Glassy behavior of light in random lasers

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A theoretical analysis [Angelani *et al.*, Phys. Rev. Lett. **96**, 065702 (2006)] predicts glassy behavior of light in a nonlinear random medium. This implies slow dynamics related to the presence of many metastable states. We consider very general equations (that also apply to other systems, like Bose-Condensed gases) describing light in a disordered nonlinear medium and through some approximations we relate them to a mean-field spin-glass-like model. The model is solved by the replica method, and replica-symmetry breaking phase transition is predicted. The transition describes a mode-locking process in which the phases of the modes are locked to random (history and sample-dependent) values. An extended discussion of possible experimental implications of our analysis is reported.

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I. INTRODUCTION

In a nutshell, laser action in a stochastic resonator (SR) defines a random laser (RL). Following the original Lethokov's article,[1] a SR is a disordered medium sustaining a large number of electromagnetic modes with overlapping resonances. The modes are not necessarily localized (in the Anderson sense), but can be extended modes in a random medium; they have typically a finite lifetime and are sometimes referred to as "quasimodes." Generally speaking, we will refer to a RL as a multimode laser system that displays some disorder; this will be described by a probability distribution and we will then consider different realizations of the system. Such a general definition not only embraces the experiments addressed below, but also include standard lasers with a disordered cavity, or integrated devices, as, for example, ordered photonic crystals[2] infiltrated by some active (i.e., doped) soft material, like liquid crystals or polymers, that induces a given amount of disorder, or even intentionally disordered photonic crystals enriched by quantum wells providing optical gain.

In the early developments, the theoretical framework at the basis of a RL has relied on light diffusion.[3–5] These studies stimulated many investigations concerning photon dynamics in a disordered medium, up to considering the quantum transport of photons.[6–12] Subsequent detailed numerical studies revealed how important for RLs is the nature and the distribution of localized modes in random amplifying media, in particular, in the strongly scattering regime.[13,14] Experiments were reported on the emerging of many coupled oscillation modes while increasing the pump energy and the consequent nontrivial dynamics of the resulting optical signals.[15–18] Coupling of modes was addressed in Ref. 18, as the fact that the maximum observed number of modes increases with the pumping intensity and with the sample volume.[19,20] Recent results pointed out new key issues concerning the physics of random lasers, as the role of extended modes,[21,22] or the presence of specific fluctuations.[23,24]

When considered from a semiclassical perspective, a multimode random laser strikingly displays those ingredients which are typical of the physics of complexity: i.e., randomness and nonlinearity. The latter is due to typical mode-interaction processes, like mode-competition and mode-locking.[25–29] Complex processes in laser physics, including nonlinear optics, are well-known and studied (see, e.g., Refs. 30 and 31), up to recent investigations in multimode systems[32–34] and successful reformulations of standard laser thermodynamics.[35–39] The extension of these approaches to a RL immediately leads to the application of the statistical theory of disordered systems, of which spin glass theory is a paradigm,[40] and which is the subject of the present paper.

We show that, in the presence of a large number of coupled modes (extended or not), a mode-locking (ML) process can be observed. ML is related to the relative phases between resonant states, which in some cases become locked at the same value. For a standard laser it can be realized by an active device, like an acusto-optic modulator, or can be self-starting as in the presence of a nonlinearly mediated mode interaction.[41] We can expect that RL-ML appears when many modes are put into oscillations, and their amplitudes are clamped at the oscillation values, which are random variables. The temporal dynamics of the emitted signal is indeed strongly related to the phases,[42] given the fact that the mode amplitudes vary on a much longer time scale. The latter circumstance favors the consideration of the mode amplitudes as "quenched" (i.e., random but slowly varying) variables, and the phases are to be taken as the relevant dynamical variables. The mode-locking process in standard lasers is now recognized as a thermodynamic phase transition;[35–39] it is expected, therefore, that the mode-locking transition for a RL takes the form of a phase transition in a disordered system.

This paper follows a recent letter[43] and furnishes extensive details on the derivation of the analytical results (including the stability analysis that was previously not reported); a discussion of the underlying working hypotheses; a discussion on the nature of the considered electromagnetic modes;

and the analysis of possible experimental frameworks where glassy behavior of light can be observed. The paper is structured as follows. In Sec. II, we will review coupled mode theory in a dielectric resonator in the presence of a nonlinear susceptibility; in Sec. III we will specialize the approach by deriving the leading model for our analysis; in Sec. IV we will apply the methods used in spin glass theory to solve the model; Sec. V is focused on a discussion of the physical meaning of our results, using real-world units, and of possible experimental setups; and conclusions are drawn in Sec. VI.

II. COUPLED MODE THEORY EQUATIONS

The physical system under consideration is an open electromagnetic cavity supporting modes at optical frequencies. The cavity is characterized by the presence of disorder; for example, randomly structured dielectrics in between a couple of mirrors, or a mirrorless system (e.g., a distribution of dielectric particles) such that there is a sufficiently high refractive index contrast, so that localized modes (which means modes belonging to the discrete spectrum of the eigenvalue problem given by the Maxwell equations, as detailed below) do exist. This is the case, for example, of a disordered distribution of TiO_2 particles, of semiconductor powders in a liquid or a glassy matrix, or of a nanostructured microcavity filled by a randomly fluctuating material like liquid crystals or soft matter. The localized modes supported by these systems can be very different, depending on the degree of localizations, e.g., they can be distributed over all the dielectric samples (as those investigated, for example, in the experiments reported in Ref. 21) or correspond to well-localized states (as those numerically analyzed in Ref. 14); this distinction can influence the properties of the interaction between modes, leading to interesting effects, as will be discussed in the following. However, the physical picture we will obtain is expected to be independent of the details of the interaction.

Models for multimode nonlinear optical cavities have been largely reported in literature (see, e.g., Refs. 2, 44, and 45). Typically they result into coupled equations for complex amplitudes, which can be obtained using various and equivalent approaches. In order to fix the notation and for the benefit of the nonexpert reader, here we will briefly report a derivation based on a multiple scale approach. The electromagnetic cavity [a dielectric resonator (DR)], is described by a (static) refractive index profile $n(\mathbf{r})$. Such a kind of system may support the existence of resonance modes, which can be either localized or distributed in the system. Maxwell's equations are written as

$$\nabla \times \mathbf{H} = \varepsilon_0 n^2(\mathbf{r}) \partial_t \mathbf{E},$$

$$\nabla \times \mathbf{E} = -\mu_0 \partial_t \mathbf{H}. \tag{1}$$

The electric and magnetic fields can be expanded in normal modes with angular frequencies ω_n and eigenvalues $\mathbf{E}_n(\mathbf{r})$ and $\mathbf{H}_n(\mathbf{r})$ as

$$\mathbf{H} = \mathrm{Re}\left[\sum_n \mathbf{H}_n(\mathbf{r}) \exp(-i\omega_n t)\right], \tag{2}$$

The latter quantities satisfy the generalized eigenvalue problem

$$\mathcal{L}\mathcal{F}_s = \omega_s \mathcal{M}\mathcal{F}_s \tag{3}$$

while being

$$\mathcal{L} = \begin{pmatrix} 0 & i\nabla \times \\ -i\nabla \times & 0 \end{pmatrix}, \tag{4}$$

$$\mathcal{M} = \begin{pmatrix} \epsilon_0 n^2(\mathbf{r}) & 0 \\ 0 & \mu_0 \end{pmatrix}, \tag{5}$$

and

$$\mathcal{F}_s = \begin{pmatrix} \mathbf{E}_s \\ \mathbf{H}_s \end{pmatrix}. \tag{6}$$

Given a volume V much wider than the DR, over which periodical (Born-Von Karman) boundary conditions are posed, and introducing the complex valued scalar product

$$(\mathbf{A}, \mathbf{B}) = \int_V \mathbf{A}^* \cdot \mathbf{B} dV \tag{7}$$

it turns out that $\mathcal{L}$ and $\mathcal{M}$ are self-adjoint operators. As a result ω_n are real valued and the eigenvectors are orthogonal with weight $\mathcal{M}$. Furthermore, since $(\mathbf{E}_n, \mathbf{H}_n)$ and $(\mathbf{E}_n^*, -\mathbf{H}_n^*)$ correspond to the same eigenvalue ω_n, $\mathbf{E}_n$ can be taken as real valued.

The average electromagnetic energy for each unnormalized mode is given by

$$\mathcal{E}_s = \frac{1}{4}\int_V \varepsilon_0 n^2(\mathbf{r})|\mathbf{E}_s|^2 + \mu_0|\mathbf{H}_s|^2 dV = \frac{1}{4}(\mathcal{F}_s, \mathcal{M}\mathcal{F}_s). \tag{8}$$

In the following the modes are normalized in such a way

$$\frac{1}{4}(\mathcal{F}_s, \mathcal{M}\mathcal{F}_q) = \delta_{sq}. \tag{9}$$

Next we consider the perturbed Maxwell equations in the presence of a nonlinear polarization $\mathbf{P}_{NL}$, such that the overall dielectric displacement vector is given by $\mathbf{D} = \varepsilon_0 n^2(\mathbf{r})\mathbf{E} + \mathbf{P}_{NL}$ and $\mathbf{J} = \partial_t \mathbf{P}_{NL}$ a generalized current:

$$\nabla \times \mathbf{H} = \varepsilon_0 n^2(r) \partial_t \mathbf{E} + \eta \mathbf{J},$$

$$\nabla \times \mathbf{E} = -\mu_0 \partial_t \mathbf{H}, \tag{10}$$

η is a bookkeeping perturbation parameter to be set equal to one at the end of the derivation. Our aim is to write the solution of the nonlinear Maxwell equations as a superposition of modes such that the leading order has the form

$$\mathbf{E} = \mathrm{Re}\left[\sum_n \sqrt{\omega_n} a_n(t) \mathbf{E}_n(\mathbf{r}) \exp(-i\omega_n t)\right],$$

$$\mathbf{H} = \mathrm{Re}\left[\sum_n \sqrt{\omega_n} a_n(t)\mathbf{H}_n(\mathbf{r})\exp(-i\omega_n t)\right], \tag{11}$$

and the complex amplitudes a_s are such that the total energy stored in the DR is

$$\mathcal{E} = \Sigma_m \mathcal{E}_m = \Sigma_m \omega_m |a_m|^2. \tag{12}$$

There are various techniques to derive the leading equations for the a_s, here we adopt the multiple scale method (see, e.g., Ref. 46). The perturbative expansion is written as (with obvious notation)

$$\mathbf{E} = \mathrm{Re}\left\{\sum_n [\sqrt{\omega_n} a_n(\eta t, \eta^2 t, \ldots)\mathbf{E}_n + \eta \mathbf{E}_n^{(1)} + \cdots] \times \exp(-i\omega_n t)\right\},$$

$$\mathbf{H} = \mathrm{Re}\left\{\sum_n [\sqrt{\omega_n} a_n(\eta t, \eta^2 t, \ldots)\mathbf{H}_n + \eta \mathbf{H}_n^{(1)} + \cdots] \times \exp(-i\omega_n t)\right\}, \tag{13}$$

