Collective modes and skyrmion excitations in graphene $SU(4)$ quantum Hall ferromagnets

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Graphene exhibits quantum Hall ferromagnetism in which an approximate $SU(4)$ symmetry involving spin and valley degrees of freedom is spontaneously broken. We construct a set of integer and fractional quantum Hall states that break the $SU(4)$ spin (valley) symmetry, and study their neutral and charged excitations. Several properties of these ferromagnets can be evaluated analytically in the $SU(4)$ symmetric limit, including the full collective-mode spectrum at integer fillings. By constructing explicit wave functions we show that the lowest-energy skyrmion states carry charge ± 1 for *any* integer filling, and that skyrmions are the lowest-energy-charged excitations for graphene Landau-level index $|n| \leqslant 3$. We also show that the skyrmion lattice states which occur near integer-filling factors support four gapless collective-mode branches in the presence of full $SU(4)$ symmetry. Comparisons are made with the more familiar $SU(2)$ quantum Hall ferromagnets studied previously.

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I. INTRODUCTION

Recent experimental work[1] has established graphene as a new two-dimensional (2D) electron system with linear Dirac-type energy-band dispersion. An important aspect of graphene physics is the twofold valley degeneracy of its low-energy band structure, which, in combination with the usual spin doublet, implies band eigenstates with degeneracy $N=4$. In a strong magnetic field, the fourfold degeneracy of Landau levels in graphene has been clearly observed in recent quantum Hall measurements.[2–4] This property is expected[5] to lead to an intriguing interplay between interaction and quantum Hall physics by introducing a rich variety of new states with different types of spontaneous symmetry breaking, and new varieties of low-lying collective modes. In this paper we explore some of the theoretical possibilities for quantum Hall ground states and collective modes that follow from the enlarged ($N=4$) Landau-level degeneracy of graphene, emphasizing similarities and differences compared to the well-studied $N=2$, spin-only circumstance[6,7] relevant to GaAs 2D systems.

One of the most theoretically interesting[8–10] and phenomenologically significant[11] consequences of quantum Hall ferromagnetism is the presence of a finite density of skyrmions in the ground state near integer-filling factors. A dense system of skyrmions constitutes an emergent set of low-energy degrees of freedom that qualitatively alters[11] NMR, optical, thermal, and transport properties of the 2D electron system. In the $N=2$ case, dense skyrmion systems occur only near Landau-level filling factor $\nu=1$. We predict that in graphene, dense skyrmion states[8,9,11] occur near many integer-filling factors, and that they have a larger number of gapless collective modes than for $N=2$.[12] For Coulombic electron-electron interactions, skyrmion lattice states occur only[13] near $\nu=1$ in the $SU(2)$ case. For graphene we predict skyrmion lattice states near $\nu=\pm l$ for all $l<14$ except for $l=2,6,10$ when all Landau-level multiplets are either full or empty, and that skyrmion lattice states in graphene have four branches of gapless collective modes at all these filling factors in the absence of symmetry-breaking perturbations.

The $N=4$ internal degrees of freedom present in graphene take on a particular significance because of the very strong Coulomb interaction energy scale (estimated to be more than 1000 K at 45 T,[4] several times larger than in GaAs), which will help make all the physics that follows from interaction-driven, spontaneous symmetry breaking (i.e., phases which break the internal symmetry associated with the degeneracy N) more accessible experimentally. The point of departure for our analysis will be an $SU(N=4)$ symmetric Hamiltonian, which allows us to obtain a number of exact results. This highly symmetric model is believed to be a good approximation of the full Hamiltonian of graphene.[5,14] For relatively low carrier densities the largest symmetry-breaking term in the Hamiltonian will be Zeeman coupling, which favors spin alignment along the field direction. We will discuss some of the important consequences of this term.

We remark that the possibility of a spontaneously broken valley symmetry is akin to orbital ordering, with the novel twist that the orbital ordering is in *momentum space*, rather than in real space. The anticipated exotic orbital order in graphene should coexist with spin-order, making graphene (along with systems such as superfluid He-3, manganites, and related systems where real space-orbital order apparently coexists with spin order) an interesting system to study novel quantum phases with interplay between spin and orbital order. The momentum space orbital ordering discussed here may also imply observable Jahn-Teller coupling effects in graphene, but we do not explore this idea further in this paper.

The consequences of $SU(4)$, or more generally, $SU(N)$ symmetry in quantum Hall physics have been discussed previously by Arovas and co-workers[15] in the context of silicon systems, which also have additional (approximate) valley degeneracies, and by Ezawa and co-workers[16,17] in the context of bilayer quantum Hall systems where the additional degeneracy comes from the *which layer* degree of freedom. Although both groups used nonlinear sigma-model descriptions they reached different conclusions on the properties of Skyrmion excitations in $SU(N)$ quantum Hall ferromagnets, and in particular, on the minimum charge a Skyrmion can carry. By explicitly constructing the *microscopic* wave functions for collective modes and Skyrmions, we demonstrate that the lowest-energy Skyrmion excitations of the system carry charge ± 1. This analysis also allows us to enumerate the internal degrees of freedom associated with an individual Skyrmion and predict the number of gapless collective modes present in $SU(4)$ Skyrmion lattice states.

