

Superconducting instability in three-band metallic nanotubes

David Carpentier and Edmond Orignac

CNRS UMR 5672 - Laboratoire de Physique de l'École Normale Supérieure de Lyon, 46, Allée d'Italie, 69007 Lyon, France

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Motivated by recent experiments on small radius nanotubes, we study the superconducting instabilities of cylindrical (5,0) nanotubes. According to band-structure calculations, these nanotubes possess three bands at the Fermi energy. Using a Fermionic renormalization-group approach and a careful Bosonization treatment, we consider the effect of different attractive interactions, mediated by phonons, within the Luttinger liquid framework. We particularly focus on a superconducting instability specific to the three-band model we consider for the description of these (5,0) cylindrical nanotubes.

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I. INTRODUCTION

Superconducting behavior of carbon nanotubes have been observed both in ropes of single walled nanotubes,¹ and in single walled² and multiwalled³ small radius nanotubes grown in a zeolite matrix. The high value of the critical temperature T_c in these last two cases raised the question of its possible relation with the small radius of the nanotubes. Indeed, for these 4-Å nanotubes, the large curvature induces a hybridization of the σ and π orbitals of the carbon atoms,⁴ leading to electron and phonon properties different from the larger nanotubes. The relation between the high T_c and these peculiarities have motivated several works, in particular on the metallic (5,0) nanotube which constitute the best candidate for the origin of the superconductivity.⁵ These previous approaches include both numerical calculations of the band structure and phonon dispersion relation,⁶⁻⁸ and renormalization-group approaches are either restricted to a subspace of the couplings,⁹ or use specific initial conditions in the full space of couplings.¹⁰

In this paper, we identify the different instabilities of the (5,0) metallic nanotubes in the presence of effective electronic attractive couplings mediated by phonons. We follow previous approaches on larger nanotubes^{5,11-20} in using the Luttinger liquid framework to describe the low-energy behavior of nanotubes. Our approach is based on the band structure for *cylindrical* (5,0) nanotubes provided by various methods such as the local-density approximation (LDA), the GW method, and tight-binding calculations, consisting in three bands at the Fermi energy.^{4,21-26} Then, we study the perturbations of this band structure induced by the residual interactions between the low-energy fermions. The nature of these interactions is constrained by the specific symmetries of the initial band structure, different from the usual the-leg Fermionic ladders previously studied in Refs. 27-36. Using a Fermionic renormalization group we identify the dominant instability corresponding to each effective attractive potential. The instabilities we find involve either electronic degrees of freedom on a single band, or on a two-band subsystem, or on the whole three-band system. We focus on this last case, which corresponds to different instabilities specific to the symmetries of the (5,0) nanotubes. This allows us to determine the momentum of the phonons responsible for the main instability, analogously to the proposal for a supercon-

ductivity induced by radial breather modes in regular two-band metallic nanotubes.³⁷ For the specific instabilities, seen as strong-coupling directions of the renormalization-group flow, we use the Abelian Bosonization formalism to identify its nature and specify the corresponding dominant correlation function. This Bosonization description requires a careful treatment of the so-called Klein factors, a crucial technical point in this three-band model. We pedagogically present this problem and its solution in Sec. IV. Finally, the remaining gapless spin modes are identified using a non-Abelian Bosonization approach.³⁸⁻⁴³

The paper is organized as follows: in Sec. II we define the Fermionic model we consider and the notations used throughout the paper. The renormalization approach is sketched in Sec. III, the details being postponed to Appendix A for readability of the manuscript. The strong-coupling phases are analyzed in Sec. IV as well as the conventions used for the Bosonization formalism. The complete Bosonized expressions of all operators and necessary correlations functions are given in Appendix B. Finally we discuss the validity of our results and our main conclusions in Sec. V. Appendix C is devoted to some peculiar technical difficulties of our model associated with the Klein factor of Abelian Bosonization.

II. MODEL

A. Band structure of (5,0) nanotube

The band structure predicted by LDA-DFT (density functional theory in the local density approximation) calculations for (5,0) metallic nanotubes is depicted schematically around the Fermi energy E_F in Fig. 1. It consists of three bands.^{4,21-23,25} For a cylindrical nanotube, rotational invariance results in the conservation of the angular momentum m , and translational invariance in the conservation of momentum along the tube k_x . The quantum numbers of the three bands are thus determined accordingly. In our specific case the two bands with angular momentum $m = \pm 1$ are degenerate and possess the same Fermi momentum k_{F_1} , smaller than the Fermi momentum k_{F_0} of the band $m = 0$. Linearizing these bands near E_F , we decompose the fermion annihilation operator into

$$\psi_{0,\sigma}(x, \phi) = e^{ik_{F_0}x} \psi_{R,0,\sigma}(x) + e^{-ik_{F_0}x} \psi_{L,0,\sigma}(x) \quad (1)$$

and

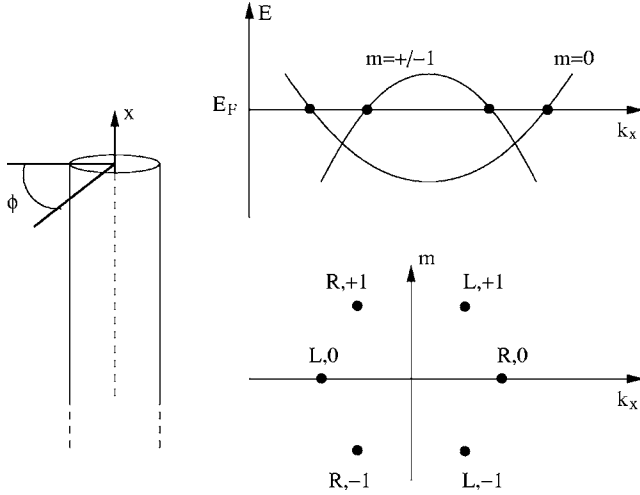


FIG. 1. Schematic representation of the band structure near E_F for the three-band nanotubes considered in this paper.

$$\psi_{m=\pm 1, \sigma}(x, \phi) = e^{-ik_{F_1} x + im\phi} \psi_{R, m, \sigma}(x) + e^{+ik_{F_1} x - im\phi} \psi_{L, m, \sigma}(x), \quad (2)$$

where $\psi_{R/L, m, \sigma}$ represent the annihilation operator for a right- (respectively left-) moving fermion of angular momentum m and spin σ . From now on x denotes the coordinate in the direction of the nanotube, and ϕ the angle along the circumference (see Fig. 1). Note that as the Fermi velocity v_{F_1} in the $m = \pm 1$ bands is negative, the right-moving fermions of these bands have a longitudinal momentum $-k_{F_1}$ as opposed to the usual situation of the band $m = 0$ where they have momentum $+k_{F_0}$ (see Fig. 1). We then describe the low-energy properties of this model by the simple Hamiltonian

$$H_0^{(0)} = - \sum_{\sigma=\uparrow, \downarrow} \sum_{m=0, \pm 1} v_{F_m} \int dx \psi_{R, m, \sigma}^\dagger \partial_x \psi_{R, m, \sigma}. \quad (3)$$

Note that as the nature of the superconducting instabilities we discuss in this paper will not depend on the differences of Fermi velocities between the bands, we will assume from now that they are all equal: $v_{F_1} = v_{F_0} = v_F$.

B. Residual interactions

Then we consider the perturbations around this band structure, which can originate either from electronic interactions not taken into account by the band-structure calculations, especially in a quasi-one-dimensional (1D) geometry, or from the effective attractive interaction originating from the coupling to phonons. An expression for these phonon-mediated couplings is given in Eq. (2) of Ref. 5. Note that a good screening of the electron-electron interaction is already assumed by the band-structure calculations via the use of a density functional for a three-dimensional electron gas. Thus to remain consistent with the calculated band structure we assume that the remaining part of the Coulomb interaction is screened and can be replaced by a perturbative short-range interaction. Such a good screening can be expected from the presence of nearby metallic gates or the screening by the

surrounding nanotubes array, similar to the case considered in Ref. 44.

Within our effective low-energy approach, we consider a minimal model possessing all the conservation laws. This results in a perturbative interaction action which can be written formally as

$$S_{\text{int}} = g_{abcd}^{(1)} \sum_{\sigma, \sigma'} \int dx \psi_{R, a, \sigma}^\dagger \psi_{L, b, \sigma'}^\dagger \psi_{R, c, \sigma'} \psi_{L, d, \sigma} + g_{abcd}^{(2)} \sum_{\sigma, \sigma'} \int dx \psi_{R, a, \sigma}^\dagger \psi_{L, b, \sigma'}^\dagger \psi_{L, c, \sigma'} \psi_{R, d, \sigma}, \quad (4)$$

where $a, b, c, d = 0, \pm 1$ stands for the band indices (angular momentum). In this expression, as is usual in 1D systems, the first part corresponds to the backscattering operators, and the second to forward-scattering operators. The forward-scattering part can be decomposed into g_4 processes and g_2 processes.^{45,46} The g_4 processes only renormalize the velocities of the particles.^{45,46} Since we have neglected the velocity differences, consistency requires us to also neglect the g_4 processes. We use the convention for the action that the partition function of the system is written as $Z = \int d\psi d\psi^\dagger \exp(-S[\psi, \psi^\dagger])$, and that repulsive (respectively, attractive) interactions between the fermions correspond to $g^{(1)}, g^{(2)} > 0$ (respectively, < 0).

To proceed, we must use the symmetry of the problem at stake to select out of all the couplings in Eq. (4) only those fulfilling the required conservation laws. As the results of band-structure calculation suggest, the Fermi wave vector k_{F_1} of the two bands ± 1 is different (and incommensurate) from the Fermi wave vector $k_{F_0} \neq k_{F_1}$ of the band with angular momentum $m = 0$. Interactions must preserve both rotational invariance and translational invariance, i.e., conserve the total angular momentum m and the total momentum k_x . To classify these interactions, we follow the notations of Refs. 47 and 10. Note that whereas this model has some superficial similarity with the three-leg ladder model,^{27,28,30,35,36} it differs from it by the symmetries as all three Fermi momenta are different, as opposed to the present case.

1. Interactions in the band $m = 0$ subsystem

The first two allowed interactions are the usual backscattering and forward-scattering interactions in the single band $m = 0$. The associated fields are denoted by $g^{(1)}$ and $g^{(2)}$ and they correspond to the action

$$S_{\text{int}}^{(0)} = -g^{(1)} \sum_{\sigma, \sigma'} \int dx \psi_{R, 0, \sigma}^\dagger \psi_{L, 0, \sigma'}^\dagger \psi_{L, 0, \sigma} \psi_{R, 0, \sigma'} + g^{(2)} \sum_{\sigma, \sigma'} \int dx \psi_{R, 0, \sigma}^\dagger \psi_{L, 0, \sigma'}^\dagger \psi_{L, 0, \sigma'} \psi_{R, 0, \sigma}. \quad (5)$$

2. Interactions in the two-band $m = \pm 1$ subsystem

The next group of interactions we consider are the forward- and backscattering couplings in the subsystem consisting in the two degenerate bands $m = \pm 1$. This corresponds

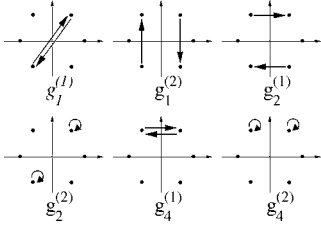


FIG. 2. Formal representation in the k_x, m plane of the considered interactions in the two-band $m = \pm 1$ subsystem.