where the amplitudes are taken to be dependent on the slow scales $t_n = \eta^n t$, as the first and higher order corrections like $\mathbf{E}_n^{(1)}$, the fastest scale is $t_0 = t$ and the temporal derivatives are written as $\partial_t = \partial_{t_0} + \eta \partial_{t_1} + \cdots$. Letting

$$\mathbf{P}_{NL} = \mathrm{Re}\left[\sum_n \mathbf{P}_n(t_1, t_2, \ldots)\exp(-i\omega_n t_0)\right] \tag{14}$$

and

$$\mathbf{J} = \mathrm{Re}\left[\sum_n \mathbf{J}_n(t_1, t_2, \ldots)\exp(-i\omega_n t_0)\right] = \partial_t \mathbf{P}_{NL} \tag{15}$$

with

$$\mathbf{P}_n = \mathbf{P}_n^{(0)} + \eta \mathbf{P}_n^{(1)} + \cdots,$$

$$\mathbf{J}_n = \mathbf{J}_n^{(0)} + \eta \mathbf{J}_n^{(1)} + \cdots, \tag{16}$$

it is

$$\mathbf{J}_n^{(0)} = -i\omega_n \mathbf{P}_n^{(0)}. \tag{17}$$

Using the previous machinery in the nonlinear Maxwell equations at the first order in η it is found for the term oscillating with $\exp(-i\omega_s t_0)$

$$\mathcal{L}\mathcal{F}_s^{(1)} - \omega_s \mathcal{M}\mathcal{F}_s^{(1)} = \mathcal{B}_s, \tag{18}$$

while having

$$\mathcal{F}_s^{(1)} = \begin{pmatrix} \mathbf{E}_s^{(1)} \\ \mathbf{H}_s^{(1)} \end{pmatrix} \tag{19}$$

and

$$\mathcal{B}_s = \begin{pmatrix} i\varepsilon_0 n^2(\mathbf{r})\sqrt{\omega_s}\dfrac{da_s}{dt_1}\mathbf{E}_s + i\mathbf{J}_s^{(0)} \\ i\mu_0\sqrt{\omega_s}\dfrac{da_s}{dt_1}\mathbf{H}_s \end{pmatrix}. \tag{20}$$

The Fredholm theorem applied to Eq. (18) states that the solvability condition is the orthogonality with the kernel solution, i.e., $\mathcal{F}_s$: $(\mathcal{F}_s, \mathcal{B}_s) = 0$. Hence

$$\sqrt{\omega_s}\frac{da_s}{dt_1} = \frac{i\omega_s}{4}(\mathbf{E}_s, \mathbf{P}_s^{(0)}). \tag{21}$$

Going back to the original variables, we have the desired result

$$\frac{da_s(t)}{dt} = -\frac{\sqrt{\omega_s}}{4i}\int_V \mathbf{E}_s^*(\mathbf{r}) \cdot \mathbf{P}_s(\mathbf{r}) dV. \tag{22}$$

III. NONLINEAR SUSCEPTIBILITY AND MODE INTERACTIONS IN ACTIVE RANDOM CAVITIES

We consider the case in which many modes are put into oscillations and interact due to the nonlinearity of the amplifying medium. In resonant systems the nonlinear optical response can be found from the density matrix equations in a two-level system, as originally investigated by Lamb.[26] The component of the nonlinear susceptibility oscillating at ω_s is modeled as usual[47] and is written as

$$P_s^\alpha = \sum_{\omega_s + \omega_p = \omega_q + \omega_r} \chi_{\alpha\beta\gamma\delta}(\omega_s; \omega_q, \omega_r, -\omega_p, \mathbf{r}) \times E_p^\beta(\mathbf{r}) E_q^\gamma(\mathbf{r}) E_r^\delta(\mathbf{r}) \sqrt{\omega_p \omega_q \omega_r} a_q a_r a_p^*, \tag{23}$$

where χ is the third order response susceptibility tensor, which in general depends on the positions in the DR. Using Eq. (23) the coupled mode theory equations (22) read as

$$\frac{da_s}{dt} = -\frac{1}{2}\sum_{pqr} g_{spqr} a_q a_r a_p^*, \tag{24}$$

while being

$$g_{spqr} = \frac{\sqrt{\omega_s \omega_p \omega_q \omega_r}}{2i}\int_V \chi_{\alpha\beta\gamma\delta}(\omega_s; \omega_q, \omega_r, -\omega_p, \mathbf{r}) \times E_s^\alpha(\mathbf{r}) E_p^\beta(\mathbf{r}) E_q^\gamma(\mathbf{r}) E_r^\delta(\mathbf{r}) dV. \tag{25}$$

A. Mode interactions and the role of localized modes

Our treatment follows early works on multimode cavities[26,28] and consistently, in the previous equations, intermode frequencies and higher harmonics are neglected because they have in general a lower Q-factor if compared to those of the supported cavity modes. Additionally, since the sum in Eq. (24) is extended to all the modes combinations satisfying the condition $\omega_s = \omega_q + \omega_r - \omega_p$, we recall that the frequencies satisfying this relation can be divided into three categories:[28] (a) $\omega_s = \omega_q$ and $\omega_r = \omega_p$; (b) $\omega_s = \omega_r$ and $\omega_q = \omega_p$; and (c) $\omega_s = \omega_q + \omega_r - \omega_p$ excluding (a) and (b). Categories (a) and (b) were shown to determine the oscillation values of the energies of the modes $\mathcal{E}_s$, and provide terms like self and cross-saturation, as also recently considered in Ref. 20 with reference to RLs. The third group are the "combination tone terms"[28] which were originally neglected, even if it was later

recognized, through numerical calculations, to have a role when the number of modes increases.[48] We are interested in the regime in which a large number of modes is put into oscillation in a limited spectral range around a given carrier wavelength ω_0 (which can be taken as the resonant angular frequency of the active medium) and we will show below that in RLs, the combination tone terms provide a complex structure to the laser dynamics, as due to the fact that for an increasing number of modes they couple almost all the cavity resonances.

In general, the resonant condition for the mode-locking processes $\omega_s=\omega_q+\omega_r-\omega_p$, does not need to be satisfied exactly but in such a way that the mode combination tone $\omega_q+\omega_r-\omega_p$ lies within the linewidth at ω_s (this is discussed, e.g., in Ref. 29 with reference to three modes mode-locking). In the presence of many modes oscillating in a small bandwidth, and such that the linewidths are overlapping, as it is typically the case for RLs (see the cited references) and (by definition) for SRs, many mode combination tones will couple to ω_s, for which we have taken $\omega_s \cong \omega_q+\omega_r-\omega_p$ in Eq. (24). This opens the way to a "mean field theory" where all the modes are coupled, i.e., the sum in Eq. (24) is over all the possible values of *pqr*. We will describe this regime, moreover, considering the thermodynamic limit as the number of modes goes to infinity.

However, it is worth to observe that the coupling g_{spqr} in Eq. (25) is related to the spatial overlap of the four modes E_s, E_p, E_q, E_r that enter in the integral. In the case of *extended* modes, all the modes will have large spatial overlap and $g_{spqr} \neq 0$ for all *spqr*, so the "mean field limit" above is expected to be a very good approximation. On the contrary, in the case of *strongly localized modes* with localization length ξ, it is reasonable to expect that only a finite number of modes will be supported in a localization volume $\sim \xi$,[3,11,14,18] so that the coupling g_{spqr} will be nonzero only for those modes which are large in the same (or in adjacent) localization volumes. In this case the interaction will be *short range*, i.e., in the sum (24) only quadruplets of "nearby" modes will appear. Many intermediate situations between the extended and the strongly localized ones might happen in random lasers[11,14,18,21] and indeed the precise nature of the modes in these systems is not completely clear.

In the short range case, the basic phenomenology of the glass transition we will find (slow dynamics, random mode-locking) remains the same, but the physics of the system is strongly affected by *activated processes* (nucleation, barrier crossing, etc.) which are negligible in the mean field limit. Indeed the nature of the glass phase of short range spin glasses is still a debated problem.[49] Note that the localization length (and thus the interaction range) may vary on many orders of magnitude and can be experimentally controlled,[11] at variance to what happens in spin glasses and molecular or colloidal glasses, where the interaction range is fixed by the property of the material and is always of the order of the interparticle distance. This observation opens the way toward the possibility of an experimental investigation of the crossover between the mean field limit and the short range case that might be crucial for the theoretical understanding of the glass phase in short range systems.

To summarize, we will assume that (i) all the lasing modes have frequency $\omega_s \cong \omega_0$, ω_0 being the resonant frequency of the active medium, so that the constraint $\omega_s=\omega_q+\omega_r-\omega_p$ can be released, and that (ii) the spatial overlap of the modes is large, so that the integral in Eq. (25) will not be negligible for any quadruple of modes. Under these hypotheses a "mean field" treatment in which all quadruples of modes interact will be a very good approximation. Nevertheless, we expect the physical picture we will find in the following to hold under much more general assumptions on the interaction between modes. Its modifications due to the violation of the hypotheses above will be very interesting for the theory of spin glass systems.

B. The "quenched" approximation: A Langevin equation for the phases

Letting $a_s(t)=A_s(t)\exp[i\varphi_s(t)]$, we take A_s as slowly varying with respect to φ_s. Indeed, the facts that the temporal variation of the phases is on a time scale faster than that of the amplitudes and that fluctuations in a cavity take place because of the random interference between modes and not because of the intensity fluctuations of individual modes are well-established from the theory of mode-locking of standard multimode lasers.[27,42,44] Previous analytical, or semianalytical, studies[26–28] (the RL case has been recently considered in Ref. 20) relied on the so-called "free run approximation," i.e., the phases are taken to be rapidly varying and independent and can be averaged out (see Appendix A). This turns out into removing all phase-dependent terms in Eq. (24) and the resulting equations determine the amplitudes A_s, and hence the energy into each mode $\mathcal{E}_s$, which stays clamped at this value after that the corresponding mode has been put into oscillation. As far as the phases can be taken as independent, the output laser signal displays small oscillations around an equilibrium value because the noises into each mode amplitude are independent. It is clear that in this approximation the combination tone terms in Eq. (24) will disappear due to the averaging over the phases. However, since the beginning[26,28] (and later also confirmed by detailed numerical investigations[48]) it has been known that this regime holds as far as beating between modes due to the mode combination tones, being negligible; and this is valid if a few modes with nonoverlapping resonances are excited. Conversely, the mode combination terms are known to be responsible of "mode locking" processes that in standard laser provide a fruitful approach to the generation of ultrashort pulses.[42,44]

Gain (described by an amplification coefficient γ_s) and radiation losses (measured by α_s) are included in the equation of motion for the complex amplitudes following a standard approach:[44]

$$\frac{da_s}{dt} = -\frac{1}{2}\sum_{pqr} g_{spqr} a_q a_r a_p^* + (\gamma_s - \alpha_s)a_s + \eta_s(t), \qquad (26)$$

having introduced, as usual, a complex noise term, mainly due to spontaneous emission (see, e.g., Refs. 50 and 51), with $\langle \eta_p(t)\eta_q(t')\rangle = \langle \eta_p^*(t)\eta_q^*(t')\rangle = 0$ and $\langle \eta_p(t)\eta_q^*(t')\rangle = 2k_B T_{bath}\delta_{pq}\delta(t-t')$, with k_B the Boltzmann constant and T_{bath} an effective temperature, whose expression will be reported in a later section. In Eq. (26), the sum has been ex-

tended over all the modes, as discussed above, and the contribution of each possible combination tone to the amplitude a_s is given by the relevant coupling coefficient g_{spqr}.

