Competing orders, nonlinear sigma models, and topological terms in quantum magnets

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A number of examples have demonstrated the failure of the Landau-Ginzburg-Wilson (LGW) paradigm in describing the competing phases and phase transitions of two-dimensional quantum magnets. In this paper, we argue that such magnets possess field theoretic descriptions in terms of their slow fluctuating orders provided certain topological terms are included in the action. These topological terms may thus be viewed as what goes wrong within the conventional LGW thinking. The field theoretic descriptions we develop are possible alternates to the popular gauge theories of such non-LGW behavior. Examples that are studied include weakly coupled quasi-one-dimensional spin chains, deconfined critical points in fully two-dimensional magnets, and two-component massless QED$_3$. A prominent role is played by an anisotropic O(4) nonlinear sigma model in three space-time dimensions with a topological theta term. Some properties of this model are discussed. We speculate that similar sigma model descriptions might exist for fermionic algebraic spin liquid phases.

Recently, there has been considerable progress in describing the competition between various kinds of ordering tendencies of quantum spin systems in two space dimensions. Possible Landau-forbidden direct second-order quantum transitions between Neel and valence bond solid (VBS) phases of spin-1/2 quantum antiferromagnets have been described in Ref. 1. The corresponding critical points have been described in terms of field theories involving gapless bosonic spinon fields coupled to a fluctuating gauge field. They have hence been dubbed "deconfined quantum critical points." Other recent work has established the stability of "algebraic spin liquid" phases of two-dimensional quantum magnets,[2] at least within systematic large-N expansions. These are critical *phases* that exhibit power-law correlations in the spin[3] and other operators.[4] In the best studied cases, the low-energy theory is described by a conformally invariant fixed point. All of the existing descriptions of such algebraic spin liquids are in terms of theories of Dirac spinon fields coupled to a fluctuating massless gauge field. Thus these may be regarded as "deconfined critical phases." Despite the utility of the spinon-gauge description, neither the spinons nor the gauge photon are good quasiparticles of the system at these deconfined critical points/phases. Indeed, it is not clear that there is *any* quasiparticle description of the low-energy fixed point theory.

A large number of open questions remain on such critical points/phases. Is a description in terms of spinons and gauge fields necessary? Given the slow power-law decay for various classical order parameters, is a description directly in terms of these orders possible? In this paper, we study a number of related questions and suggest some interesting answers to these questions.

In contrast to two dimensions, competing orders in one-dimensional spin systems are extremely well-understood: the classic example is the one-dimensional spin-1/2 antiferromagnetic (AF) chain. These have a number of properties reminiscent of their two-dimensional counterparts. It is well known that the power-law phase in such spin chains can be described by a number of different equivalent field theories.

In a semiclassical description, an O(3) nonlinear sigma model with a topological θ term[5] is obtained (with $\theta=\pi$). The O(3) vector has the physical interpretation of being the Neel order parameter. More useful is a field theory in terms of an SU(2) matrix U with a Wess-Zumino-Witten (WZW) term.[6] The SU(2) matrix U again has a simple interpretation. Indeed $U=D+i\vec{N}\cdot\vec{\sigma}$ with $\vec{N}$ the Neel vector and D the dimerization (i.e., VBS or spin-Peierls) order parameter. The VBS order parameter also has the same power-law correlations as the Neel vector in the spin-1/2 AF chain. Thus the WZW field theory is written precisely in terms of the classical order parameters that have slow correlations in this algebraic spin liquid phase. However, the action necessarily involves the nontrivial "topological" WZW term. Finally, a gauge description of the spin-1/2 chain in terms of fermionic Dirac spinons coupled to fluctuating U(1) or SU(2) gauge fields is also possible.[7,8] It is readily seen using bosonization techniques that this is exactly equivalent to the SU(2) level-1 WZW theory.

Motivated by this, we may speculate that two-dimensional quantum magnets (including possible deconfined critical points and/or phases) also possess field theoretic descriptions in terms of their slow fluctuating orders provided certain topological terms are included in the action. Here we will collect some evidence supporting this suggestion.

We first study quasi-one-dimensional systems of weakly coupled one-dimensional spin-1/2 chains. If the interchain exchanges are unfrustrated, it is expected that two-dimensional long-range Neel order will develop. Frustrating interchain exchanges promote VBS ordering of the columnar dimer pattern with a twofold degenerate ground state. One approach to think about the competition between these two distinct ground states is in terms of a (2+1)-dimensional O(3) nonlinear sigma model with appropriate Berry phases for the hedgehog topological defects.[9] It is known that the hedgehogs are doubled in a continuum description, and their proliferation leads to VBS order in the paramagnet.[10] In this approach, the Neel order parameter is simply represented as

the O(3) vector. On the other hand, the VBS order parameter has a more complicated representation and is identified as the hedgehog topological defect. Thus this approach treats the Neel and VBS orders on an unequal footing. Here we will show how a "superspin" description in terms of a four-component order parameter field may be set up. Three of these will simply correspond to the Neel vector while the fourth will be the VBS order parameter (which is a scalar in this case with rectangular symmetry). A superspin O(4) sigma model [with some weak O(3) anisotropy] will be derived. Interestingly, this sigma model contains a topological θ term (at $\theta = \pi$). We will show how the O(4) sigma model together with this topological term reproduces the known physics of the coupled spin-1/2 chains.