Our paper is organized as follows. In Sec. II we identify exact eigenstates of $SU(N)$ quantum Hall ferromagnets at integer-filling factors, which have broken symmetry. We argue that these states are the ground states for any physically sensible repulsive interaction between the particles, and obtain exact results for their elementary particle-hole excitations. We find that in the absence of symmetry-breaking perturbations there are $M(N-M)$ gapless collective modes when M of the N members of a Landau-level multiplet are occupied. In Sec. III we discuss the properties of skyrmions in graphene. We find that for Coulombic electron-electron interactions, skyrmions are much more robust in systems like graphene with a Dirac band structure, than for the more familiar parabolic band systems. We predict that dense-skyrmion systems will be ubiquitous in the quantum Hall regime of graphene, and that at even integer fillings skyrmion lattice states will have four branches of gapless collective modes, two with k^1 dispersion, one with $k^{3/2}$ dispersion, and one with k^2 dispersion. Finally, in Sec. IV we briefly present some elementary considerations on broken symmetry states in graphene at fractional-filling factors. We conclude in Sec. V with a brief summary of our results and some final comments.

II. SYMMETRY-BREAKING GROUND STATES AND COLLECTIVE-MODE SPECTRA

For simplicity we ignore disorder and mixing between different Landau levels. To keep the discussion general, we assume electrons have N internal states and that the electron-electron interaction is independent of these internal states, so that the Hamiltonian is $SU(N)$ symmetric. Thus the total degeneracy of each Landau level (LL), including the internal degeneracy, is $N \times N_\Phi$, where N_Φ is the number of flux quanta enclosed in the system. We start by considering the case where the filling factor of the valence LL (with LL index n) is an integer,

$$\nu_n = N_e/N_\Phi = M \leqslant N, \tag{1}$$

where N_e is the number of electrons occupying the valence LL. We note that in the case of graphene $N=4$ and the four-fold degenerate $n=0$ LL is half filled (corresponding to $\nu_0 = 2$) at zero doping; thus the Hall conductance is

$$\sigma_{xy} = \nu e^2/h, \tag{2}$$

with

$$\nu = 4n - 2 + \nu_n. \tag{3}$$

In the absence of interactions, the quantum Hall effect occurs only when each of the four fold-degenerate Landau levels is completely full ($\nu_n = N = 4$) or completely empty ($\nu_n = 0$) and the quantized Hall conductance is

$$\sigma_{xy} = (4n+2)e^2/h, \tag{4}$$

where n is the index of the highest-*filled* Landau level. Quantum Hall ferromagnetism will[5] lead to quantum Hall pleateaus characterized by intermediate integers and by fractions. In the following, we neglect interaction-induced mixing between orbitals with different n. For different values of n the properties of the $SU(N)$ quantum Hall ferromagnet are different. In the following, the dependence on n is sometimes left implicit to avoid notational clutter.

Because of the $SU(N)$ symmetry of the system, the following single Slater-determinant state (in which only electronic states in the valence LL are explicitly described) is an *exact* eigenstate of the Hamiltonian:

$$|\Psi_0\rangle = \prod_{1 \leqslant \sigma \leqslant M} \prod_k c^\dagger_{k,\sigma} |0\rangle. \tag{5}$$

Here $c^\dagger$ is the electron-creation operator, $|0\rangle$ is the vacuum state, σ is the index of the internal state that runs from 1 to N, and k is an intra-LL orbital index; for example, in the Landau gauge it is the wave vector along the plane-wave direction, while in the symmetric gauge it is the angular-momentum quantum number.

For a broad class of repulsive interactions, we expect $|\Psi_0\rangle$ to be the exact *ground state* of the system; for the case of $\nu_n = M = 1$ this can be proved rigorously for short-range repulsion (see below). In this paper we use this assumption as the starting point of our discussion. Obviously, the ground state $|\Psi_0\rangle$ breaks the $SU(N)$ symmetry spontaneously since an $SU(N)$ rotation transforms the M spontaneously chosen occupied single-electron orbitals to another different set. It therefore represents the ground state of an $SU(N)$ ferromagnet, and is expected to support ferromagnetic spin waves. In the following, we construct the *exact* single-spin-wave states and determine their spectrum in a manner similar to that of Kallin and Halperin.[18] Consider the following Landau gauge states, with $\hat{y}$ in the direction of the plane waves:

$$|\mathbf{k}\rangle_{\sigma_1\sigma_2} = |k_x, k_y\rangle_{\sigma_1\sigma_2} = \frac{1}{\sqrt{N_\Phi}} \sum_{k'} e^{ik_x k' \ell^2} c^\dagger_{k'+k_y,\sigma_2} c_{k',\sigma_1} |\Psi_0\rangle. \tag{6}$$

