Ferromagnetic Potts model under an external magnetic field: An exact renormalization group approach

Sérgio Coutinho,[1,*] Welles A. M. Morgado,[2] Evaldo M. F. Curado,[3] and Ladário da Silva[4]
[1]*Departamento de Física, Universidade Federal de Pernambuco, CEP 50670-901, Recife, Pernambuco, Brazil*
[2]*Departamento de Física, Pontifícia Universidade Católica do Rio de Janeiro, CP 38071, CEP 22452-970, Rio de Janeiro, Brazil*
[3]*Centro Brasileiro de Pesquisas Físicas, Rua Xavier Sigaud 150, CEP 22290-180, Rio de Janeiro, Brazil*
[4]*Escola Naval, Marinha do Brasil, Av. Almte. Silvio de Noronha s/n, CEP 20021-010, Rio de Janeiro, Brazil*
(Received 2 June 2006; revised manuscript received 2 August 2006; published 27 September 2006)

The q-state ferromagnetic Potts model under a nonzero magnetic field coupled with the zeroth Potts state was investigated by an exact real-space renormalization group approach. The model was defined on a family of diamond hierarchical lattices of several fractal dimensions d_F. On these lattices, the renormalization group transformations became exact for such a model when a correlation coupling that singles out the zeroth Potts state was included in the Hamiltonian. The rich criticality presented by the model with $q=3$ and $d_F=2$ was fully analyzed. Apart from the Potts criticality for the zero field, an Ising-like phase transition was found whenever the system was submitted to a strong reverse magnetic field. Unusual characteristics such as cusps and dimensional reduction were observed on the critical surface.

DOI: 10.1103/PhysRevB.74.094432 PACS number(s): 75.10.Nr, 05.50.+q, 05.70.Jk, 05.45.Df

I. INTRODUCTION

The q-state Potts model is one of the most studied models in statistical mechanics due to its wide theoretical interest and practical applications.[1–12] The pure ferromagnetic version of this model in a zero field is known to exhibit a phase transition from the high-temperature disordered paramagnetic phase to the low-temperature ferromagnetic phase for $q>1$ and Bravais lattices with dimensions $d\geqslant 2$.[2,7]

From a theoretical viewpoint, the q-state Potts model in the absence of an external magnetic field and defined on many kinds of Bravais and fractal lattices has been extensively studied by mean-field-like and numerical approaches for ferromagnetic, antiferromagnetic, and disordered couplings. However, few exact results have been obtained for such models.[7,13–15] Exact results are very important as a guide for gaining an insight into many complicated points of the phase diagram.

The q-state Potts model under an external magnetic field in its turn, due to its complexity, has been studied by only a few authors,[16–19] and several aspects of its physical behavior have remained poorly established.

This paper studies the role of a magnetic field on the physical properties of the ferromagnetic q-state Potts model defined on a family of fractal lattices, called diamond hierarchical lattices (DHL's). The zero-field version of this model was previously studied by the authors.[13] One important issue for the study of spin models on such lattices arises from the scale invariance property which is presented and which leads to exact and tractable solutions. Generally speaking, in qualitative terms, they compare favorably to similar ones obtained by other approximate methods. Indeed, the study of spin models on such lattices can be viewed as the counterpart of the real-space renormalization group approximation for the corresponding models defined on Bravais lattices.[20,21] Therefore, they provide an alternative framework for studying the critical behavior of more realistic systems whenever approaches considering translational invariant lattices lead to untractable procedures.

To benefit from the scale invariance property of the DHL, an exact real-space renormalization group (RG) scheme was applied in order to study the q-state Potts model on such lattices. Closed renormalization transformations in the coupling parameter space were achieved when a zero-state correlation coupling interaction was introduced into the Hamiltonian, as well as the constants of the magnetic field and ferromagnetic coupling. Under such a zero-state correlation interaction pairs of spins in the zeroth Potts state, aligned with the magnetic field, are singled out. The particular model with $q=3$ Potts states defined on a DHL with a scaling factor of 2 and fractal dimension of $d_F=2$ was exhaustively studied. The full flow diagram of the closed RG equations in three-dimensional (3D) parameter space was obtained exhibiting the stable fixed points associated with the zero-field ferromagnetic and paramagnetic Potts phases, as well as those associated with an Ising-like phase induced by a strong reverse magnetic field. The critical surface delineating the basin of attraction of the Ising-like fixed point was sketched, revealing unusual characteristics such as cusps and dimensional reduction.

This paper is organized as follows: the Hamiltonian model and the general renormalization equations for the model with q-state Potts states defined on a general DHL with an arbitrary fractal dimension are presented and discussed in Sec. II, and its deduction is displayed in the Appendix. The particular model for $q=3$ and $d_F=2$ is studied in Sec. III, the corresponding renormalization equations set in Sec. III A, while Sec. III B is devoted to presenting the critical points in the coupling constant parameter space and to analyzing the full renormalization flow and unusual characteristics appearing on the critical surface. Finally, the discussion is presented and the conclusions are summarized in Sec. IV.