The tensor g is a quantity symmetric with respect to the exchange of $s \leftrightarrow p$, $q \leftrightarrow r$, while under $\{s,p\} \leftrightarrow \{q,r\}$ one has $g_{spqr} = g^*_{qrsp}$, see Eq. (23) and Refs. 28 and 47. Introducing the real-valued potential function

$$H = \frac{1}{4}\mathrm{Re}\left[\sum_{spqr} g_{spqr} a_q a_r a_p^* a_s^*\right] = \frac{1}{4}\sum_{spqr} g^R_{spqr} a_q a_r a_p^* a_s^* - \frac{1}{4i}\sum_{spqr} g^I_{spqr} a_q a_r a_p^* a_s^*, \quad (27)$$

and letting $\mathcal{H} = \Sigma_s(\alpha_s - \gamma_s)|a_s|^2 + H$, the resulting model (26) is rewritten as

$$\frac{da_s}{dt} = -\frac{\partial \mathcal{H}}{\partial a_s^*} + \eta_s(t), \quad (28)$$

where

$$\frac{\partial}{\partial a^*} = \frac{1}{2}\left[\frac{\partial}{\partial a^R} + i\frac{\partial}{\partial a^I}\right]. \quad (29)$$

The previous equation can be cast in the form of a standard Langevin equation for a system of N "particles" moving in $2N$ dimensions (represented by $\{a_s^R, a_s^I\}_{s=1,\ldots,N}$)[36,51] and its invariant measure is given by $\exp(-\mathcal{H}/k_B T_{bath})$.

The simplest case is attained when g can be taken as real valued. Indeed, considering Lamb theory for a two level system,[26] which is the only approach providing an explicit expression for the susceptibility tensor χ, one can show that the imaginary part of g vanishes as all the resonant frequencies are packed around a given value ω_0. The generalization to a complex g is discussed in Appendix A.

Finally, the phases φ_s can be taken as the relevant dynamic variables, due to the quenched approximation for the amplitudes A_s, see Appendix A, and the Hamiltonian is written as

$$\mathcal{H}(G,\varphi) = \mathcal{H}_o + \sum_{\{sp\},\{qr\}} G_{spqr}\cos(\varphi_s + \varphi_p - \varphi_q - \varphi_r), \quad (30)$$

where $\mathcal{H}_o = \Sigma_s(\alpha_s - \gamma_s)A_s^2$ is an irrelevant constant term (as long as the amplitudes A_s are constant) and $G_{spqr} = 2g_{spqr}A_s A_p A_q A_r$ is the real-valued coupling. Note that the couplings G_{spqr} are symmetric under internal permutations of the sets $\{s,p\}$ and $\{q,r\}$ and also under exchange $\{s,p\} \leftrightarrow \{q,r\}$. Indeed the couplings have the same symmetry of the interaction term $\cos(\varphi_s + \varphi_p - \varphi_q - \varphi_r)$. To count each term only once, the sum $\Sigma_{\{sp\},\{qr\}}$ in Eq. (30) has been restricted only to the values of $spqr$ which are not related by the symmetries above, and correspondingly a factor of 8 has been added in the coupling.

Hereafter we will consider these G coefficients as "quenched" (due to the slow t dependence of A_s), and the relevant phase space is reduced to that spanned by φ_s. The pump energy which controls the average energy into each mode (and hence the amplitudes A_s) fixes the amplitude of G.

C. Gaussian random couplings

If the cavity is realized by a random medium, as described above, the coupling coefficients g are random variables, i.e., they will depend on the specific sample one is considering. For a given cavity realization, the values of the coupling coefficients g are determined by the specific nonlinear mechanism, i.e., by the function χ in Eq. (23), the mode frequencies ω_s and amplitudes A_s, and by the mode profiles $E_s(\mathbf{r})$ as expressed in Eq. (25). All these are sources of randomness in the computation of g. However, as we cannot compute the properties of the model for a specific choice of the g, we will assume that the couplings are drawn from a given probability distribution and compute average properties of the system with respect to this distribution. It is possible to show that, in the thermodynamic limit, these average quantities will be representative of many of the properties of a given sample (e.g., the free energy), see the discussion in the next sections and Ref. 40.

Given the fact that the fields are real-valued functions with positive and negative values at each point in the medium, and additionally, parity of the modes may eventually make some coupling vanish, a possible choice, in order to simplify the problem as much as possible, is to consider the signs of these coupling coefficients (those corresponding to the mode-combination tones) as random and treat them as Gaussian independent variables with zero mean, as detailed below. This choice is further supported by the fact that, in general, the mode frequencies are symmetrically distributed with respect to the resonant frequency ω_0, and correspondingly the sign of the nonlinear susceptibility largely varies.[47] The hypothesis of zero mean can be removed by generalizing the treatment reported below,[40] leading to a very rich phase diagram;[52] different distributions of the couplings can also be investigated but the problem becomes more difficult.

Additionally, it is important to point out the scaling properties of the Hamiltonian (30). Recalling that $E_s^\alpha = O(V^{-1/2})$ (due to the normalization) and $\chi_{\alpha\beta\gamma\delta} = O(1)$ are random variables, one has, from Eq. (25), $g \sim V^{-2}\int_V R(\mathbf{r})dV \sim V^{-3/2}$, as $R(\mathbf{r}) = V^2 \chi EEEE$ is an $O(1)$ random variable whose integral scales as $V^{1/2}$. The coupling G's then scale as $\langle A^2\rangle^2 g_0 V^{-3/2}$. By a simple rescaling, the invariant measure can be written as $\exp[-\beta H(J,\varphi)]$, where $J_{spqr} = G_{spqr}/(g_0\langle A^2\rangle^2)$ has standard deviation $1/V^{3/2} \propto 1/N^{3/2}$, as the number of modes is proportional to the volume of the cavity, see, e.g., Ref. 18: then, conventionally we will set $\langle J^2\rangle = 8/N^3$ and include all the system-dependent constants in the definition of β. Note that this scaling of the J's guarantees that the Hamiltonian is extensive, i.e., the average energy is proportional to volume.[40,53] The parameter that controls the phase transition is then $\beta = 1/T = \mathcal{P}^2/k_B T_{bath}$, where $\mathcal{P}^2 = \langle A^2\rangle^2 g_0$. We recall that $\omega_0\langle A^2\rangle$ measures the average energy per mode, while g_0 is a material-dependent constant. Then $\mathcal{P}$ is proportional to the energy stored on average into each mode. Hence the relevant parameter for the lasers model is the adimensional

"temperature" T: lowering T can be obtained both lowering the bath temperature T_{bath} (e.g., acting on the noise, as done in recent experiments on the thermodynamics of standard lasers[37]) or increasing "the pumping rate" $\mathcal{P}$.

IV. REPLICA ANALYSIS OF THE MODEL

Our interest here is to draw a mean-field statistical description for random lasers, which enables one to go beyond the "free run approximation" and unveil the complex structures of the states of these systems, due to "random mode-locking" processes. We can do this by computing the thermodynamic properties of the Hamiltonian (30) describing the stationary states of the system.

Summing up, the random laser studied in the above sections is described by the disordered mean-field Hamiltonian

$$H = \sum_{\{pr\},\{sp\}} J_{spqr} \cos(\varphi_s + \varphi_p - \varphi_q - \varphi_r), \tag{31}$$

where $\{\varphi_s\}$ are angular variables, $\varphi_s \in [0, 2\pi)$, and J_{spqr} are independent Gaussian random variables with zero mean and variance $\overline{J^2} = \sigma_J^2 = 8/N^3$. The sum in Eq. (31) is restricted to the values of $spqr$ that are not related by the symmetries of the interaction term, see the discussion after Eq. (30). Our purpose is to study the thermodynamics of this model. In particular we are interested in average properties of the model considering the variables J's as *quenched*: that means we will average the free energy, and not the partition function, over the distribution of the couplings: averaging the partition function correspond to considering the J's as dynamical variables evolving on the same time scale of the phases. The average of the free energy can be done by means of the replica trick.[40] Here we report the calculation in full detail, even if it is very similar to the replica calculation for the p-spin model, see, e.g., Ref. 53.

A. Replicated partition function

The partition function is

$$Z_N(\beta, J) = \int d\{\varphi\} e^{-\beta H(J,\varphi)}, \tag{32}$$

with free energy

$$f_N(\beta, J) = -\frac{T}{N} \ln Z_N(\beta, J). \tag{33}$$

We want to calculate the free energy averaged over the disorder

$$-\beta f(\beta) = \lim_{N\to\infty} \frac{1}{N} \overline{\ln Z_N(\beta, J)} = \lim_{N\to\infty} \lim_{n\to 0} \frac{\overline{[Z_N(\beta, J)]^n} - 1}{nN}, \tag{34}$$

where the overbar denotes the average over the random coupling J's and, as usual in the replica method, one uses the formula $\ln Z = \lim_{n\to 0}(Z^n - 1)/n$, introducing the partition function of n independent replicas $Z^n(J) \equiv [Z_N(\beta, J)]^n$ with the same random couplings J's. It is possible to show[40] that the free energy is *self-averaging*, i.e., one has

$$\lim_{N\to\infty} f_N(\beta, J) = f(\beta) \tag{35}$$

with probability one with respect to the distribution of the couplings J's; in other words, in the thermodynamic limit the free energy of a given sample is given, with probability 1, by the average of the free energy over the disorder that we are able to compute.

In the following we will neglect all the multiplicative constants growing as powers of N in the partition function as they do not contribute to the free energy. The J's have distribution $P(J) = \sqrt{N^3/16\pi} \exp(-J^2 N^3/16)$: by the relation

$$\int dJ P(J) e^{AJ} = \exp\left[\frac{4}{N^3} A^2\right], \tag{36}$$

for integer n, one has

$$\overline{Z^n(J)} = \int \left(\prod_{a=1}^{n} d\{\varphi^a\}\right) e^{\beta^2/2 H_{eff}(\varphi_i^a)}, \tag{37}$$

with

$$\begin{aligned} H_{eff} &= \frac{8}{N^3} \sum_{a,b} \sum_{\{sp\},\{qr\}} \cos(\varphi_s^a + \varphi_p^a - \varphi_q^a - \varphi_r^a) \\ &\quad \times \cos(\varphi_s^b + \varphi_p^b - \varphi_q^b - \varphi_r^b) \\ &= \frac{8N^{-3}}{2\left(\frac{4}{2!}\right)^2} \sum_{a,b} \sum_{spqr}^{1,N} \cos(\varphi_s^a + \varphi_p^a - \varphi_q^a - \varphi_r^a) \\ &\quad \times \cos(\varphi_s^b + \varphi_p^b - \varphi_q^b - \varphi_r^b) = \frac{N}{2} \sum_{a,b} [|Q_{ab}|^4 + |R_{ab}|^4], \end{aligned} \tag{38}$$

where in the second line the constraints on the sets $\{s, p\}$, $\{q, r\}$ have been released. In the above expression we have introduced the quantities:

$$Q_{ab} = N^{-1} \sum_i e^{i(\varphi_i^a - \varphi_i^b)}, \tag{39}$$

$$R_{ab} = N^{-1} \sum_i e^{i(\varphi_i^a + \varphi_i^b)}. \tag{40}$$

Note that $Q_{aa} \equiv 1$ and $Q_{ba} = Q_{ab}^*$, while $R_{ba} = R_{ab}$. Also note that the effective Hamiltonian H_{eff} depends only on the global variables Q_{ab} and R_{ab}.