These spin-wave states are labeled by a two-dimensional wave vector $\mathbf{k}$ and two internal indices: $1 \leqslant \sigma_1 \leqslant M$ and $M < \sigma_2 \leqslant N$. $\ell = \sqrt{\hbar c/eB}$ is the magnetic length. Physically they can be understood as single particle-hole states formed from the σ_2 and σ_1 internal states, i.e., as magnetic excitons. Since

there are M choices for σ_1 and $N-M$ choices for σ_2, the total number of these spin-wave modes is $M(N-M)$. It follows from translational invariance that $\mathbf{k}$ is a good quantum number, from $SU(N)$ invariance that excitons with distinct (σ_2,σ_1) labels are uncoupled, and that $|\mathbf{k}\rangle_{\sigma_1,\sigma_2}$ is therefore an exact eigenstate of the Hamiltonian. The exact $SU(N)$ quantum Hall ferromagnet magnetoexciton-dispersion relation is

$$E(k) = \langle \mathbf{k}|\hat{V}|\mathbf{k}\rangle - \langle \Psi_0|\hat{V}|\Psi_0\rangle = \frac{1}{2\pi}\int_0^\infty qV(q)[F_n(q)]^2 e^{-q^2\ell^2/2}[1 - J_0(qk\ell^2)]dq, \tag{7}$$

where $V(q)$ is the Fourier transform of electron-electron interaction, J_0 is the Bessel function, and

$$F_n(q) = \tfrac{1}{2}[L_{|n|}(q^2\ell^2/2) + L_{|n|-1}(q^2\ell^2/2)] \tag{8}$$

is the Landau-level structure factor appropriate for Dirac fermions.[5,14] (In Eq. (8) $L_n(x)$ is a Laguerre polynomial. The dependence of graphene-quantum Hall-ferromagnet properties on n enters only through F_n.) The second term in Eq. (7) is due to particle-hole attraction[18] and vanishes for $k\to\infty$ where $E(k)$ approaches the energy of a widely separated electron-hole pair. For the case of a $1/r$ Coulomb interaction and $n=0$ $[F_0(x)\equiv 1]$, we have

$$E(k) = \frac{e^2}{\epsilon\ell}\sqrt{\frac{\pi}{2}}[1 - e^{-k^2\ell^2/4}I_0(k^2\ell^2/4)], \tag{9}$$

where ϵ is the effective dielectric constant and I_0 is the modified Bessel function. In the long-wavelength limit $E(k)\propto k^2$ as expected for ferromagnetic spin waves.

It is worth pointing out here that among the various sources of perturbations that break the $SU(4)$ symmetry in graphene, the simplest but most important one is the Zeeman splitting of electron spin,

$$H_z = -g\mu_B B S^Z_{tot}, \tag{10}$$

where g is the electron-spin g factor and μ_B is the Bohr magneton. Due to the fact that the total spin along the magnetic-field direction S^Z_{tot} commutes with the $SU(4)$-invariant Hamiltonian, it only shifts the energies of individual eigenstates without changing the states themselves. In the presence of H_Z, electrons will choose to occupy spin-up states in the ground state, Eq. (5), until all such states are filled; the internal states as labeled by σ are thus eigenstates of S^Z. The single spin-wave states take the same form as in Eq. (6); the spectrum remains the same as Eq. (7) if σ_1 and σ_2 are of the same spin orientation, while it gets shifted to $E(k)+g\mu_B B$ if they are of opposite spin orientations.

Returning to the $SU(N)$ symmetric case, we note that the nature of the symmetry breaking as realized in the ground state (5), as well as the resultant gapless spin-wave excitations, may be understood from the following group-theoretical analysis. The state (5), while not invariant under a general $SU(N)$ transformation, is invariant under a subgroup of $SU(N)$, $SU(M)\times SU(N-M)$. Physically the $SU(M)\times SU(N-M)$ subgroup represents $SU(M)$ transformations among the M-occupied levels, and $SU(N-M)$ transformations among the $N-M$ unoccupied levels; these transformations do *not* change the state (5). Thus the order parameter as represented by the symmetry-breaking state (5) forms a coset space of $SU(N)/SU(M)\times SU(N-M)\times U(1)$ (where the last $U(1)$ represents an overall phase difference between the occupied and unoccupied levels), or equivalently, $U(N)/U(M)\times U(N-M)$. Since a $U(N)$ transformation is parametrized by N^2 parameters (its number of generators), we find the dimensionality (or the number of independent fields) of the coset space to be $N^2-M^2-(N-M)^2=2M(N-M)$. Because we are dealing with a ferromagnetic state (whose effective action contains a dynamic term with a single time derivative, see below), half of these fields are the conjugate momenta of the other half; we thus expect $M(N-M)$ independent spin-wave modes, in agreement with the microscopic construction (6). Very similar analyses were performed in Refs. 15 and 16 that led to the same conclusion.

