# Dynamics of a two-level system coupled with a quantum oscillator: The very strong coupling limit

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The time-dependent behavior of a two-level system interacting with a quantum oscillator system is analyzed in the case of a coupling larger than both the energy separation between the two levels and the energy of the quantum oscillator ($\Omega<\omega<\lambda$, where $\Omega$ is the frequency of the transition between the two levels, $\omega$ is the frequency of the oscillator, and $\lambda$ is the coupling between the two-level system and the oscillator). Our calculations show that the amplitude of the expectation value of the oscillator coordinate decreases as the two-level system undergoes the transition from one level to the other, while the transfer probability between the levels is staircaselike. This behavior is explained by the interplay between the adiabatic and nonadiabatic regimes encountered during the dynamics with the system acting as a quantum counterpart of the Landau-Zener model. The transition between the two levels occurs as long as the expectation value of the oscillator coordinate is driven close to zero. On the contrary, if the initial conditions are set such that the expectation values of the oscillator coordinate are far from zero, the system will remain locked on one level.

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One of the most studied quantum-mechanical models is the two-level system interacting with a quantum oscillator. It is used in a wide range of phenomena, especially in atomic physics where it describes a two-level atom coupled to a quantized electromagnetic field.[1,2] A challenging enterprise is to extend the model to "artificial atoms" in a condensed-matter environment. These small solid-state devices like flux lines threading a superconducting loop, charges in Cooper pair boxes, and single-electron spins exhibit quantum-mechanical properties which can be manipulated by currents and voltages.[3–5]

The solid-state devices offer wider regimes for the coupling strength between the two-level system and the oscillator. Typically, the coupling strength in atomic systems is $\lambda/\omega=10^{-7}-10^{-6}$.[2] Similar dipolar coupling in Cooper pair boxes and Josephson charge qubits is 3–4 orders of magnitude larger than the coupling in atomic systems.[6,7] In contrast to the dipolar coupling, the capacitive and inductive couplings show even larger coupling strengths.[8–10] Recently it has been argued that it is possible to achieve values $\lambda/\omega\approx 1$ experimentally.[11] In this Brief Report we examine the regime $\lambda/\omega>1$. We will show that, although the oscillator dynamics "follows" the dynamics of the two-level system as in the case studied in Ref. 11, the general features are different: the system undergoes the transition from one level to the other with a sudden change in the transition probability, whenever the expectation value of the oscillator coordinate is close to zero.

The Hamiltonian of the system is written as ($\hbar=1$)

$$H=\frac{p^2+\omega^2q^2}{2}+\lambda q\sigma_z+\Omega\sigma_x, \tag{1}$$

where $\sigma_z$ and $\sigma_x$ are the spin operator matrices, and $p$ and $q$ are the oscillator coordinates. The essential parameters are $\omega$, $\lambda$, and $\Omega$, associated with the frequency of the oscillator, the coupling strength of the two-level system with the oscillator, and the splitting frequency of the two-level system, respectively (our parameter $\lambda$ is scaled up by a factor of $2\sqrt{2}$ with respect to parameter $\lambda$ in Ref. 11). In Ref. 11 it was assumed that the splitting frequency is much smaller than the frequency of the oscillator; thus the problem can be cast into the displaced oscillator basis which is the basis for the first two terms of the Hamiltonian (1). This displaced oscillator basis is found by applying the unitary transformation $U=\exp\left(\frac{i\lambda p}{\omega^2}\sigma_z\right)$ to the Hamiltonian (1). The transformed Hamiltonian becomes[12]

$$H'=\frac{p^2+\omega^2q^2}{2}+\frac{\Omega}{2}\left[\sigma^+\exp\left(-i\frac{\lambda p}{\omega^2}\right)+\text{H.c.}\right]-\frac{1}{8}\frac{\lambda^2}{\omega^2}, \tag{2}$$

with $\sigma^+$ ($\sigma^-$) as the spin-$\frac{1}{2}$ creation (annihilation) operator; then time-dependent or/and time-independent perturbation calculations can be performed as long as $\Omega/\omega\ll 1$. A similar path was followed by Schweber.[13] Perturbation calculations on the energy spectrum have been performed and compared with the exact calculations in Fig. 3 of Ref. 11. The authors extended the comparison to the regime $\Omega/\omega\geq 1$ and they found some quantitative and qualitative resemblance between the exact solution and their approximate solution [Figs. 3(b) and 3(d) in their paper]. The quantitative and qualitative agreement shown in Fig. 3 of Ref. 11 can be explained in simple terms as follows. The net effect of $\exp\left(\frac{i\lambda p}{2\omega^2}\right)$ on the wave function is to displace it by the amount $\frac{\lambda}{2\omega^2}$. Thus, the effective splitting given by the second term of the Hamiltonian (2) will be quenched, such that perturbation calculations on the Hamiltonian (2) can be extended to larger values of $\Omega$.

