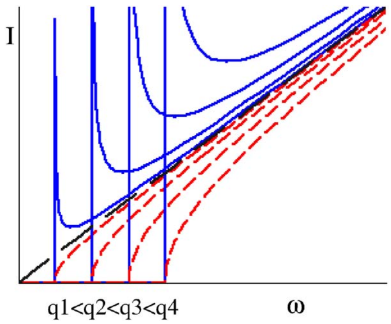


FIG. 3. (Color online) The spectral functions associated with the second sound (full lines) and condensate (dashed lines) pieces of the velocity propagator in the critical regime for various momenta [Eqs. (40) and (41)]. At $q=0$ both critical continua become degenerate and linear in $\omega$ reflecting the simple correlation function exponent $\eta_A=1$ associated with the dual gauge fields. However, at finite momenta it is seen that the second sound continuum diverges at threshold while the condensate piece is actually suppressed, although both continue to be governed by the same scaling dimension. At finite momenta this different behavior of the critical continua has to be present in order for them to be consistent with the momentum dependence of the pole strengths of the propagating disorder excitations appearing at the moment one moves from the critical coupling.

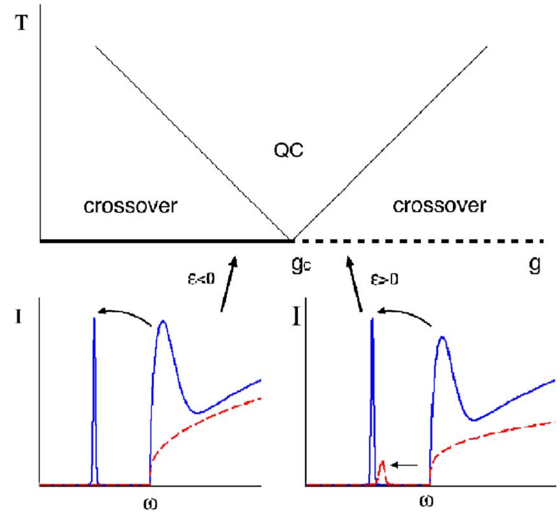


FIG. 4. (Color online) Cartoon of the appearance of the second sound (full lines) and dual condensate (dashed lines) contributions to the velocity spectral functions at finite momentum in the close vicinity of the quantum critical point, both on the ordered (left) and disordered (right) side. Although the critical continua are expected to have a very similar appearance, on the ordered side only a second sound pole is found. However, on the disordered side the system scales to dual superconducting order with the effect that one finds both a propagating second sound and condensate excitations with strengths governed by the $XY$ correlation length exponent. However, the way their pole strengths develops as a function of momentum tracks the Gaussian result shown in Fig. 2(b) and it turns out that the momentum dependence of the critical continua is just the right kind to be consistent with the behavior of the disorder poles (Sec. VI).

(longitudinal and transversal photons) excitations discussed so far still make sense because they will appear as bound states pulled out from the low energy side of the continuum of critical modes, with a pole-strength and binding energy diminishing upon approaching the critical coupling. The missing link at this point is the appearance of the continua of critical modes as picked up by the velocity propagator. After some preliminaries we will derive their form resting on the large body of knowledge on the 3D $XY$ critical state. These critical continua turn out to behave in a quite surprising way, with the second sound and condensate contributions showing a completely different behavior away from $q=0$ (see Figs. 3 and 4). We will subsequently focus in on the detailed way the quasiparticle poles (second sound, the excitons) develop as a function of the distance from the critical point, making the case that the critical continua have to be as they appear in order to be consistent with the quasiparticle poles.

In order to describe the system close to and right at criticality, we introduce renormalized parameters and critical exponents. The role of reduced temperature is taken by the quantity $\epsilon=\frac{g-g_c}{g_c}$, which is the reduced coupling constant. If $\epsilon<0$ or $\epsilon>0$, we approach the quantum critical point at $g_c$ from the ordered and disordered side, respectively. Pending if we approach the critical point from the order or disorder side, the system will scale either to the stable fixed points associated with phase order and noninteracting second sound ($g=0$) or with noninteracting rotors ($g=\infty$). The reduced coupling constant $\epsilon$ is therefore a relevant operator with scaling dimension $y_\epsilon>0$. Another relevant field, which plays the role of the magnetic field in the standard scaling analysis, is the generating functional field $\mathcal{J}_\mu$. Since it is relevant at the transition, its scaling dimension is also positive $y_{\mathcal{J}}>0$.