exactly to a two-leg ladder with degenerate bands.^{29,48–50} Note that even this two-band subsystem differs from the usual description of larger nanotubes, which possess right- (or left-) moving fermions at both $+k_F$ and $-k_F$.⁶¹ The interactions are depicted schematically in Fig. 2. The explicit part of the interacting action is

$$\begin{aligned}
S_{\text{int}}^{(\pm 1)} = & \sum_{\sigma, \sigma'} \int dx - g_1^{(1)} [\psi_{R,+1,\sigma}^\dagger \psi_{L,-1,\sigma'}^\dagger \psi_{L,-1,\sigma} \psi_{R,+1,\sigma'} \\
& + (+1 \leftrightarrow -1)] + g_1^{(2)} [\psi_{R,+1,\sigma}^\dagger \psi_{L,-1,\sigma'}^\dagger \psi_{L,+1,\sigma'} \psi_{R,-1,\sigma} \\
& + (+1 \leftrightarrow -1)] - g_2^{(1)} [\psi_{R,+1,\sigma}^\dagger \psi_{L,-1,\sigma'}^\dagger \psi_{L,+1,\sigma} \psi_{R,-1,\sigma'} \\
& + (-1 \leftrightarrow -1)] + g_2^{(2)} [\psi_{R,+1,\sigma}^\dagger \psi_{L,-1,\sigma'}^\dagger \psi_{L,-1,\sigma'} \psi_{R,+1,\sigma} \\
& + (+1 \leftrightarrow -1)] - g_4^{(1)} [\psi_{R,+1,\sigma}^\dagger \psi_{L,+1,\sigma'}^\dagger \psi_{L,+1,\sigma} \psi_{R,+1,\sigma'} \\
& + (+1 \leftrightarrow -1)] + g_4^{(2)} [\psi_{R,+1,\sigma}^\dagger \psi_{L,+1,\sigma'}^\dagger \psi_{L,+1,\sigma'} \psi_{R,+1,\sigma} \\
& + (+1 \leftrightarrow -1)]. \quad (6)
\end{aligned}$$

3. Interactions between the $m=0$ band, and the two-band $m = \pm 1$

The last group of interactions, specific to the model we consider, corresponds to the interactions between the band $m=0$ and the two-band subsystem $m = \pm 1$. With our conventions for the signs of the coupling, they read

$$\begin{aligned}
S_{\text{int}}^{(0/\pm 1)} = & \sum_{\sigma, \sigma'} \int dx - f^{(1)} [\psi_{R,0,\sigma}^\dagger \psi_{L,+1,\sigma'}^\dagger \psi_{L,+1,\sigma'} \psi_{R,0,\sigma} \\
& + (+1 \leftrightarrow -1) + \text{H.c.}] + f^{(2)} \\
& \times [\psi_{R,0,\sigma}^\dagger \psi_{L,+1,\sigma'}^\dagger \psi_{L,+1,\sigma'} \psi_{R,0,\sigma} \\
& + (+1 \leftrightarrow -1) + \text{H.c.}] + u [\psi_{R,0,\sigma}^\dagger \psi_{L,0,\sigma'}^\dagger \psi_{L,+1,\sigma'} \psi_{R,-1,\sigma} \\
& + (+1 \leftrightarrow -1) + \text{H.c.}] - v [\psi_{R,+1,\sigma}^\dagger \psi_{L,-1,\sigma'}^\dagger \psi_{L,0,\sigma} \psi_{R,0,\sigma'} \\
& + (+1 \leftrightarrow -1) + \text{H.c.}] \quad (7)
\end{aligned}$$

All these couplings are depicted schematically in Fig. 3.

III. RENORMALIZATION-GROUP STUDY

A. Derivations of the scaling equations

Having defined explicitly the action describing our model, we will now study its low-energy behavior using the

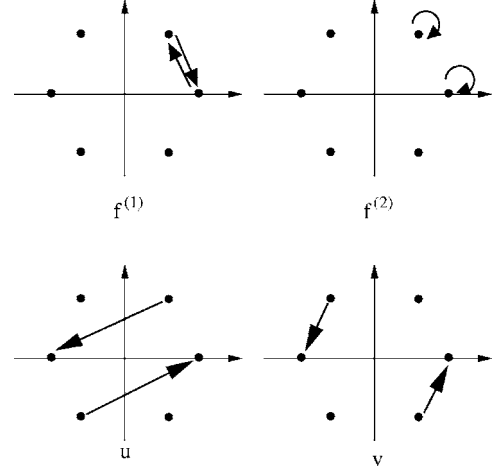


FIG. 3. Formal representation in the k_x, m plane of the considered interactions between the $m=0$ band and the two-band $m = \pm 1$ subsystem.

renormalization-group formalism. The standard procedure for one-dimensional Fermionic model is implemented by using the operator product expansion formalism (see, e.g., Ref. 62, Chap. 5). The product expansion of the four-fermion operators appearing in the perturbative expansion of the partition function reads formally

$$\begin{aligned}
& \langle \psi_{R,a}^\dagger \psi_{L,b}^\dagger \psi_{L,c} \psi_{R,d} \psi_{R,e}^\dagger \psi_{L,f}^\dagger \psi_{L,g} \psi_{R,h} \rangle \\
& = \frac{\delta_{ah} \delta_{bg}}{4\pi^2 z_a \bar{z}_b} \langle \psi_{R,e}^\dagger \psi_{L,f}^\dagger \psi_{L,c} \psi_{R,d} \rangle - \frac{\delta_{ah} \delta_{cf}}{4\pi^2 z_a \bar{z}_c} \langle \psi_{R,e}^\dagger \psi_{L,b}^\dagger \psi_{L,g} \psi_{R,d} \rangle \\
& - \frac{\delta_{bg} \delta_{de}}{4\pi^2 z_d \bar{z}_b} \langle \psi_{R,a}^\dagger \psi_{L,f}^\dagger \psi_{L,c} \psi_{R,h} \rangle \\
& + \frac{\delta_{cf} \delta_{de}}{4\pi^2 z_d \bar{z}_c} \langle \psi_{R,a}^\dagger \psi_{L,b}^\dagger \psi_{L,g} \psi_{R,h} \rangle, \quad (8)
\end{aligned}$$

where we have used mixed labels a, b, c, d, e, f, g, h for the band m and spin σ . In this expression, z_a stands for $x - iv_a \tau$. Within the approximation $v_{F_a} = v_F$, all the prefactors will produce the constant

$$\frac{1}{4\pi^2 v_F} \int_{a < |z| < ae^{dl}} \frac{dz d\bar{z}}{z \bar{z}} = \frac{dl}{2\pi v_F}, \quad (9)$$

where a is a real-space ultraviolet cutoff. Thus specifying the operator product expansion (8) to the interactions of our model, we obtain the renormalization-group equations to second order in the couplings $g_i^{(j)}, f_i, u, v$. They are given explicitly in formula (A3) in Appendix A. They only differ from those of Ref. 10 by an extra term $2(u^2 + v^2)$ in the equation for $\partial_l g_2^{(2)}$.

B. Renormalization flow integration

The scaling equations (A3) admit asymptotic solutions of the form

$$g_i^{(j)}(l) = \frac{c_{ij}}{(l^* - l)^{\mu_{ij}}} + \mathcal{O}[(l - l^*)^{-\mu_{ij}}]. \quad (10)$$

However, a direct analytical solution in the full parameter space is not tractable. Hence we have numerically integrated these equations for different initial values of the couplings. Our strategy was to choose some reasonable perturbative initial point in the parameter space, and to study the instability from this starting point occurring upon increasing the strength of a given attractive interaction. We have done this for all the possible attractive interactions, i.e. the different phonons coupling the different electronic branches of the model: $g^{(1)}$, $f^{(1)}$, $g_1^{(1)}$, $g_1^{(2)}$, $g_2^{(1)}$, $g_4^{(1)}$, u , v . We have also checked that the results were independent of the initial perturbative point chosen.

Each of these instabilities corresponds to a strong-coupling direction where at least some of the couplings $g_i^{(j)}$, $f^{(i)}$, u , v diverge at a finite scaling length l^* . Thus we characterize each strong-coupling direction by the subset of the most diverging couplings, namely those with the largest power μ_{ij} in Eq. (10). Indeed, we have found that for the considered directions, while the dominant couplings always diverge with an exponent $\mu_{ij}=1$, there exist other couplings diverging with smaller exponents $\mu_{ij}<1$. These couplings with weaker divergences are expected to give rise to anomalous scaling^{63,64} but not to modify the strong-coupling phases.

The nature of the instability corresponding to a given strong-coupling fixed point will be identified in Sec. IV, by Bosonizing the model in the subspace consisting of the dominant diverging couplings. We will focus particularly on the instabilities specific to the three-band model.

1. Single-band or two-band model instabilities

We first list the instabilities of the band $m=0$, and two-band $m=\pm 1$ subsystem. These instabilities are not specific to the present model, and have been previously studied (see, e.g., Refs. 65 and 66, and references therein). The first instability is obtained for a negative g_1 , i.e., an attractive interaction in the $m=0$ band. The corresponding asymptotic fixed point, at which g_1 and g_2 both diverge to $-\infty$, is the well-known instability of the Luther-Emery model.^{45,67,68}

Three different instabilities affect only the bands $m=\pm 1$. Upon decreasing $g_1^{(1)}$ to negative values, we find that the dominant diverging couplings are $g_1^{(1)}$, $g_1^{(2)}$, $g_2^{(1)}$, $g_2^{(2)}$, and $g_4^{(2)}$. While $g_1^{(1)}$ and $g_2^{(2)}$ flow towards $-\infty$, $g_1^{(2)}$, $g_2^{(1)}$, and $g_4^{(2)}$ flow towards $+\infty$. The phase associated with a negative coupling $g_1^{(2)}<0$ is described by dominant divergence of $g_2^{(1)}\rightarrow+\infty$ and $g_2^{(2)}$, $g_4^{(1)}$, $g_4^{(2)}\rightarrow-\infty$. Finally a negative $g_4^{(1)}<0$ induces the phase $g_2^{(1)}$, $g_2^{(2)}$, $g_4^{(1)}$, $g_4^{(2)}\rightarrow-\infty$. All these strong-coupling fixed points correspond to the superconducting phase of a Fermionic two-leg ladder, associated with different symmetries.^{29,48–50,52–54,56,58}

2. Three-bands instabilities

We now focus on the phases induced by attractive interactions specific to the three-band nanotube model, namely $f^{(1)}$, u , and v . In all three cases, the dominant divergent cou-

plings are $g^{(2)}$, $f^{(1)}$, $f^{(2)}$, $g_1^{(2)}$, $g_2^{(2)}$, $g_4^{(2)}$, v . We have identified two pairs of asymptotic directions, one induced by negative $f^{(1)}$ or u , and the second by v . These two pairs of strong-coupling directions differ only by the sign of the asymptotic $v(l^*)$. In both cases, $g^{(2)}$, $f^{(1)}$, $f^{(2)}$, $g_2^{(2)}$ flow towards $-\infty$, $g_4^{(2)}$ towards $+\infty$ and $g_1^{(2)}\rightarrow\pm\infty$. The first direction corresponds to $v\rightarrow\infty$, and the second to $v\rightarrow-\infty$. Two numerical flows obtained by slowly decreasing either u or v to negative values are shown in Figs. 4 and 5.

It is instructive to analyze further the renormalization flow by focusing in the subspace of the dominant couplings $g^{(2)}$, $f^{(1)}$, $f^{(2)}$, $g_1^{(2)}$, $g_2^{(2)}$, $g_4^{(2)}$, v . Indeed this subspace is stable under the renormalization-group (RG) equations (A3). When restricted to this subspace, these equations read

$$\partial_l \tilde{g}^{(2)} = -2\tilde{v}^2, \quad (11a)$$

$$\partial_l \tilde{g}_1^{(2)} = -2\tilde{g}_1^{(2)}\tilde{g}_2^{(2)} + 2\tilde{g}_1^{(2)}\tilde{g}_4^{(2)} - \tilde{v}^2, \quad (11b)$$

$$\partial_l \tilde{g}_2^{(2)} = -(\tilde{g}_1^{(2)})^2 - \tilde{v}^2, \quad (11c)$$

$$\partial_l \tilde{g}_4^{(2)} = (\tilde{g}_1^{(2)})^2, \quad (11d)$$

$$\partial_l \tilde{f}^{(1)} = -2(\tilde{f}^{(1)})^2 - 2\tilde{v}^2, \quad (11e)$$

$$\partial_l \tilde{f}^{(2)} = -(\tilde{f}^{(1)})^2, \quad (11f)$$

$$\partial_l \tilde{v} = -(4\tilde{f}^{(1)} - 2\tilde{f}^{(2)} + \tilde{g}_1^{(2)} + \tilde{g}^{(2)} + \tilde{g}_2^{(2)})\tilde{v}. \quad (11g)$$

These equations possess two scaling invariants: $C=2\tilde{g}_4^{(2)} + 2\tilde{g}_2^{(2)} - \tilde{g}^{(2)}$ and $D=2\tilde{f}^{(2)} - \tilde{f}^{(1)} + \tilde{g}^{(2)}$. Let us start by considering the RG flow in the subspace $\tilde{v}=0$. Introducing the variable $Y=\tilde{g}_4^{(2)} - \tilde{g}_2^{(2)}$, the RG equations reduce to those of Kosterlitz and Thouless:

$$\partial_l Y = 2(\tilde{g}_1^{(2)})^2; \quad \partial_l \tilde{g}_1^{(2)} = 2\tilde{g}_1^{(2)}Y, \quad (12a)$$

$$\partial_l \tilde{f}^{(1)} = -2(\tilde{f}^{(1)})^2 \quad (12b)$$

and $\tilde{g}^{(2)}$ is a flow constant. The asymptotic solutions are thus the two directions (A and B in Fig. 6) $Y(l) \simeq 1/[2(l^* - l)]$ and $\tilde{g}_1^{(2)} = \pm 1/[2(l^* - l)]$ and the line (C) $\tilde{g}_1^{(2)}=0$, coupled to the solutions $\tilde{f}^{(1)}=0$ or $\tilde{f}^{(1)}=-1/[2(l^* - l)]$. The solutions corresponding to $\tilde{g}_1^{(2)}=0$ or $\tilde{f}^{(1)}=0$ are easily found to be unstable when introducing a small v .