The analysis above suggests the following matrix parametrization of the ferromagnetic order parameter, appropriate for the symmetry breaking corresponding to $U(N)/U(M)\times U(N-M)$:[15]

$$Q(\mathbf{r},t) = U^\dagger(\mathbf{r},t)\hat{S}U(\mathbf{r},t), \tag{11}$$

where $U(\mathbf{r},t)$ is the (space-time dependent) $N\times N$ $U(N)$ transformation matrix, $\hat{S}_{ij}\propto\delta_{ij}$ is a diagonal matrix with $\hat{S}_{ii}=1$ for $0<i\leq M$, and $\hat{S}_{ii}=-1$ for $M<i\leq N$. The $N\times N$ matrix field $Q(\mathbf{r},t)$ is the order parameter. Ezawa and co-workers[16] use a different, but presumably equivalent, parametrization of the order parameter.

In terms of the matrix field $Q(\mathbf{r},t)$, the long-wavelength, low-energy effective action of the system takes the form[15]

$$S[Q(\mathbf{r},t)] = \int dtd\mathbf{r} \times\left\{\alpha\,\mathrm{tr}\,A(Q)\partial_t Q + \frac{1}{4}\rho_s\,\mathrm{tr}(\nabla Q)\cdot(\nabla Q) + \cdots\right\}, \tag{12}$$

where the first term is the Berry phase term[15] that encodes the commutation relations between different components of the order-parameter field, and the second term describes the energy cost when the order parameter is nonuniform. The choice of the prefactor $1/4$ (instead of $1/2$) for the second term is for later convenience as it compensates for the fact that the trace of the square of Pauli matrices is two. Terms involving higher orders of either time or spatial derivatives, as well as possible symmetry-breaking perturbations are represented by $\cdots$. Thanks to the knowledge of the exact ground state, the order-parameter stiffness ρ_s may be determined exactly in a manner similar to that of Ref. 19: We first construct a state by performing a slow, $\mathbf{r}$-dependent $SU(N)$ rotation on $|\Psi_0\rangle$, then project it to the appropriate LL, calculate its energy, and perform a gradient expansion of the energy functional. The result is

$$\rho_s = \frac{1}{32\pi^2}\int_0^\infty q^3 V(q)[F_n(q)]^2 e^{-q^2\ell^2/2} dq, \tag{13}$$

and for the case of $n=0$ and $1/r$ Coulomb interaction, $\rho_s = e^2/(16\sqrt{2\pi}\epsilon\ell)$.[15] ρ_s is independent of both N and M; Eq. (13) is identical to the $N=2$ and $\nu_n=M=1\rho_s$ expression derived in earlier work.[8,19] This finding is not surprising since any infinitesimal rotation in the $U(N)/U(M)\times U(N-M)$ subgroup of $SU(N)$ can be decomposed into combinations of $SU(2)$ rotations between occupied and unoccupied levels. It is easy to verify that the action (12) reproduces the spin-wave spectrum (7) in the long-wavelength limit.

III. SINGLE-SKYRMION STATES AND COLLECTIVE MODES OF SKYRMION LATTICES

The matrix field Q supports topologically nontrivial spatial configurations, which can be parametrized by an integer-valued topological quantum number called the Pontryagon index,

$$q = \frac{i}{16\pi}\int d^2\mathbf{r}\epsilon^{\mu\nu}\,\mathrm{tr}[Q\partial_\mu Q\partial_\nu Q], \tag{14}$$

where $\epsilon^{\mu\nu}$ is the antisymmetric tensor. Field configurations with nonzero q carry topological charge and are called skyrmions. As in the $N=2$, $\nu_n=M=1$ case,[8,19] quantum Hall ferromagnet skyrmions also carry an *electric* charge that is equal to its topological charge. It follows from the above considerations, as concluded in earlier work,[15] that skyrmions with charge ±1 exist within the lowest Landau level. This remarkable property implies that when skyrmions are the lowest-energy-charged excitations, they will be present[8–10,20] in the ground state of the system when ν is close to, but not equal to, an integer. Skyrmions thus appear as emergent low-energy degrees of freedom and influence all observable properties.

To determine whether or not skyrmions are the lowest-energy-charged excitation for a particular n, we need to compare the energy of a skyrmion (antiskyrmion) pair,