The model we consider is relativistic, with dynamical critical exponent $z=1$, and its critical behavior will coincide with that of the 3D $XY$ model. We use the state of the art for the exponents, based on analytic methods (high-temperature expansion,[24] vortex-loop scaling,[25] one-loop renormalization group[6]) as well as numerical results from Monte Carlo simulations.[8,26,27] The critical exponent $\eta$ for the order parameter propagator $\langle e^{i\phi_j}e^{-i\phi_i}\rangle$ has been studied in great detail.[3,7,8,28–30] However, our interest is in the velocity correlation function Eq. (9) which is not straightforwardly related to the vertex correlator away. Instead, we will use the knowledge of the scaling dimensions of the dual field $\eta_A$ to derive the form of the velocity propagator in the critical regime.

Let us first analyze the model and its propagators right at the critical point $g=g_c$. The exponent $\eta_A$ is usually defined as the critical exponent of the gauge fields correlation function right at the critical point $g=g_c$, i.e., $\langle\langle AA\rangle\rangle \propto 1/p^{2-\eta_A}$. To be consistent with the literature,[6,8] we have to change the gauge fix from "our" Coulomb/unitary gauge fix to the Lorentz gauge fix ($\partial_\mu A_\mu=0$, i.e., the vector potential is purely transversal). In this gauge fix, the gauge field can be projected onto a 3D linearly polarized basis (defined as $\mathbf{e}_0=\frac{\mathbf{p}}{p}$, $\mathbf{e}_{-1}=-\mathbf{e}_T$ and $\mathbf{e}_{+1}=\mathbf{e}_{-1}\times\mathbf{e}_0$). The component $A_0$ is set to zero by the gauge fix, with only the space-time transversal components of the fields being physical. The spatially transversal photon (second sound) degree of freedom $A_T$ is now represented by $A_{-1}$. The remaining component $A_{+1}$ that admixes the Coulomb and the longitudinal photons plays the role of the vortex phase degree of freedom in this particular gauge fix. On the Gaussian level of the previous sections, the propagators for the gauge fields within the Lorentz gauge fix degenerate and are given by

$$\langle\langle A_h^\dagger | A_{h'} \rangle\rangle = \rho_s \frac{\delta_{h,h'}}{\omega_n^2 + c^2 q^2 + \Omega^2}. \tag{34}$$

In the Coulomb phase one finds the same propagator with a vanishing gap, $\Omega=0$. The indices are taking "transversal" values $h,h'=\pm 1$. The coupling constant in the prefactor is expressed in terms of the superfluid stiffness $\rho_s=1/g$ which is a quantity which does renormalize. It follows that the residues of the quasiparticle poles (order/disorder excitations) are also renormalized which would not be the case if the prefactor would correspond to the bare coupling $g$. The overall prefactor in the expression for the velocity propagators corresponds to $g_b^2 \rho_s$ in this scheme. The $g_b^2$ is the bare critical coupling since the relation between the dual and original propagators Eq. (17) is an exact relation from the Legendre transformation, which is also valid in the critical regime. Accordingly, both the second sound of the ordered side and the excitons of the disordered side lose their pole strength approaching the critical point and this is governed by the renormalization of the superfluid density $\rho_s$ which we will deduce starting from the known critical behavior of the dual gauge field propagators.

Herbut and Tešanović[6] analyzed the charged $XY$ model which is equivalent to the dual action Eq. (2). From their expression for the $\beta$-function governing the renormalization of the electrical charge, it follows that at the fixed point

$$0 = \hat{e}_0^2 (D - 4 + \eta_A). \tag{35}$$

Assuming that the charge scales to a finite value $\hat{e}_0$, it follows that $\eta_A = 4 - D \equiv 1$. The same result was obtained by Hove and Sudbø[8] using Monte Carlo simulations to determine the exponent $\eta_A$ from the vortex correlations at the critical point. They introduced a relation between the correlation function of the vortex tangle $G(p)$ and the dual gauge field propagator,