Using the notation $\delta(z) = \delta(z^R)\delta(z^I)$, the partition function can be written as

$$\overline{Z^n(J)} = \int \prod_{a>b} dq_{ab} \prod_{a\geqslant b} dr_{ab} e^{\beta^2/2 H_{eff}(q_{ab},r_{ab})} \times \int \prod_{a=1}^{n} d\{\varphi^a\} \prod_{a>b} \delta(q_{ab} - Q_{ab}) \prod_{a\geqslant b} \delta(r_{ab} - R_{ab}), \tag{41}$$

and the second integral can be easily evaluated introducing the integral representation for the complex δ-function

$$\delta(z) = \int \frac{d\lambda}{(2\pi)^2} e^{\mathrm{Re}(z\lambda^*)}, \quad d\lambda = d\lambda^R d\lambda^I \tag{42}$$

and the integral is done on the imaginary axis of the complex λ^R, λ^I planes (i.e., one has to consider both λ^R and λ^I as complex numbers). With some algebra one gets, with the convention that $[ab] \to a \geqslant b$ and $(ab) \to a > b$, and summing over the repeated indexes,

$$\int d\{\varphi^a\} \prod_{(ab)} \delta(q_{(ab)} - Q_{(ab)}) \prod_{[ab]} \delta(r_{[ab]} - R_{[ab]}) = \int d\lambda_{(ab)} d\mu_{[ab]} \exp[N \,\mathrm{Re}(\lambda^*_{(ab)} q_{(ab)} + \mu^*_{[ab]} r_{[ab]}) + N \ln Z(\lambda_{(ab)}, \mu_{[ab]})], \tag{43}$$

where

$$Z(\lambda_{(ab)}, \mu_{[ab]}) = \int d[\varphi^a] \exp[-\mathrm{Re}(\lambda^*_{(ab)} e^{i(\varphi^a - \varphi^b)} + \mu^*_{[ab]} e^{i(\varphi^a + \varphi^b)})] \tag{44}$$

(note the difference between $d\{\varphi^a\} = \Pi_{a,i} d\varphi^a_i$ and $d[\varphi^a] = \Pi_a d\varphi^a$). The partition function then has the form

$$\overline{Z^n(J)} = \int dq_{(ab)} d\lambda_{(ab)} dr_{[ab]} d\mu_{[ab]} \exp[-N h(q,\lambda,r,\mu)], \tag{45}$$

where the function h is given by

$$h(q,\lambda,r,\mu) = -\frac{\beta^2}{4}\left[\sum_a |r_{aa}|^4 + n + 2\sum_{(ab)} [|q_{ab}|^4 + |r_{ab}|^4]\right] - \mathrm{Re}(\mu^*_{aa} r_{aa} + \lambda^*_{(ab)} q_{(ab)} + \mu^*_{(ab)} r_{(ab)}) - \ln Z(\lambda_{(ab)}, \mu_{[ab]}). \tag{46}$$

We have extracted the diagonal part in the effective Hamiltonian reminding that $q_{aa} \equiv 1$ is fixed (this is why we did not include the relative δ-function).

The integral (45) can be evaluated at the saddle-point. The derivatives with respect to q and r yield

$$\lambda_{(ab)} = -2\beta^2 |q_{(ab)}|^2 q_{(ab)}, \quad \mu_{aa} = -\beta^2 |r_{aa}|^2 r_{aa}, \quad \mu_{(ab)} = -2\beta^2 |r_{(ab)}|^2 r_{(ab)}. \tag{47}$$

Substituting these equations into Eq. (46), h is

$$h(q,r) = -\frac{n\beta^2}{4} + \frac{3\beta^2}{4}\left[\sum_a |r_{aa}|^4 + 2\sum_{(ab)} |q_{ab}|^4 + |r_{ab}|^4\right] - \ln Z(q_{(ab)}, r_{[ab]}). \tag{48}$$

To perform the analytic continuation to $n \to 0$ one has to make an *ansatz* on the structure of the matrices q and r. The free energy is then computed using Eq. (34). Using the relation

$$\lim_{N\to\infty} \overline{Z^n} \sim e^{-N \min[h(q,r)]} \tag{49}$$

and assuming that $\min[h(q,r)] \sim n$ (i.e., it is small) we have for the free energy

$$\beta f = -\lim_{n\to 0} \frac{e^{-N \min[h(q,r)]} - 1}{nN} \sim \lim_{n\to 0} \frac{N \min[h(q,r)]}{nN} = \min[\lim_{n\to 0} n^{-1} h(q,r)] = \min[\beta\phi(q,r)]. \tag{50}$$

Note that the limits $n \to 0$ and $N \to \infty$ have been exchanged.[40] A reasonable *ansatz* that we will make is that $r_{ab} \equiv 0$ at the saddle point. Indeed, we are looking for disordered states that are usually characterized by a nonzero overlap q and a vanishing magnetization, i.e., the rotational symmetry is not broken. As r is not invariant under rotations, we will set $r = 0$ in the following. Then we have to minimize

$$\beta\phi(q) = -\frac{\beta^2}{4} + \frac{3\beta^2}{2n} \sum_{a>b} |q_{ab}|^4 - n^{-1} \ln Z(q),$$

$$Z(q) = \int d\varphi^a \exp\left[\mathrm{Re} \sum_{a>b} 2\beta^2 |q_{ab}|^2 q^*_{ab} e^{i(\varphi^a - \varphi^b)}\right], \tag{51}$$

for $n \to 0$. This function is very similar to the one that describes the Ising p-spin glass.

The vanishing of the derivative with respect to q_{ab} of $\beta\phi(q)$ gives the saddle point equation

$$q_{ab} = \langle e^{i(\varphi^a - \varphi^b)} \rangle, \tag{52}$$

where the average is on the measure that defines $Z(q)$. Indeed, performing the derivative with respect to the real and imaginary part of q_{ab}, one obtains two equations that can be written in a single saddle point equation for q_{ab}

$$3q_{ab} = \langle e^{i(\varphi^a - \varphi^b)} \rangle + 2\frac{q_{ab}}{|q_{ab}|^2} \mathrm{Re}\, q^*_{ab} \langle e^{i(\varphi^a - \varphi^b)} \rangle. \tag{53}$$

It is easy to show that Eq. (52) is the solution of Eq. (53).

The replica symmetric (RS) solution corresponds to $q_{ab} \equiv q$; in particular $q = 0$ is a solution of the saddle point equations and is the stable one in the high temperature phase. Another solution appears at low temperatures but is always unstable (see below). The RS free energy is simply $f_{RS} = -\beta/4$ as in the Ising p-spin glass (neglecting irrelevant constants).

B. One step replica symmetry breaking (1RSB)

The 1RSB *ansatz* is the following: we divide the matrix q_{ab} in n/m blocks of side m. The elements in the off-diagonal

blocks are set to 0 while in the diagonal blocks RS is assumed and $q_{ab}=q$. The simplest choice is to assume that q is real. This is very reasonable due to the rotational symmetry, see, e.g., the discussion in Ref. 54, p. 894, and moreover in this way the constraint $q_{ab}=q_{ba}^{*}$ is respected. For instance, we have for $n=6$ and $m=3$

$$(q_{ab})=\begin{pmatrix} \begin{pmatrix} 1 & q & q \\ q & 1 & q \\ q & q & 1 \end{pmatrix} & 0 \\ 0 & \begin{pmatrix} 1 & q & q \\ q & 1 & q \\ q & q & 1 \end{pmatrix} \end{pmatrix}. \tag{54}$$

Then one has

$$\lim_{n\to 0} n^{-1}\sum_{a>b} q_{ab}^{4}=\frac{1}{2}(m-1)q^{4}, \tag{55}$$

and, as the variables φ^a in different blocks become uncorrelated,

$$\begin{aligned} Z(q) &= \left[\int \prod_{a=1}^{m} d\varphi^a e^{\beta^2 q^3 \Sigma_{a\neq b} e^{i(\varphi^a-\varphi^b)}}\right]^{n/m} \\ &= \left[\int \prod_{a=1}^{m} d\varphi^a e^{\beta^2 q^3 [|\Sigma_a e^{i\varphi^a}|^2 - m]}\right]^{n/m} \\ &= \left[e^{-m\beta^2 q^3}\int \prod_{a=1}^{m} d\varphi^a \int \mathcal{D}\zeta e^{\beta\sqrt{2q^3}\ \mathrm{Re}\ \zeta^* \Sigma_a e^{i\varphi^a}}\right]^{n/m}, \end{aligned} \tag{56}$$

where ζ is a complex variable and $\mathcal{D}\zeta=\frac{d\,\mathrm{Re}\,\zeta\, d\,\mathrm{Im}\,\zeta}{2\pi}e^{-1/2\zeta\zeta^*}$. Defining also $\lambda=\sqrt{2q^3}$ one has

$$n^{-1}\ln Z(q) = -\beta^2 q^3 + m^{-1}\ln\int \mathcal{D}\zeta\left(\int d\varphi e^{\beta\lambda\ \mathrm{Re}\ \zeta^* e^{i\varphi}}\right)^{m}. \tag{57}$$

Introducing the modified Bessel function of the first kind of order 0

$$I_0(\beta\lambda|\zeta|)=\frac{1}{2\pi}\int_0^{2\pi} d\varphi e^{\beta\lambda\ \mathrm{Re}\ \zeta^* e^{i\varphi}}, \tag{58}$$

and noting that it depends on the modulus $z=|\zeta|$ one finally gets (apart from constant terms)

$$\begin{aligned} \beta\phi_{1RSB}(m,T) &= -\frac{\beta^2}{4}[1+3(1-m)q^4-4q^3] \\ &\quad -\frac{1}{m}\ln\int_0^{\infty}\mathcal{D}z I_0^m(\beta\lambda z), \end{aligned} \tag{59}$$

where $\mathcal{D}z=ze^{-z^2/2}dz$. The value of q is determined by the condition $\partial_q\phi_{1RSB}=0$ that gives (see Appendix B for details)

$$q=\frac{\int_0^{\infty}\mathcal{D}z I_0^m(\beta\lambda z)\left[\frac{I_1(\beta\lambda z)}{I_0(\beta\lambda z)}\right]^2}{\int_0^{\infty}\mathcal{D}z I_0^m(\beta\lambda z)}, \tag{60}$$

where $I_1(x)=I_0'(x)$ is the modified Bessel function of order 1. This expression is similar to the 1RSB free energy for the p-spin model with $p=4$, the only difference being the presence of the Bessel functions instead of the hyperbolic cosine in the integrals, the domain of integration in z, and a z in the integrand.