II. HAMILTONIAN MODEL AND METHODOLOGY

DHL lattices were constructed commencing from a basic unit (called first generation or hierarchy) and replaced all of

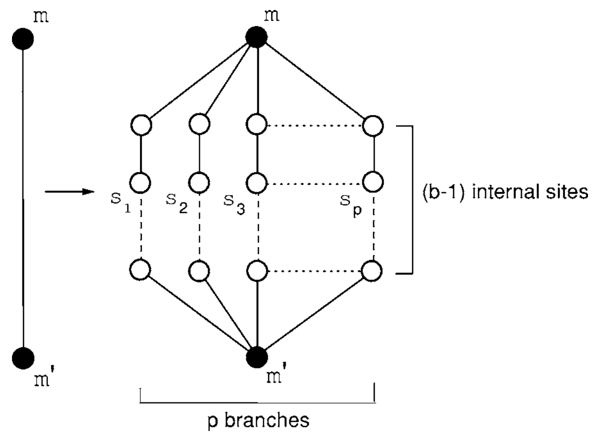

FIG. 1. Inflation process for a general diamond hierarchical lattice with scaling factor b and p branches.

its single bonds with the basic unit itself to build the second generation. Such a replacing procedure is sketched in Fig. 1 for the general DHL, with a scaling factor b and p connecting branches, hereafter referred to as (b,p)-DHL. This inflation procedure is recursively applied leading to a two-root lattice with fractal dimension, the number of sites, and the number of bonds given by $d_F=1+\ln p/\ln b$, $N_S^n=2+(b-1)p[(bp)^n-1]/(bp-1)$, and $N_B^n=(bp)^n$, respectively, with n as the number of generations or hierarchies.

The general q-state Potts Hamiltonian $\mathcal{H}^{(n)}$ on a n-generation (b,p)-DHL can be written as

$$-\beta\mathcal{H}^{(n)}=\sum_{\langle i,j\rangle} qK_{ij}^{(n)}\delta_{\sigma_i\sigma_j}+\sum_i qH_i^{(n)}\delta_{\sigma_i,0}+\sum_{\langle i,j\rangle} qL_{ij}^{(n)}\delta_{\sigma_i 0}\delta_{\sigma_j,0}, \tag{1}$$

where $\beta=1/kT$, T is the absolute temperature, and $\delta_{\sigma,\sigma'}$ is the Kronecker delta function. The energy of the ferromagnetic interaction coupling the nearest-neighbor spins σ's ($\sigma=0,1,2,\ldots,q-1$) is described by the first term of Eq. (1), while the energy of the interaction of the external magnetic field aligned to the zeroth state is given by the second term. Finally, the above-mentioned correlation coupling interaction, which bolsters the energy of the pairs of spins in the zeroth state is accounted for by the third term.

The Hamiltonian function for the pure (homogeneous) ferromagnetic Potts model obtained from Eq. (1) when all coupling constant interactions are considered independent of the site position—that is, $K_{ij}^{(n)}\equiv K^{(n)}$, $L_{ij}^{(n)}\equiv L^{(n)}$, and $H_{ij}^{(n)}\equiv H^{(n)}$—is simplified as

$$-\beta\mathcal{H}^{(n)}=qK^{(n)}\sum_{\langle i,j\rangle}\delta_{\sigma_i\sigma_j}+qH^{(n)}\sum_i\delta_{\sigma_i,0}+qL^{(n)}\sum_{\langle i,j\rangle}\delta_{\sigma_i 0}\delta_{\sigma_j,0}. \tag{2}$$

The pure q-state Potts model, defined on the (b,p)-DHL with n generations, can be exactly transformed by renormalization in an equivalent model defined on the (b,p)-DHL with $(n-1)$ generations as long as the $p(b-1)$ internal spin variables within each basic unit are decimated.

The new Hamiltonian coupling constants are obtained by a formal set of renormalization equations given by

$$K^{(n-1)}=F_1(K^{(n)},L^{(n)},H^{(n)}),$$

$$H^{(n-1)}=F_3(K^{(n)},L^{(n)},H^{(n)}),$$

$$L^{(n-1)}=F_2(K^{(n)},L^{(n)},H^{(n)}),$$

where the functions F_1, F_2, and F_3, in general, are determined by the decimation process.[22,23]

From this point, appropriate compact variables (*transmissivities*) introduced by Tsallis and Levy[24] were considered:

$$t_n=\frac{e^{qK^{(n)}}-1}{e^{qK^{(n)}}+(q-1)}, \tag{3}$$

$$u_n=\frac{e^{qH^{(n)}}-1}{e^{qH^{(n)}}+(q-1)}, \tag{4}$$

$$v_n=\frac{e^{qL^{(n)}}-1}{e^{qL^{(n)}}+(q-1)}. \tag{5}$$

The equivalent decimation relations for the above variables (t,u,v), rather than (K,L,H), were obtained for the particular $(2,p)$-DHL family with the scaling factor $b=2$. The choice of such a hierarchical lattice family (scaling factor $b=2$) is appropriated for studying the ferromagnetic model. To investigate the antiferromagnetic model, for instance, hierarchical lattices with odd scaling factor (odd b) should be required to preserve the ground-state configuration under renormalization. The deduction of the $b=2$ decimation relations is presented in the Appendix and leads formally to closed renormalization-group-coupled equations given by