The scaling behavior of \tilde{v} can be deduced from inspection of Eqs. (11). Except for extremely large initial values of $\tilde{g}_4^{(2)}$, \tilde{v} will always end up diverging to $\pm\infty$, in the direction given by its initial sign. The scaling equation (11b) shows that an increasing $v(l)$ leads to an instability of the asymptotic direction A. The only remaining scaling direction driven by v corresponds to point B in Fig. 6. The other possibility corresponds to strong initial intraband repulsion $g_4^{(2)}$, leading to a large and positive $g_1^{(2)}$. These strong interactions naturally forbid superconducting interactions, and we will not consider them in the following.

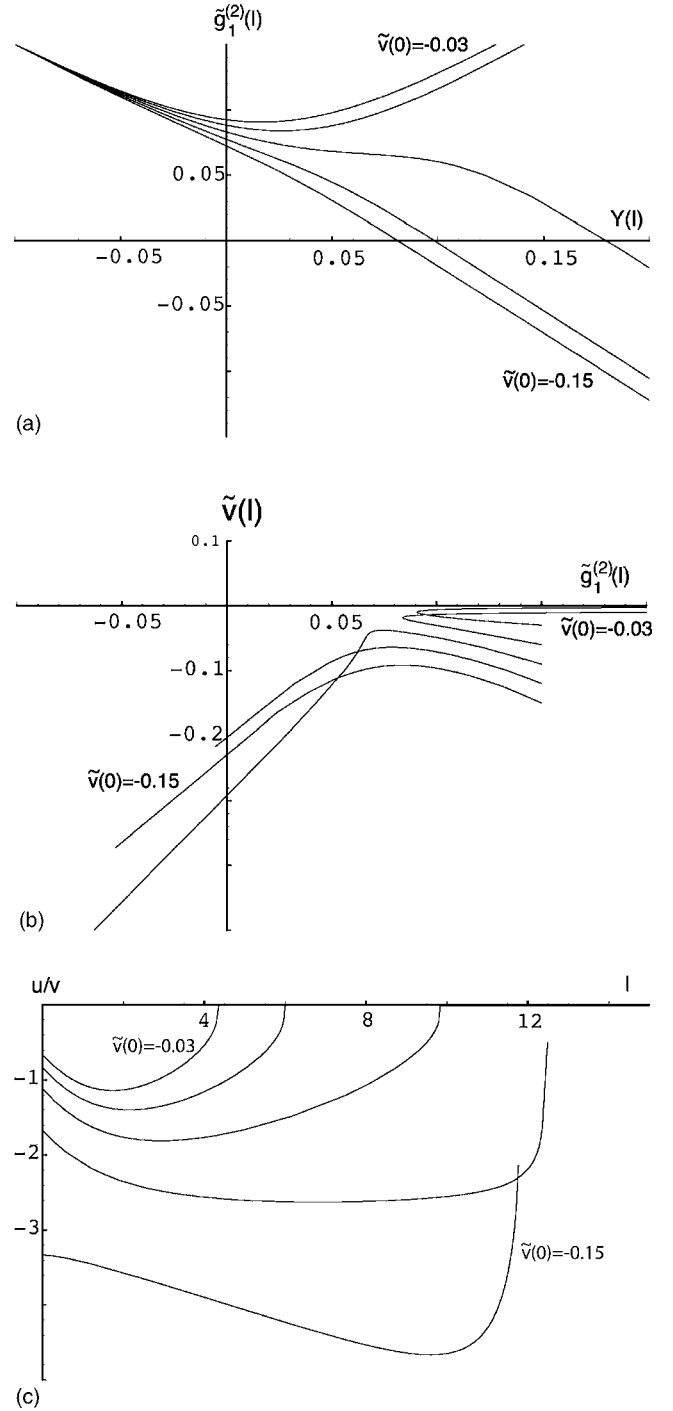
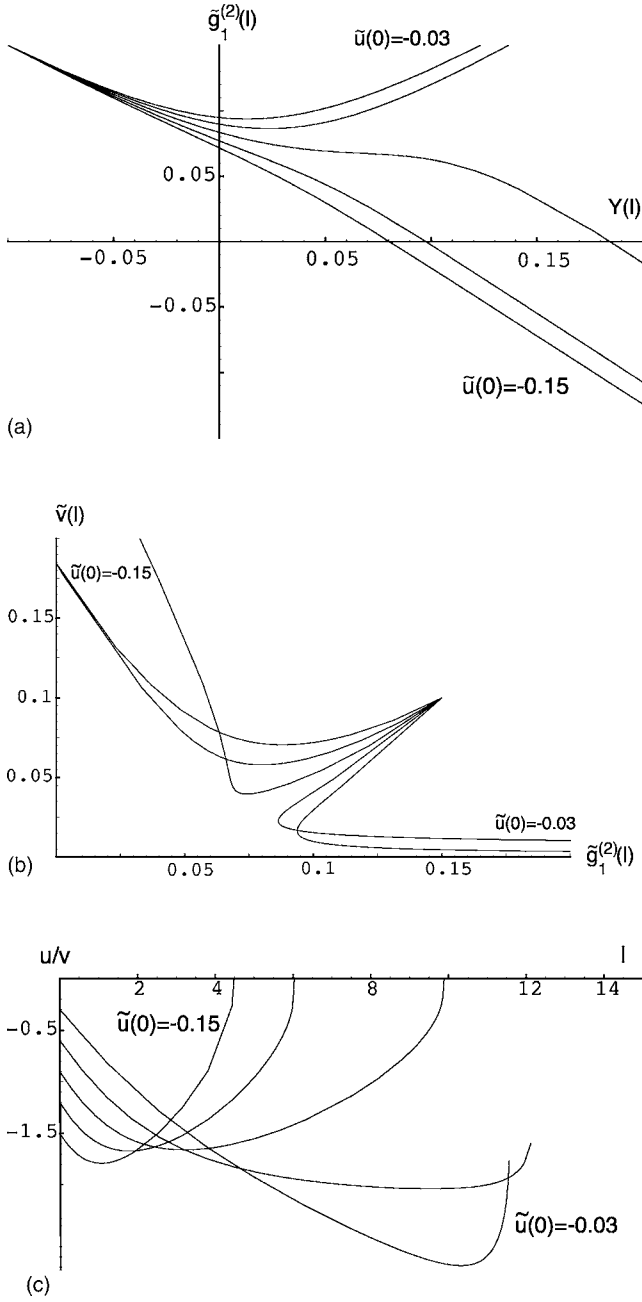


FIG. 4. Numerical renormalization-group flow showing the instability induced by a negative u in the $Y = \tilde{g}_4^{(2)} - \tilde{g}_2^{(2)}, \tilde{g}_1^{(2)}$ plane (left), $\tilde{g}_1^{(2)}, \tilde{v}$ plane (middle), and the ratio $u(l)/v(l)$ as a function of l (right). These flows corresponds to the following initial values $\tilde{g}^{(1)} = 0.17, \tilde{g}^{(2)} = 0.3, \tilde{f}^{(1)} = 0.1, \tilde{f}^{(2)} = 0.09, \tilde{g}_1^{(1)} = 0.13, \tilde{g}_1^{(2)} = 0.15, \tilde{g}_2^{(1)} = 0.05, \tilde{g}_2^{(2)} = 0.2, \tilde{g}_4^{(1)} = 0.08, \tilde{g}_4^{(2)} = 0.1,$ and $\tilde{v} = 0.1$. \tilde{u} was taken to negative values in steps of 0.03: $\tilde{u} = -0.03i$ for $i = 1, 5$.

To complement the above analysis perturbative in v , we have analytically solved Eqs. (11) with the ansatz $g(l) \simeq A_g / (l - l^*)$ for all couplings. We have found that besides the $\tilde{v} = 0$ fixed points, there exist three strong-coupling directions corresponding to $\tilde{g}^{(2)}, \tilde{f}^{(1)}, \tilde{f}^{(2)}, g_2^{(2)} \rightarrow -\infty, g_4^{(2)} \rightarrow +\infty,$ and $\tilde{g}_1^{(2)} \rightarrow \pm\infty$ (together with $\tilde{v} \rightarrow \pm\infty$). The first direction corresponds to the direction $\tilde{g}_1^{(2)} = 1/[2(l^* - l)]$ (point A in Fig. 6) which is the point induced by a very strong initial $g_4^{(2)}$ dis-

FIG. 5. Numerical renormalization-group flow showing the instability induced by a negative v in the $Y = \tilde{g}_4^{(2)} - \tilde{g}_2^{(2)}, \tilde{g}_1^{(2)}$ plane (left), $\tilde{g}_1^{(2)}, \tilde{v}$ plane (middle), and the ratio $u(l)/v(l)$ as a function of l (right). These flows correspond to the same initial values as in Fig. 4, except that \tilde{u} was held to $\tilde{u} = 0.1$ and $\tilde{v} = -0.03i$ for $i = 1, 5$.

cussed above. Both the last two directions correspond to the same limiting sign of the coupling constants. They differ only by the numerical value of the ansatz parameters, and correspond to the main instability discussed in the following. We have thus identified a different instability, specific to the three band model we consider. We now turn to the bosonization formalism to identify the nature of the phase corre-

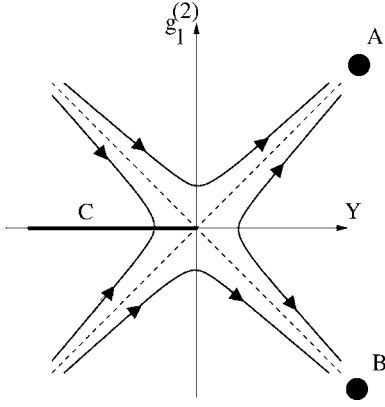


FIG. 6. Schematic renormalization-group flow in the $\tilde{g}_1^{(2)}$, $Y = \tilde{g}_4^{(2)} - \tilde{g}_2^{(2)}$ plane, for $\tilde{v}=0$.

sponding to this renormalization flow direction.

IV. BOSONIZATION AND NATURE OF THE INSTABILITIES

The purpose of this section is to identify the nature of the instability for the previously identified strong-coupling direction listed in Table I. This will be achieved using the Bosonization formalism within the subspace corresponding to the dominant couplings. We will pay special attention to the proper definition of the so-called Klein factor. We first start by defining our conventions on the noninteracting three-band model defined in Sec. II.