$$\langle \mathbf{A}^\dagger \mathbf{A} \rangle = \frac{2\beta}{p^2}\left(1 - \frac{2\beta\pi^2 G(p)}{p^2}\right) \tag{36}$$

valid for the case of the uncharged original/charged dual action. Notice that in Ref. 8, $\mathbf{h}$ is used for the dual gauge fields and $\mathbf{A}$ for the original gauge fields. At the critical point the vortex correlator is given by $\lim_{p\to 0} 2\beta\pi^2 G(p) = p^2 - C_3(g) p^{2+\eta_A} + \cdots$ and using Eq. (36) it follows that

$$p^2 \langle \mathbf{A}^\dagger \mathbf{A} \rangle = C_3(g) p^{\eta_A} + \cdots . \tag{37}$$

According to their numerical simulation, Eq. (37) shows a linear behavior and Hove and Sudbø[8] conclude that $\eta_A=1$. This result is in agreement with the expectations due to the following simple argument by Tešanović. The dual of the uncharged $XY$ model Eq. (3) is the charged $XY$ model Eq. (2) and vice versa.[31] Since the twice applied duality yields the same theory, i.e., the "duality squared" is the identity, this implies that the critical exponent has to be $\eta_A=1$ (Refs. 6 and 32) (see also Ref. 33).

The critical propagator of the dual gauge fields, Eq. (37), can be used to establish the form of the velocity propagator Eq. (9) in the critical regime. Comparing Eq. (37) with our form of the gauge field propagator Eq. (34), and bearing in mind the degeneracy, we conclude that each field component propagator corresponds to one-half of the propagator Eq. (37),

$$\langle\langle A_h^\dagger A_{h'} \rangle\rangle = \frac{C_3}{2} p^{\eta_A - 2} \delta_{h,h'} + \cdots . \tag{38}$$

We can now use again the universal Zaanen-Mukhin relation Eq. (17) to obtain the velocity propagator

$$\langle\langle v_i | v_j \rangle\rangle \sim P_{ij}^L \left[ \frac{-\omega_n^2}{p^{2-\eta_A}} + \cdots \right] + P_{ij}^T [p^{\eta_A} + \cdots] \tag{39}$$

right at $g=g_c$. This is the first main result of this section. The dots represent constant terms with no imaginary parts as well as short distance corrections. At least deep in the critical regime, the Wick rotation to real time is simple[9] because scale invariance implies that Euclidean propagators are power laws, turning into branch cuts in real frequency. With $\eta_A=1$, right at the criticality, the spectral function has two quite different branch cuts in the longitudinal and the transversal channel

$$\mathrm{Im}\langle\langle v_i | v_j \rangle\rangle_L \sim \theta(\omega^2 - c^2 q^2) \frac{\omega^2}{\sqrt{\omega^2 - c^2 q^2}}, \tag{40}$$

$$\mathrm{Im}\langle\langle v_i | v_j \rangle\rangle_T \sim \theta(\omega^2 - c^2 q^2) \sqrt{\omega^2 - c^2 q^2}, \tag{41}$$

and we sketch both pieces of the velocity correlator in Fig. 3. $\theta(x)$ is the Heaviside unit step function.

This is quite an unexpected result. At $q=0$ we find both spectral functions to be simply proportional to frequency, a simple behavior which of course originates in $\eta_A=1$. Upon increasing momentum, the "sound" and "condensate" spectral functions start to behave very differently near the threshold $\omega=cq$ although at large $\omega$ they merge together again. The sound part shows the usual[9] divergence $\omega^2/(\omega-cq)^{2-\eta_A} \sim \omega^2/(\omega-cq)$ while the condensate piece develops like $(\omega^2 - c^2q^2)^{\eta_A/2} = \sqrt{\omega^2 - c^2 q^2}$. The degeneracy of the two contributions at $q=0$ rings a bell: at infinite wavelength it should be that the critical fluctuations are eigenstates of rotor angular momentum, "equalizing" the condensate and sound contributions as we found for the propagating excitations. To understand better why these contributions should become different at finite momenta, we should first analyze in more detail what happens with the quasiparticle poles close to the critical point.