$$t_{n-1}=\frac{T(t_n,u_n)-1}{T(t_n,u_n)+q-1}, \tag{6}$$

$$u_{n-1}=\frac{U(t_n,u_n,v_n)-1}{U(t_n,u_n,v_n)+q-1}, \tag{7}$$

$$v_{n-1}=\frac{V(t_n,u_n,v_n)-1}{V(t_n,u_n,v_n)+q-1}, \tag{8}$$

where

$$T(t,u)=\left[1+\frac{qt^2(1-u)}{(1-t)(1+t-2tu)}\right]^p, \tag{9}$$

$$U(t,u,v)=\left[\frac{1+(q-1)u}{1-u}\right]\left[\frac{N_2}{D}\right]^p, \tag{10}$$

$$V(t,u,v)=\left[\frac{N_3D^2}{N_1N_2^2}\right]^p, \tag{11}$$

with

$$D=\frac{q(1+t-2tu)}{(1-t)(1-u)}, \tag{12}$$

$$N_1 = \frac{qt[2+(q-2)t]}{(1-t)^2} + \frac{q}{1-u}, \tag{13}$$

$$N_2 = q - 2 + \frac{1+(q-1)t}{1-t} - \frac{[1+(q-1)t][1+(q-1)u][1+(q-1)v]}{(1-t)(1-u)(1-v)}, \tag{14}$$

$$N_3 = q - 1 + \frac{[1+(q-1)t]^2[1+(q-1)u][1+(q-1)v]^2}{(1-t)^2(1-u)(1-v)^2}. \tag{15}$$

Previous results for the zero-field pure q-state Potts model were reobtained[13] when the zero-field ($u=0$) and zero-correlation-coupling ($v=0$) limits are imposed on Eqs. (6)–(8)—that is,

$$t_{n-1} = \frac{2t_n^2 + (q-2)t_n^4}{1+(q-1)t_n^4}.$$

III. $q=3$ STATE POTTS MODEL ON A (2,2)-DHL WITH FIELDS

The particular model with $q=3$ and $b=p=2$ is investigated in the present section. Such a model, which corresponds to the simplest one defined on a hierarchical lattice with integer fractal dimension $d_F>1$, is suitable for comparision with the corresponding model defined on the square lattice in the framework of the real-space renormalization group approximation,[20,21] even though it presents a rich criticality with unusual characteristics on the critical surface. The study of other cases should follow straightforwardly from the present calculation although with much heavier algebraic computational efforts.

A. Renormalization transformations

The renormalization equations for the three-state Potts model defined on the (2,2)-DHL, which has the fractal dimension $d_F=2$, written in terms of the transmissivity variables, can be directly obtained from Eqs. (6)–(15) by setting $q=3$ and $p=2$. After some algebraic rearrangements the following coupled renormalization equations are obtained:

$$t_{n-1} = \frac{T(t,u)-1}{T(t,u)+2}, \tag{16}$$

$$u_{n-1} = \frac{U(t,u,v)-1}{U(t,u,v)+2}, \tag{17}$$

$$v_{n-1} = \frac{V(t,u,v)-1}{V(t,u,v)+2}, \tag{18}$$

where

$$T(t,u) = \frac{[1+(2-u)t^2-2tu]^2}{(1-t)^2[1+(1-2u)t]^2}, \tag{19}$$

TABLE I. Fixed points of the renormalization flow diagram of the three-state Potts model under nonzero field. The t, u, and v variables are defined by Eqs. (3)–(5).

t	u	v	Description
Zero-field Potts model: $H=L=0$ or line $u=v=0$			
0	0	0	Potts paramagnetic phase
$\frac{1}{2}$	0	0	Potts transition
1	0	0	Potts ferromagnetic phase
Negative infinite field $H=-\infty$ or plane $u=-0.5$			
0	$-\frac{1}{2}$	0	Ising paramagnetic phase
0.442687…[a]	$-\frac{1}{2}$	−0.397513…	Ising like transition
1	$-\frac{1}{2}$	$-\frac{1}{2}$	Ising ferromagnetic phase
Zero-temperature invariant plane: $T=0$ or plane $t=1$			
1	1	$-\frac{1}{2}$	
1	0	0	Potts ferromagnetic phase
1	$-\frac{1}{2}$	$-\frac{1}{2}$	Ising ferromagnetic phase

[a]Exact value $=-(4/19)+(17766-3078\sqrt{33})^{1/3}/57+[2(329+57\sqrt{33})]^{1/3}/19$.