A. ‘‘Condensed matter’’ Bosonization dictionary

In the standard ‘‘condensed-matter’’ Bosonization procedure, we represent the annihilation operators of right- and left-moving fermions, defined in Eq. (1), as^{69–71}

$$\psi_{R,m,\sigma}(x) = \eta_{R,m,\sigma} \frac{1}{\sqrt{2\pi a}} e^{-i\Phi_{R,m,\sigma}(x)}, \quad (13a)$$

$$\psi_{L,m,\sigma}(x) = \eta_{L,m,\sigma} \frac{1}{\sqrt{2\pi a}} e^{i\Phi_{L,m,\sigma}(x)}, \quad (13b)$$

where we introduced Majorana fermion operators (the so-called Klein factors) $\eta_{R/L,m,\sigma}$ that satisfy

$$\{\eta_{R,m,\sigma}, \eta_{R,m',\sigma'}\}_+ = 2\delta_{m,m'}\delta_{\sigma,\sigma'}, \quad (14a)$$

$$\{\eta_{L,m,\sigma}, \eta_{L,m',\sigma'}\}_+ = 2\delta_{m,m'}\delta_{\sigma,\sigma'}, \quad (14b)$$

TABLE I. Table of the dominant couplings and their asymptotic directions corresponding to the three-band instability on which we focus.

Backscattering				Forward scatt.		
$g^{(2)}$	$f^{(2)}$	$g_1^{(2)}$	$g_2^{(2)}$	$g_4^{(2)}$	$f^{(1)}$	v
$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$+\infty$	$\pm\infty$

$$\{\eta_{R,m,\sigma}, \eta_{L,m',\sigma'}\}_+ = 0. \quad (14c)$$

Note that in this convention, we introduce one Klein factor per set of quantum numbers $(\pm k_F, m, \sigma)$. With these anticommutation relations, the proper anticommutation relations for the fermion operators defined in Eq. (13) are satisfied with the following commutation relation of the fields $\Phi_{R/L}$:

$$[\Phi_{R,m,\sigma}(x), \Phi_{R,m',\sigma'}(x')] = i\pi\delta_{m,m'}\delta_{\sigma,\sigma'} \operatorname{sgn}(x-x'),$$

$$[\Phi_{L,m,\sigma}(x), \Phi_{L,m',\sigma'}(x')] = -i\pi\delta_{m,m'}\delta_{\sigma,\sigma'} \operatorname{sgn}(x-x'),$$

$$[\Phi_{L,m,\sigma}(x), \Phi_{R,m',\sigma'}(x')] = 0. \quad (15)$$

With these conventions, the Bosonized noninteracting Hamiltonian reads:^{45,68–71}

$$H = \sum_{m=0,\pm 1} \int \frac{dx}{4\pi} v_F [(\nabla\Phi_{R,m,\sigma})^2 + (\nabla\Phi_{L,m,\sigma})^2]. \quad (16)$$

$\sigma=\uparrow,\downarrow$

Finally, the densities of right-moving and left-moving fermions read, respectively,^{69–71}

$$\rho_{R,m,\sigma} = -\frac{\nabla\Phi_{R,m,\sigma}}{2\pi}, \quad (17a)$$

$$\rho_{L,m,\sigma} = -\frac{\nabla\Phi_{L,m,\sigma}}{2\pi}. \quad (17b)$$

Keeping only the g_2 processes defined in Sec. II B, we express the forward-scattering part of the interactions in terms of the above densities (17):

$$\begin{aligned} H_{\text{forward}} = & g_4^{(2)} \sum_{\sigma,\sigma'} (\rho_{R,1,\sigma}\rho_{L,1,\sigma'} + \rho_{R,-1,\sigma}\rho_{L,-1,\sigma'}) \\ & + f^{(2)} \sum_{\sigma,\sigma'} [(\rho_{R,1,\sigma} + \rho_{R,-1,\sigma})\rho_{L,0,\sigma'} + (\rho_{L,1,\sigma} \\ & + \rho_{L,-1,\sigma})\rho_{R,0,\sigma'}] + g_2^{(2)} \sum_{\sigma,\sigma'} (\rho_{R,1,\sigma}\rho_{L,-1,\sigma'} \\ & + \rho_{R,-1,\sigma}\rho_{L,1,\sigma'}) + g^{(2)} \sum_{\sigma,\sigma'} \rho_{R,0,\sigma}\rho_{L,0,\sigma'}. \end{aligned} \quad (18)$$

Using Eq. (17), these expressions can be reduced to quadratic expressions in the fields $\Phi_{R/L,m,\sigma}$. Note that we will treat the $g_1^{(2)}$ term below with the backscattering part of the Hamiltonian: although it appears as a forward-scattering term, it cannot be reduced to a density-density coupling and its treatment closely follows the one for the backscattering couplings.

B. Backscattering interactions

1. Klein factors problem

Whereas the Bosonized forward-scattering part of the Hamiltonian is function solely of the densities, and thus does not depend on the convention chosen for the Klein factor (14) and fields (15), the situation is different for the back-

scattering part of the action (4). Quite generally, these back-scattering operators can be written as

$$\psi_{R,m_1,\sigma}^\dagger \psi_{L,m_2,\sigma'}^\dagger \psi_{L,m_3,\sigma'} \psi_{R,m_4,\sigma} = \frac{1}{(2\pi a)^2} \mathcal{N}_{\{m\},\sigma,\sigma'} \mathcal{O}_{\{m\},\sigma,\sigma'}, \quad (19)$$

where we assumed the transverse momentum conservation $m_1 + m_2 = m_3 + m_4$ and we defined the product of Majorana fermions

$$\mathcal{N}_{\{m\},\sigma,\sigma'} = \eta_{R,m_1,\sigma} \eta_{L,m_2,\sigma'} \eta_{L,m_3,\sigma'} \eta_{R,m_4,\sigma} \quad (20)$$

and the product of vertex operators

$$\mathcal{O}_{\{m\},\sigma,\sigma'} = e^{i\Phi_{R,m_1,\sigma}} e^{-i\Phi_{L,m_2,\sigma'}} e^{i\Phi_{L,m_3,\sigma'}} e^{-i\Phi_{R,m_4,\sigma}}. \quad (21)$$

The usual strategy is to find a representation such that the operators $\mathcal{O}_{\{m\},\sigma,\sigma'}$ commute with each other, and similarly for the products of four Majorana fermions $\mathcal{N}_{\{m\},\sigma,\sigma'}$. It is important to note that the only case discussed in Ref. 29 is the one in which all the vertex operators $\mathcal{O}_{\{m\},\sigma,\sigma'}$ are already commuting so that no redefinition of the fields is necessary. When determining the ground state, it is then possible to consider the Sine-Gordon form of the Hamiltonian, obtained by replacing the operators $\mathcal{N}_{\{m\},\sigma,\sigma'}$ by their eigenvalues.

Obviously, while the four fermion operators (19) always commute with each other, the above condition of independent commutation of the $\mathcal{N}_{\{m\},\sigma,\sigma'}$ and $\mathcal{O}_{\{m\},\sigma,\sigma'}$ becomes more and more difficult to fulfill with an increasing number of fermion species. This is particularly true for our three-band model, corresponding to 12 Fermionic species (two spins and six Fermi points), and we can check that this condition cannot be satisfied for the operators we consider within the convention defined by Eqs. (13)–(15). Indeed, the products of four Majorana fermion operators are commuting when they have an even number of Majorana fermions in common and anticommuting otherwise. In the second case, which occurs for our model (see, e.g., the operator v), the corresponding operators $\mathcal{O}_{\{m\},\sigma,\sigma'}$ also contain an odd number of vertex operators in common. Since the vertex operators associated with fermions are anticommuting when they correspond to the same fermion species, we recover in the Bosonization formalism the commutation of the four Fermi operators, but the independent commutations of the $\mathcal{O}_{\{m\},\sigma,\sigma'}$ and $\mathcal{N}_{\{m\},\sigma,\sigma'}$ is not possible. We thus need to change our Bosonization convention for this particular model.

2. “Field theory” convention

The problem we have to deal with is thus whether it is possible to redefine the Majorana fermion operators (14) and the commutation relations of the fields (15) that appear in the vertex operators in such a way that all the new products of four Majorana fermion operators are commuting with each other and simultaneously all the new products of vertex operators are also commuting with each other. Another possible convention, different from the above “condensed-matter” convention (left- or right-moving chiral field of the same band commuting with each other, and one Majorana fermion

per Fermi point⁷¹), consists in what we will call the “quantum field theory” convention. We now introduce a single Majorana fermion for a pair of right and left fermions which need not to belong to the same band. Correspondingly, the chiral fields for this pair have a nonzero commutator. In this representation, the fermion operators are now expressed as⁹⁹

$$\psi_{R,m,\sigma}(x) = \frac{1}{\sqrt{2\pi a}} e^{-i\tilde{\Phi}_{R,m,\sigma}(x)} \eta_{m,\sigma}, \quad (22a)$$

$$\psi_{L,P(m),\sigma}(x) = \frac{1}{\sqrt{2\pi a}} e^{i\tilde{\Phi}_{L,m,\sigma}(x)} \eta_{m,\sigma}, \quad (22b)$$

where P is a permutation of the band indices (fermion species). Now the field commutations relations are modified into

$$[\tilde{\Phi}_{R,m,\sigma}(x), \tilde{\Phi}_{R,m',\sigma'}(x')] = i\pi \delta_{m,m'} \delta_{\sigma,\sigma'} \operatorname{sgn}(x - x'),$$

$$[\tilde{\Phi}_{L,m,\sigma}(x), \tilde{\Phi}_{L,m',\sigma'}(x')] = -i\pi \delta_{m,m'} \delta_{\sigma,\sigma'} \operatorname{sgn}(x - x'),$$

$$[\tilde{\Phi}_{R,m,\sigma}(x), \tilde{\Phi}_{L,m',\sigma'}(x')] = i\pi \delta_{m,m'} \delta_{\sigma,\sigma'} \quad (23)$$

and the Majorana fermion operators $\eta_{m,\sigma}$ satisfy

$$\{\eta_{m,\sigma}, \eta_{m',\sigma'}\} = 2\delta_{m,m'} \delta_{\sigma,\sigma'}. \quad (24)$$

We discuss the equivalence of these two representations in Appendix C. Let us now apply this convention to the present model. We have found that the suitable (necessary) permutation P of band indices in Eq. (22) is simply a permutation of the bands +1 and -1 and leaves the band 0 untouched.

The remaining interactions $g_1^{(2)}$, $f^{(1)}$, v are conveniently expressed in terms of the following non chiral fields:⁴⁶

$$\begin{aligned} \theta_{m,\sigma} &= \frac{1}{2} (\tilde{\Phi}_{L,m,\sigma} - \tilde{\Phi}_{R,m,\sigma}), \\ \phi_{m,\sigma} &= \frac{1}{2} (\tilde{\Phi}_{L,m,\sigma} + \tilde{\Phi}_{R,m,\sigma}). \end{aligned} \quad (25)$$

These fields satisfy $[\phi_{m,\sigma}(x), \phi_{m',\sigma'}(x')] = [\theta_{m,\sigma}(x), \theta_{m',\sigma'}(x')] = 0$ and $[\phi_{m,\sigma}(x), \theta_{m',\sigma'}(x')] = i\frac{\pi}{2} \delta_{m,m'} \delta_{\sigma,\sigma'} \operatorname{sgn}(x' - x)$. Taking the derivative with respect to x' and introducing $\Pi_{m,\sigma}(x) = \frac{1}{\pi} \partial_x \theta_{m,\sigma}$ one finds $[\phi_{m,\sigma}(x), \Pi_{m,\sigma}(x')] = i\delta(x - x')$, showing that the fields $\Pi_{m,\sigma}$ and $\phi_{m,\sigma}$ are canonically conjugate. It is convenient to introduce the total and relative “charge” and “spin” fields:⁴⁶

$$\theta_{c,m} = \frac{1}{\sqrt{2}} (\theta_{m,\uparrow} + \theta_{m,\downarrow}), \quad (26a)$$

$$\theta_{s,m} = \frac{1}{\sqrt{2}} (\theta_{m,\uparrow} - \theta_{m,\downarrow}), \quad (26b)$$

$$\phi_{c,m} = \frac{1}{\sqrt{2}} (\phi_{m,\uparrow} + \phi_{m,\downarrow}), \quad (26c)$$

$$\phi_{s,m} = \frac{1}{\sqrt{2}}(\phi_{m,\uparrow} - \phi_{m,\downarrow}). \quad (26d)$$