The equilibrium value of m is the solution of $\partial_m\phi_{1RSB}=0$. At high temperature the solution $q=0$ and $m=1$ (*paramagnetic state*) is the stable one, while for $T<T_c$ a new solution with $q\neq 0$ and $m<1$ (*spin glass*) becomes stable. The temperature T_c (also called Kauzmann temperature T_K) marks the appearance of the thermodynamic glassy phase.

C. Phase space structure of the model

Starting from Eq. (59) one can repeat the analysis of Ref. 55 to derive the full phase space structure of the model at the 1RSB level. Again, we will reproduce in some details the original derivations for the reader who is not familiar with these methods.

In this class of mean field disordered models, at low temperature, the phase space in disconnected in many *metastable states*, i.e., local minima of the free energy. The number of states of given free energy density f is $\Omega(f)=\exp N\Sigma(f)$. The function $\Sigma(f)$ vanishes continuously at $f=f_{min}$ and drops to zero above $f=f_{max}$ (see, e.g., Ref. 53). The main peculiarity of these models is that an *exponential number* of metastable states is present at low enough temperature.

One can write the partition function Z, at low enough temperature and for $N\to\infty$, in the following way:

$$Z=e^{-\beta N F(T)}\sim\sum_{\alpha}e^{-\beta N f_\alpha}=\int_{f_{min}}^{f_{max}}df e^{N[\Sigma(f)-\beta f]}\sim e^{N[\Sigma(f^\star)-\beta f^\star]}, \tag{61}$$

where $f^\star\in[f_{min},f_{max}]$ is such that $\Phi(f)=f-T\Sigma(f)$ is minimum, i.e., it is the solution of

$$\frac{d\Sigma}{df}=\frac{1}{T}, \tag{62}$$

provided that it belongs to the interval $[f_{min},f_{max}]$. Starting from high temperature, one encounters three temperature regions.

(1) For $T>T_d$, the free energy density of the paramagnetic state is smaller than $f-T\Sigma(f)$ for any $f\in[f_{min},f_{max}]$, so the paramagnetic state dominates [in this region the decomposition (61) is meaningless].

(2) For $T_d\geqslant T\geqslant T_c$, a value $f^\star\in[f_{min},f_{max}]$ is found, such that $f^\star-T\Sigma(f^\star)$ is equal to f_{para}. This means that the paramagnetic state is obtained from the superposition of an *exponential number* of states of *higher* individual free energy density $f^\star$. The phase space is disconnected in this exponen-

tial number of regions: however, no phase transition happens at T_d because of the equality $f^\star - T\Sigma(f^\star) = f_{para}$ which guarantees that the free energy is analytic on crossing T_d.

(3) For $T < T_c$, the partition function is dominated by the lowest free energy states, $f^\star = f_{min}$, with $\Sigma(f_{min}) = 0$ and $F(T) = f_{min} - T\Sigma(f_{min}) = f_{min}$. At T_c a phase transition occurs, corresponding to the one-step replica symmetry breaking transition found in the replica computation.

In the range of temperatures $T_d > T > T_c$, the phase space of the model is disconnected in an exponentially large number of states, giving a contribution $\Sigma(T) \equiv \Sigma[f^\star(T)]$ to the total entropy of the system. This means that the entropy per particle $S(T)$ for $T_d > T > T_c$ can be written as

$$S(T) = \Sigma(T) + S_{vib}(T), \tag{63}$$

$S_{vib}(T)$ being the individual entropy of a state of free energy $f^\star$. The task is then to compute the function $\Sigma(f)$ at fixed T and the *equilibrium complexity* $\Sigma(T) = \Sigma[f^\star(T)]$.

To this aim, the idea[56] is to consider m copies of the original system, coupled by a small attractive term added to the Hamiltonian. The coupling is then switched off after the thermodynamic limit has been taken. For $T < T_d$, the small attractive coupling is enough to constrain the m copies to be in the same state. At low temperatures, the partition function of the replicated system is then

$$Z_m = e^{-\beta N \Phi(m,T)} \sim \sum_\alpha e^{-\beta N m f_\alpha} = \int_{f_{min}}^{f_{max}} df e^{N[\Sigma(f) - \beta m f]} \sim e^{N[\Sigma(f^\star) - \beta m f^\star]}, \tag{64}$$

where now $f^\star(m,T)$ is such that $\Phi(m,f) = mf - T\Sigma(f)$ is minimum and satisfies the equation

$$\frac{d\Sigma}{df} = \frac{m}{T}. \tag{65}$$

If m is allowed to assume real values by an analytical continuation, the complexity can be computed from the knowledge of the function $\Phi(m,T) = mf^\star(m,T) - T\Sigma[f^\star(m,T)]$. Indeed, it is easy to show that

$$f^\star(m,T) = \frac{\partial \Phi(m,T)}{\partial m},$$

$$\Sigma(m,T) = \Sigma(f^\star(m,T)) = m^2 \frac{\partial [m^{-1}\beta\Phi(m,T)]}{\partial m} = m\beta f^\star(m,T) - \beta\Phi(m,T). \tag{66}$$

Thus the function $\Sigma(f)$ can be reconstructed from the parametric plot of $f^\star(m,T)$ and $\Sigma(m,T)$ by varying m at fixed temperature. The equilibrium complexity is simply $\Sigma(T) = \Sigma(m=1,T)$.

Using the replica trick to compute the free energy,

$$\Phi(m,T) = -\frac{T}{N}\overline{\log Z_m} = -\frac{T}{N}\lim_{n\to 0}\frac{\overline{(Z_m)^n} - 1}{n} = -\frac{T}{N}\lim_{n\to 0}\frac{\overline{Z^{mn}} - 1}{n}, \tag{67}$$

one obtains the partition function of nm copies of the system, with the constraint that each block of m replicas has to be in the same state, i.e., the replicas must have nonzero overlap. This leads naturally to the 1RSB structure for the overlap matrix (with m fixed), see Eq. (54), and

$$\Phi(m,T) = -\frac{T}{N}\lim_{n\to 0}\{\exp[-\beta n m N \phi_{1RSB}(m,\bar{q},T)] - 1\}/n = m\phi_{1RSB}(m,T). \tag{68}$$

Note that the hypothesis that the m replicas are in the same state implies that for any value of (m,T) one has to substitute in ϕ_{1RSB} the nonzero solution of Eq. (60), $q^\star(m,T)$. Above T_d this solution disappears as a vanishing coupling cannot constrain the replicas to stay close to each other.

Using Eqs. (66) and (68) the complexity as a function of m is

$$T\Sigma(m,T) = m^2\partial_m[m^{-1}\Phi(m,T)] = m^2\partial_m\phi_{1RSB}(m,q^\star,T), \tag{69}$$

and the equilibrium complexity is

$$\Sigma(T) = \Sigma(1,T) = -\frac{3\beta^2(q^\star)^4}{4} + \ln\int_0^\infty \mathcal{D}z I_0(\beta\lambda^\star z) - \frac{\int_0^\infty \mathcal{D}z I_0(\beta\lambda^\star z)\ln I_0(\beta\lambda^\star z)}{\int_0^\infty \mathcal{D}z I_0(\beta\lambda^\star z)}, \tag{70}$$

where $\lambda^\star = \sqrt{2(q^\star)^3}$.

D. Phase diagram in the (m,T) plane

For a given value of T, we can identify four relevant values of m. They are reported in Fig. 1 and are defined as follows.

(1) A solution $q^\star(m,T) \neq 0$ of Eq. (60) is present for $m \geqslant m_{min}(T)$. Thus in the region $m \geqslant m_{min}(T)$, $\Phi(m,T)$ is well-defined and we can compute the complexity $\Sigma(m,T)$ and the free energy $f^\star(m,T)$ using Eq. (66).

(2) The value $m^\star(T)$ such that $\Sigma(m,T) = 0$ corresponds to the solution of the thermodynamics and $f^\star(m^\star,T) = \phi_{1RSB}(m^\star,T)$ is the free energy in the spin glass phase. The temperature T_c is defined by $m^\star(T_c) = 1$.

(3) The function $f^\star(m,T)$ has a maximum for $m = m_d(T)$. This means that $f^\star(m_d,T)$ is the maximum possible free energy $f_{max}(T)$. The states with $f = f_{max}$ are called *threshold states*.[53,55]

(4) Finally, one can investigate the stability of the 1RSB solution with respect to further steps of replica symmetry breaking, following the analysis of Ref. 57. We report the

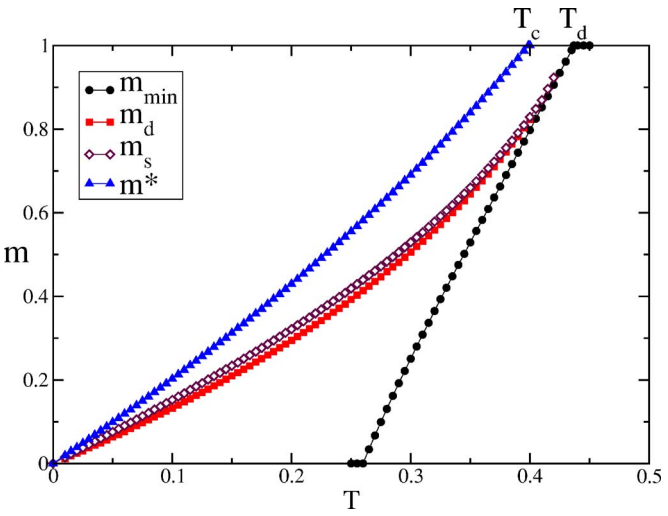

FIG. 1. (Color online) Phase diagram of the model in the (m,T) plane (see Sec. IV D in the text).

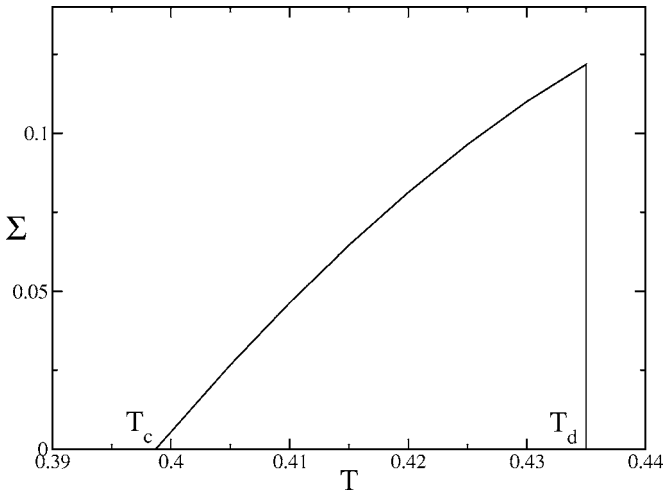

FIG. 2. The equilibrium complexity $\Sigma(T)$ as a function of the temperature.

details of the calculation in Appendix C. It turns out that the 1RSB solution is stable toward further steps of replica symmetry breaking for $m \geqslant m_s(T)$.