Irish and co-workers[11] studied the collapse and revival of the wave function for $\Omega/\omega\ll 1$ and $\lambda/\omega<1$. In the following we will explore the dynamics of the very strong coupling regime ($\lambda/\omega>1$ and $\Omega/\omega<1$). The Hamiltonian (2) would be able, in principle, to explain the dynamics in the very strong coupling regime because the effective coupling be-

 

tween the two wells generated by the displaced oscillators is quenched by the separation of these two potential wells. The effective coupling between the two wells decreases exponentially[11] with increasing strength of $\lambda$ as can be shown also by analyzing Eq. (2). However, to gain additional insight into the dynamics of the very strong coupling regime, we follow a different approach. We perform another unitary transformation[12,14] $U=\exp[i\Lambda(q)\sigma_y]$ on the Hamiltonian (1), with $\tan[\Lambda(q)]=-\frac{\Omega}{\lambda q}$, to obtain adiabatic motions which are valid for either very strong coupling $\lambda \gg \Omega, \omega$ (case A) or large level splitting $\Omega \gg \lambda, \omega$ (case B). The intuitive picture of the transformation is a $q$-dependent rotation around the $y$ axis that brings the effective field experienced by the two-level system along the $z$ axis. The Hamiltonian resulting from the above unitary transformation is

$$H' = \frac{1}{2}(p^2+\omega^2 q^2) + \frac{1}{8}\frac{\Omega^2\lambda^2}{(\Omega^2+\lambda^2 q^2)^2} + \sigma_z\sqrt{\Omega^2+\lambda^2 q^2} + \frac{\lambda\Omega}{2}\sigma_y\left(p\frac{1}{\Omega^2+\lambda^2 q^2} + \frac{1}{\Omega^2+\lambda^2 q^2}p\right). \tag{3}$$

The above Hamiltonian (3) will be used below for our investigation. The last term ($\sigma_y$ term) in Eq. (3) is small as long as $\lambda q \gg \Omega$ (which is supposed to be satisfied in case A) or $\lambda q \ll \Omega$ (which is satisfied in case B). The inequalities $\lambda q \gg \Omega$ and $\lambda q \ll \Omega$ should be understood as operator inequalities in the sense that they have to be satisfied as inequalities for matrix elements in a certain basis. The $\sigma_y$ term is the nondiagonal term and it accounts for the nonadiabaticity. Without the $\sigma_y$ term, the Hamiltonian (3) reveals an adiabatic motion, with one part (either the two-level system or the oscillator) becoming fast, while the other part becomes slow.[14] Thus, the adiabatic motion is generated by the first three terms,

$$H'_{ad} = \frac{1}{2}(p^2+\omega^2 q^2) + \frac{1}{8}\frac{\Omega^2\lambda^2}{(\Omega^2+\lambda^2 q^2)^2} + \sigma_z\sqrt{\Omega^2+\lambda^2 q^2}. \tag{4}$$

In the very strong coupling case ($\lambda \gg \Omega, \omega$), there are two adiabatic potential sheets coupled by the $\sigma_y$ term.[12,14] The shape of the adiabatic potentials is presented in Fig. 1. Dynamically, the oscillator is fast and the two-level system is slow.[14] The lower adiabatic sheet has two minima located at

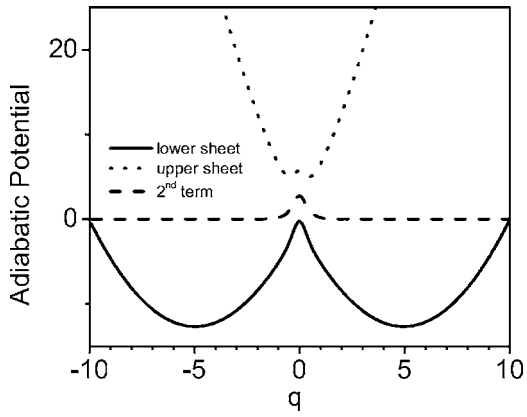


FIG. 1. The shape of the two sheets of the adiabatic potential generated by the Hamiltonian in Eq. (4). The solid line is the lower sheet and the dotted line is the upper sheet. The second term in Eq. (4) is sharply peaked around the origin and it is plotted with a dashed line.