Finally, the following rotation for the charge modes will simplify the expressions of the interactions:

$$\begin{pmatrix} \phi_{c,+} \\ \phi_{c,A} \\ \phi_{c,B} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} \phi_{c,1} \\ \phi_{c,-1} \\ \phi_{c,0} \end{pmatrix}. \quad (27)$$

The same rotation is performed for $\theta_{c,i}$ so that canonical commutation relations are preserved. A different rotation must be performed for the spin modes:

$$\begin{pmatrix} \phi_{s,+} \\ \phi_{s,-} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \phi_{s,1} \\ \phi_{s,-1} \end{pmatrix}. \quad (28)$$

With these notations and within the above field-theoretic representation, the Bosonized expression of the $g_1^{(2)}$ term becomes

$$\frac{g_1^{(2)}}{(\pi a)^2} \int dx \cos 2\theta_{c,A} (\cos 2\theta_{s-} + \cos 2\phi_{s-} \eta_{1,\uparrow} \eta_{1,\downarrow} \eta_{-1,\downarrow} \eta_{-1,\uparrow}). \quad (29)$$

After some simple algebra, we find for the $f^{(1)}$ part of the Hamiltonian the simplified expression

$$\begin{aligned} & -\frac{f^{(1)}}{(2\pi a)^2} \sum_{\sigma} [e^{-i\sigma\sqrt{2}(\theta_{s,0}-\phi_{s,0})} \eta_{0,\sigma} \eta_{0,-\sigma} e^{i\sigma(\theta_{s,+}+\phi_{s,+})} \\ & \times (e^{i\sigma(\theta_{s,-}+\phi_{s,-})} \eta_{-1,-\sigma} \eta_{-1,\sigma} + e^{-i\sigma(\theta_{s,-}+\phi_{s,-})} \eta_{1,-\sigma} \eta_{1,\sigma}) \\ & + e^{-i\sigma\sqrt{2}(\theta_{s,0}+\phi_{s,0})} \eta_{0,\sigma} \eta_{0,-\sigma} e^{i\sigma(\theta_{s,+}-\phi_{s,+})} (e^{i\sigma(\theta_{s,-}-\phi_{s,-})} \eta_{1,-\sigma} \eta_{1,\sigma} \\ & + e^{-i\sigma(\theta_{s,-}-\phi_{s,-})} \eta_{-1,-\sigma} \eta_{-1,\sigma})], \end{aligned} \quad (30)$$

and for the only remaining v coupling, the Bosonization expressions reads

$$\begin{aligned} & \frac{2v}{(\pi a)^2} \{ \eta_{1\uparrow} \eta_{1\downarrow} \eta_{0\downarrow} \eta_{0\uparrow} [\cos(\sqrt{3}\theta_{c,B}) \cos \theta_{c,A} \cos \phi_{s-} \cos(\sqrt{2}\phi_{s,0} \\ & + \phi_{s,+}) + \sin(\sqrt{3}\theta_{c,B}) \sin \theta_{c,A} \sin \phi_{s-} \sin(\sqrt{2}\phi_{s,0} + \phi_{s,+})] \\ & - [\cos(\sqrt{3}\theta_{c,B}) \cos \theta_{c,A} \cos \theta_{s-} \cos(\sqrt{2}\theta_{s,0} - \theta_{s,+}) \\ & - \sin(\sqrt{3}\theta_{c,B}) \sin \theta_{c,A} \sin \theta_{s-} \sin(\sqrt{2}\theta_{s,0} - \theta_{s,+})] \}. \end{aligned} \quad (31)$$

We immediately observe that the change of Bosonization convention results in an important simplification. In this field-theoretic representation, the Majorana fermion product in the $g_1^{(2)}$ term is commuting with the Majorana fermion products that appear in the $f^{(1)}$ and v terms. In the usual representation, where the same Majorana fermion is associated with the right-moving and left-moving fermions belonging to the same band, the products of Majorana fermion

operators would be anticommuting, and this anticommutation would be compensated by the anticommutation of the vertex operators. By our choice of right- and left-moving pairs, we have eliminated this double anticommutation, and we are in position to apply the scheme of Ref. 29. The commutation of the products of Majorana fermion operators allows for their simultaneous diagonalization. Representing the two Majorana fermion products as pseudospins, $\eta_{1,\uparrow} \eta_{1,\downarrow} = i\tilde{\sigma}_1^z$, $\eta_{-1,\uparrow} \eta_{-1,\downarrow} = i\tilde{\sigma}_{-1}^z$, $\eta_{0,\uparrow} \eta_{0,\downarrow} = i\tilde{\sigma}_0^z$, and choosing the +1 eigenvalues of the $\tilde{\sigma}_m^z$, we obtain the final Bosonized action, which takes the generalized sine-Gordon form. The first $g_1^{(2)}$ term reads

$$\frac{g_1^{(2)}}{(\pi a)^2} \int dx \cos 2\theta_{c,A} (\cos 2\theta_{s-} + \cos 2\phi_{s-}), \quad (32)$$

the $f^{(1)}$ term simplifies into

$$\begin{aligned} & -\frac{f^{(1)}}{(\pi a)^2} \{ \cos[(\theta_{s,+} + \phi_{s,+}) - \sqrt{2}(\theta_{s,0} - \phi_{s,0})] \cos(\theta_{s,-} + \phi_{s,-}) \\ & + \cos[(\theta_{s,+} - \phi_{s,+}) - \sqrt{2}(\theta_{s,0} + \phi_{s,0})] \cos(\theta_{s,-} - \phi_{s,-}) \}, \end{aligned} \quad (33)$$

and finally the v term can be written as

$$\begin{aligned} & \frac{2v}{(\pi a)^2} [\cos(\sqrt{3}\theta_{c,B}) \cos \theta_{c,A} \cos \phi_{s-} \cos(\sqrt{2}\phi_{s,0} + \phi_{s,+}) \\ & + \sin(\sqrt{3}\theta_{c,B}) \sin \theta_{c,A} \sin \phi_{s-} \sin(\sqrt{2}\phi_{s,0} \\ & + \phi_{s,+}) \cos(\sqrt{3}\theta_{c,B}) \cos \theta_{c,A} \cos \theta_{s-} \cos(\sqrt{2}\theta_{s,0} - \theta_{s,+}) \\ & - \sin(\sqrt{3}\theta_{c,B}) \sin \theta_{c,A} \sin \theta_{s-} \sin(\sqrt{2}\theta_{s,0} - \theta_{s,+})]. \end{aligned} \quad (34)$$

C. Analysis of the strong-coupling fixed points: superconducting instability

Having obtained the above expressions (32)–(34), we are now ready to characterize the phases corresponding to the strong-coupling directions identified in the renormalization study of Sec. III B 2. We will only focus on the new instability specific to a three-band model. This instability corresponds to a divergence of $g_1^{(2)} \rightarrow -\infty$, $f^{(1)} \rightarrow +\infty$, and $v \rightarrow \pm\infty$ (see Table I). The two signs of v are possible, depending on the driving attractive perturbations (e.g., u or v). From Eq. (32), we find that large negative values of $g_1^{(2)}$ induce a locking of the field $\theta_{c,A} = 0$. θ_{s-} and ϕ_{s-} being dual to each other, no further information on the spin part can be gained at this point. The $f^{(1)}$ interaction being a current-current interaction part, we will postpone its analysis to the next section. And finally, plugging the result $\theta_{c,A} = 0$ into Eq. (34), we find that large values of v will induce a locking of the charge field $\theta_{c,B}$ to

$$\theta_{c,B} = \begin{cases} \frac{\pi}{\sqrt{3}} & \text{if } v > 0 \\ 0 & \text{if } v < 0 \end{cases}. \quad (35)$$

Thus we find that in the ground state corresponding to this instability, the fields $\theta_{c,A}$ and $\theta_{c,B}$ are locked so as to mini-

mize the condensation energy. This implies that the corresponding charge degrees of freedom develop a gap. The total charge remains gapless as a result of the global U(1) symmetry. Thus only a single charge degree of freedom remains. The analysis of the spin degrees of freedom is more difficult, and is done in the next section. However, knowing which charge modes are gapped already enables us to determine some of the order parameters, and find the corresponding nature of the instability.

In any case, the long-range ordering of the charge fields $\theta_{c,B}$ and $\theta_{c,A}$ has important consequences. Indeed, it is seen from the Bosonized expressions (B12)–(B18) of the charge-density wave operators that none of them can develop quasi-long-range order. On the other hand, superconducting fluctuations are strongly reinforced by the ordering of the charge fields, as can be seen on the corresponding expressions (B4) derived in Appendix B. Hence we have analyzed this instability as being driven by superconducting fluctuations.

It is worthwhile to contrast our results with those previously obtained in three-leg ladders.²⁷ In our notations, it was found that in the three-leg ladder system, the only charge field developing long-range order was $\theta_{c,A}$. Here, by contrast, we find that two charge fields are developing a long-range order, $\theta_{c,A}$ and $\theta_{c,B}$. The difference of behavior of the three-band nanotube and the three-leg ladder is a consequence of the equality of the Fermi wave vectors of the band of angular momentum ± 1 which is itself a consequence of the rotational symmetry of the tube. This equality of wave vector allows extra interactions between the bands ± 1 such as $g_1^{(2)}$ and between the two bands ± 1 and the band 0 (such as u or v). The existence of these interactions is driving the system to a different fixed point. As one more charge mode is gapped in the nanotube compared with the three-leg ladder, the reinforcement of superconducting fluctuations is expected to be a stronger effect in the nanotube.

D. Effective low-energy spin theory for the instabilities

The charge modes of the nanotubes being gapped, apart from the global decoupled charge mode, the corresponding low-energy description of the instability we consider consists only of the spin modes. Further progress in the understanding of this theory can be made by introducing the pseudo-fermion creation and annihilation operators:

$$\begin{aligned}\Psi_{R,+} &= \frac{\eta_+}{\sqrt{2\pi a}} e^{i(\theta_{s,+} - \phi_{s,+})}, \\ \Psi_{L,+} &= \frac{\eta_+}{\sqrt{2\pi a}} e^{i(\theta_{s,+} + \phi_{s,+})}, \\ \Psi_{R,-} &= \frac{\eta_-}{\sqrt{2\pi a}} e^{i(\theta_{s,-} - \phi_{s,-})}, \\ \Psi_{L,-} &= \frac{\eta_-}{\sqrt{2\pi a}} e^{i(\theta_{s,-} + \phi_{s,-})},\end{aligned}\quad (36)$$

and the associated Majorana fermion operators ($\nu=R,L$):

$$\begin{aligned}\Psi_{\nu,+} &= \frac{1}{\sqrt{2}} (-\zeta_{\nu,1} - i\zeta_{\nu,2}), \\ \Psi_{\nu,-} &= \frac{1}{\sqrt{2}} (\zeta_{\nu,3} + i\zeta_{\nu,0}).\end{aligned}\quad (37)$$

The interaction term proportional to $g_1^{(2)}$ is then rewritten as

$$2i \frac{g_1^{(2)}}{\pi a} \int dx \cos 2\theta_{c,A} \zeta_{R,0} \zeta_{L,0}, \quad (38)$$

we see that the interaction term proportional to $g_1^{(2)}$ gives a nonzero mass to the $\zeta_{R/L,0}$ Majorana fermions, while leaving the $\zeta_{R/L,3}$ fermions massless. This Ising criticality is a consequence of the self-dual^{72,73} character of the interaction (32). In the context of two-leg ladders, these self-dual interactions have been discussed in Refs. 54 and 74.