$$\Delta_{SK} = 8\pi\rho_s, \tag{15}$$

with the energy of an ordinary Hartree-Fock theory particle-hole pair,

$$\Delta_{PH} = E(k\to\infty). \tag{16}$$

Table I compares results for graphene Dirac-band and the ordinary parabolic-band cases. In the Dirac-band cases both positive and negative values of n occur whereas the Landau-level indices of parabolic systems are non-negative integers. In both cases, the increase in cyclotron orbit radius with $|n|$ is reflected in the quantum form factor F_n; for the parabolic case the form factor $F_n = L_n^2(q^2\ell^2/2)$. Since the cyclotron-orbit radius $R_c\sim\ell\sqrt{n}$, it is clear simply on dimensional grounds that for Coulomb interactions and large n, $\Delta_{PH}\sim e^2/R_c\sim e^2/\ell\times 1/\sqrt{n}$, whereas $\Delta_{SK}\sim e^2R_c/\ell^2\sim e^2/\ell\times\sqrt{n}$. This difference in the large-n behavior is already apparent in both cases in Table I. For sufficiently large n then, Δ_{SK} will exceed Δ_{QP}, the lowest-energy-charged excitations will be ordinary Hartree-Fock quasiparticles, and the ground state near integer-filling factors will not have low-energy skyrmion degrees of freedom. The quantitative calculations summarized in Table I show that ordinary quasiparticles are already energetically preferred for $n=1$ in the parabolic band case, a result obtained first by Wu and Sondhi.[13] Interestingly, the crossover to ordinary quasiparticles does not occur until $n=4$ in the Dirac-band case; thus skyrmion physics will occur within $n=0$, $n=1$, $n=2$, and $n=3$ Landau levels in graphene.

TABLE I. Hartree-Fock quasiparticle (Δ_{QP}) and Skyrmion (antiskyrmion) (Δ_{SK}) particle-hole excitation gaps for the Dirac (D) bands of graphene and for parabolic bands (P). The two cases are distinquished by different dependencies of form factor F_n on Landau level (LL) index n. These results are for Coulomb interactions in an ideal two-dimensional electron system without finite-thickness corrections and energies are in units of $e^2/\epsilon\ell\sqrt{\pi/2}$. In graphene, the effective value of ϵ depends on the dielectric screening environment provided by the substrate but is typically less than 2, whereas in GaAs and other common heterojunction systems $\epsilon\sim10$.

|LL index|	$\Delta_{QP}^{(D)}$	$\Delta_{SK}^{(D)}$	$\Delta_{QP}^{(P)}$	$\Delta_{SK}^{(P)}$
0	1	1/2	1	1/2
1	11/16	7/32	3/4	7/8
	(0.6875)	(0.2188)	(0.75)	(0.875)
2	145/256	169/512	41/64	145/128
	(0.5664)	(0.3301)	(0.6406)	(1.1328)
3	515/1024	839/2048	147/256	687/512
	(0.5029)	(0.4097)	(0.5742)	(1.3418)
4	0.4608	0.4754	0.5279	1.5522
5	0.4298	0.5328	0.4927	1.6834

We note in passing that we have so far compared only quasiparticle- (quasihole-) and skyrmion- (antiskyrmion-) pair excitation energies. In order to conclude that skyrmions and antiskyrmions are present in the ground state on both sides of integer-filling factors, we need to demonstrate that the skyrmion energy and the antiskyrmion energy are separately smaller than the quasiparticle and quasihole energies, respectively, whenever the pair-excitation energies are smaller. In the case of $N=2$ and $\nu_n=M=1$, this property follows[21] from particle-hole symmetry. We demonstrate below that the pair-excitation-energy criterion also applies for graphene, although the justification is subtly different for $\nu_n=1$ and $\nu_n=3$ cases compared to the $\nu_n=2$ case.

In earlier work Ezawa[16] and co-workers concluded that for $\nu_n=M>1$, lowest-Landau-level (LLL) skyrmions must have a charge that is a multiple of M. To address this discrepancy, we explicitly construct LLL skyrmion wave functions and demonstrate that it is indeed possible to have charge ±1. Our microscopic single-Slater-determinant skyrmion wave function is constructed in a manner similar to that of Ref. 9. For definiteness we discuss a holelike skyrmion with $q=-1$,

$$|\Psi_{sky}\rangle = \prod_{m=0}^{N_\Phi-2} [u_m c^\dagger_{M,m+1} + v_m c^\dagger_{M+1,m}] \times \prod_{m=0}^{N_\Phi-1} \left[\prod_{1\leq\sigma<M} c^\dagger_{\sigma,m} \right] |0\rangle, \tag{17}$$

where the internal-state labels match those of the symmetry-broken ground state. Hartree-Fock skyrmion states are obtained by minimizing the expectation value of the Hamiltonian with respect to u_m and v_m. Skyrmion states with larger topological charges may be constructed in a similar manner. These single Slater-determinant (or Hartree-Fock) skyrmion states correspond to classical skyrmions of the field theory (12), although the Coulomb self-interaction energy of skyrmions must be included in the field theory[8] in order to describe small skyrmions.