Next we consider spin-1/2 quantum magnets on an isotropic two-dimensional square lattice. In an interesting recent work, Tanaka and Hu[11] have shown how a "superspin" nonlinear sigma model may be derived to describe the Neel-VBS competition in this system. Such a sigma model again contains a topological term. Here we revisit their derivation from a slightly different perspective, and show the equivalence between the resulting sigma model theory and the spinon-gauge descriptions of Ref. 1. This can be accomplished in full for the case with easy plane anisotropy for the Neel order parameter.

Finally, we consider massless QED$_3$ with N two-component Dirac fermions. For large enough N, this is the low-energy theory of the algebraic spin liquid phases shown recently to be stable quantum phases of SU(N) quantum magnets. The case $N=4$ is of direct interest to SU(2) magnets. Here we study $N=2$ and show that this is again equivalent to the O(4) sigma model with a topological term at $\theta = \pi$. The possible duality of $N=2$ QED$_3$ to the global O(4) model without a topological term was conjectured recently in Ref. 12. Our derivation provides partial support for this conjecture but reveals the presence of the topological term.

Based on all these results, we suggest that similar descriptions are possible for massless QED$_3$ at any N. If correct, this would be a possibly useful alternate to the spinon-gauge descriptions that are currently available.

I. ALGEBRAIC SPIN LIQUID IN ONE DIMENSION

The classic example of a gapless "algebraic" spin liquid is the ground state of the nearest-neighbor spin-1/2 chain. This state has slow power-law correlations for both the Neel and VBS correlations. In this brief section, we provide a quick overview of its field theoretic description that will be useful for us later. The low-energy physics of the spin-1/2 chain may be described by the following field theory:

$$S = S_0 + S_{\mathrm{WZW}}, \tag{1}$$

$$S_0 = \int d^2x \frac{1}{2g} \mathrm{tr}(\partial_i U^\dagger \partial_i U), \tag{2}$$

$$S_{\mathrm{WZW}} = i\Gamma. \tag{3}$$

Here $U \in \mathrm{SU}(2)$. The number Γ is defined as follows. The field U defines a map from S^2 to S^3. The volume in S^3

bounded by the surface traced out by U defines Γ. Formally,

$$\Gamma = \frac{1}{12\pi} \int d^3y \epsilon_{ijk} \mathrm{tr}(U^{-1}\partial_i U U^{-1}\partial_j U U^{-1}\partial_k U). \tag{4}$$

Here the two-dimensional space-time S^2 is regarded as the boundary of a solid ball B, and $y=(y_1,y_2,y_3)$ are coordinates of B. The matrix U has been extended to the ball B in such a way that at the boundary it has the correct value for the two-dimensional field theory. We may write U in the form

$$U = \phi_0 + i\vec{\phi} \cdot \vec{\sigma} \tag{5}$$

with $\phi_0^2 + (\vec{\phi})^2 = 1$, i.e., in terms of a four-component unit vector. Then the action may be written

$$S_0 = \frac{1}{g}(\partial_i \hat{\phi})^2, \tag{6}$$

$$S_{\mathrm{WZW}} = \frac{i}{6\pi} \int d^3y \epsilon_{ijk} \epsilon_{\alpha\beta\gamma\delta} \phi_\alpha \partial_i \phi_\beta \partial_j \phi_\gamma \partial_k \phi_\delta. \tag{7}$$

The Neel and VBS order parameters are simply determined in terms of the unit vector $\hat{\phi}$. We have $\vec{N} \sim \vec{\phi}$ and $D \sim \phi_0$. The WZW field theory has global SO(4) symmetry due to which the Neel vector can be rotated into the VBS order parameter. This symmetry that emerges as a property of the low-energy fixed point guarantees that the Neel and VBS order parameters have the same long-distance correlations. Thus the WZW field theory provides a "superspin" field theory of the two dominant competing orders of the quantum spin-1/2 chain. Note that the topological WZW term is required to correctly reproduce the known physics of the spin chain in this superspin description.

The WZW theory must be contrasted with another popular field theory for the spin-1/2 chain: the O(3) sigma model with a topological term. This representation treats the Neel and VBS orders on unequal footing even though they eventually have the same low-energy behavior. Indeed, the VBS order parameter is represented rather nontrivially as the topological charge density, and its connections to the Neel vector are not at all obvious in this field theory.

Finally, it is possible to examine the spin-1/2 chain in a slave particle description of the spins by writing the spin operator at site r as

$$\vec{S}_r = \frac{f_r^\dagger \vec{\sigma} f_r}{2} \tag{8}$$

with f_r a spinful fermionic "spinon" field at each lattice site. Following standard techniques, the spin Hamiltonian may be recast as a $U(1)$ [or SU(2)] gauge theory of these spinons coupled to a fluctuating gauge field; see Refs. 7 and 8. A suitable continuum approximation can then be made to obtain a theory of massless Dirac spinons coupled to a fluctuating gauge field [massless QED$_2$ in the $U(1)$ case]. The gauge fluctuations can then be integrated out exactly after bosonization and the resulting theory is equivalent to the WZW theory above. In particular, as emphasized in Ref. 8, the slave spinon is quite distinct from the true spinon known to exist in the spectrum of the spin-1/2 chain. Thus in this

one-dimensional algebraic spin liquid, the spinon-gauge description is unnecessary and a description in terms of the slow competing orders is possible.