Moreover, the interaction term proportional to $f^{(1)}$ can also be reexpressed in terms of the fermion fields $\zeta_{R/L,1,2,3}$. Indeed, we have the relations

$$\begin{aligned}\frac{e^{-i(\theta_{s,+} + \phi_{s,+})}}{\pi a} \cos(\theta_{s,-} + \phi_{s,-}) &= -i(\zeta_{L,2} \zeta_{L,3} + i\zeta_{L,3} \zeta_{L,1}), \\ \frac{e^{-i(\theta_{s,+} - \phi_{s,+})}}{\pi a} \cos(\theta_{s,-} - \phi_{s,-}) &= -i(\zeta_{R,2} \zeta_{R,3} + i\zeta_{R,3} \zeta_{R,1}).\end{aligned}\quad (39)$$

In fact, this representation is well known⁷⁵ and has been used to study the two-leg spin ladder^{76,77} and the two-channel Kondo effect.^{71,78} Using the equivalence between Majorana fermions and the two-dimensional Ising model, it is also possible to re-express the interaction v using order and disorder parameters of the quantum Ising model.^{79–83} Indeed, one has the relations⁸⁴

$$\begin{aligned}\cos \phi_{s,+} &= \mu_1 \mu_2, \\ \cos \theta_{s,+} &= i\kappa_1 \sigma_1 \mu_2, \\ \sin \theta_{s,+} &= -i\kappa_2 \mu_1 \sigma_2, \\ \sin \phi_{s,+} &= -i\kappa_1 \kappa_2 \sigma_1 \sigma_2,\end{aligned}\quad (40)$$

and similar relations for $\phi_{s,-}$ and $\theta_{s,-}$ with $(\mu_1, \sigma_1) \rightarrow (\mu_3, \sigma_3)$ and $(\mu_2, \sigma_2) \rightarrow (\mu_0, \sigma_0)$. In Eq. (40), $\kappa_{1,2}$ are Klein factors that result from a careful treatment of the commutation relation of the Ising order and disorder fields.⁸⁴ With these relations, we easily find that

$$\begin{aligned}\cos \phi_{s,-} e^{i\phi_{s,+}} &= \mu_3 \mu_0 [\mu_1 \mu_2 + \kappa_1 \kappa_2 \sigma_1 \sigma_2], \\ \cos \theta_{s,-} e^{i\theta_{s,+}} &= i\kappa_3 \sigma_3 \mu_0 [i\kappa_1 \sigma_1 \mu_2 + \mu_1 \kappa_2 \sigma_2].\end{aligned}\quad (41)$$

Noting that $g_1^{(2)} \rightarrow -\infty$ in Eq. (38) implies that μ_0 develops long-range order for $\langle \theta_{c,A} \rangle = 0$, we find that in the low-energy limit the expressions in Eq. (41) reduce to

$$\begin{aligned}\cos \phi_{s,-} e^{i\phi_{s,+}} &\sim \mu_3 [\mu_1 \mu_2 + \kappa_1 \kappa_2 \sigma_1 \sigma_2], \\ \cos \theta_{s,-} e^{i\theta_{s,+}} &\sim i\kappa_3 \sigma_3 [i\kappa_1 \sigma_1 \mu_2 + \mu_1 \kappa_2 \sigma_2].\end{aligned}\quad (42)$$

Introducing the Pauli spin matrices $\tau_c = -\frac{i}{2}\epsilon_{abc}\kappa_b\kappa_c$ these expressions are easily seen to reduce to the expression of the spin-1/2 primary fields of the $SU(2)_2$ Wess-Zumino-Novikov-Witten (WZNW) model^{38–40,75,77} in terms of Ising fields. Moreover, the expression of the spin currents (39) also reduces to the $SU(2)_2$ form.

Thus the theory describing the spin excitations at low energy reduces to a $SU(2)_1$ WZNW model (that describes the spin excitations of the band 0) coupled with a $SU(2)_2$ WZNW model (that describes the spin excitations of the bands ± 1) by a term

$$\lambda \int dx \text{tr}(g_1\sigma)(x) \cdot \text{tr}(g_2\sigma)(x), \quad (43)$$

and a marginal current-current interaction term. Power counting shows that the term (43) is relevant with RG dimension $5/4$. Therefore it is reasonable to treat first this relevant term, as was done in the case of two-leg spin ladders.⁷⁶ For analyzing the effect of the interaction (43) on the spin spectrum, is convenient to introduce a coset representation:⁸⁵ $SU(2)_1 \times SU(2)_2 \sim SU(2)_3 \times \text{TIM}$ where TIM stands for the tricritical Ising model.¹⁰⁰ With the coset decomposition, we can rewrite the WZNW fields as

$$\begin{aligned} g_1 &= \epsilon_{\text{TIM}} g_3, \\ g_2 &= \sigma_{\text{TIM}} g_3, \end{aligned} \quad (44)$$

where g_k is the spin-1/2 $SU(2)_k$ field, ϵ_{TIM} is the energy operator of the tricritical Ising model of dimension $\frac{1}{10}$, σ_{TIM} is the spin operator of the tricritical Ising model of dimension $\frac{3}{80}$. Using the operator product expansion of the TIM $\epsilon_{\text{TIM}}\sigma_{\text{TIM}} \sim (\sigma + \sigma')_{\text{TIM}}$ from Ref. 86 (p. 224) we can rewrite the interaction (43) as

$$\lambda' \int dx \sigma_{\text{TIM}} \text{tr}(g_3\sigma)(x) \cdot \text{tr}(g_3\sigma)(x), \quad (45)$$

where only the most relevant term has been kept. The interaction is now brought to the form of a self-coupling for the $SU(2)_3$ WZNW model. Now, we simplify this self-coupling by using a second coset⁸⁷ representation, $SU(2)_3 \sim U(1) \times Z_3$ where Z_3 represents the critical three-state Potts model (or equivalently the three-state clock model)⁷² and $U(1)$ represents a free Bosonic field described by the Hamiltonian

$$H = \int \frac{dx}{2\pi} \left[v_F K (\pi\Pi)^2 + \frac{v_F}{K} (\partial_x \phi)^2 \right], \quad (46)$$

where $K=1$. This coset representation was used in Ref. 88 in a study of the Haldane gap in spin- S chains, and from now on our treatment follows this work closely. We write the components of the fundamental field as $g_3^{[m,m']}$ with $m, m' = \pm 1/2$. We have from Refs. 75 and 87 the relations

$$g_3^{[1/2,1/2]} = e^{-i\sqrt{2/3}\theta} \sigma_1, \quad (47)$$

$$g_3^{[-1/2,1/2]} = e^{-i\sqrt{2/3}\phi} \mu_1, \quad (48)$$

where σ_1, μ_1 are the order and disorder parameters of the three-state clock model. With this, we can rewrite the interaction as

$$\begin{aligned} \lambda'' \int dx \sigma_{\text{TIM}} & [\sigma_1(\sigma_1)^\dagger + \mu_1 \mu_1^\dagger / 2 + e^{-2i\sqrt{2/3}\phi} \mu_1^2 \\ & + e^{-2i\sqrt{2/3}\phi} (\mu_1^\dagger)^2]. \end{aligned} \quad (49)$$

Then we use the properties of the three-state clock model:⁸⁷ $\sigma_1^\dagger = \sigma_2, \sigma_2^\dagger = \sigma_1$ and similarly with $\mu \leftrightarrow \sigma$, together with the operator product expansion $\sigma_1 \sigma_2 \sim \epsilon$. This allows us to reduce the above interaction term to

$$\lambda'' \int dx \sigma_{\text{TIM}} (\epsilon_{Z_3} + e^{-2i\sqrt{2/3}\phi} \mu_2 + e^{-2i\sqrt{2/3}\phi} \mu_1). \quad (50)$$

Now let us make the assumption that the TIM develops a long-range order and so does the three-state clock model. Let us assume further that the three-state clock model is in the low-temperature phase, with $\mu_{1,2}$ disordered. Only the Bosonic field ϕ can *a priori* remain gapless. In order to determine whether ϕ indeed remains gapless, we have to consider the perturbations generated by the disordered operators $\mu_{1,2}$. It is straightforward to see that these terms yield a perturbation

$$\lambda_0 \cos 2\sqrt{6}\phi \quad (51)$$

for the $U(1)$ theory. The operators of the $SU(2)_3$ theory then reduce to:⁸⁸

$$n^+ = \text{tr}(g\sigma^+) \sim e^{-i\sqrt{2/3}\theta}, \quad (52)$$

$$\begin{aligned} n^z = \text{tr}(g\sigma^z) & \sim (e^{-i\sqrt{2/3}\phi} \mu_1 + e^{i\sqrt{2/3}\phi} \mu_2) \times (e^{-2i\sqrt{2/3}\phi} \mu_2 \\ & + e^{-2i\sqrt{2/3}\phi} \mu_1) \sim e^{-i\sqrt{6}\phi} + e^{i\sqrt{6}\phi}. \end{aligned} \quad (53)$$

The $SU(2)$ symmetry of the system imposes that n^+ and n^z have the same scaling dimension. Therefore at this new fixed point, one must have $K=1/3$. Hence after a rescaling $\phi = \tilde{\phi}/\sqrt{3}$ and $\theta = \sqrt{3}\tilde{\theta}$, $\tilde{K}=3$ K, the expressions (46) and (52) reduce to the ones of the $SU(2)_1$ case,⁶⁵ with a perturbation (51) which is marginal. Two regimes are possible depending on whether λ_0 is marginally relevant or marginally irrelevant. In the first case, a spin gap is obtained. In the second case, no spin gap is obtained and the system has the same spin correlation as a free $SU(2)_1$ model up to logarithmic corrections.^{45,89,90} In order to predict which phase is realized, we have to consider the flow of the Luttinger exponent of ϕ . If the fixed point is approached from the side where the perturbation (51) is irrelevant, then we can expect the fixed point to be stable. Since at the origin the Luttinger exponent is $K=1$, and at the fixed point it is $K=1/3$, the flow is indeed on the side where eq. (51) is marginally irrelevant. Thus we find gapless spin modes at the fixed point.

If we assume that μ_1 is ordered, then we find that ϕ is also long-range ordered. As a result, n^+ is short-range ordered while n^z or $\epsilon = \text{tr}(g_3)$ is long-range ordered. Since the system has to be rotationally symmetric, the only solution is

to have ϵ long-range ordered and n^z short-range ordered. In that case, the system has a spin gap, and only the singlet superconducting order parameters exhibits quasi-long-range order.

E. Superconducting fluctuations

In the preceding section, we have seen that the spin excitations at the fixed point are described by a single $SU(2)_1$ WZNW model or a spin gapped state. As a result, the operators describing superconducting fluctuations (B4) become equivalent to each other in the limit of low energy. Physically, this means that the interaction v is an internal Josephson coupling between the three bands, resulting in a single superconducting order parameter for low energy. Note that this effective attractive interaction v can be mediated by an optical phonon of momentum $\pm|k_{F0}-k_{F1}|$ and angular momentum $m=\pm 1$.

The resulting state is a one-dimensional analog of the superconducting BCS state with overlapping bands.⁹¹ The absence of such state in the three-leg ladders results from the difference between the Fermi wave vectors of the three bands touching the Fermi energy which prevents the presence of an interaction of the v type at low energy. When the low-energy spin excitations are gapless, both the triplet and the singlet superconducting order parameters exhibit power-law correlations. However, the logarithmic corrections induced by the marginally irrelevant perturbation at the $SU(2)_1$ fixed point are known^{45,89,90} to lead to dominant triplet superconductivity fluctuations. Triplet superconductivity is thus naturally expected in the present case. In this respect, we note that a similar situation arises in single chain systems where the renormalization group predicts dominant triplet superconducting fluctuations in the vicinity of a spin-density wave phase.⁴⁵ Since in Refs. 2 and 3 the superconductivity appears to be sensitive to the application of a magnetic field, it is likely that either the intertube coupling tends to better stabilize the singlet superconductivity with respect to the triplet one or the system is in the spin gapped phase.

V. CONCLUSION

In conclusion, by means of Fermionic renormalization, Abelian and non-Abelian Bosonization, we have analyzed the low-energy properties of a three-band one-dimensional model deduced from the band structure of cylindrical small radius (5,0) nanotubes. We have found that this system possesses a specific instability, besides the usual single-band and two-band model instabilities. This instability corresponds to the development of superconducting fluctuations in the nanotube. Within our approach, in the absence of a spin gap, triplet superconductivity fluctuations are expected to be dominant due to logarithmic corrections, with subdominant singlet superconductivity fluctuations.

This interesting instability is tightly related to the symmetry of our three-band model, and more precisely to the couplings u and v . In our model, in the presence of these couplings, the spin excitations are either fully gapped leaving only a C1S0 phase as in the two-leg ladder^{29,48–50,52–56,58} or

they are described by a $SU(2)_1$ WZNW model leading to a C1S1 phase as in a single chain Hubbard model. This is in contrast to previous studies of a three-leg model with different symmetries, which included only two-band couplings as opposed to the three-band couplings u or v : in these models, a C2S1 phase was found.^{27,28,30} Technically, this difference lies in the ordering of the field $\theta_{c,B}$, directly related to the presence of the v coupling. Note that the three-band nature of the v coupling also induced the technical problem of the Klein factor discussed in this paper.