$$U(t,u,v) = \frac{(1+2u)[1+2uv+t(1+u+v+3uv)]^2}{(1-u)[1+t(1-2u)]^2(1-v)^2}, \tag{20}$$

$$V(t,u,v) = \frac{W(t,u,v)}{9[(1-t)^2+t(2+t)(1-u)]^2} \times \frac{[1+t(1-2u)]^4}{[1+2uv+t(1+u+v+3uv)]^4}, \tag{21}$$

$$W(t,u,v) = [2(1-t)^2(1-u)(1-v)^2 + (1+2t)^2(1+2u)(1+2v)^2]^2. \tag{22}$$

B. Renormalization flow diagram and critical points

The 3D renormalization parameter space in the variables (t,u,v) has the advantage of being confined to the region $(-0.5\leqslant t\leqslant 1,\ -0.5\leqslant u\leqslant 1,\ -0.5\leqslant v\leqslant 1)$, the bounds corresponding to the $\pm\infty$ values of the respective former variables (K,H,L). However, the region $-0.5\leqslant t<0$ corresponding to $K<0$, although allowed by the variable transformation (3), must be excluded for the following conceptual reason: the $(2,p)$-DHL family (scaling factor 2) is not suitable for studying the antiferromagnetic systems ($K<0$) since the RG decimation process does not preserve the antiferromagnetic ground-state symmetry under renormalization. DHL's with an odd scaling factor must be considered in order to study antiferromagnetic models within real space RG approaches.

The fixed points of the corresponding recursion equations (16)–(18) are shown in Table I and sketched in Fig. 2.

Three manifolds in the 3D cubic parameter space (t,u,v) can be distinguished.

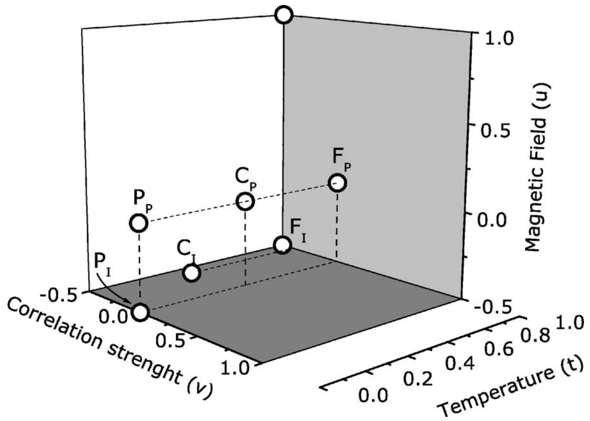

FIG. 2. 3D plot displaying the fixed points described in Table I. P_P and F_P denote the fixed points associated with paramagnetic and ferromagnetic Potts phases, while C_P label the unstable fixed point. P_I, F_I, and C_I indicate the corresponding fixed points associated with the Ising-like transition, respectively.

(a) The zero-field and zero-correlation coupling Potts model fixed points located on the line $u=v=0$, which contains the stable fixed points (0,0,0) and (1,0,0) associated with the paramagnetic and the ferromagnetic phases, respectively, as well as the unstable fixed point (0.5,0,0) associated with the corresponding phase transition, the latter obtained by the exact solution of $t^*=2t^{*2}/(1+2t^{*4})$.

(b) The $u=-1/2$ $(H\rightarrow-\infty)$ plane, which corresponds to the Ising model without magnetic field. The reduction of the number of states from $q=3$ to $q=2$ occurred because the zeroth state became unreachable. For the general q-state Potts ferromagnetic model, whenever $u=-1/2$ $(H\rightarrow-\infty)$ the zeroth state becomes unreachable, leading to a $(q-1)$-state Potts model without magnetic field on such manifold. The renormalization flow in the $u=-0.5$ plane is depicted in Fig. 3. The Ising-like paramagnetic $(0,-0.5,0)$ and the ferromagnetic $(1,-0.5,-0.5)$ stable fixed points as well as the unstable phase transition points $(0.442687\ldots,-0.5,-0.397513\ldots)$ are found within this plane. Exact values may be obtained as indicated in Table I. The renormalization flow is confined in the plane $u=-0.5$ and given by

$$t' = \frac{3t^2(2+t)^2}{4+8t-4t^3+19t^4}, \tag{23}$$

$$v' = \frac{2A-B}{A+B}, \tag{24}$$

with

$$A = 16[(1+2t)(1-t)]^4,$$

$$B = [2(1-t)^2+3(2+t)]^2(2+t)^4.$$

(c) The zero-temperature invariant plane $(t=1)$, which contains both the Potts and Ising fixed points (1,0,0) and $(1,-0.5,-0.5)$, respectively, is displayed in the Fig. 4.