II. WEAKLY COUPLED SPIN CHAINS

Ordered phases and transitions

Here we consider a two-dimensional system obtained as an infinite array of one-dimensional spin-1/2 chains coupled together by short-ranged antiferromagnetic interactions. The entire system then has rectangular symmetry in two dimensions. In general, such coupling tends to stabilize ordering of one of the slow fluctuating order parameters of the decoupled chain. For unfrustrated interchain couplings, collinear Neel ordering at (π,π) is stabilized. With frustration that suppresses Neel order, columnar VBS ordering at $(\pi,0)$ often gets stabilized as shown in recent work by Starykh and Balents.[13] The competition between the Neel state and the columnar VBS state of this rectangular lattice may be understood in the following terms. Consider the limit of reasonably strong two-dimensional coupling. Then a $D=2+1$ O(3) nonlinear sigma model description of the Neel vector fluctuations is useful. This model must be supplemented with appropriate Berry phases. These Berry phases only affect singular hedgehog configurations of the Neel vector in space-time.[9] Due to the Berry phases, these hedgehog insertions have the same symmetry as the VBS order parameter and hence may be identified with them.[10] With rectangular symmetry, it is known that in any putative continuum description the hedgehogs are doubled (note the contrast to a square lattice, where they are quadrupled). Neel order is destroyed if the hedgehogs proliferate. Their Berry phases then induce VBS order. Here the resulting state is twofold degenerate due to the doubling of hedgehogs and may be identified with the twofold degenerate columnar VBS pattern. The Neel-VBS transition in this case is presumably first order due to the likely relevance of doubled monopoles at the critical fixed point of the hedgehog-free theory.[14,15] (However, there will be an interesting deconfined multicritical point.)

Now we show how these results may be reproduced in an alternate approach that directly attempts a "superspin" description of the two competing orders (Neel and VBS) in terms of a four-component field. As already illustrated by the analysis in $d=1$, this will require incorporating an appropriate topological term. Before supplying the derivation, let us first understand the structure of the allowed superspin field theory. With a four-component "order parameter" field, we might attempt a description in terms of an O(4) nonlinear sigma model with some O(3) anisotropy. Ignoring the anisotropy for the time being, the action for such a theory is defined by

$$S_0 = \int d^2x d\tau \frac{1}{t}(\partial_i \hat{\phi})^2, \qquad (9)$$

where $\hat{\phi}$ is a four-component unit vector. The Neel and VBS order parameters are related to $\hat{\phi}$ in the same way as in the $d=1$ case. The order parameter clearly lives in S^3. In three space-time dimensions (with, for instance, boundary condi-

tions that identify space-time with S^3), configurations of the $\hat{\phi}$ field may be classified by an integer "winding number" corresponding to $\Pi_3(S^3)=Z$. The winding number is expressed in terms of the $\hat{\phi}$ field through

$$Q = \frac{1}{12\pi^2} \int d^2x d\tau \epsilon_{ijk}\epsilon_{\alpha\beta\gamma\delta}\phi_\alpha\partial_i\phi_\beta\partial_j\phi_\gamma\partial_k\phi_\delta. \qquad (10)$$

It is then possible to consider adding a topological Berry phase term to the action S_0, which gives different weights to the different topological sectors parametrized by Q. Thus consider

$$S = S_0 + i\theta Q. \qquad (11)$$

The physics is clearly periodic under $\theta \rightarrow \theta+2\pi$. The system is parity and time reversal invariant for $\theta=0$ or $\theta=\pi$.

Now we will show that a derivation of such a theory with a four-component field for the coupled spin chains naturally leads to the sigma model at $\theta=\pi$. Our derivation will closely follow Haldane's well-known derivation of the O(3) sigma model field theory with the θ term for *one-dimensional* spin chains starting from the microscopic path integral for a single quantum spin. Here we will start instead from the WZW theory for each spin chain, and then derive a two-dimensional continuum field theory for the field $\hat{\phi}$.

We start from the decoupled chain limit in which each chain is described by the SU(2) level 1 theory. We then include interchain couplings for the slow modes of this theory. Consider, therefore, the following action for the coupled chains:

$$S = S_1 + S_2, \qquad (12)$$

$$S_1 = \sum_y S[U(x,\tau,y)], \qquad (13)$$

$$S_2 = \int dx d\tau \sum_y [u\vec{\phi}(x,\tau,y) \cdot \vec{\phi}(x,\tau,y+1) \qquad (14)$$

$$- v\phi_0(x,\tau,y)\phi_0(x,\tau,y+1)]. \qquad (15)$$

Here x is the continuous coordinate along the chain direction, y is an integer labeling the different chains, and U is the SU(2) matrix field that enters the description of each chain. The action $S[U]$ is the SU(2) level 1 WZW action summed over all the chains. The term S_2 in the action is the interchain coupling. It has been written in terms of the four-component vector $\hat{\phi}$. The couplings $u,v>0$. We have assumed there is an antiferromagnetic coupling of the Neel order parameter ($\vec{\phi}$) between two neighboring chains, and a ferromagnetic coupling between the VBS orders (ϕ_0). This is completely appropriate to describe the competition between (π,π) Neel order and $(\pi,0)$ columnar VBS order in the two-dimensional problem.