Let us finally relate our results to previous studies on the (5,0) nanotubes. In Ref. 9, only a subset of the couplings of the present model was considered, which did not include the u and v term. Hence this superconducting instability was not discussed. In Ref. 10, Gonzalez and Peretto studied the same model as ours, by means of a renormalization-group procedure. The nature of the phase was determined *via* the scaling behavior of correlations functions, as opposed to the Bosonization procedure used in this paper. In Ref. 10, it was found that the dominant instability would be a charge-density wave coupling the bands ± 1 , with subdominant spin-density wave fluctuations, whereas we find that charge-density wave fluctuations are suppressed. The origin of this discrepancy is that in Ref. 10, only specific initial conditions were considered, with initial values of some couplings so large that they render a one-loop renormalization-group approach questionable. It appears likely that the initial conditions chosen in Ref. 10 strongly favor a two-band instability between the band ± 1 . Indeed, a divergent $g_1^{(1)}$ as we found in Sec. III B 1 indeed leads to a reinforcement of charge-density wave fluctuations between the bands ± 1 .

Along these lines, let us mention that it is difficult to determine which of the possible mechanisms, including the one proposed in this paper, actually takes place in a (5,0) nanotube. Indeed, as opposed to theoretical approaches of conventional larger nanotubes, our one-dimensional electronic model is using the band structure provided by *ab initio* calculations as an input. Since these methods already include a renormalization of the band structure by a fraction of the electronic interactions, an estimate of the bare coupling in our one-dimensional model based on an unrenormalized Coulomb interaction as in Ref. 11 is likely to lead to misleading results by overestimating the effect of some interactions. As a result, we can only propose a classification of the various fixed points at weak coupling and characterize the possible scenarios, with the usual hypothesis in one-dimensional systems that the weak-coupling behavior and the strong-coupling behavior are continuously connected.⁴⁶ A related remark is that the gaps calculated in any weak-coupling approximation are generally very small. However, in the real system, where interaction strength can be expected to be comparable to the bandwidth, since Luttinger exponents are found in the range⁹² 0.2–0.5, the real gaps can be much higher than those estimated in a weak-coupling treatment. Thus the present treatment cannot lead to a realistic estimate of critical temperatures.

Another aspect of the physics to consider is the possibility of a pseudo-Peierls transitions in this small radius nanotubes.⁶ Indeed, the approach of this paper is based on the band structure of numerical approach which did not con-

sider the possibility of a cylindrical geometry breaking. If such a phenomenon was to happen, an indication of strong electron-phonon coupling in the system, other mechanisms for superconductivity could occur, but their description is beyond the scope of the present paper. Finally, let us mention that the experimental results on the superconducting transition in these small nanotubes^{2,3} suggest that a real three-dimensional superconducting phase transition takes place. A complete understanding of these results must also include a coupling between the nanotubes to stabilize the superconducting fluctuations at nonzero temperature.^{93,94} However, since the gap in the zeolite matrix is of order 4 eV, an intertube Josephson coupling term would be *a priori* strongly suppressed by the presence of the insulator between the tubes.

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APPENDIX A: DERIVATION OF THE RG EQUATIONS

To express the RG equations in a simpler form, we first define rescaled couplings as

$$\tilde{g}_i^{(j)} = \frac{1}{2\pi v_F} g_i^{(j)}; \quad \tilde{f}^{(i)} = \frac{1}{2\pi v_F} f^{(i)}, \quad (\text{A1})$$

$$\tilde{u} = \frac{1}{2\pi v_F} u; \quad \tilde{v} = \frac{1}{2\pi v_F} v. \quad (\text{A2})$$

The scaling equations read in terms of these couplings

$$\partial_l \tilde{g}^{(1)} = -2(\tilde{g}^{(1)})^2 - 4\tilde{u}\tilde{v}, \quad (\text{A3a})$$

$$\partial_l \tilde{g}^{(2)} = -(\tilde{g}^{(2)})^2 - 2\tilde{u}^2 - 2\tilde{v}^2, \quad (\text{A3b})$$

$$\partial_l \tilde{g}_1^{(1)} = -2(\tilde{g}_1^{(1)})^2 - 2\tilde{g}_1^{(2)}\tilde{g}_2^{(1)} - 2\tilde{u}\tilde{v}, \quad (\text{A3c})$$

$$\partial_l \tilde{g}_1^{(2)} = -2\tilde{g}_1^{(1)}\tilde{g}_2^{(1)} - 2\tilde{g}_1^{(2)}\tilde{g}_2^{(2)} + 2\tilde{g}_1^{(2)}\tilde{g}_4^{(2)} - \tilde{u}^2 - \tilde{v}^2, \quad (\text{A3d})$$

$$\partial_l \tilde{g}_2^{(1)} = -2\tilde{g}_1^{(1)}\tilde{g}_1^{(2)} - 2\tilde{g}_2^{(1)}\tilde{g}_2^{(2)} + 2\tilde{g}_1^{(2)}\tilde{g}_4^{(1)} - 4\tilde{g}_2^{(1)}\tilde{g}_4^{(1)} + 2\tilde{g}_2^{(1)}\tilde{g}_4^{(2)} - 2\tilde{u}\tilde{v}, \quad (\text{A3e})$$

$$\partial_l \tilde{g}_2^{(2)} = -(\tilde{g}_1^{(1)})^2 - (\tilde{g}_1^{(2)})^2 - (\tilde{g}_2^{(1)})^2 - \tilde{u}^2 - \tilde{v}^2, \quad (\text{A3f})$$

$$\partial_l \tilde{g}_4^{(1)} = +2\tilde{g}_1^{(2)}\tilde{g}_2^{(1)} - 2(\tilde{g}_2^{(1)})^2 - 2(\tilde{g}_4^{(1)})^2, \quad (\text{A3g})$$

$$\partial_l \tilde{g}_4^{(2)} = (\tilde{g}_1^{(2)})^2 - (\tilde{g}_4^{(1)})^2, \quad (\text{A3h})$$

$$\partial_l \tilde{f}^{(1)} = -2(\tilde{f}^{(1)})^2 + 2\tilde{u}\tilde{v} - 2\tilde{v}^2, \quad (\text{A3i})$$

$$\partial_l \tilde{f}^{(2)} = -(\tilde{f}^{(1)})^2 + \tilde{u}^2, \quad (\text{A3j})$$

$$\partial_l \tilde{u} = (2\tilde{f}^{(2)} - \tilde{g}_1^{(2)} - \tilde{g}^{(2)} - \tilde{g}_2^{(2)})\tilde{u} - (\tilde{g}^{(1)} + \tilde{g}_1^{(1)} + \tilde{g}_2^{(1)})\tilde{v}, \quad (\text{A3k})$$

$$\partial_l \tilde{v} = -(-2\tilde{f}^{(1)} + \tilde{g}^{(1)} + \tilde{g}_1^{(1)} + \tilde{g}_2^{(1)})\tilde{u} - (4\tilde{f}^{(1)} - 2\tilde{f}^{(2)} + \tilde{g}_1^{(2)} + \tilde{g}^{(2)} + \tilde{g}_2^{(2)})\tilde{v}. \quad (\text{A3l})$$

APPENDIX B: BOSONIZATION EXPRESSION OF ORDER PARAMETERS

In this Appendix, we provide the detailed expressions of the various order parameters of our three-band model within the field-theoretical convention used and defined in the text. We use the field-theoretical convention for Bosonization, defined in Sec. IV B 2.

1. Superconductivity

We consider the following order parameters for the formation of singlet superconductivity in the nanotube:

$$O_0(x) = \sum_{\sigma} \psi_{R,0,\sigma} \psi_{L,0,-\sigma}, \quad (\text{B1})$$

$$O_1(x) = \sum_{\sigma} \psi_{R,1,\sigma} \psi_{L,-1,-\sigma}, \quad (\text{B2})$$

$$O_{-1}(x) = \sum_{\sigma} \psi_{R,-1,\sigma} \psi_{L,1,-\sigma}. \quad (\text{B3})$$

Using the Bosonization decomposition introduced in the text, we can express these order parameters for superconductivity into Bosonized variables:

$$O_0(x) = \frac{i}{\pi a} e^{-i\sqrt{2}[\theta_{c,+/\sqrt{3}-\sqrt{2/3}\theta_{c,B}}]} \sin \sqrt{2} \phi_{s,0}, \quad (\text{B4a})$$

$$O_1(x) = \frac{i}{\pi a} e^{i\sqrt{2}(\theta_{c,+/\sqrt{3}+\theta_{c,A}/\sqrt{2}+\theta_{c,B}/\sqrt{6})} \sin(\phi_{s,+} + \phi_{s,-}), \quad (\text{B4b})$$

$$O_{-1}(x) = \frac{i}{\pi a} e^{i\sqrt{2}(\theta_{c,+/\sqrt{3}-\theta_{c,A}/\sqrt{2}+\theta_{c,B}/\sqrt{6})} \sin(\phi_{s,+} - \phi_{s,-}). \quad (\text{B4c})$$

Note that we have not considered the triplet superconductivity order parameters. Indeed, they naturally possess the same charge part as the singlet superconductivity operators, and only differ by their spin part. Since the spin part is more conveniently treated within the non-Abelian Bosonization formalism (see Sec. IV D), it is not necessary to give an explicit expression of the triplet operators here, since they can be obtained from the expression of the operators (B4) in non-Abelian Bosonization by the substitution $\text{tr}(g) \rightarrow \text{tr}(g\sigma)$.

2. Charge-density waves

Besides superconductivity, one can also expect to observe charge-density wave order to develop at low temperature in a quasi-1D system. The various charge-density wave order operators, labeled by their ordering momentum vector, are defined as

$$O_{(2k_{F_0},0)}(x) = \sum_{\sigma} \psi_{R,0,\sigma}^{\dagger} \psi_{L,0,\sigma}, \quad (\text{B5})$$

$$O_{(2k_{F_1},0)}(x) = \sum_{\sigma} \psi_{R,1,\sigma}^{\dagger} \psi_{L,1,\sigma}, \quad (\text{B6})$$

$$O_{(2k_{F_{-1}},0)}(x) = \sum_{\sigma} \psi_{R,-1,\sigma}^{\dagger} \psi_{L,-1,\sigma}, \quad (\text{B7})$$

$$O_{(k_{F_0}-k_{F_1},K_y)}(x) = \sum_{\sigma} \psi_{R,0,\sigma}^{\dagger} \psi_{L,-1,\sigma}, \quad (\text{B8})$$

$$O_{(k_{F_0}-k_{F_1},-K_y)}(x) = \sum_{\sigma} \psi_{R,0,\sigma}^{\dagger} \psi_{L,1,\sigma}, \quad (\text{B9})$$

$$O_{(2k_{F_1},2K_y)}(x) = \sum_{\sigma} \psi_{R,1,\sigma}^{\dagger} \psi_{L,+1,\sigma}, \quad (\text{B10})$$

$$O_{(2k_{F_1},-2K_y)}(x) = \sum_{\sigma} \psi_{R,1,\sigma}^{\dagger} \psi_{L,-1,\sigma}. \quad (\text{B11})$$

We have considered only the charge-density wave operators of the form $\psi_{R,m,\sigma}^{\dagger} \psi_{L,m',\sigma}$, as those of the form $\psi_{R,m,\sigma}^{\dagger} \psi_{R,m',\sigma}$ are currents that cannot develop quasi-long-range order.