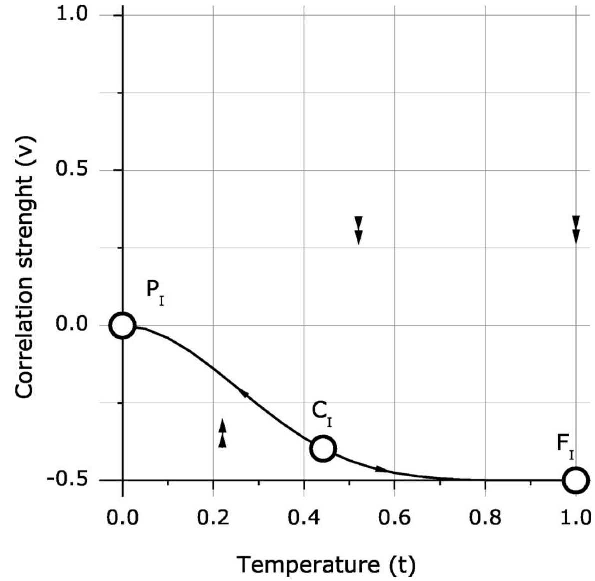

FIG. 3. Flow diagram $(t\times v)$ corresponding to an Ising-like model in the invariant plane $u=-0.5$. The open circles mark the loci of the fixed points. P_I and F_I label the stable-fixed points of the paramagnetic (P) and ferromagnetic (F) phases while C_I labels the unstable-fixed point (critical point). The arrows indicate the flow direction, the double arrows signifying a direct jump to the flow line (solid line).

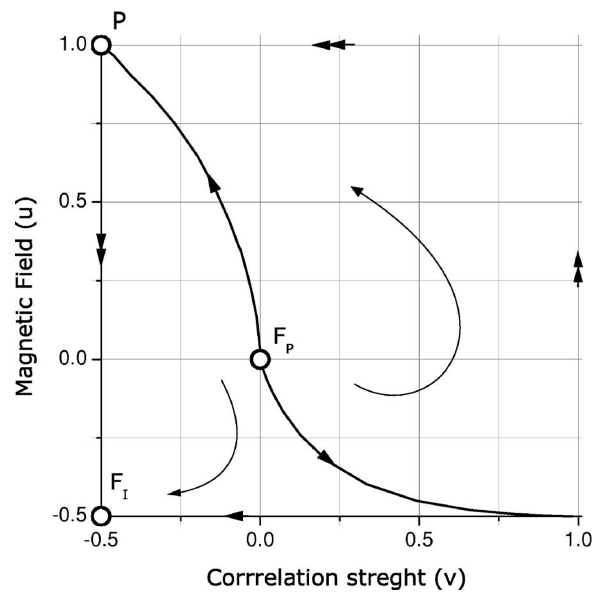

FIG. 4. Flow diagram $(v\times u)$ corresponding to the zero-temperature invariant plane $t=1$. Open symbols mark the fixed points, F_P labels the stable fixed point of the Potts ferromagnetic phase, and F_I labels the stable fixed point of the Ising ferromagnetic phase, while P indicates the stable fixed-point associated paramagnetic phase (infinite H field). The arrows indicate the flow direction, the double arrows signifying a direct jump to the end point. The solid plot corresponds to the intersection of the critical surface with the $t=1$ plane. Note the singularity in the derivative of the plot at the point F_P (0,0).

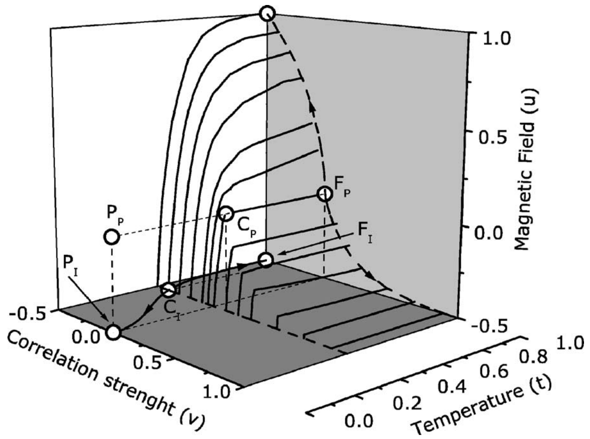

FIG. 5. Profiles of intersections of the critical surface with the planes $v=-0.5$, -0.395, -0.3, -0.2, -0.1, -0.05, 0.0, 0.1, 0.2, 0.3, 0.5, 0.7, and 0.9 (solid lines), from top to bottom. Open circles mark the loci of the fixed points, $\mathsf{F_P}$, $\mathsf{P_P}$, and $\mathsf{C_P}$ label ferromagnetic, paramagnetic, and Potts transition fixed points, while $\mathsf{F_I}$, $\mathsf{P_I}$, and $\mathsf{C_I}$ indicate the corresponding Ising-like fixed points at the plane $u=-0.5$.

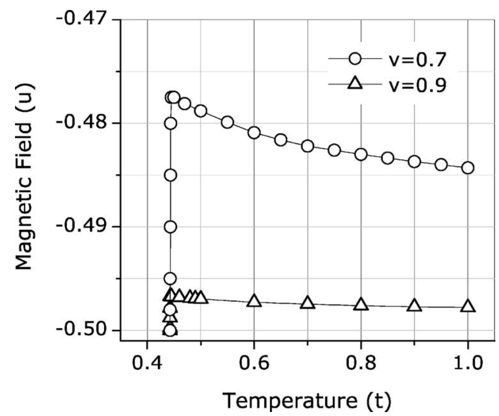

FIG. 6. Details of intersections of the critical surface with the planes $v=0.7$ and $v=0.9$, from top to bottom. The vertical scale was expanded to emphasize the cusps for lower values of t.