To analyze the behavior of the quasiparticle poles in the ordered and disordered phase close to the critical coupling we need hyperscaling. Although one has to be careful,[34] recent numerical simulations[27] show that there is none or a very small violation of the hyperscaling for 3D $XY$. Let us first repeat the standard hyperscaling arguments applied to the velocity-velocity propagators. We denote the propagators of the gauge field in real space as $G_A(x,\epsilon)$. It is generated by the term $\mathcal{J}_h A_h$ in the action and this generating functional of the gauge fields $\mathcal{J}_h$ plays a role similar to a magnetic field. It is a relevant field that scaling dimension $y_{\mathcal{J}}$. Hyperscaling requires that such fields act on a block of $b^{d+1}$ points in space time, treated as a single variable. After a scale transformation, the new propagator is related to the original one by

$$G_A\left(\frac{r}{b},\mathcal{J}'\right) = \frac{\partial^2}{(\partial \mathcal{J}')^2}\ln Z[\mathcal{J}] \sim \frac{b^{2(d+1)}}{\lambda_{\mathcal{J}}^2}G(r,\mathcal{J}), \qquad (42)$$

where $\lambda_{\mathcal{J}}=e^{y_{\mathcal{J}}}$ is the scaling factor of the generating functional for the gauge fields.

Repeating the scale transformation $n$ times in the vicinity of the critical point, we obtain

$$G_A(r,\epsilon) = \frac{\lambda_{\mathcal{J}}^{2n}}{b^{2n(d+1)}}G(r/b^n,\lambda_\epsilon^n\epsilon), \qquad (43)$$

with the scaling factor $\lambda_\epsilon=e^{y_\epsilon}$ associated with the reduced coupling constant. Choosing $n$ such that $(\lambda_\epsilon)^n\epsilon=\text{const}$ from Eq. (43) it follows that the propagator behaves universally on both sides of the critical point as

$$G_A(r,\epsilon) \propto |\epsilon|^{2/y_\epsilon(d+1-y_{\mathcal{J}})}\Phi_\pm(r/|\epsilon|^{-1/y_\epsilon}). \qquad (44)$$

The functions $\Phi_\pm$ are universal functions associated with the ordered and disordered sides of the critical regime, and given in terms of $G(r,\text{const})$. The denominator in its argument is the correlation length that diverges at the critical point with exponent $\nu$, implying the familiar relation $\nu=1/y_\epsilon$. The relation of the "magnetic field" exponent $y_{\mathcal{J}}$ to the scaling exponent $\eta_A$ follows when we set $\epsilon=0$ in Eq. (43),

$$y_{\mathcal{J}} = \frac{d+3-\eta_A}{2} \to 2 \qquad (45)$$

using the known value $\eta_A=1$. Together with the relation for $\nu$, Eq. (44) can be written as

$$G_A(r,\epsilon) \propto |\epsilon|^{\nu(d-1+\eta_A)}\Phi_\pm(r/|\epsilon|^{-\nu}) \to |\epsilon|^{2\nu}\Phi_\pm(r/|\epsilon|^{-\nu}). \qquad (46)$$

This is just the familiar result that the behavior of the correlation function close to the critical point is governed by the exponents $\nu$ and $\eta$ (with $\eta=1$ in the present case), and the crossover functions $\Phi_\pm$.

Let us first approach the critical point from the disordered side, i.e., $\epsilon \to 0^+$. This phase is characterized by the gap Eq. (27), which we can call (compare Ref. 9) $\Delta_+=\Omega$. This gap is proportional to the inverse correlation length of the vortex tangle $\xi=c/\Omega$. Upon approaching the critical point, both the correlation length vanishes and the gap diverges with characteristic exponent $\nu$ as $\xi \propto \epsilon^{-\nu}$ and $\Delta_+ \propto \epsilon^{z\nu}$, where $z=1$ is the dynamical exponent which equals one in this specific case. The 3D $XY$ correlation length exponent $\nu=0.66-0.67\approx\frac{2}{3}$ according to a large body of work.[24–27] Given that there are two dynamical fields in the problem ($A_T$ and the vortex phase field phase $\phi_V$) one could be tempted to think that there are two correlation lengths in the problem, but this is not the case, the problem is effectively Lorentz invariant, consistent with the numerical work[7,17,29] and an argument[28] linking it to the anomalous dimension of the gauge field $\eta_A$.[6,8]