$$q_{\min} = \mp\sqrt{\frac{\lambda^2}{4\omega^4} - \frac{\Omega^2}{\lambda^2}}, \tag{5}$$

with the value

$$V(q_{\min}) = -\frac{\omega^2\Omega^2}{2\lambda^2} - \frac{\lambda^2}{8\omega^2}, \tag{6}$$

and its second derivative at the minimum points

$$\omega^2_{\min} = \omega^2\left(1 - \frac{4\Omega^2\omega^4}{\lambda^4}\right). \tag{7}$$

The lower adiabatic sheet is very close to the unperturbed ($\Omega=0$) displaced harmonic potential sheets as long as $\lambda \gg \Omega, \omega$. Therefore one might expect similar dynamic behavior as that studied in Ref. 11. However, the dynamics is different for very strong coupling. In Fig. 2 we show the dynamics of the expectation values of the $q$ coordinate and $\sigma_z$. The details of numerical integration are given in Ref. 14. We compare the exact dynamics with the dynamics of the adiabatic potential [Eq. (4)], and with the dynamics of a displaced oscillator ($\Omega=0$). One can notice in Fig. 2 that the amplitude of the expectation value of the $q$ coordinate decreases with time. It occurs as the system undergoes the transition from one level to another. At the same time, the trans-

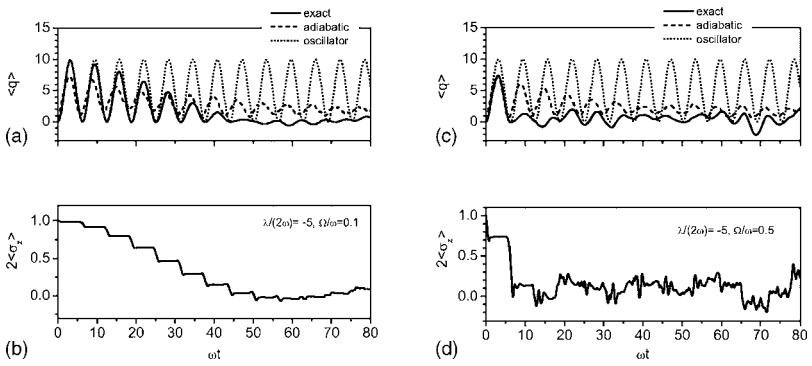


FIG. 2. Time-dependent behavior of the expectation value of (a) $q$ coordinate, with $\Omega/\omega=0.1$ and $\lambda/2\omega=-5$; (b) $2\sigma_z$, with $\Omega/\omega=0.1$ and $\lambda/2\omega=-5$; (c) $q$ coordinate, with $\Omega/\omega=0.5$ and $\lambda/2\omega=-5$; (d) $2\sigma_z$, with $\Omega/\omega=0.5$ and $\lambda/2\omega=-5$. The solid line denotes the exact dynamics, the dashed line represents the adiabatic dynamics, and the dotted line shows the corresponding displaced oscillator. For convenience, we have chosen negative values of $\lambda$. We notice that the strength of the coupling is, actually, $\lambda/2$.

fer probability is staircaselike. Moreover, the slope of the amplitude decrease depends on $\Omega$ but the frequency of the oscillations does not change significantly. Although it is weaker, the same quenching of the $q$ coordinate appears for the adiabatic motion generated by Eq. (4). It is weakly dependent on $\Omega$ and it occurs as the system undergoes the transition from one potential well to the other potential well. This was pointed out by Wagner[12] who showed that the transition rate from one well to the other is dependent on $\Omega$ in second order in the adiabatic approximation. In contrast to his paper,[12] Fig. 2 shows clearly that the adiabatic motion is not an accurate description of the full dynamics. This behavior will be explained below as interplay between the adiabatic and the nonadiabatic regimes encountered during the dynamics.

In order to explain the dynamics, we employ the equation of motion in the Heisenberg picture for an operator, $\frac{d}{dt}A = i[H,A]$. Taking the expectation value, the equation becomes $\frac{d}{dt}\langle A\rangle = i\langle[H,A]\rangle$. We apply this last equation to the Hamiltonian represented by Eq. (1). The corresponding equations are

$$\frac{d}{dt}\langle q\rangle = \langle p\rangle,$$

$$\frac{d}{dt}\langle p\rangle = -\omega^2\langle q\rangle - \lambda\langle\sigma_z\rangle,$$

$$\frac{d}{dt}\langle\sigma_z\rangle = \Omega\langle\sigma_y\rangle,$$

$$\frac{d}{dt}\langle\sigma_x\rangle = -\lambda\langle q\sigma_y\rangle,$$

$$\frac{d}{dt}\langle\sigma_y\rangle = \lambda\langle q\sigma_x\rangle - \Omega\langle\sigma_z\rangle,$$

$$\vdots . \tag{8}$$

Equation (8) is an infinite chain of coupled ordinary differential equations. The chain can be broken by making assumptions like

$$\langle q\sigma_y\rangle \cong \langle q\rangle\langle\sigma_y\rangle,$$

$$\langle q\sigma_x\rangle \cong \langle q\rangle\langle\sigma_x\rangle. \tag{9}$$