IV. DISCUSSION AND CONCLUSIONS

The critical properties of the q-state ferromagnetic Potts model defined on a fractal hierarchical lattice in the presence of a magnetic field were investigated through a real-space renormalization group technique. The inclusion of a new interaction term in the Hamiltonian that boosted the coupling of pairs of spins in the zeroth state (the state chosen to be singled out by the external field) led to a closed—and exact—coupled RG transformation. The full renormalization flow diagram in an appropriate compact parameter space is analyzed for the particular model with $q=3$ Potts states, defined on the diamond hierarchical lattice with a fractal dimension $d_F=2$. The inclusion of such a correlation coupling breaks the symmetry for the pair correlation with respect to the Potts state singled out by the external field. Whenever $q \geq 3$ pairs of spins with states parallel to the external field have different energies with respect to all pairs of parallel spins in other states. Note that such a correlation coupling has no relevance for the Ising model $(q=2)$ leading only to rescale the energy of the ferromagnetic pair interaction.

New and interesting features are found within the global phase diagram.

(a) A dimensional reduction of the order parameter at a strong (infinite) reverse magnetic field, which depopulates the zeroth Potts state: within the plane $(u=-1/2)$ the system is reduced to a $(q-1)$-state Potts like model. For the studied case $(q=3)$ the reduced model corresponds to an effective Ising ferromagnet. Within the 3D parameter space flow diagram the stable fixed point associated with the Ising condensed phase is located at zero temperature $(t=1)$ and infinite reverse magnetic field $(u=-1/2)$.

(b) The basin of attraction of such a point is separate from that of the paramagnetic phase by a complex boundary surface, which contains the locus of the zero-field q-state Potts phase transition. Both the unstable and stable fixed points associated with the Potts phase belong to the boundary surface along the line $(u=v=0, t \geq 1/2)$.

(c) Along such a line, the boundary surface has a sharp cusp. Other lines with a sharp cusp also appears in the $v \geq 0$ portion of the surface.

Figure 5 displays several plots of the intersection of the critical surface with planes (t,u) for chosen values of v, showing all the above-mentioned features. Figure 6 exhibits in the appropriate scale the details of sharp cusps appearing in the plots of the intersection of the critical surface with the planes $v=0.7$ and $v=0.9$. Whether such features are present in the q-state Potts model defined in other kinds of lattices deserves further investigations.

The eigenvalues and eigenvectors associated with the Potts and Ising unstable fixed points were calculated. Table II displays such result. The α and ν critical exponents at the fixed points associated with the Potts and Ising transitions, previously reported in the literature,[13] were recovered. The general nature of the transitions studied in this work (for $q=3$) is found to be of a second order. Other authors[14] have claimed that, at higher values of q, the inherent inhomogeneity of the model might induce a change to first-order transition, which provides another question for further investigation.

In closing, to the current knowledge of the authors, the study reported throughout this paper is one of a few exact

TABLE II. Eigenvalues λ's and the corresponding eigenvectors $\vec{v}$'s associated with the transition Potts and Ising fixed points.

Potts transition point (0.5, 0, 0)
$\lambda_t=2.26938 \rightarrow \vec{v}_t=(-0.18388, 0.813562, 0.551639)$
$\lambda_u=1.77777 \rightarrow \vec{v}_u=(1,0,0)$
$\lambda_v=1.17506 \rightarrow \vec{v}_v=(-0.159264, -0.863917, 0.477792)$
Ising transition point $(0.442687, -0.5, -0.397513)$
$\lambda_t=1.678573 \rightarrow \vec{v}_t=(0.912186, 0, -0.409776)$
$\lambda_u=0.419643 \rightarrow \vec{v}_u=(0.0499605, 0.99851, -0.0219619)$
$\lambda_v=0 \rightarrow \vec{v}_v=(0,0,1)$

renormalization group approaches to the Potts model under an external magnetic field in a growing body of literature on the subject.

ACKNOWLEDGMENTS

Work partially supported by Brazilian granting agencies CNPq, CAPES, FAPERJ, and FACEPE.

APPENDIX: EXACT RENORMALIZATION EQUATIONS FOR THE q-STATE POTTS MODEL ON $(2,p)$-DHL

The Hamiltonian for a single basic unit with one internal site $b=2$ can be written as

$$-\beta\mathcal{H} = qK\sum_{i=1}^{p}(\delta_{\sigma_i\mu_1}+\delta_{\sigma_i,\mu_2}) + qL\sum_{i=1}^{p}(\delta_{\sigma_i0}\delta_{\mu_10}+\delta_{\sigma_i0}\delta_{\mu_20}) + qH\sum_{i=1}^{p}[\delta_{\sigma_i0}+\delta_{\mu_10}+\delta_{\mu_20}], \quad \text{(A1)}$$

which can be rewritten for later convenience as

$$-\beta\mathcal{H} = \sum_{i=1}^{p}\mathcal{H}_i,$$

with

$$\mathcal{H}_i = qK(\delta_{\sigma_i\mu_1}+\delta_{\sigma_i\mu_2}) + qL(\delta_{\sigma_i0}\delta_{\mu_10}+\delta_{\sigma_i0}\delta_{\mu_20}) + qH\delta_{\sigma_i0} + \frac{q}{p}H\delta_{\mu_10} + \frac{q}{p}H\delta_{\mu_20}. \quad \text{(A2)}$$