The scaling dimension of the superfluid density can be deduced from Eq. (34). After Fourier transformation to space time, the Gaussian propagator Eq. (34) behaves like

$$\langle\langle A_{h'}^\dagger|A_{h'}\rangle\rangle = \rho_s\frac{1}{x^{d-1}}\Psi_+\left(\frac{x}{\xi}\right) = \rho_s\xi^{-(d-1)}\Phi_+\left(\frac{x}{\xi}\right) = \rho_s\epsilon^{\nu(d-1)}\Phi_+\left(\frac{x}{\xi}\right). \qquad (47)$$

Comparing it with the hyperscaling form for the gauge field propagator Eq. (44) we conclude that the superfluid density scales as

$$\rho_s \propto |\epsilon|^{\nu(2-\eta_A)} \to |\epsilon|^\nu \qquad (48)$$

at the disordered side of the critical point.

We have now arrived at a point where we can determine the behavior of the two quasiparticle poles upon approaching the critical point from the disordered side. Using Eq. (39), the fact that $g\to g_b^2\rho_s$ and the scaling of both $\rho_s$ and $\Omega$, we conclude that the vortex-condensate pole $\sim P^T$ has a strength proportional to $\rho_s\Omega^2 \propto \epsilon^{2z\nu+\nu(2-\eta_A)} \to \epsilon^{3\nu}$, vanishing upon approaching the critical point with an exponent $3\nu\cong 2$ while its strength disappears in the critical continuum as indicated in Fig. 4(c). Turning now to the second sound pole $\sim P^L$, we observe that at long wavelength ($q\to 0$) its strength behaves exactly like the condensate pole. This has to be because eventually, at large enough distances, one should recover the fact that these excitons correspond to the exact rotor angular momentum eigenstates. However, for increasing momenta the term in the numerator $\sim c^2q^2$ takes over, and the strength of the large momentum second sound pole is scaling more slowly to zero upon approaching the critical point, governed now by the superfluid density exponent $\nu(2-\eta_A)\cong\frac{2}{3}$. This is of course not different from what we found on the Gaussian level, with the second sound pole overtaking the condensate pole when the vortex condensate is "losing its grib," governed by the Higgs mass $\Omega$.

To complete the picture, let us finally consider what happens with the second sound pole approaching the critical point from the ordered side. This is straightforward: as before, we should substitute $g\to g_b^2\rho_s$ in the Gaussian result Eq. (9) and $\rho_s\sim|\epsilon|^\nu$ because $\rho_s$ renormalizes in the same way on both sides of the transition.[33] In other words, the strength of the second sound pole on the ordered side coincides with its behavior at large momenta on the disordered side.

Not surprisingly, we have found that the "order poles" behave quite like the results we found on the Gaussian level in the previous sections except that renormalized mass scales and quasiparticle residues have to be used, all governed by the same correlation exponent $\nu$, because $\eta_A$ "magically" drops out. We can now use this knowledge to comprehend why the critical continua of Fig. 3 behave the way they do. We already argued that at energies far away from the threshold $\omega=cq$ the second sound and vortex condensate pieces picked up by the velocity correlator merge in the same linear $I\sim\omega$ behavior. At finite $q$ the differences between the two are large near threshold. With the knowledge regarding the behavior of the quasiparticle poles at hand this now makes sense. $\rho_s$ being a relevant operator, its influence at high energies is small while growing when times get longer. A bit away from the critical point, it takes over at a length $\sim\xi$ where the system gets under control of the stable fixed points at zero or infinite coupling, which are also in charge of protecting the quasiparticle poles. Surely, the quasiparticles close to the critical point can be viewed as bound states pulled out of the critical continuum due to the effect of the relevant operators (Fig. 4). However, because of the way the latter scale, the quasiparticles are formed from the *low energy end* of the critical continuum. What does this mean for our velocity propagator, "watching" the true critical excitations through the "duality filter?" We derived some clear rules for how the weights should be distributed over the quasiparticles: the condensate and sound poles of the disordered state should have equal weight at $q=0$, but the former should lose its weight rapidly for increasing momentum. Inspecting now the low energy end of the critical continua for various momenta we see this rule also at work (Fig. 3). We notice that this "weight-matching" of the critical continua and the quasiparticle poles is to an extent even quantitative. For this purpose we inspect the pole strength ratio Eq. (33) close to the critical point. For fixed $q$, due to the gap in the denominator, we learned already that the ratio diverges like $\sim q^2/\epsilon^{2\nu}$. However, the prefactor of the second sound pole strength is proportional to $q^2$. Comparing it with the ratio of the spectral responses right at the critical point and near threshold ($\omega\approx cq$)

$$\left(\frac{I_L}{I_T}\right)_{g_c} = \frac{\omega^2}{\omega^2 - c^2q^2} \xrightarrow{\omega\approx cq} q^2 \times \text{" divergent part . "} \qquad (49)$$