It is useful to first consider the simple limit where $u=v$ and then to perturb around this limit. When $u=v$, the action has extra global O(4) symmetry that is broken to O(3)$\times Z_2$ when $u \neq v$. At $u=v$, we can write S_2 as

$$S_2 = -\frac{u}{2}\int_{x,\tau}\sum_y \text{tr}[U(y+1)U(y)].$$ (16)

This tends to prefer a "staggered" ordering of the U perpendicular to the chains where U for even chains is the inverse of that for the odd chains. We therefore write

$$U(y) = g(y), \quad y = 2n,$$ (17)

$$= g^{\dagger}(y), \quad y = 2n + 1$$ (18)

and take the continuum limit with g assumed to vary slowly in both spatial directions. The first term in S_1 has a simple continuum form,

$$S_0 = \frac{1}{t}\int d^2x d\tau\, \text{tr}(\partial_i g^{\dagger}\partial_i g).$$ (19)

The continuum form of the WZW term requires some work. Substituting Eq. (17), we get

$$\sum_y (-1)^y i\Gamma[g(y)].$$ (20)

This alternating sum can now be evaluated analogous to Haldane's calculation in $d=1$. First write it as

$$\sum_{y=2n} i\{\Gamma[g(y)] - \Gamma[g(y-1)]\}.$$ (21)

Each term is the difference in volume in S^3 bounded by the surfaces traced out by $g(y)$ and $g(y-1)$. With g smoothly varying, the full sum over y can then be related to the total volume in S^3 swept out by $g(x,y,\tau)$. Specifically, we get

$$\int dx dy d\tau \frac{i}{24\pi}\epsilon_{ijk}\text{tr}(g^{-1}\partial_i g g^{-1}\partial_j g g^{-1}\partial_k g).$$ (22)

This is precisely the O(4) model at $\theta = \pi$ as may be seen by writing the SU(2) matrix g in terms of the four-component $\hat{\phi}$ field.

Taking $u \neq v$ introduces some anisotropy in the model, which lowers the symmetry to O(3)$\times Z_2$. Let us now study this model in the presence of such weak anisotropy. Consider defect configurations of the O(3) vector field $\vec{\phi}$. In three space-time dimensions, these are hedgehogs. In the core of the hedgehog, the magnitude of $\vec{\phi}$ is suppressed. In the present model, this means that the $\hat{\phi}$ vector points along the 0 direction in the core. Clearly we can distinguish two different kinds of hedgehogs depending on whether ϕ_0 is positive or negative in the core. These hedgehogs are very analogous to meron vortices in O(3) models with weak easy plane anisotropy. Therefore, we will refer to these as meron hedgehogs. Each meron hedgehog may be regarded as one-half of a point defect of the O(4) model with nonzero Q. Thus for elementary meron hedgehogs we have $Q = \pm 1/2$ depending on the sign of ϕ_0 in the core. The topological θ term then gives a Berry phase $e^{\pm i(\pi/2)}$ associated with each meron hedgehog.

Let us now study various phases of the anisotropic O(4) model. When the O(3) vector orders (the Neel phase), the meron hedgehogs or their Berry phases play a very little role

in the low-energy physics. Consider now phases where the Neel vector is disordered. This may be usefully discussed in terms of a three-dimensional Coulomb gas of meron hedgehogs that interact with each other through $1/r$ interactions and where appropriate Berry phase factors are associated with the hedgehogs. The action for this Coulomb gas takes the form

$$S = S_c + S_B + S_{\text{int}},$$ (23)

$$S_c = u\sum_r (m_{+r}^2 + m_{-r}^2),$$ (24)

$$S_B = i\frac{\pi}{2}\sum_r (m_{+r} - m_{-r}),$$ (25)

$$S_{\text{int}} = \sum_{rr'} m_r V(r-r')m_{r'}.$$ (26)

Here r is the site of some three-dimensional space-time lattice. $m_{\pm r}$ are integers corresponding to the number of the two kinds of $\pm$ hedgehogs at site r. The first term is a hedgehog core action. The second term is the Berry phase. In the last term, $m_r = m_{+r} + m_{-r}$ is the total hedgehog number at any site r. This term describes the interhedgehog three-dimensional Coulomb interaction with $V(r-r') \sim 1/(r-r')$. As usual, an equivalent sine-Gordon theory is readily obtained by decoupling the interaction with a potential χ and summing over the integers $m_{\pm}$. It is easy to see that the resulting action takes the form

$$S = \int d^3x K(\nabla\chi)^2 - \sum_n \lambda_n \cos(2n\chi).$$ (27)

Here $e^{i\chi}$ creates a strength-1 hedgehog. We thus see that only even strength hedgehogs appear in this sine-Gordon theory. The Berry phases have led to a doubling of the hedgehogs. The doubled hedgehogs proliferate at long scales in this paramagnetic phase. This leads to a twofold degenerate ground state (the two ground states being distinguished by the expectation value of the single hedgehog operator). Thus phases that have unbroken O(3) symmetry of the anisotropic O(4) model at $\theta = \pi$ necessarily have a twofold ground-state degeneracy.

To understand better the nature of these O(3) symmetric phases in the particular microscopic realization of weakly coupled chains, let us study the role of various discrete symmetries. In particular, we must distinguish between parity transformations P_x, P_y along the chain direction (the x direction) and the one perpendicular to the chains. Microscopically, the P_x symmetry involves $x \to -x$ *together* with $\phi_0 \to -\phi_0$. For the hedgehogs this then implies $m_+ \to -m_-$, $m_- \to -m_+$, $\chi \to -\chi$. Under P_y we have $y \to -y$, $m_{\pm} \to -m_{\pm}$, $\chi \to -\chi + \pi$.