The thermodynamic phase diagram of the model is easily deduced from Fig. 1. The paramagnetic solution is stable for $T > T_c$. The temperature T_c is the thermodynamic glass transition temperature, at which a downward jump of the specific heat is observed, as in standard first-order transition. Below this temperature the 1RSB spin glass solution is stable, and remains such down to $T=0$, as $m^\star(T) > m_s(T)$ for all T. Thus in this model no Gardner transition (transition to a full RSB solution[57]) is observed.

Some information on the dynamics of the model can also be obtained from Fig. 1. Indeed, at T_d [defined by $m_d(T)=1$] a *dynamical transition* takes place:[53] correlation functions are expected to develop an infinitely long *plateau* and the system becomes dynamically trapped in one of the exponentially large number of states that appears at T_d, as discussed above.

Between the static transition temperature T_c and the dynamic one T_d the phase space has a nontrivial shape: it is disconnected in an exponential number of states $\mathcal{N}(T) = \exp N\Sigma(T)$, where Σ is the configuration entropy (or complexity) of the system. In Fig. 2 the quantity Σ is reported as a function of T. The point T_c at which the complexity vanishes $\Sigma(T_c)=0$ signals the appearance of the thermodynamic phase transition.

However, it is not clear what is the dynamic behavior of the model if quenched from $T=\infty$ to $T<T_d$. Indeed, as already observed in the Ising p-spin glass model, the threshold states always lie into the 1RSB-unstable region, i.e., $m_d < m_s$ for $T<T_d$. This means that the 1RSB ansatz is unable to give the correct prediction for these states below T_d and one need to consider further steps of replica symmetry breaking. The investigation of the dynamics of the model after a quench below T_d will be the object of future investigation.

V. ROUTE TO THE EXPERIMENTAL OBSERVATION OF THE GLASSY TRANSITION

The investigated model exhibits a dynamical transition at T_d and a thermodynamic phase transition at a lower temperature T_c (characterized by a one-step replica symmetry breaking scenario), very similar to p-spin glass models. Let us turn our attention to the physical interpretation of these transitions. We recall that the "temperature" T introduced in the model is defined as the ratio between the *bath* temperature T_{bath}, measuring the optical noise in the system, and the pumping rate $\mathcal{P}$, so we can vary the latter to explore the phase diagram of the system.

Starting from low pump intensity (high temperature) and increasing $\mathcal{P}$ (decreasing T), different interesting phenomena take place. The relevant quantities to look at in experiments are correlation functions in time domain, i.e., the self-correlation functions of a specific frequency (ω_m) component of the electric field in the cavity (for example, via heterodyne experiments, see below):

$$C(t,\omega_m) = \langle a_m(t+\tau)a_m^*(\tau)\rangle_\tau = A_m^2\langle \exp\{i[\varphi_m(t+\tau) - \varphi_m(\tau)]\}\rangle_\tau, \tag{71}$$

where the $\langle\cdots\rangle_\tau$ is the average over the time origin τ. At high T, because of the fast dynamics of the phases, $C(t,\omega_m)$ decays to zero on short times. On lowering T the dynamics of phase variable $\varphi_m(t)$ becomes slower and slower and $C(t,\omega_m)$ is expected to decay towards zero in longer and longer times. At the dynamic transition point T_d, the dynamics of the φ's becomes nonergodic, they are no longer able to explore the whole phase space and the function $C(t,\omega_m)$ decays towards a plateau: the mode's phases $\varphi_m(t)$ are locked to some fixed random values ("random mode-locking") and oscillate around these values. The fact that the complexity Σ is different from zero at the dynamic transition point implies that there is an exponentially large number of possible values for the locked phases and, correspondingly, many different time structures of the electric field in the random laser.

More technically, the region between T_d and T_c is dynamically not-accessible, due to the mean-field character of the interactions. Only for short range models, where activated processes become possible, this interesting region can be explored. We can expect that, if the mean-field approximation is not fully verified by the random lasers, see the discussion in Sec. III A, the system would be able to enter in the *activated* (or Vogel-Fulcher) regime (to use a terminology familiar in the liquid-glassy physics) and the correlation functions decay to zero at long times after the plateau. In this region the relaxation time is found to scale as $\tau=\tau_0 \exp[D/(T-T_c)]$ and only at the thermodynamic phase transition point T_c the system is really locked in the *ideal* glassy state (the relaxation time becomes infinite). Similarly to what happens in p-spins and structural glasses, interesting phenomena as aging, memory effects, and history dependent responses are expected to take place for $T_c<T<T_d$, see, e.g., Refs. 49, 58, and 59 for recent reviews.

It is worth remarking that in the region where these phenomena are expected to happen, the relaxation time of the system is larger, by many orders of magnitude, than the typical microscopic time scales. For instance, in molecular glasses where the typical time scale is $\tau_0 \sim 10^{-12}$ s, the relaxation time of the system can be as large as 100 s already for $T_g \sim 1.5 T_c$. These systems cannot be equilibrated close to T_c because of the exponential divergence of the relaxation time for $T \to T_c$. In random lasers, the typical time scale for photon dynamics is $\tau_0 \sim 10^{-14}$ s, so one will gain orders of magnitude in time. This should allow one to equilibrate the system closer to T_c. Even if the decrease in $T-T_c$ that one can achieve in this way will not be substantial, it could be enough to test the predictions of some recent theories of the glass transition in the presence of short range interactions.[49,58]

Moreover, we recall that, as discussed in Sec. III A, in random lasers it is possible, in principle, to tune the interaction range (by tuning the localization length), and thus the relevance of the activated processes. This is similar to what has been done theoretically by considering the Kac limit for spin glasses[60] and might allow one to explore the crossover between the mean field and the activated regime and to shed light on some debated issues concerning the nature of the spin glass phase in real systems. Before concluding, it is important to provide an order of magnitude estimate of physical quantities involved in the experiments and to address some possible experimental frameworks.

A. Order of magnitudes

For the sake of concreteness we will focus, as an example, on recent experiments in random lasers realized by scatterers (e.g., zinc oxide powder dispersed in a solvent doped with a dye, as, e.g., in Ref. 21). These experiments employed pumped source beams (see below), for the moment we show that the glassy transition can be expected for the currently employed pump power levels. We stress once again that our model is sufficiently general to embrace a much wider variety of disordered amplifying systems.

We start from the coupling coefficients, which are given by Eq. (25); the fields are real valued numbers, their modulus scale as $|E| \cong 2V^{-1/2}/\sqrt{\epsilon_0 n_0^2}$ due to the normalization (n_0 is an average refractive index); their sign can be "embedded" in the sign of the χ coefficients; additionally $\omega_s \propto \omega_0 \equiv 2\pi c/\lambda$. Hence, omitting indexes, $\langle g \rangle = 0$ and

$$g \simeq \frac{8\omega_0^2}{V^2 \epsilon_0^2 n_0^4} \int_V \chi(\mathbf{r}) dV. \tag{72}$$

We need $\langle g^2 \rangle$, and we assume

$$\langle \chi(\mathbf{r}) \chi(\mathbf{r}') \rangle = \chi_0^2 L_r^3 \hat{\delta}(\mathbf{r}-\mathbf{r}') \tag{73}$$

with χ_0 a typical nonlinear susceptibility value, L_r a characteristic length for the disorder, and $\hat{\delta}$ the *coarse grained* Dirac delta. Thus

$$\langle g^2 \rangle \simeq \left(\frac{\omega_0^2}{2V^2 \epsilon_0^2 n_0^4} \right)^2 \left\langle \int_V \chi(\mathbf{r}) dV \int_V \chi(\mathbf{r}') dV' \right\rangle = \left(\frac{8\omega_0^2}{V^2 \epsilon_0^2 n_0^4} \right)^2 \chi_0^2 L_r^3 V. \tag{74}$$

We have then for the standard deviation

$$\sqrt{\langle g^2 \rangle} \simeq \frac{8\omega_0^2 \chi_0 L_r^{3/2}}{\epsilon_0^2 n_0^4} \frac{1}{V^{3/2}}. \tag{75}$$

For the sake of simplicity, let us consider a box with volume V: the number of modes per unit of volume and unit of frequency is given by (this is just an estimate, as in nanostructured optical cavities the density of modes can be enhanced or depressed with respect to a standard box[2])

$$\rho(\nu) = \frac{8\pi n_0^3 \nu^2}{c^3}, \tag{76}$$

hence for N modes in a wavelength range $\Delta\lambda$ ($\lambda = c/\nu$) it is

$$N = \frac{8\pi n_0^3 \Delta\lambda}{\lambda^4} V. \tag{77}$$

Equation (77) is used in Eq. (75) and gives

$$\sqrt{\langle g^2 \rangle} \simeq \frac{8^{5/2} \pi^{3/2} n_0^{1/2} \omega_0^2 \chi_0 L_r^{3/2}}{\epsilon_0^2 n_0^4} \frac{(\Delta\lambda)^{3/2}}{\lambda^6} \frac{1}{N^{3/2}}, \tag{78}$$

which scales as $N^{-3/2}$ as anticipated. Next we have to consider the coefficients $G_{spqr} = g_{spqr} A_s A_p A_q A_r$:

$$\sqrt{\langle G^2 \rangle} \simeq \frac{8^{5/2} \pi^{3/2} \omega_0^2 \chi_0 L_r^{3/2} n_0^{1/2}}{\epsilon_0^2} \frac{(\Delta\lambda)^{3/2}}{\lambda^6} \frac{\langle A^2 \rangle^2}{N^{3/2}}, \tag{79}$$

and the coefficients $J = G/(g_0 \langle A^2 \rangle^2)$ with variances $8/N^3$. We can hence determine g_0 as

$$g_0 \cong \frac{8\pi^{3/2} n_0^{1/2} \omega_0^2 \chi_0 L_r^{3/2}}{\epsilon_0^2} \frac{(\Delta\lambda)^{3/2}}{\lambda^6} \tag{80}$$

and, finally, the adimensional $\beta = 1/T$ is given by

$$\beta \cong \frac{8\pi^{3/2}\omega_0^2\chi_0 L_r^{3/2}}{\epsilon_0^2}\frac{(\Delta\lambda)^{3/2}}{\lambda^6}\frac{\langle A^2\rangle^2}{k_B T_{bath}}. \quad (81)$$

Remembering that the average energy per mode is $\omega_0\langle A^2\rangle$, we obtain, at the transition, the threshold value (denoting $T_d \sim 0.435$ the normalized temperature at the dynamic transition, see Fig. 2)

$$\mathcal{E}_{RSB} = \omega_0\langle A^2\rangle = \sqrt{\frac{k_B T_{bath}\epsilon_0^2\lambda^6}{8\pi^{3/2}n_0^{1/2}T_d\chi_0(L_r\Delta\lambda)^{3/2}}}. \quad (82)$$

The noise temperature can be taken as due to the spontaneous emission, which is typically the dominant contribution: following Ref. 42, $2k_B T_{bath}=\hbar[N_2/(N_2-N_1)]_t/\tau \cong \hbar/\tau$, with τ the average lifetime per mode, and $[N_2/(N_2-N_1)]_t$ the population inversion at lasing threshold. Taking typical values from the reported experiments ($\Delta\lambda=100$ nm, $\lambda=630$ nm, $n_0=2$, $L_r=10$ nm, $\tau=100$ fs) and for the susceptibility $\chi_0=10^{-27}$ CmV^{-3},[47] it is $\mathcal{E}_{RSB}\cong 10^{-16}$ J. Assuming a pumping beam with peak power $P_{RSB}\cong N\mathcal{E}_{RSB}/\tau$, gives $P_{RSB}\cong 0.1$ W with $N=100$, which focused on the typical area of 100 μm^2 provides the typical values for the peak pump intensities used in the experiments ($\simeq 100$ kW/cm^2). Thus we expect that the "glass transition" can be observed within the currently available experimental framework.