The corresponding Bosonized expressions of these charge-density wave order parameters are easily obtained and read

$$O_{(2k_{F_0},0)}(x) = \frac{-i}{\pi a} e^{i\sqrt{2}[\phi_{c+}/\sqrt{3}-2\phi_{cA}/\sqrt{6}]} \cos \sqrt{2}\phi_{s,0}, \quad (\text{B12})$$

$$O_{(2k_{F_1},0)}(x) = \frac{1}{2\pi a} e^{i[2\phi_{c+}/\sqrt{3}-\theta_{cA}+\sqrt{2/3}\phi_{cB}]} \sum_{\sigma} e^{i\sigma(\phi_{s,+}-\theta_{s,-})} \eta_{1,\sigma} \eta_{-1,\sigma}, \quad (\text{B13})$$

$$O_{(2k_{F_{-1}},0)}(x) = \frac{1}{2\pi a} e^{i[2\phi_{c+}/\sqrt{3}+\theta_{cA}+\sqrt{2/3}\phi_{cB}]} \sum_{\sigma} e^{i\sigma(\phi_{s,+}+\theta_{s,-})} \eta_{-1,\sigma} \eta_{1,\sigma}, \quad (\text{B14})$$

$$O_{(-k_{F_0}-k_{F_1},K_y)}(x) = \frac{1}{2\pi a} e^{i[2\phi_{c+}/\sqrt{6}-(1/\sqrt{2})(\theta_{cA}+\phi_{cA})+(3/\sqrt{6})(\theta_{cB}-\phi_{cB})]} \sum_{\sigma} e^{i\sigma[-(1/2)(\theta_{s,+}+\theta_{s,-}-\phi_{s,+}-\phi_{s,-})-(\theta_{s,0}-\phi_{s,0}/\sqrt{2})]} \eta_{0,\sigma} \eta_{-1,\sigma}, \quad (\text{B15})$$

$$O_{(-k_{F_0}-k_{F_1},-K_y)}(x) = \frac{1}{2\pi a} e^{i[2\phi_{c+}/\sqrt{6}+(1/\sqrt{2})(\theta_{cA}+\phi_{cA})+(3/\sqrt{6})(\theta_{cB}-\phi_{cB})]} \sum_{\sigma} e^{i\sigma[-(1/2)(\theta_{s,+}-\theta_{s,-}-\phi_{s,+}+\phi_{s,-})-(\theta_{s,0}-\phi_{s,0}/\sqrt{2})]} \eta_{0,\sigma} \eta_{1,\sigma}, \quad (\text{B16})$$

$$O_{(2k_{F_1},2K_y)} = \frac{i}{2\pi a} e^{i\sqrt{2}[\phi_{c+}/\sqrt{3}+\phi_{cA}/\sqrt{2}+\phi_{cB}/\sqrt{6}]} \sum_{\sigma} e^{i\sigma(\phi_{s,+}+\phi_{s,-})}, \quad (\text{B17})$$

$$O_{(2k_{F_1},-2K_y)} = \frac{i}{2\pi a} e^{i\sqrt{2}[\phi_{c+}/\sqrt{3}-\phi_{cA}/\sqrt{2}+\phi_{cB}/\sqrt{6}]} \sum_{\sigma} e^{i\sigma(\phi_{s,+}-\phi_{s,-})}. \quad (\text{B18})$$

APPENDIX C: EQUIVALENCE OF BOSONIZATION CONVENTIONS

In this Appendix, we discuss a general case of 1D fermions with N “flavors” since the results are of more general applicability than the nanotube with three bands at the Fermi level that we have considered in this paper. The Bosonized representation of fermion operators used in condensed-matter literature amounts to writing:^{29,69,71,95}

$$\psi_{R,n} = \frac{1}{\sqrt{2\pi a}} e^{-i\Phi_{R,n}(x)} \eta_{R,n}, \quad (\text{C1})$$

$$\psi_{L,n} = \frac{1}{\sqrt{2\pi a}} e^{i\Phi_{L,n}(x)} \eta_{L,n}, \quad (\text{C2})$$

where

$$[\Phi_{R,n}(x), \Phi_{R,n'}(x')] = i\pi \delta_{n,n'} \text{sgn}(x-x'), \quad (\text{C3})$$

$$[\Phi_{R,n}(x), \Phi_{R,n'}(x')] = -i\pi \delta_{n,n'} \text{sgn}(x-x'), \quad (\text{C4})$$

$$[\Phi_{R,n}(x), \Phi_{L,n'}(x')] = 0, \quad (\text{C5})$$

and the Majorana fermion operators $\eta_{\nu,n}$ satisfy

$$\{\eta_{\nu,n}, \eta_{\nu',n'}\} = 2\delta_{\nu,nu'} \delta_{n,n'}. \quad (\text{C6})$$

In the field-theoretical literature, an apparently different Bosonized representation is used:^{80,96,97}

$$\psi_{R,n} = \frac{1}{\sqrt{2\pi a}} e^{-i\tilde{\Phi}_{R,n}(x)} \eta_n, \quad (\text{C7})$$

$$\psi_{L,n} = \frac{1}{\sqrt{2\pi a}} e^{i\tilde{\Phi}_{L,n}(x)} \eta_n, \quad (\text{C8})$$

where this time

$$[\tilde{\Phi}_{R,n}(x), \tilde{\Phi}_{R,n'}(x')] = i\pi \delta_{n,n'} \operatorname{sgn}(x-x'), \quad (\text{C9})$$

$$[\tilde{\Phi}_{L,n}(x), \tilde{\Phi}_{L,n'}(x')] = -i\pi \delta_{n,n'} \operatorname{sgn}(x-x'), \quad (\text{C10})$$

$$[\tilde{\Phi}_{R,n}(x), \tilde{\Phi}_{L,n'}(x')] = i\pi \delta_{n,n'} \quad (\text{C11})$$

and the Majorana fermion operators η_n satisfy

$$\{\eta_n, \eta_{n'}\} = 2\delta_{n,n'}. \quad (\text{C12})$$

While in the condensed-matter Bosonization for a given flavor the right and left Bosonic fields are made commuting and there is one Majorana fermion associated with the left mover and another Majorana fermion associated with the right mover, in the field-theoretical representation, there is only one Majorana fermion for each flavor. The price to pay for this is to make the commutator of the chiral fields nonzero. An application of this representation in condensed-matter physics is the derivation of the Bosonized form of the doubled Ising model.^{80,81,83}

and one has

$$\begin{aligned} e^{i\sqrt{\pi/2}(Q_n-P_n)} \eta_n e^{i\sqrt{\pi/2}(Q_n+P_n)} \eta_n &= e^{i\sqrt{\pi/2}(Q_n+P_n)} \eta_n e^{i\sqrt{\pi/2}(Q_n-P_n)} \eta_n e^{-(\pi/2)[Q_n+P_n, Q_n-P_n]}, \\ &= -e^{i\sqrt{\pi/2}(Q_n+P_n)} \eta_n e^{i\sqrt{\pi/2}(Q_n-P_n)} \eta_n. \end{aligned} \quad (\text{C20})$$

Therefore we can define a set of Majorana fermion operators

$$\eta_{R,n} = e^{i\sqrt{\pi/2}(Q_n-P_n)} \eta_n, \quad (\text{C21})$$

$$\eta_{L,n} = e^{i\sqrt{\pi/2}(Q_n+P_n)} \eta_n \quad (\text{C22})$$

which satisfy the commutation relations (C6). As a result, the representation (C18) and the representation (C7) are equivalent to the representation (C1).

It is well known that one can also define Bosonization using nonchiral fields⁶⁵ θ_n, ϕ_n given by

$$\phi_n = \frac{1}{2}(\Phi_{L,n} + \Phi_{R,n}), \quad (\text{C23})$$

To show that these two representations are in fact equivalent, let us introduce the conjugate variables Q_n and P_n such that

$$[Q_n, \tilde{\Phi}_{v,n'}(x)] = 0, \quad (\text{C13})$$

$$[P_n, \tilde{\Phi}_{v,n'}(x)] = 0, \quad (\text{C14})$$

$$[Q_n, P_m] = i\pi \delta_{n,m}, \quad (\text{C15})$$

and let us write

$$\bar{\Phi}_{R,n} = \tilde{\Phi}_{R,n} - \sqrt{\frac{\pi}{2}}(Q_n - P_n), \quad (\text{C16})$$

$$\bar{\Phi}_{L,n} = \tilde{\Phi}_{L,n} + \sqrt{\frac{\pi}{2}}(Q_n + P_n). \quad (\text{C17})$$

We have $[\bar{\Phi}_{R,n}, \bar{\Phi}_{L,n}] = i\pi - \pi([Q_n, P_n] - [P_n, Q_n])/2 = 0$. Then the fermion operators are rewritten as

$$\psi_{R,n} = \frac{1}{\sqrt{2\pi a}} e^{-i\bar{\Phi}_{R,n}(x)} e^{i\sqrt{\pi/2}(Q_n-P_n)} \eta_n, \quad (\text{C18})$$

$$\psi_{L,n} = \frac{1}{\sqrt{2\pi a}} e^{i\bar{\Phi}_{L,n}(x)} e^{i\sqrt{\pi/2}(Q_n+P_n)} \eta_n, \quad (\text{C19})$$

$$\theta_n = \frac{1}{2}(\Phi_{L,n} - \Phi_{R,n}). \quad (\text{C24})$$

In the case of the Bosonization procedure (C1), the nonchiral fields have the commutation relation

$$[\phi_n(x), \phi_m(x')] = 0, \quad (\text{C25})$$

$$[\theta(x)_n, \theta_m(x')] = 0, \quad (\text{C26})$$

$$[\theta(x)_n, \phi_m(x')] = -i\frac{\pi}{2} \delta_{n,m} \operatorname{sgn}(x-x'). \quad (\text{C27})$$

In the case of the bosonization procedure (C7), the nonchiral fields can be defined similarly but they have the commutation relation $[\tilde{\theta}(x)_n, \tilde{\phi}_m(x')] = -i\pi \delta_{n,m} \Theta(x-x')$, where Θ is the Heaviside function. The relation between the two sets of chiral fields is

$$\tilde{\phi}_n(x) = \phi_n(x) + \sqrt{\frac{\pi}{2}} Q_n, \quad (\text{C28})$$

$$\tilde{\theta}_n(x) = \theta_n(x) + \sqrt{\frac{\pi}{2}} P_n. \quad (\text{C29})$$

An interpretation of Q_n is that it is proportional to an operator counting the number of fermions (both right and left moving) of type n . P_n is the proportional to the phase conjugate to this fermion number, and thus must be compactified.

Note that if we perform a rotation on the fields θ_n and ϕ_n , this rotation preserves the commutation relations (C25). This is a crucial property. It also preserves the commutation relations in the case of field-theoretic Bosonization, and obviously it preserves the relation of commutation of the zero modes P and Q . Therefore one always goes from the condensed-matter convention to the field-theoretical convention by using the same shift of the nonchiral fields (C28) by zero modes.

When converting products of two fermion operators from the condensed-matter convention to the field-theoretical convention, the following rules apply: $\eta_{R/L,n} \eta_{R/L,n'} \rightarrow \eta_n \eta_{n'}$ for different species $n \neq n'$ and for a given species n :

$$e^{-i\Phi_{R,n}} \eta_{R,n} e^{i\Phi_{L,n}} \eta_{L,n} \rightarrow i e^{i(\tilde{\Phi}_{L,n} - \tilde{\Phi}_{R,n})} = i e^{i2\tilde{\phi}_n},$$

$$e^{i\Phi_{R,n}} \eta_{R,n} e^{i\Phi_{L,n}} \eta_{L,n} \rightarrow -i e^{i(\tilde{\Phi}_{L,n} - \tilde{\Phi}_{R,n})} = -i e^{i2\tilde{\theta}_n}. \quad (\text{C30})$$

We close this section with two remarks. First, the field-theoretical representation can be understood naturally on a semi-infinite system extending to $+\infty$, with an open boundary condition at the origin:⁹⁸ the origin is then sent to $-\infty$ to recover an infinite system. The nonzero value of the commutator of the right- and left-moving field possesses then the physical interpretation of left movers being reflected into right movers at $-\infty$. As a second remark let us note that when applying the condensed-matter Bosonization to the double Ising model, one finds that the operator $\cos \phi$ does not have the same commutation with the fermion field as the products of Ising disorder fields,⁸¹ whereas this relation is satisfied in the field-theoretical Bosonization. The reason for this difference can be inferred from our first remark: the construction of the product of disorder fields as a string is made for a system with an open boundary condition at the origin. This boundary condition is included from the start in field-theoretical Bosonization but not in the condensed-matter Bosonization. Thus when using condensed-matter Bosonization, Klein factors must appear in the expression of the products of the Ising disorder fields as a function of $\cos \phi$.

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