In the smooth ground states of the sine-Gordon model the possible values of χ depend on the λ_n. If λ_1 is the dominant coupling, then $\lambda_1 < 0$ prefers $e^{i\chi} = \pm i$ so that P_x is broken while P_y is preserved. We may identify this with the $(\pi, 0)$ columnar dimer state. On the other hand, $\lambda_1 > 0$ prefers $e^{i\chi}$

$=\pm 1$ so that P_y is broken while P_x is preserved. This may actually be identified with the $(0,\pi)$ columnar dimer state (i.e., vertical dimers between the chains). It is interesting that though we derived the sigma model focusing on the competition between Neel and the $(\pi,0)$ VBS state, it is still capable of describing the $(0,\pi)$ VBS state as well.

All of this is completely consistent with results known from the earlier direct analysis of the O(3) sigma model with Berry phases appropriate for a two-dimensional lattice with rectangular symmetry. Thus the superspin O(4) formulation with the θ term correctly captures the Neel-VBS competition in this system.

Finally, we note that spin-1/2 magnets on lattices with rectangular symmetry are not expected to have trivial featureless paramagnetic phases that also preserve all lattice symmetries. Nontrivial paramagnetic phases such as spin liquids with topological order are, however, possible. This then implies that the O(4) model at $\theta=\pi$ in the presence of $O(3)\times Z_2$ anisotropy will also not possess simple phases that break no symmetry. Any such symmetry unbroken phase must necessarily also have some hidden order (such as topological order).

III. DECONFINED CRITICALITY IN THE TWO-DIMENSIONAL SQUARE LATTICE

We now turn our attention to spin-1/2 quantum magnets on isotropic two-dimensional square lattices. Here apart from the well-known Neel state, paramagnetic VBS states are again possible. In contrast to the quasi-two-dimensional case, here the VBS state will have a fourfold groundstate degeneracy. A superspin description of the Neel-VBS competition was recently derived by Tanaka and Hu.[11] Here we first very briefly review their derivation from a slightly different perspective. In the presence of some easy plane anisotropy for the Neel vector, we explicitly show the equivalence to the spinon-gauge field theory proposed in Ref. 1 for the deconfined critical point between these two ordered phases. The appropriate spinon-gauge field theory involves gapless bosonic spinons coupled to a gapless noncompact $U(1)$ gauge field and has been dubbed the noncompact CP^1 (NCCP^1) model.[15]

A. Tanaka-Hu superspin field theory

Consider a half-filled Hubbard model on a two-dimensional square lattice with π flux through each square plaquette described by the Hamiltonian

$$H = -\sum_{\langle rr'\rangle} t_{rr'}(c_r^\dagger c_{r'} + \text{H.c.}) + U\sum_r n_r(n_r - 1). \quad (28)$$

Here $c_{r\alpha}$ are spinful electron operators at sites i of a square lattice. The $t_{rr'}$ are chosen to have π flux through each plaquette. A specific choice is $t_{rr'}=-(-1)^y t$ for horizontal bonds and $t_{rr'}=t$ for vertical bonds. The U term is the usual on-site Hubbard repulsion. At $U=0$, the band structure is well known and consists of four Fermi points [at $(\pm\frac{\pi}{2},\pm\frac{\pi}{2})$] in the Brillouin zone of the square lattice. The electron dispersion is linear in the vicinity of these points. The low-

energy physics may be described in terms of a continuum Dirac theory that focuses on the modes near these Fermi points. For $r=(x,2y)$, write

$$c_{\vec{r}} = e^{i\vec{K}_1\cdot\vec{r}}\psi_{12} + e^{i\vec{K}_2\cdot\vec{r}}\psi_{21}, \quad (29)$$

$$c_{\vec{r}+\hat{y}} = i(e^{i\vec{K}_1\cdot\vec{r}}\psi_{11} + e^{i\vec{K}_2\cdot\vec{r}}\psi_{22}) \quad (30)$$

with $\vec{K}_1=(\frac{\pi}{2},\frac{\pi}{2})$ and $\vec{K}_2=(-\frac{\pi}{2},\frac{\pi}{2})$. Then the continuum Hamiltonian reads

$$H \approx \int d^2x\,\psi^\dagger(-i\partial_x\tau^z - i\partial_y\tau^x)\psi. \quad (31)$$

Here $\psi=\psi_{a\alpha}$ and each $\psi_{a\alpha}$ is a two-component Dirac spinor. The Pauli matrices $\vec{\tau}$ act on the Dirac index. The index $a=1,2$ labels the node and α labels the physical spin. The corresponding action (after letting $y\to -y$) is

$$S = \int d^2x\,d\tau\,\bar{\psi}(-i\tau_y\partial_\tau - i\partial_x\tau^x - i\partial_y\tau^z)\psi \quad (32)$$

$$\equiv \int d^3x\,\bar{\psi}(-i\tau_i\partial_i)\psi, \quad (33)$$

where we have defined $\bar{\psi}=i\psi^\dagger\tau_y$.