B. Continuous-wave random lasers

All the theory reported in this paper makes reference to continuous-wave (cw) RLs, hence we start discussing this kind of systems. Experimental investigations of cw-RL were already reported in nanopowders (see Ref. 61 and references therein); alternative experimental geometries include disordered photonic crystals[2] as membranes or multilayered systems with gain provided, e.g., by quantum wells in semiconductor materials. In these integrated high-index contrast geometries multimode random-cavity lasers can in principle operate in the cw regime, and this opens the way to a comprehensive experimental analysis of the dynamics of the laser emission. The experimental setup can follow previous investigations of the noise figure of standard semiconductor lasers (see, e.g., Ref. 62). RL emission is collected and filtered in a narrow band in order to select one or few modes. At the mode locking transition the intensity signal $I(t)$ is expected to switch from a random noise superimposed to a cw value, to a largely modulated line-shape displaying a random sequence of disordered pulses. Heterodyne measurements are employed to extract phase and amplitude noise from which the dynamics of the amplitudes of the modes and their phase are extracted, as well as the coherence function $C(t,\omega)$. As discussed above, the fact that amplitude fluctuations make a negligible contribution to the field autocorrelation is well-known from laser theory,[62] hence $C(t,\omega)$ gives information on the phase-dynamics.

Specifically, before the glassy transition (low pumping rate) the linewidth of the laser modes is wide and the autocorrelation $C(t,\omega)$ of the mode signal is expected to decay with a single exponential time constant, corresponding to a Lorentzian line shape of the noise spectrum of the field. At the glass transition the laser dynamics slows down, and this results into a slower decaying of $C(t,\omega)$, with the appearance of multiple time scales, and eventually to an ergodicity breaking corresponding to a plateau in the $C(t,\omega)$ signal [as sketched in Fig. 3(a)].

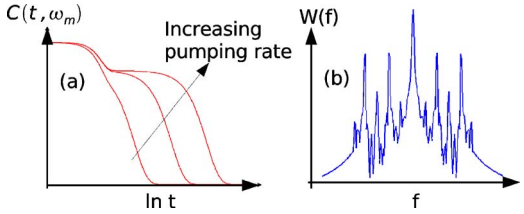

FIG. 3. (Color online) Sketch of two experimental signatures of the onset of a glassy transition of light in random lasers: (a) The mode coherence function develops a plateau at high pumping rates, denoting ergodicity breaking. (b) The power density spectrum of the mode instantaneous frequency displays modulation sidebands; this corresponds to the transition from random-mode-phases to phase-locking in one of the metastable states; the sidebands are due to small oscillations in these minima that result in frequency shifts of the mode resonances.

Further information is retrieved by phase demodulation (e.g., by employing the usual combination of a limiter and a discriminator[62]) whose output is the instantaneous frequency $d\phi_m/dt$, whose power density spectrum can be determined by a spectrum analyzer. When the phase of the filtered modes are locked, they oscillate around one of the many equilibrium values. Correspondingly, the spectrum of the phase noise for each mode is expected to switch from a wide line to a narrow one displaying modulation sidebands, which are due to the fact that the phases display small oscillations around one of the many phase-locked states [as sketched in Fig. 3(b)].

C. Pulsed random lasers and speckle patterns

Most of the reported experiments on RLs have been done by using pumped laser beams, with pulse duration from tens of picoseconds to tens of nanoseconds. One could argue if the mentioned mode-locking transition can be actually observed in these regimes.

First of all we observe that our theory deals with the phase-dynamics of the (quasi-)modes of a RL. As far as the laser reaches a steady state for the amplitudes (and this is expected to happen in the leading edge of a nano-second pump pulse, taking into account typical lifetimes; see, e.g., Ref. 20 for an extensive discussion of the mode amplitude dynamics), the phases are expected to vary on the $\tau_0 \sim 10$ fs time scale. Indeed, in the framework of the Lamb's two level theory, they are affected by the dynamics of the resonant medium polarization, i.e., by the time-constant of the off-diagonal density matrix elements, whose inverse is the "dipole dephasing rate" which is around 10^{14} s^{-1} (see, e.g., Ref. 34). The latter time scale is the "elementary" time scale for the dynamics of the phases, that corresponds in molecular systems to the typical time scale of atomic vibrations $\tau_0 \sim 10^{-12}$ s.

In a first approximation, the arrival of a pulse produces a fast variation of the "temperature" from $T\sim\infty$ (before the pulse) to a value of $T<T_d$ (subsequently after the arrival of the pulse). This corresponds, in the language of molecular glasses, to an instantaneous (i.e., on the scale of τ_0) *quench* of the system from infinite (or very high) temperature to below T_d. On a very general ground, it is expected that the system will then start to *age*[49,59] in the sense that the relaxation time τ of $C(t)$ will increase with the time t_w elapsed after the arrival of the pulse (the quench). For systems that are in the class of the p-spin model it is generally found that, for $t_w \gg \tau_0$, $\tau(t_w)\sim t_w$ (see as a striking example Fig. 5 in Ref. 63). This means that for a pulse duration $t_w \gg \tau_0 \sim 10$ fs, the relaxation time of the phases will be of the same order of the pulse duration and they will appear to be frozen on this time scale. Hence the mode-locking process should be observable in a standard random laser for *ns* pump pulses. Note that, incidentally, the validity of the cw approximation has been thoroughly discussed in a recent paper for a nanosecond pulsed laser.[24]

Additionally, the pulsed regime favors the investigation of the correlation between different laser shots with fixed disorder. In the presence of many modes, due to mode interferences and complex phase modulations, the peaks in the spectrum are expected to largely vary from pulse to pulse. This is also due to the fact that with a pulsed pump, in a thermodynamic language, the system is first "cooled" (i.e., the average energy per mode is increased as the pump power increases) and then "heated" (as in the trailing edge of the pump pulse). Between two subsequent pulses the system is at infinite temperature and will rapidly loose memory of the previous state: as a result, in the presence of an exponentially large number of thermodynamically equivalent states, the system settles in a different minima from pulse to pulse and correspondingly the phases will largely vary from shot to shot. During the laser oscillation, the phase-modulation corresponding to the mode-locking process should be visible. Similar phenomena (namely a large variation of the emitted spectrum from pulse to pulse) were already reported in the literature (as in Ref. 21).

A note on the speckle pattern of the emitted light is in order. Indeed the speckle pattern is determined by the phase difference between the modes, hence the spatial distribution of the emitted light is expected to largely vary from one shot to another of the random laser, when the pump power is above the threshold for the glass transition. It is important to emphasize that other authors predicted an exponentially large number of speckle patterns in nonlinear random media,[64] however, in that case, the leading mechanism was the nonresonant Kerr effect (incidentally, our model Eqs. (31) also applies for Kerr media as will be discussed elsewhere).

VI. CONCLUSIONS

We derived a statistical model for the mode phases in a random laser. The obtained Hamiltonian resembles that of some spin glass model (the p-spin model with $p=4$[55,57,65,66]), and can also be thought of as a generalization to the disordered case of a toy model Hamiltonian recently studied (the k-trigonometric model[67]). The relevant parameter for exploring the phase diagram is a scaled temperature T, the ratio between the "true" bath temperature and the square pumping rate (the energy stored on average in each mode). Using standard statistical mechanics techniques of disordered systems (i.e., replica method), we predict the existence of a dynamic transition at T_d and of a thermodynamic phase transition at T_c, characterized by a one-step symmetry breaking scenario. Between the two temperatures, the appearance of an exponentially large number of states is expected. This corresponds to the existence of a *random-mode locking* transition in random lasers: looking at self-correlation functions of a specific frequency component of the electric field in the cavity, one should observe a nonergodic behavior at T_d, i.e., the decay towards a *plateau*, where the phases are locked at random values. The mode-locking will happen in a configuration of the phases that depends on the history for a given sample because the system will reach a different metastable state depending on the initial state, with large sample-to-sample fluctuations. This can be observed by looking at the frequency fluctuation spectrum or at the speckle pattern of the emitted light.

The logarithm of the number of these possible random configurations of the phases is given by the complexity (see Fig. 2) times the number of active modes. As long as the physical realization of the random laser is well-described by the mean-field Hamiltonian, the system is not dynamically able to explore the phase space for temperature $T<T_d$ (or pumping rate higher than that corresponding to the dynamic transition) and remains trapped always in the state reached at T_d. However, taking into account the fact that the mean-field scenario could be a too "crude" approximation for real random lasers, we expect that the system will be able to explore on a long time scale $T_c<T<T_d$ region, where aging, memory effects, and history dependent responses are expected to take place.

The expert reader in nonlinear optics or Bose-Einstein condensation will certainly recognize in our model a typical system for many-modes interaction processes (e.g., solitons, parametric processes, supercontinuum generation,…); we believe indeed that our results are also relevant in many branches of modern nonlinear physics involving disordered systems. Spin glasses have been defined as the "most complex kind of condensed state,"[68] we are convinced that there is no difficulty in accepting the emission of random lasers as the "most complex kind of light."

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APPENDIX A: EQUATIONS FOR THE AMPLITUDES AND FOR THE PHASES

Here we will show that the Hamiltonian (30) can be derived also directly from the equation of motion for the com-

plex amplitudes a_s, Eq. (26), by assuming that the dynamics of the phases is much faster than that of the A_s.