The Hartree-Fock (or semiclassical) skyrmion state (17) has a rich internal structure, which may be analyzed in a way similar to the analysis performed earlier on the ground state (5). The state (17) is invariant only under unitary transformations among the first $M-1$ labels or the final $N-M-1$ labels. It follows that the family of $SU(N)$ transformations, that actually transforms the skyrmion states, form a coset space of dimension $N^2-(N-M-1)^2-(M-1)^2=2M(N-M)+2(N-1)$. The set of transformations of dimension $2M(N-M)$ corresponds to the order-parameter fields of the ground state itself, and the additional $2(N-1)$-dimensional space to $(N-1)$ skyrmion internal-complex degrees of freedom. These must be specified in addition to location and size (determined by minimizing the energy with respect to u_m and v_m) in order to completely characterize a classical skyrmion. In other words, the skyrmion has $N-1$ internal (complex) degrees of freedom. A more intuitive way to understand this point is the following. Let us assume the ground-state order-parameter configuration is fixed (either spontaneously or by symmetry-breaking perturbations). We now introduce a single skyrmion at the origin and minimize its energy. This fixes the size of the skyrmion and the *magnitudes* of u_m and v_m. On the other hand, u_m is a $[CP(M-1)]$ vector that lives in an $SU(M)$ space spanned by the M-occupied levels of the ground state, so there are $M-1$ remaining internal (complex) degrees of freedom associated with it. Similarly there are $N-M-1$ remaining internal degrees of freedom associated with v_m. Adding the relative overall phase between u_m and v_m, we find the total internal degrees of freedom is $N-1$. As a matter of fact, u_m and v_m may be combined and viewed as a $CP(N-1)$ vector in the original $SU(N)$ space; thus the $N-1$ internal (complex) degrees of freedom of a skyrmion may be viewed as those of a $CP(N-1)$ superspin.[24]

Once we move away from $\nu_n=M$, we have a finite density of skyrmions in the ground state, provided that they are indeed the lowest-energy-charged excitations. For graphene we are able to judge the energetic ordering of charged excitations based on the energetic ordering of particle-hole excitations. To justify this statement at $\nu_n=M=1$, we must appeal to Zeeman coupling which is always present experimentally. Zeeman coupling selects a fully spin-polarized state and also selects fully spin-polarized skyrmions. We can therefore simply ignore the spin degree of freedom and appeal to the same arguments[21] used for $\nu_n=M=1$ when $N=2$. For $\nu_n=M=3$ we can then appeal to particle-hole symmetry in the $N=4$ Landau-level multiplet, which suggests that the situation is the same as $\nu_n=M=1$. Finally, for $\nu_n=M=2$ we can generalize the argument of Ref. 21 by appealing directly to particle-hole symmetry at this filling.

In their classical ground state, skyrmions will form a lattice. (Quantum corrections to the classical ground state become more important as the skyrmion density increases.[12,22]) In this case the internal degrees of freedom associated with skyrmions on different sites will interact and the classical energy will be minimized by an arrangement with long-range order. When quantized, fluctuations in the internal degrees of freedom will result in $N-1$ spin-wavelike modes, which will be present in addition to the single phonon mode associated with fluctuation in skyrmion positions. (In a strong magnetic field transverse- and longitudinal-position fluctuations are canonically conjugate leading[23] to phonons with $k^{3/2}$ dispersion.) In the presence of full $SU(N)$ symmetry these internal modes are gapless, and some (or all) of them may remain gapless in the presence of symmetry-breaking perturbations, under appropriate conditions (see below for examples). For the $SU(2)$ case we find $N-1=1$ internal mode, which is known previously;[12] here we provide a more general understanding of this result.

For graphene $N=4$, and the case $\nu_n=M=2$ is particularly interesting. Weak Zeeman coupling will select a *unique* fully spin-polarized, valley-singlet ground state. All collective modes are therefore gapped for ν_n exactly equal to 2, i.e., in the absence of skyrmions. In this case the $SU(2)$ valley symmetry is *preserved* by the ground state. We nevertheless predict that the skyrmion lattice state near $\nu_n=2$ will have phonons and 3 additional *gapless* modes. Of these three internal modes, we predict that one has *quadratic* dispersion, while the other two will have linear dispersions. The quadratic mode may be understood in the following manner. The skyrmion lattice state has a finite pseudospin (valley) magnetization, and is thus a spontaneous valley ferromagnet; the quadratic mode is simply the ferromagnetic spin wave in the valley channel. The two linear modes are the Goldstone modes corresponding to the two additional spontaneously broken $U(1)$ symmetries; these symmetries are broken by the fixed relative phases between u_m and v_m with the same valley index. Adding the phonon mode, we thus find one quadratic mode, one $k^{3/2}$ mode, and two linear modes, all gapless.

The situation is very different at $\nu_n=M=1$. In this case, the ground state is still fully spin polarized due to the Zeeman coupling, but is also a *spontaneous* valley ferromagnet that breaks the $SU(2)$ valley symmetry spontaneously. Thus we expect a single gapless mode with quadratic dispersion in the absence of skyrmions, the $SU(2)$ valley pseudospin waves. The lowest-energy skyrmion states are thus pseudospin textures in the valley degree of freedom,[25] and the skyrmion lattice states at low temperature will be fully spin polarized. As a result, we expect only one gapless internal mode with quadratic dispersion, which does not involve spin flips, while the other two internal modes, which do involve spin flip will have a gap of $g\mu_B B$. Thus the only new gapless mode of the skyrmion lattice state is the phonon mode in this case.