The corresponding restricted partition function for a fixed configuration of the root sites:

$$z(\mu_1,\mu_2) = \mathrm{Tr}_{\{\sigma_i\}}e^{-\beta\mathcal{H}} = \mathrm{Tr}_{\{\sigma_i\}}\prod_{i=1}^{p}e^{\mathcal{H}_i} = [\mathrm{Tr}_{\{\sigma_i\}}e^{\mathcal{H}_i}]^p = \left\{\sum_{\sigma=0}^{q-1}\exp\left[qK(\delta_{\sigma\mu_1}+\delta_{\sigma\mu_2}) + qL(\delta_{\sigma0}\delta_{\mu_10} + \delta_{\sigma0}\delta_{\mu_2,0}) + \frac{q}{p}H(p\delta_{\sigma0}+\delta_{\mu_10}+\delta_{\mu_20})\right]\right\}^p. \quad \text{(A3)}$$

The renormalized bond Hamiltonian is defined by

$$-\beta\mathcal{H}' = qK'\delta_{\mu_1\mu_2} + qL'\delta_{\mu_10}\delta_{\mu_20} + qH'(\delta_{\mu_10}+\delta_{\mu_20}), \quad \text{(A4)}$$

while the renormalized restricted partition function results as

$$z'(\mu_1,\mu_2) = e^{-\beta\mathcal{H}'} = \exp[qK'\delta_{\mu_1\mu_2} + qL'\delta_{\mu_10}\delta_{\mu_20} + qH'(\delta_{\mu_10} + \delta_{\mu_20})]. \quad \text{(A5)}$$

The partition functions Z' and Z are compared to obtain the renormalization equations for the coupling constants K', L', and H' as a function of K, L, and H—that is,

$$Z \equiv AZ', \therefore \sum_{\{\mu_1,\mu_2\}} z(\mu_1,\mu_2) = A\sum_{\{\mu_1,\mu_2\}} z'(\mu_1,\mu_2), \quad \text{(A6)}$$

where A is a constant to be determined as a function of K, L, and H, which can be used to fix the origin of the energy scale. The expansions of the configurations of the root sites in the sums of Eq. (A6) are performed and four kinds of possible configurations may be distinguished: namely, (i) $\mu_1=\mu_2=0$, one configuration; (ii) $\mu_1=0$ and $\mu_2\neq 0$ and vice versa, two configurations; (iii) $\mu_1=\mu_2\neq 0$, leading to $(q-1)$ configurations; and (iv) $\mu_1\neq\mu_2$, $\mu_1\mu_2\neq 0$, leading to $(q-1)(q-2)$ configurations.

The contributions for the partition functions are equal by symmetry for each case. The comparison term by term led to the following relations:

$$Ae^{qK'+qL'+2qH'} = e^{2qH}(e^{2qK+2qL+qH}+q-1)^p, \quad \text{(A7)}$$

$$Ae^{qH'} = e^{qH}(e^{qK}+e^{2qH+qL}+q-2)^p, \quad \text{(A8)}$$

$$Ae^{qK'} = (e^{qH}+e^{2qK}+q-2)^p, \quad \text{(A9)}$$

$$A = (e^{qH}+2e^{qK}+q-3)^p, \quad \text{(A10)}$$

which can be solved for the coupling constants K', L', and H', leading to

$$e^{qK'} = \left(\frac{e^{2qK}+e^{qH}+q-2}{2e^{qK}+e^{qH}+q-3}\right)^p, \quad \text{(A11)}$$

$$e^{qH'} = e^{qH}\left(\frac{e^{qK}+e^{qK+qL+qH}+q-2}{2e^{qK}+e^{qH}+q-3}\right)^p, \quad \text{(A12)}$$

$$e^{qL'} = \left[\frac{(e^{2qK+2qL+qH}+q-1)(2e^{qK}+e^{qH}+q-3)^2}{(e^{2qK}+e^{qH}+q-2)(e^{qK}+e^{qK+qL+qH}+q-2)^2}\right]^p. \quad \text{(A13)}$$