The approximations made in Eq. (9) are valid if one part (the oscillator) is fast and the other part (the two-level system) is slow as in the usual adiabatic approximation.[15,16] Thus, combining Eqs. (8) and (9), one can show that Eq. (8) can be approximated and then recast as

$$\frac{d}{dt}\langle q\rangle = \langle p\rangle,$$

$$\frac{d}{dt}\langle p\rangle = -\omega^2\langle q\rangle - \frac{\lambda}{2}(|f_1|^2 - |f_2|^2),$$

$$i\frac{d}{dt}f_1 = \frac{\lambda}{2}\langle q\rangle f_1 + \frac{\Omega}{2}f_2,$$

$$i\frac{d}{dt}f_2 = -\frac{\lambda}{2}\langle q\rangle f_2 + \frac{\Omega}{2}f_1, \tag{10}$$

with $f_1$ ($f_2$) being the probability function of the level 1 (2). In other words, $|f_1|^2$ ($|f_2|^2$) is the probability of the system to be on the level 1 (2).

Equation (10) sheds a better light on the relationship between the oscillator and the two-level system. The first two equations in (10) are the equations of the classical harmonic oscillator displaced by the amount $\frac{\lambda}{2}(|f_1|^2 - |f_2|^2)$. This implies that the time variation of $(|f_1|^2 - |f_2|^2) = 2\langle\sigma_z\rangle$ modulates the amplitude of the oscillator as can be seen in Fig. 2. The last two equations in (10) explain the time-dependent behavior of $(|f_1|^2 - |f_2|^2)$. First, let us assume that $\lambda\langle q\rangle \gg \Omega$, which is the adiabatic condition and is supposed to be true most of the time for $\lambda \gg \Omega, \omega$. The last two equations in (10) will be approximated by $i\frac{d}{dt}f_1 \cong \frac{\lambda}{2}\langle q\rangle f_1$ and $i\frac{d}{dt}f_2 \cong -\frac{\lambda}{2}\langle q\rangle f_2$. This means that the probability functions $f_1$ and $f_2$ acquire just a phase factor. Therefore, there is no mixing between $f_1$ and $f_2$, and $\langle\sigma_z\rangle$ is constant in time. Now, let us assume that the reverse is true, $\lambda\langle q\rangle < \Omega$. Then, the last two equations in (10) will be approximated by $i\frac{d}{dt}f_1 \cong \frac{\Omega}{2}f_2$ and $i\frac{d}{dt}f_2 \cong \frac{\Omega}{2}f_1$, i.e., $f_1$ and $f_2$ will mix, and $\langle\sigma_z\rangle$ will change. These assertions are

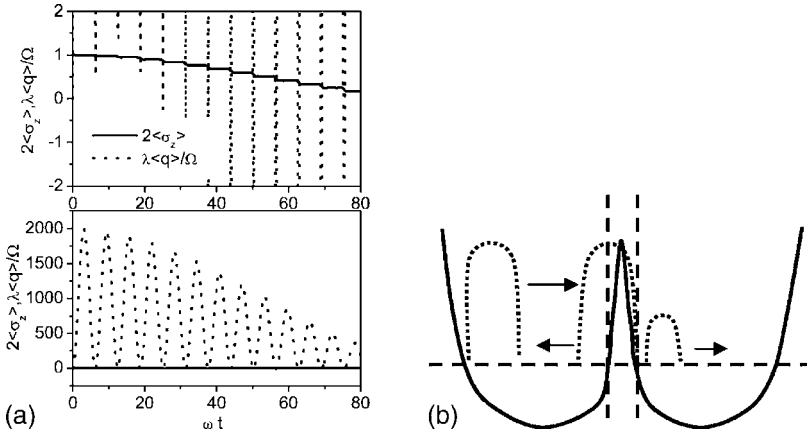


FIG. 3. (a) Comparison between the dynamics of $\langle\sigma_z\rangle$ (solid line) and $\lambda\langle q\rangle/\Omega$ (dotted line). It shows that $\langle\sigma_z\rangle$ changes whenever $\lambda\langle q\rangle/\Omega$ is close to 0. The upper panel shows the scale around the origin, while the lower panel shows the full scale of the plots. (b) Pictorial explanation of the dynamics of the two-level system in this very strong coupling case. The transition from the left well to the right well occurs at the origin where the adiabatic condition ($\lambda\langle q\rangle \gg \Omega$) is not satisfied.