After Eq. (26), using $g_{spqr}=g^R_{spqr}+ig^I_{spqr}$ and $\eta_s=\eta^R_s+i\eta^I_s$, the equations for the amplitudes are

$$\frac{dA_s}{dt}=-\frac{1}{2}\sum_{pqr}A_pA_qA_r[g^R_{spqr}\cos(\varphi_s+\varphi_p-\varphi_q-\varphi_r)+g^I_{spqr}\sin(\varphi_s+\varphi_p-\varphi_q-\varphi_r)]+(\gamma_s-\alpha_s)A_s+\eta^R_s\cos(\varphi_s)+\eta^I_s\sin(\varphi_s). \tag{A1}$$

By assuming the phases as rapidly varying with respect to the amplitudes, these can be averaged out and the "free run approximation" equations[20,26,28] are retrieved, providing the average energy in each mode $\mathcal{E}_s$:

$$\frac{d\mathcal{E}_s}{dt}=2(\gamma_s-\alpha_s)\mathcal{E}_s-\frac{g^R_{ssss}}{\omega_s}\mathcal{E}_s^2+\mathcal{E}_s\sum_r\frac{g^R_{srsr}+g^R_{ssrr}}{\omega_r}\mathcal{E}_r. \tag{A2}$$

Note that Eq. (A1) takes into account random cross- and self-saturation effects, by the terms weighted by $g^R_{srsr}+g^R_{ssrr}$ and g^R_{ssss}, respectively, as well as the fact that the decay rates and gains are expected to be different for each mode. We model this circumstance in the text by taking the amplitude dependent G-coefficients as Gaussianly distributed.

Next, we find the ruling equation for the phases after Eq. (26):

$$A_s\frac{d\varphi_s}{dt}=-\frac{1}{2}\sum_{pqr}A_pA_qA_r\{g^I_{spqr}\cos(\varphi_s+\varphi_p-\varphi_q-\varphi_r)-g^R_{spqr}\sin(\varphi_s+\varphi_p-\varphi_q-\varphi_r)\}+\eta^I_s\cos(\varphi_s)-\eta^R_s\sin(\varphi_s). \tag{A3}$$

Denoting $\mathcal{E}_0=\omega_0\langle A^2\rangle$ the average energy per mode, we multiply Eq. (A4) by ω_0A_s and take $\omega_0A_s^2\cong\mathcal{E}_0$ (while being $\omega_s\cong\omega_0$, as outlined above).

The equation for the phases (A3) are equivalent to the following:

$$\frac{d\varphi_s}{dt}=-\frac{\partial H_\varphi}{\partial\varphi_s}+\eta^{(\varphi)}_s, \tag{A4}$$

with $\eta^{(\varphi)}_s\equiv[\eta^I_s\cos(\varphi_s)-\eta^R_s\sin(\varphi_s)]/A_s$ and

$$H_\varphi=\frac{1}{8}\sum_{spqr}\frac{\omega_0A_sA_pA_qA_r}{\mathcal{E}_0}[g^I_{spqr}\sin(\varphi_s+\varphi_p-\varphi_q-\varphi_r)+g^R_{spqr}\cos(\varphi_s+\varphi_p-\varphi_q-\varphi_r)], \tag{A5}$$

where we exploited the symmetries for the real part which are the same as discussed above, and where the imaginary part satisfies $g^I_{spqr}=-g^I_{qrsp}$.

Equation (A5) is cast in the form $H_\varphi=\omega_0H/2\mathcal{E}_0$ with

$$H=\frac{1}{4}\sum_{spqr}A_sA_pA_qA_r[g^R_{spqr}\cos(\varphi_s+\varphi_p-\varphi_q-\varphi_r)-g^I_{spqr}\sin(\varphi_s+\varphi_p-\varphi_q-\varphi_r)]=\frac{1}{4}\mathrm{Re}\left[\sum_{spqr}g_{spqr}a_qa_ra_p^*a_s^*\right]. \tag{A6}$$

Equation (A4) is a Langevin system for the phases, and being (due to the fact that the noise η is assumed to vary on a much faster scale than the phases φ) $\langle\eta^{(\varphi)}_p(t)\eta^{(\varphi)}_q(t')\rangle=\omega_0k_BT_{bath}\delta_{pq}\delta(t-t')/\mathcal{E}_0$, its invariant measure is $\exp(-2H_\varphi\mathcal{E}_0/\omega_0k_BT_{bath})=\exp(-H/k_BT_{bath})$, which is identical to the previous one. In the case of a complex coupling the Hamiltonian will include also the second term in Eq. (A6). The replica analysis of the generalized model is a generalization of the one concerning Eq. (27) (see Sec. IV) and provides the same outcome for what concerns the existence of a RSB transition.

APPENDIX B: SELF-CONSISTENCY EQUATION OF 1RSB SOLUTION

Here we derive the self-consistency equation for q, Eq. (60). It is obtained imposing that the derivative of the free energy in Eq. (59) with respect to q vanishes, $\partial_q\beta\times\phi_{1RSB}(m,q)=0$:

$$3\beta^2q^2[(1-m)q-1]=-\frac{1}{m}\frac{\int_0^\infty\mathcal{D}z\partial_qI_0^m(\beta\lambda z)}{\int_0^\infty\mathcal{D}zI_0^m(\beta\lambda z)}. \tag{B1}$$

Now, $\partial_qI_0^m=mI_0^{m-1}\partial_qI_0$ and $\partial_qI_0=(\partial_\alpha I_0)(\partial_q\alpha)=I_1\partial_q\alpha$, where $\alpha=\beta\lambda z=\beta\sqrt{2}q^{3/2}z$. Then we can write

$$m^{-1}\int_0^\infty\mathcal{D}z\partial_qI_0^m=\frac{3\beta q^{1/2}}{\sqrt{2}}\int_0^\infty\mathcal{D}zzI_0^{m-1}I_1=\frac{3\beta q^{1/2}}{\sqrt{2}}\int_0^\infty dze^{-z^2/2}\partial_z(zI_0^{m-1}I_1), \tag{B2}$$

having used the identity $ze^{-z^2/2}=-\partial_ze^{-z^2/2}$ and integrating by part. Performing the derivatives and using $\partial_z=(\partial_z\alpha)\partial_\alpha$ and the property of Bessel functions $\partial_\alpha I_1=I_0-I_1/\alpha$, we have

$$m^{-1}\int_0^\infty\mathcal{D}z\partial_qI_0^m=3\beta^2q^2\int_0^\infty\mathcal{D}zI_0^m\left[1+(m-1)\frac{I_1^2}{I_0^2}\right]. \tag{B3}$$

Substituting in Eq. (B1) we then obtain the self-consistency equation:

$$q=\frac{\int_0^\infty\mathcal{D}zI_0^m(\beta\lambda z)\left[\frac{I_1(\beta\lambda z)}{I_0(\beta\lambda z)}\right]^2}{\int_0^\infty\mathcal{D}zI_0^m(\beta\lambda z)}. \tag{B4}$$

APPENDIX C: STABILITY OF 1RSB SOLUTION

In this Appendix we discuss the stability of the 1RSB solution. First we have to compute the Hessian of $\phi(q)$ evaluated in a solution that verifies the saddle point equations $q_{ab}=\langle\cos(\varphi^a-\varphi^b)\rangle$, where we assume that q_{ab} is real. This assumption is motivated by the analysis of Ref. 54, p. 894, where it is shown for the case of a two spin interaction that even the full RSB solution verifies q_{ab} real for all ab. Still this is an assumption as in principle there could be solutions such that q_{ab} has an imaginary part for $a\neq b$, see again Ref. 54.

Considering a perturbation δq_{ab} around the solution, one has, differentiating Eq. (51):

$$G_{ab,cd}=\frac{d^2\beta\phi}{dq_{ab}dq_{cd}}=\frac{6\beta^2}{n}q_{ab}^2[\delta_{ab,cd}-6\beta^2q_{cd}^2 \times[\langle\cos(\varphi^a-\varphi^b)\cos(\varphi^c-\varphi^d)\rangle-q_{ab}q_{cd}]]. \quad \text{(C1)}$$

The condition $q_{ab}=q_{ba}^*$ implies $\delta q_{ab}=\delta q_{ba}$. The matrix G is symmetric under the exchanges $a\leftrightarrow b$, $c\leftrightarrow d$. When the matrices G are evaluated in the 1RSB solution, it is easy to see that it becomes a block matrix that has nonvanishing elements only if (ab) and (cd) belong to the same diagonal block related to one of the diagonal blocks of the matrix q_{ab}. Thus we can restrict one to consider perturbations of one single block. With this restriction, substituting the 1RSB structure of q_{ab}, and neglecting the irrelevant prefactor $6\beta^2n^{-1}q^2$, G has the following elements:

$$P\equiv G_{ab,ab}=1-3\beta^2q^2 \times\left[1+\frac{\int\mathcal{D}\zeta I_0(\beta\lambda|\zeta|)^{m-2}I_2(\beta\lambda|\zeta|)^2}{\int\mathcal{D}\zeta I_0(\beta\lambda|\zeta|)^m}-2q^2\right],$$

$$Q\equiv G_{ab,ad}=-3\beta^2q^2 \times\left[q+\frac{\int\mathcal{D}\zeta I_0(\beta\lambda|\zeta|)^{m-3}I_2(\beta\lambda|\zeta|)I_1(\beta\lambda|\zeta|)^2}{\int\mathcal{D}\zeta I_0(\beta\lambda|\zeta|)^m}-2q^2\right],$$

$$R\equiv G_{ab,cd}=-6\beta^2q^2\left[\frac{\int\mathcal{D}\zeta I_0(\beta\lambda|\zeta|)^{m-4}I_1(\beta\lambda|\zeta|)^4}{\int\mathcal{D}\zeta I_0(\beta\lambda|\zeta|)^m}-q^2\right]. \quad \text{(C2)}$$

Following the analysis of Ref. 69, the relevant eigenvalue of G that eventually becomes unstable is $\Lambda=P-2Q+R$, i.e., the stability condition is

$$\Lambda=1-6\beta^2q^2\frac{\int\mathcal{D}\zeta I_0(\beta\lambda|\zeta|)^m\left\{\frac{1}{2}\left[1-\frac{I_1(\beta\lambda|\zeta|)^2}{I_0(\beta\lambda|\zeta|)^2}\right]^2+\frac{1}{2}\left[\frac{I_2(\beta\lambda|\zeta|)}{I_0(\beta\lambda|\zeta|)}-\frac{I_1(\beta\lambda|\zeta|)^2}{I_0(\beta\lambda|\zeta|)^2}\right]^2\right\}}{\int\mathcal{D}\zeta I_0(\beta\lambda|\zeta|)^m}>0, \quad \text{(C3)}$$

that is

$$\frac{1}{6\beta^2q^2}>\frac{\int\mathcal{D}\zeta I_0(\beta\lambda|\zeta|)^m\left\{\frac{1}{2}\left[1-\frac{I_1(\beta\lambda|\zeta|)^2}{I_0(\beta\lambda|\zeta|)^2}\right]^2+\frac{1}{2}\left[\frac{I_2(\beta\lambda|\zeta|)}{I_0(\beta\lambda|\zeta|)}-\frac{I_1(\beta\lambda|\zeta|)^2}{I_0(\beta\lambda|\zeta|)^2}\right]^2\right\}}{\int\mathcal{D}\zeta I_0(\beta\lambda|\zeta|)^m}. \quad \text{(C4)}$$

Once the solution of the saddle point Eq. (B4) is substituted in the expression above, one obtains the condition $m>m_s(T)$, see Fig. 1.

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