We find a perfect match—the strengths of the spin-wave and the condensate excitations are proportional to the strengths of their respective critical continua where they have their "origin."

Surely, this does not explain everything, and to a degree Eqs. (40) and (41) are a result which stands on its own. However, given the simple integer $\eta_A$ exponent, it appears to us to be a unique analytical form which obeys the general requirements of scale invariance and Wick rotation, having at the same time the right form to be consistent with the evolution of the spectral weights in the quasiparticle poles.

## VII. CONCLUSION

What have we proven by this exercise? We have taken the simple example of phase dynamics at zero chemical potential to predict the form of the superfluid velocity correlator in the ordered, disordered, and critical regimes, exploiting the vortex duality. Although the order parameter (of the ordered phase) vanishes in the disordered phase, the dual order of the disordered side does manifest itself when the system is interrogated with "orderly means." In our example, one of the two degenerate excitons of the phase disordered/Mott insulating state can as well be called the longitudinal photon associated with the phase-rigidity of the dual superconductor. Although less obvious in the present simple example, this might be used as a technical convenience. Quite generally, it is easier to compute the excitation spectrum of the system "around" the ordered state, helped by the Goldstone theorem, the Higgs mechanism, etc., while disorder is not a convenient starting point when the interactions are strong. It is actually so that this work was originally inspired on problems encountered in the study of quantum liquid crystals where this "dual route" to the spectral functions associated with measurable quantities seems to be the only way available.[11]

This dual route also tells another story which is far less obvious. This can be summarized as: "studying the disordered state with order operators, one recovers the signal of the ordered state at energies and momenta where the dual order parameter loses control." In our specific example, the dual condensate piece of the exciton doublet of the disordered state fades away when momentum is increased and at large momentum only second sound remains. We argued that this same "mechanism" is even at work in the critical state, being ultimately responsible for the rather odd appearance of the continuum of critical modes as measured by the velocity correlator.

Within the field theory this is surely correct—is is based on controlled calculations. Another issue is, can we literally apply the field theory to condensed matter problems for this particular purpose? We are actually not sure. One way to read the effect of the previous paragraph is as follows. It is assumed in the field theory that the lattice constant is vanishing. As applied to any problem with a finite lattice cutoff, this means that the field theory can only be taken literally when the distance between vortices is large compared to the lattice constant—the small fugacity limit is implicitly taken. If this is the case, by zooming in one will eventually get at length scales which are smaller than the intervortex distance, and here one will rediscover the implacable order and its dynamical implication in the form of its Goldstone mode. The way this limit is reached is a bit more sophisticated than suggested with these words, but this we discussed at length in this paper.

The problem is that in real condensed matter systems the vortex (or, in general, "dual") matter is actually rather dense when density is measured in units of the lattice constant. Although the field theory is fine in the scaling limit, as helped by universality, to what extent can the effects we discussed in this paper become noticeable? We leave this as an open question, in the hope that others might get motivated to have a closer look.