Isolated individual *quantum*-skyrmion states have well-defined total-orbital angular momentum and internal $SU(N)$ quantum numbers, which can be analyzed for $\nu_n=M=1$ following the strategy of Ref. 20. These are the quantum counterparts of classical $CP(N-1)$ skyrmions. In this case the single-Slater-determinant state (5) is the exact ground state for δ-function interaction between electrons, since it has exactly zero energy. Holelike skyrmion states may be identified as all zero-energy states for the case $N_\Phi=N_e+1$. As emphasized in Ref. 20 (see also Ref. 26), all such states may be written in the form

$$\Psi(z,\chi)=\left[\prod_{i<j}(z_i-z_j)\right]\Psi_B(z,\chi), \tag{18}$$

where the antisymmetric Jastrow factor $\Pi_{i<j}(z_i-z_j)$ ensures zero energy, while $\Psi_B(z,\chi)$ is a *bosonic* wave function that is *symmetric* under the exchange of spatial and internal coordinates of two particles. We can classify the skyrmion states based on the properties of this bosonic wave function.

Due to the fact $N_\Phi=N_e+1$, the bosons can only be in $m=0$ or $m=1$ orbital states. Letting these occupation numbers be n_0 and n_1, respectively, the total angular momentum of a state (18) measured from that of the ground state is $\Delta L=n_1$, thus $n_0=N_e-\Delta L$. We now classify all the skyrmion states for a fixed ΔL based on the $SU(N)$ representations they form (in the familiar $SU(2)$ case these representations are labeled by a single quantum number, the total spin[20]). Because bosons occupying the same orbital have totally symmetric orbital wave functions, their internal wave function must also be totally symmetric. It follows that the internal states of the n_0 bosons in orbital $m=0$ form a totally symmetric representation of $SU(N)$, as do the n_1 bosons in orbital $m=1$. In terms of Young tableau, they form two horizontal-row representations, with n_0 and n_1 horizontal boxes, respectively. More generally, each irreducible representation of the $SU(N)$ group as represented by the Young tableau[27] can be labeled by a set of $N-1$ non-negative integers in descending order: $[l_1,l_2,\ldots,l_{N-1}]$, where each integer represents the number of boxes in each row. Thus the two representations formed by the bosons in $m=0$ and $m=1$ orbitals form representations $[n_0,0,\ldots]$ and $[n_1,0,\ldots]$, respectively. Now we take the direct product of these two representations and decompose them into irreducible representations of $SU(N)$,

$$\begin{aligned}&[n_0,0,\ldots]\otimes[n_1,0,\ldots]\\&\quad=[n_0+n_1,0,0,\ldots]\oplus[n_0+n_1-1,1,0,\ldots]\\&\quad\oplus\ \ldots\ \oplus[n_0,n_1,0,\ldots]=[N_e,0,0,\ldots]\\&\quad\oplus[N_e-1,1,0,\ldots]\oplus\ \ldots\ \oplus[N_e-\Delta L,\Delta L,0,\ldots].\end{aligned} \tag{19}$$

In the above equation we have assumed that $n_0\geqslant n_1$ or $N_e\geqslant 2\Delta L$. If the opposite is true, the positions of n_0 and n_1 need to be interchanged. The dimensionality of these representations may be found in the literature.[27] This procedure classifies the holelike skyrmion states at $\nu_n=1$ based on their angular-momentum quantum number and the irreducible $SU(N)$ representations they form.

IV. $SU(N)$ FERROMAGNETS AT FRACTIONAL-FILLING FACTORS

We now turn our discussion to possible fractional quantum Hall (FQH) states (which have not yet been observed in graphene), many of which are also $SU(N)$ ferromagnets. Many FQH states may be constructed by starting with integer quantum Hall (IQH) states[28] and using the composite fermion (CF) flux-attachment *ansatz*. In this construction, we start from an IQH state with filling factor $\nu_{n,CF}=m$, and attach an even $2n'$ flux quanta to the CFs to turn them into electrons; after the flux is spread out the electron-filling factor in the valence LL becomes

$$\nu_n=\frac{m}{2n'm\pm 1}. \tag{20}$$

This is, of course, the familiar Jain's sequence.[28,29] The difference here is that in the presence of the internal degeneracy and $SU(N)$ symmetry, the CF IQH state is an $SU(N)$ ferromagnet as long as m is not a multiple of N. We thus expect the $SU(N)$ symmetry properties to be reflected in the FQH states. The number of collective modes and the number of branches of skyrmion excitations should be the same as that of the IQH states at $\nu_n=M$, if we identify $M=\mathrm{mod}(m,N)$. On the other hand if m is a multiple of N, we obtain an $SU(N)$ singlet, and there is no spontaneous symmetry breaking; the system will be fully gapped. For cases with $m\leqslant N$, we display below Laughlin-Halperin type trial wave functions for the FQH states with the expected $SU(N)$-symmetry properties in first quantization,

$$\psi_{n',m}(z)=\left[\prod_{i<j}^{N_e}(z_i-z_j)^{2n'}\right]\left[A\prod_{\sigma=1}^{m}\prod_{k<l}^{N_\Phi}(z_{k\sigma}-z_{l\sigma})\right]. \tag{21}$$