At small U, the interactions renormalize to zero and a semimetallic state is stable. As U is increased, there will be a metal-insulator transition where the fermions acquire a charge gap. In the resulting insulator, it is expected that in some parameter ranges a simple Neel state will emerge. In other parameter ranges, a valence bond solid state may well be stabilized. Either insulator may conveniently be described starting from the Hubbard model in a Hartree-Fock description that builds in a nonzero value for the appropriate order parameter. To describe the competition between these two kinds of ordered insulators, we will introduce fields that couple to either order parameter and integrate out the fermions to obtain effective sigma model descriptions. It is easy to see that the Neel vector corresponds to the fermion bilinear $i\bar{\psi}\vec{\sigma}\mu_y\psi$, where $\vec{\mu}$ are Pauli matrices in the node index. The VBS order parameter (v_x,v_y) corresponds to $v_x=i\bar{\psi}\mu_x\psi$, $v_y=i\bar{\psi}\mu_z\psi$. To access either insulating phase, we therefore include the terms

$$u_N(\bar{\psi}\vec{\sigma}\mu_y\psi)^2 + u_v[(\bar{\psi}\mu_x\psi)^2 + (\bar{\psi}\mu_z\psi)^2]. \quad (34)$$

To begin with, we will assume that $u_N=u_v=u$ for simplicity. Later we will add anisotropy between the Neel and VBS fluctuations. This term may be decoupled with a five-component Hubbard-Stratonovich field $\vec{\phi}=(\vec{N},\phi_{vy},\phi_{vx})$. We therefore get

$$S = \int d^3x\,\bar{\psi}(-i\tau_i\partial_i + i\vec{\phi}\cdot\vec{\Gamma})\psi + \frac{|\phi|^2}{2u}. \quad (35)$$

Here $\vec{\Gamma}$ are a set of five 4×4 matrices defined through

$$\Gamma_{1,2,3} = \sigma_{x,y,z}\mu_y, \quad (36)$$

$$\Gamma_4 = \mu_z, \tag{37}$$

$$\Gamma_5 = \mu_x. \tag{38}$$

Note that $[\Gamma_\alpha, \Gamma_\beta]_+ = 2\delta_{\alpha\beta}$ and $\Gamma_5 = -\Gamma_1\Gamma_2\Gamma_3\Gamma_4$. Equivalently, the phase diagram and universal aspects of the various phases will be preserved by restricting the ϕ vector to have unit magnitude. We thus consider the action

$$S = \int d^3x \, \bar{\psi}(-i\tau_i\partial_i + im\hat{\phi} \cdot \vec{\Gamma})\psi. \tag{39}$$

The Hartree-Fock aproximation to the insulating phases consists of giving $\hat{\phi}$ some nonzero mean value that gaps out the fermions. To access these insulators while including fluctuation effects, we integrate out the ψ assuming that m is large. The resulting fermion determinant has been evaluated by Abanov and Wiegmann[16] and gives

$$S[\hat{\phi}] = \int d^3x \frac{1}{2g}(\partial_i\hat{\phi})^2 - 2\pi i\Gamma[\hat{\phi}]. \tag{40}$$

This takes the form of a nonlinear sigma model for the superspin vector $\hat{\phi}$ that combines the Neel and VBS order parameters. The crucial feature is the presence of the second term—this is a Wess-Zumino-Witten term for the five-component unit vector field $\hat{\phi}$ and is defined as follows. The field $\hat{\phi}$ defines a map from S^3 to S^4. The fraction of the total volume of S^4 that is bounded by the hypersurface traced out by $\hat{\phi}$ defines Γ. Specifically, let $\hat{\phi}(x,u)$ be any smooth extension of $\hat{\phi}(x)$ such that $\hat{\phi}(x,0)=(1,0,0,0,0)$ and $\hat{\phi}(x,1) = \hat{\phi}(x)$. Then

$$\Gamma = \frac{3}{8\pi^2} \int_0^1 du \int d^3x \, \epsilon_{\alpha\beta\gamma\delta\kappa} \phi_\alpha \partial_x \phi_\beta \partial_y \phi_\gamma \partial_\tau \phi_\delta \partial_u \phi_\kappa. \tag{41}$$

Thus the Neel-VBS competition for spin-1/2 magnets on an isotropic two-dimensional square lattice is described by this SO(5) superspin nonlinear sigma model with the extra topological WZW term. This sigma model must be supplemented with an anisotropy that reduces the symmetry to SO(3) $\times U(1)$, where the SO(3) corresponds to spin rotation and the $U(1)$ rotates between the two components of the VBS order parameter. Further microscopically this $U(1)$ symmetry must have further anisotropy that reduces the symmetry to Z_4.

In their original derivation, Tanaka and Hu viewed the π flux state as a fermionic spinon mean-field theory for Heisenberg spin magnets. Their subsequent calculations are identical to what we described above. However, from that point of view it becomes important to include gauge fluctuations before establishing a firm connection to the original quantum magnet.[20] Here we have sidestepped this issue by considering the Hubbard model on the square lattice with π flux as our starting point. There is no question of introducing extra fluctuating gauge fields from this point of view. Note that once the system enters the insulating phase, the spin physics at energy scales below the charge gap may be described (in principle) in terms of a short-ranged spin model of spin-1/2 moments. Thus the π-flux Hubbard model as a microscopic starting point is very convenient to derive an effective theory to describe the Neel-VBS competition in two-dimensional quantum magnets.