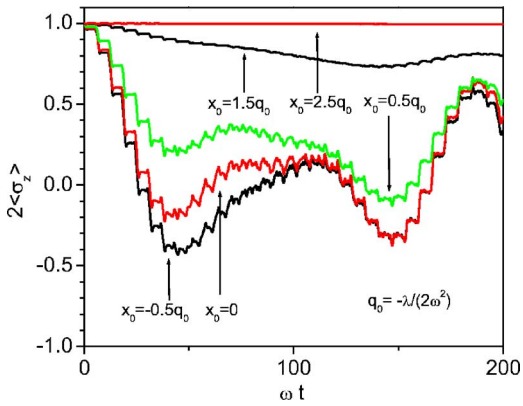


FIG. 4. (Color online) Dynamics of $\langle\sigma_z\rangle$ with different initial conditions. The arrows indicate various initial conditions in terms of the displaced ground state wave function of the initial oscillator. The oscillator displacements are $-0.5q_0$ (it produces the faster transition), 0, $0.5q_0$, $1.5q_0$, and $2.5q_0$ (it locks the system on level 1), with $q_0=-\lambda/2\omega^2$. $\Omega/\omega=0.2$ and $\lambda/2\omega=3$.

proven numerically in Fig. 3(a), where one can see that $\langle\sigma_z\rangle$ changes whenever $\lambda\langle q\rangle/\Omega\cong 0$.

In Fig. 3(b), we give a simple explanation of the dynamics of the transition from level 1 to level 2. Suppose that the wave packet of the system is in the left well which is associated with level 1. As soon as the wave packet reaches regions close to the origin it "sheds" a part of itself into the right well and the rest of it returns into the left well. Basically, the transition from level 1 to level 2 occurs during short periods of time when the system is out of its adiabatic regime. The picture presented here is closely related to Landau-Zener theory,[17,18] which treats a quantum two-level system placed in a slowly varying external field. Near the crossing point the adiabaticity is violated and the system can escape from the state it occupied initially to another one. In the present case, the coupling is purely quantum through the quantum oscillator; hence the oscillator explores all possible trajectories (in the sense of Feynman's path integral) and the ones that explore the region at the level crossing can induce a nonzero flipping probability for the two-level system.

In Fig. 4 we show the dynamics of $\langle\sigma_z\rangle$ as depending on initial conditions. We consider as initial conditions various displaced ground state wave functions of the harmonic oscillator. The displacements are given in terms of $q_0=-\frac{\lambda}{2\omega^2}$, which is the displacement of the unperturbed oscillator ($\Omega=0$) corresponding to the level 1. Assuming that the system is initially on the level 1, its $q$ coordinate $\langle q\rangle$ will tend to oscillate around $q_0$, starting from the initial displacement. Figure 4 shows that the system undergoes faster transitions from level 1 to level 2 for those initial conditions which generate dynamics that satisfies the condition $\lambda\langle q\rangle \ll \Omega$ for a longer time. In addition to that, the system can be locked on level 1 as long as the initial expectation value of $q$ is set to be close to $q_0$.

In conclusion, we investigated the dynamics of a two-level system interacting with a quantum harmonic oscillator in the regime of very strong coupling ($\lambda > \omega > \Omega$). This regime generates an adiabatic motion defined by the condition $\lambda\langle q\rangle \gg \Omega$. We have found that the amplitude of the expectation value of the oscillator coordinate $\langle q\rangle$ varies similarly to the expectation value of $2\langle\sigma_z\rangle$ (the difference between the occupation probabilities of the two-level system). In addition, the difference $2\langle\sigma_z\rangle$ is staircaselike. This behavior is explained by the interplay between the adiabatic (whenever $\lambda\langle q\rangle \gg \Omega$) and nonadiabatic (whenever $\lambda\langle q\rangle < \Omega$) regions. The transition from one level to another occurs during short periods of time when the system is out of its adiabatic region. Thus, the system can be locked on one level if it is prepared to satisfy the adiabatic condition all the time.

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