New appropriated compact variables t, u, and v, defined in the interval $(-1/(q-1),1)$ instead of the K, H, and L defined in $(-\infty,\infty)$, are considered, respectively, by

$$t = \frac{e^{qK}-1}{e^{qK}+q-1}, \quad \text{(A14)}$$

$$u = \frac{e^{qH}-1}{e^{qH}+q-1}, \quad \text{(A15)}$$

$$v = \frac{e^{qL}-1}{e^{qL}+q-1}. \quad \text{(A16)}$$

The inverse relations give

$$e^{qK} = \frac{1+(q-1)t}{1-t}, \quad \text{(A17)}$$

$$e^{qH} = \frac{1+(q-1)u}{1-u}, \quad \text{(A18)}$$

$$e^{qL} = \frac{1+(q-1)v}{1-v}. \tag{A19}$$

Equations (A17)–(A19) are substituted into Eqs. (A11)–(A13), which by turns are substituted into Eqs. (A14)–(A16) after changing $(t \to t')$. The result is

$$t' = \frac{T(t,u)-1}{T(t,u)+q-1}, \tag{A20}$$

$$u' = \frac{U(t,u,v)-1}{U(t,u,v)+q-1}, \tag{A21}$$

$$v' = \frac{V(t,u,v)-1}{V(t,u,v)+q-1}, \tag{A22}$$

where

$$T(t,u) = \left[1 + \frac{qt^2(1-u)}{(1-t)(1+t-2tu)}\right]^p, \tag{A23}$$

$$U(t,u,v) = \left[\frac{1+(q-1)u}{1-u}\right]\left[\frac{N_2}{D}\right]^p, \tag{A24}$$

$$V(t,u,v) = \left[\frac{N_3 D^2}{N_1 N_2^2}\right]^p, \tag{A25}$$

with

$$D = \frac{q(1+t-2tu)}{(1-t)(1-u)}, \tag{A26}$$

$$N_1 = \frac{qt[2+(q-2)t]}{(1-t)^2} + \frac{q}{1-u}, \tag{A27}$$

$$N_2 = q - 2 + \frac{1+(q-1)t}{1-t} - \frac{[1+(q-1)t][1+(q-1)u][1+(q-1)v]}{(1-t)(1-u)(1-v)}, \tag{A28}$$

$$N_3 = q - 1 + \frac{[1+(q-1)t]^2[1+(q-1)u][1+(q-1)v]^2}{(1-t)^2(1-u)(1-v)^2}. \tag{A29}$$

Equations (A20)–(A29) give the complete set renormalization equations for the q-state Potts model defined on a general diamond hierarchical lattice with scaling factor $b=2$ and fractal dimension $d_F = 1 + \ln p / \ln 2$, whose Hamiltonian is given by Eq. (2).

*Electronic address: sergio@lftc.ufpe.br

[1] R. B. Potts, Proc. Cambridge Philos. Soc. **48**, 106 (1952).
[2] F. Y. Wu, Rev. Mod. Phys. **54**, 235 (1982).
[3] S. N. Dorogovtsev, A. V. Goltsev, and J. F. F. Mendes, Eur. Phys. J. B **38**, 177 (2004).
[4] F. Igloi and L. Turban, Phys. Rev. E **66**, 036140 (2002).
[5] C. Day, Phys. Today **58**(11), 19 (2005).
[6] R. Baxter, Phys. Today **58**(12), 15 (2005).
[7] R. J. Baxter, *Exactly Solved Models in Statistical Mechanics* (Academic Press, London, 1982).
[8] M. Bretz, Phys. Rev. Lett. **38**, 501 (1977).
[9] T. C. Lubensky and J. Isaacson, Phys. Rev. Lett. **41**, 829 (1978).
[10] D. S. Robinson and M. B. Salamon, Phys. Rev. Lett. **48**, 156 (1982).
[11] T. Halpin-Healy and M. Kardar, Phys. Rev. B **31**, 1664 (1985).
[12] D. C. Parks, N. A. Clark, D. M. Walba, and P. D. Beale, Phys. Rev. Lett. **70**, 607 (1993).
[13] L. da Silva, E. M. F. Curado, S. Coutinho, and W. A. M. Morgado, Phys. Rev. B **53**, 6345 (1996).
[14] P. N. Timonin, J. Exp. Theor. Phys. **99**, 1044 (2004).
[15] R. J. Baxter, Phys. Rev. Lett. **94**, 130602 (2005); J. Stat. Phys. **120**, 1 (2005).
[16] F. A. Kassan-Ogly and B. N. Filippov, J. Magn. Magn. Mater. **258-259**, 219 (2003).
[17] F. Karsch and S. Stickan, Phys. Lett. B **488**, 319 (2000).
[18] S.-Y. Kim, J. Korean Phys. Soc. **45**, 302 (2004).
[19] Z. Glumac and K. Uzelac, J. Phys. A **27**, 7709 (1994).
[20] A. A. Migdal, Zh. Eksp. Teor. Fiz. **69**, 810 (1975); [Sov. Phys. JETP**42**, 413 (1976)]; **69**, 1457 (1975); [**42**, 743 (1976)].
[21] L. P. Kadanoff, Ann. Phys. (N.Y.) **100**, 353 (1977).
[22] C. Tsallis and A. C. N. de Magalhães, Phys. Rep. **268**, 305 (1996).
[23] W. A. M. Morgado, S. Coutinho, and E. M. F. Curado, J. Stat. Phys. **61**, 913 (1991).
[24] C. Tsallis and S. V. F. Levy, Phys. Rev. Lett. **47**, 950 (1981).