B. Equivalence to NCCP^1 field theory

We now show the equivalence of the superspin sigma model with the WZW term to the NCCP^1 model in Ref. 1 for the deconfined critical point between the Neel and VBS states. We will do this in the specific case in which there is large easy plane anisotropy on the Neel vector so that it points primarily in the xy plane in spin space. To handle this case, we write

$$\hat{\phi} \approx (0,\hat{\Pi}) \tag{42}$$

with $\hat{\Pi} = (\vec{N}_\perp, \phi_{vy}, \phi_{vx})$ a four-component unit vector. Here $\vec{N}_\perp$ is a two-component vector that denotes the direction of the Neel vector in spin space. With this restriction, the Wess-Zumino term in the action can be evaluated straightforwardly and simply becomes

$$S_{WZW} = -i\pi Q, \tag{43}$$

where Q is the by now familiar index describing the winding of the unit four vector field $\hat{\Pi}$ in three space-time dimensions. Thus we appear to be back to the O(4) sigma model at $\theta = \pi$ but now with O(2)$\times$O(2) anisotropy for the $\hat{\Pi}$ field.[21]

The topological term again plays a crucial role. To expose its role, let us warm up by considering some simple configurations of the $\hat{\Pi}$ field. It will be useful to view it as a two-component complex vector $z=(z_1, z_2)$ with $z_{1,2}$ complex numbers satisfying $|z_1|^2 + |z_2|^2 = 1$. The topological index Q is readily writen in terms of derivatives of z. An equivalent form is obtained by considering the "vector potential"

$$a_i = -iz^\dagger \partial_i z. \tag{44}$$

Some algebra shows that

$$Q = \frac{1}{2\pi} \int d^3x \, \epsilon_{ijk} a_i \partial_j a_k. \tag{45}$$

As a topological index, Q is invariant under smooth continuous deformations of $z(x)$. As a particular example, a smooth gauge transformation $z \rightarrow e^{i\theta(x)}z$ changes a_i to $a_i + \partial_i\theta$. The integral for Q above is clearly invariant under such a gauge transformation.

Consider first configurations where $z_1 = |z_1|e^{i\theta_1(x)}$ with θ_1 a smooth single-valued function of x, and z_2 arbitrary. This means that that there are no vortices anywhere in the complex field z_1. Then the gauge transformation $z \rightarrow e^{-i\theta_1(x)}z$ makes z_1 real everywhere. It is clear then that for such configurations $Q=0$ (as with z_1 real one component of $\hat{\Pi}$ vanishes everywhere). The same result obviously also holds for configurations where there are no vortices anywhere in z_2 but z_1 is arbitrary. Thus Q is nonzero only if there are vortices in both z_1 and z_2.

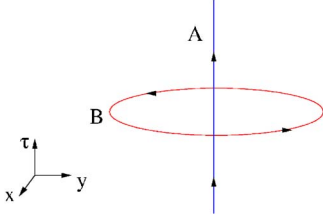To construct a configuration with nonzero Q, consider the map

FIG. 1. (Color online) Configuration with $Q=1$ showing the location of the vortices in z_1 and z_2. The curve A is the vortex in z_2 and B is the vortex in z_1.

$$\hat{\Pi}(r,\theta,\phi) = (\cos[\alpha(r)], \sin[\alpha(r)e_r]) \qquad (46)$$

with (r,θ,ρ) the spherical coordinates for space-time and $\alpha(r)$ some smooth function of r satisfying $\alpha(0)=0$ and $\alpha(\infty)=\pi$. Here $\hat{e}_r$ is a radial unit vector defined by $(\cos\theta,\sin\theta\cos\phi,\sin\theta\sin\phi)$ (the first component is along the τ direction). This configuration clearly has $Q=1$ and is a simple generalization of the familiar O(3) skyrmion in two dimensions. Let us identify the location of the vortex lines of $z_{1,2}$ in this configuration. When $\theta=0$, $z=(e^{i\alpha},0)$. Similarly, when $\theta=\pi$, $z=(e^{-i\alpha},0)$. Thus along the line A (see Fig. 1) which runs along the τ axis at the spatial origin, $z=(e^{i\theta_1},0)$ with θ_1 varying from $-\pi$ to π. Similarly at r such that $\alpha(r)=\frac{\pi}{2}$ and $\theta=\frac{\pi}{2}$ (i.e., along the curve B), we have $z=(0,e^{i\phi})$. Thus we may identify A with a vortex line in z_2 and B with a vortex line in z_1. In this configuration, these two oriented vortex lines have a nontrivial linking number 1.

These considerations illustrate a general mathematical result that we will henceforth use: The topological index Q may be identified with the linking number between oriented vortex loops in z_1 and those in z_2. For the $\theta=\pi$ model, this implies that whenever a z_1 vortex moves fully around a z_2 vortex (or vice versa), there is a phase factor of π. In other words, the z_1 and z_2 vortices are mutually nonlocal and see each other as sources of π-flux.