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[1] H. A. Kramers and G. H. Wannier, Phys. Rev. **60**, 252 (1941).
[2] R. Savit, Rev. Mod. Phys. **52**, 453 (1980).
[3] M. P. A. Fisher, P. B. Weichman, G. Grinstein, and D. S. Fisher, Phys. Rev. B **40**, 546 (1989).
[4] H. Kleinert, *Gauge Fields in Condensed Matter*, Vol. I: *Superflow and Vortex Lines, Disorder Fields, Phase Transitions* (World Scientific, Singapore, 1989).
[5] L. P. Kadanoff and H. Ceva, Phys. Rev. B **3**, 3918 (1971).
[6] I. F. Herbut and Z. Tesanovic, Phys. Rev. Lett. **76**, 4588 (1996); **78**, 980 (1997).
[7] P. Olsson and S. Teitel, Phys. Rev. Lett. **80**, 1964 (1998).
[8] J. Hove and A. Sudbø, Phys. Rev. Lett. **84**, 3426 (2000).
[9] S. Sachdev, *Quantum Phase Transitions* (Cambridge University Press, Cambridge, England, 1999).
[10] S. W. Heinekamp and R. A. Pelcovits, Phys. Rev. B **32**, 4528 (1985).
[11] J. Zaanen, S. I. Mukhin, and Z. Nussinov, Ann. Phys. (N.Y.) **310/1**, 181 (2004).
[12] A. Zee, Physica A **281**, 442 (2000).
[13] C. Dasgupta and B. I. Halperin, Phys. Rev. Lett. **47**, 1556 (1981).
[14] M. P. A. Fisher and D. H. Lee, Phys. Rev. B **39**, 2756 (1989).
[15] R. Fazio and H. van der Zant, Phys. Rep. **355**, 235 (2001).
[16] T. Banks, B. Meyerson, and J. Kogut, Nucl. Phys. B **129**, 493 (1977).
[17] A. K. Nguyen and A. Sudbø, Phys. Rev. B **60**, 15307 (1999); A. K. Nguyen and A. Sudbø, Europhys. Lett. **46**, 780 (1999).
[18] T. Senthil and M. P. A. Fisher, Phys. Rev. B **62**, 7850 (2000).
[19] M. Franz and Z. Tesanovic, Phys. Rev. Lett. **87**, 257003 (2001).
[20] G. Mozurkewich, M. B. Salamon, and S. E. Inderhees, Phys. Rev. B **46**, 11914 (1992); S. Kamal, D. A. Bonn, Nigel Goldenfeld, P. J. Hirschfeld, Ruixing Liang, and W. N. Hardy, Phys. Rev. Lett. **73**, 1845 (1994).
[21] P. W. Higgs, Phys. Rev. Lett. **13**, 508 (1964).
[22] P. W. Anderson, Phys. Rev. **110**, 827 (1958); **130**, 439 (1963).
[23] N. Bogolyubov, N. Tolmachev, and D. Shirkov, *New Methods in the Theory of Superconductivity* (Academy of Sciences of the U.S.S.R., Moscow, 1958); V. M. Galitskii, Sov. Phys. JETP **34**, 1011 (1958).
[24] J. C. Le Guillou and J. Zinn-Justin, Phys. Rev. B **21**, 3976 (1980).
[25] S. R. Shenoy, Phys. Rev. B **40**, 5056 (1989).
[26] M. Hasenbusch and T. Török, J. Phys. A **32**, 6361 (1999).
[27] A. P. Gottlob and M. Hasenbusch, cond-mat/9305020 (unpublished).
[28] P. Olsson, Europhys. Lett. **58**, 705 (2002).
[29] J. Hove, S. Mo, and A. Sudbø, Phys. Rev. Lett. **85**, 2368 (2000).
[30] D. S. Fisher, M. P. A. Fisher, and D. A. Huse, Phys. Rev. B **43**, 130 (1991).
[31] Z. Tesanovic, Phys. Rev. B **51**, 16204 (1995); **59**, 6449 (1999).
[32] Z. Tesanovic, notes from lectures at The Boulder Summer School in Condensed Matter and Materials Physics 2000 (available at http://research.yale.edu/boulder/Boulder-2000/transparencies/tesanovic-lecture1/vortices.pdf).
[33] Notice that the Josephson correlation length scales according to

$$\xi_J = c/\rho_s \propto |\epsilon|^{-\nu(2-\eta_A)} \xrightarrow{\eta_A=1} |\epsilon|^{-\nu}.$$

It is well known (Ref. 35) that $\xi_J$ is governed by the same exponent $\nu$ in charge of the mass on the disordered side. According to the above equation, this can only be the case when $\eta_A=1$, once again highlighting the remarkable consistency of the duality construction.
[34] K. Binder, Phys. Rev. Lett. **47**, 693 (1981); K. Binder, M. Nauenberg, V. Privman, and A. P. Young, Phys. Rev. B **31**, 1498 (1985).
[35] M. E. Fisher, M. N. Barber, and D. Jasnow, Phys. Rev. A **8**, 1111 (1972).