Here A represents antisymmetrization of all coordinates, and we have neglected the common exponential factors for LL wave functions. The second factor is the first quantized wave function for (5), while the first factor reflects flux attachment. The wave functions for $m>N$ is more complicated, as in this case some of the CFs occupy higher LLs, and LLL projection is necessary.[28] We note that this type of construction was found to be generally reliable in predicting the spin structure in the $SU(2)$ case without Zeeman splitting.[28]

The low-energy physics of the FQH $SU(N)$ ferromagnets are also described by field theory (12). In this case we do not have exact knowledge about the parameters (like the stiffness ρ_s) of the theory, but ρ_s has been calculated numerically for the members of the Laughlin sequence $\nu=1/3,1/5$ for $1/r$ interaction based on the Laughlin trial wave function in the $SU(2)$ case;[19] they are $9.23\times10^{-4}(e^2/\epsilon\ell)$ and $2.34\times10^{-4}(e^2/\epsilon\ell)$, respectively. Using the same arguments as for the IQH case discussed above, we expect ρ_s to take the same two values at the same filling factors in graphene. These values can be used to determine the energies of skyrmion-antiskyrmion pairs, which may be compared with transport measurements should FQH states be observed in graphene in future experiments. We caution, however, that at fractional-filling factors, we can no longer appeal to particle-hole symmetry properties so that it is possible,[30] in general, for

Laughlin-type fractionally charged quasiparticles to be present in the ground state on one side of an incompressible filling factor and fractionally charged skyrmions on the other.

V. CONCLUDING REMARKS

In closing we comment on the possible effect of single-particle valley splitting on our $SU(4)$-based analysis of graphene. We first note that the valley degeneracy of graphene is rather robust; for example, unlike in silicon, simple strain cannot lift the degeneracy. Electron-electron interactions are likely to provide the most important source of Hamiltonian matrix elements that break the valley portion of the $SU(4)$ symmetry. Our considerations should nevertheless largely apply for weak valley-symmetry breaking, as long as the characteristic energy scale of these terms is much weaker than the Coulomb interaction scale. We mention that among the possible extrinsic sources of four-fold degeneracy lifting in the single-particle Hamiltonian are edge effects and intervalley scattering. For large extrinsic splittings, the $SU(4)$ symmetry will be reduced to $SU(2)\times U(1)\times U(1)$, where the remaining $SU(2)$ symmetry corresponds to spin, which is further reduced by Zeeman splitting as discussed earlier. At the lowest temperatures and energy scales the physics may be be similar to that of a bilayer system,[31] which may support other types of broken symmetry states.[32,33] The question of whether our predicted interaction-driven spontaneous breaking of valley degeneracy is playing the key role in the observed valley splitting at high field[4] can be decided by careful measurements of zero-field valley splitting. If the experimental zero-field valley splitting is negligibly small, then it seems certain that $SU(4)$ quantum Hall ferromagnetism, associated with the spontaneous breaking of valley and spin degeneracy, is already playing a role in the high-field quantum Hall experiments in graphene. (It seems clear that many-body physics does not have a large influence on quasiparticle valley splitting in the absence of a field.) Direct observations of skyrmions and associated collective excitations in graphene then take on a particular experimental relevance. We further note that both Zeeman and valley splittings are single-electron effects; electron-electron interactions also have weak symmetry-breaking effects due to physics at lattice scale, as discussed recently.[14]

In summary, we have identified exact broken-symmetry eigenstates of $SU(N)$-invariant Hamiltonians in the quantum Hall regime of graphene. We argue that these states are ground states for any physically sensible repulsive interaction between the particles, and for Coulomb interactions between electrons in particular. Given $SU(N)$ invariance we were able to obtain exact results for the elementary collective-excitation spectrum. We found that in the absence of symmetry breaking there are $M(N-M)$ gapless collective modes when M of the N members of a Landau-level multiplet are occupied. In addition, we have shown that for Coulombic electron-electron interactions, skyrmions are much more robust in systems like graphene with a Dirac-band structure, than for the more familiar parabolic-band systems. We predict that dense-skyrmion systems will be ubiquitous in the quantum Hall regime of graphene and that skyrmion lattice states near even-integer filling factors will have four branches of gapless collective modes, two with k^1 dispersion, one with $k^{3/2}$ dispersion, and one with k^2 dispersion. The identification of probes that can study skyrmion physics in graphene layers is therefore an attractive challenge for experiment.

Note added. While this paper was being written, a preprint[34] authored by Alicea and Fisher appeared, which has some overlap with the present paper. Among other contributions, Alicea and Fisher[34] studied collective modes in the long-wavelength limit, and single-skyrmion states at certain specific integer fillings. Where overlap occurs, our results are consistent with those of Alicea and Fisher. We thank Jason Alicea for informative conversations about this work.

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