It is now straightforward to see the equivalence to the easy-plane NCCP^1 action. Indeed, the vortices in z_1 are just the 2π vortices in the easy plane Neel order parameter. The π phase shift that these vortices acquire when taken around the vortices in z_2 suggests that the latter may be thought of as spin-1/2 spinons. This is indeed consistent with the expected structure of the vortices in the VBS order parameter.[17] To explicitly demonstrate the equivalence to the NCCP^1 action, let us go to a dual representation in terms of the vortices in z_1 and z_2. A lattice action that incorporates the nontrivial mutual statistics of these two vortices is readily written down,

$$S = S_{t1} + S_{t2} + S_a + S_{CS}, \qquad (47)$$

$$S_{t1} = -t \sum_{\langle ij \rangle} \sigma_{ij} \cos(\vec{\nabla}\phi_1 - \vec{a}_1), \qquad (48)$$

$$S_{t2} = -t \sum_{\langle IJ \rangle} \mu_{IJ} \cos(\vec{\nabla}\phi_2 - \vec{a}_2), \qquad (49)$$

$$S_a = K \sum_P (\vec{\nabla}\times\vec{a}_1)^2 + (\vec{\nabla}\times\vec{a}_2)^2, \qquad (50)$$

$$S_{CS} = i \sum_{\langle IJ \rangle} \frac{\pi}{4}(1-\mu_{IJ})\left(1 - \prod_P \sigma\right). \qquad (51)$$

Here $i,j,\dots$ are the sites of a cubic space-time lattice and $I,J,\dots$ the sites of the corresponding dual lattice. The fields $\phi_{1,2}$ are the phases of the vortices in z_1 and z_2, respectively. These vortices are coupled minimally to their respective noncompact $U(1)$ gauge fields $a_{1,2}$. The σ_{ij},μ_{IJ} are Z_2 gauge variables that take values ±1. The last term is an Ising mutual Chern-Simons term[18] that incorporates the mutual π statistics of the two vortices. This will be made explicit below. The symbol $\Pi_P\sigma$ represents a product of σ_{ij} over the four links of the space-time plaquette pierced by the dual link $\langle IJ \rangle$.

To check that the S_{CS} term correctly implements the mutual statistics of the two vortices, it is convenient to go to a Villain representation. For instance, for the 2-vortex the hopping term becomes

$$S_{t2} \to \sum_{\langle IJ \rangle} u(\vec{j}_2)^2 + i\vec{j}_2 \cdot \left(\vec{\nabla}\phi_2 - \vec{a}_2 + \frac{\pi}{2}(1-\mu_{IJ})\right). \quad (52)$$

A sum over the integer vortex currents $\vec{j}_2$ is implied. The integration over ϕ_2 as usual implies the current conservation condition

$$\vec{\nabla}\cdot\vec{j}_2 = \mathbf{0}. \qquad (53)$$

The sum over the Ising variable μ_{IJ} may also be performed and leads to the constraint

$$(-1)^{j_2} = \prod_P \sigma. \qquad (54)$$

Thus the presence of a 2-vortex is seen as π flux by the 1-vortex. Thus the term S_{CS} precisely implements the mutual π statistics of the two vortices.

In the Neel phase, the vortex $e^{i\phi_1}$ is gapped while the 2-vortex is condensed (with the reverse being true in the VBS phase). At a putative second-order transition between the two, we might expect that both vortices stay critical. A standard duality transformation on one of the two vortex degrees of freedom enables us to demonstrate that this action is indeed in the easy plane NCCP^1 universality class near this transition. This is elaborated in Appendix A.

Note that the easy plane deconfined critical fixed point also has O(2)×O(2) symmetry[1] consistent with the superspin field theory discussed in this section. One of these O(2) symmetries is realized simply as a uniform phase rotation of both the matter fields $e^{i\theta_\pm}$ of Appendix A. The other O(2) is realized nontrivially as a topological symmetry associated with the conservation of the photon.

IV. SIGMA MODEL DESCRIPTION OF MASSLESS QED$_3$

In this section, we will consider massless QED$_3$ in $D=(2+1)$ dimensions. This consists of N two-component

Dirac fermions ψ coupled to a fluctuating noncompact $U(1)$ gauge field a with the action

$$S = \int d^3x \bar{\psi}[\tau_i(-i\partial_i - a_i)]\psi + \frac{1}{2e^2}(\epsilon_{ijk}\partial_j a_k)^2. \quad (55)$$

The τ_i are Pauli matrices. This action has a global SU(N) symmetry asociated with unitary rotations of ψ and a hidden global $U(1)$ symmery associated with gauge flux conservation. For $N=4$, this is the low-energy description of the dRVB algebraic spin liquid of SU(2) invariant spin-1/2 models that has been much discussed recently. Neither the Dirac fermions nor the gauge photon are well-defined quasiparticle excitations of the system when its low energy properties are controlled by a scale invariant fixed point (as happens generically for large enough N and at special multicritical points for small N). For any $N\geq 2$, this scale invariant theory has slow power-law correlations for a set of gauge invariant fermion bilinears that transform as adjoints under the SU(N). Can we dispense with the gauge description completely in favor of some sort of sigma model decription in terms of these slow fluctuations? As discussed in previous sections, this is certainly possible in various other analogous problems but required inclusion of a topological term in the sigma model action. Here we study the specific case $N=2$ in $D=2+1$ and show that a sigma model description is again possible provided topological terms are included.

For $N=2$, the fermions transform with spin-1/2 under the global SU(2) symmetry. It is expected that the gauge invariant SU(2) vector $\bar{\psi}\sigma\psi$ will have slow correlations, and perhaps might even generically order. To expose its effects, let us add the following term to the action:

$$S_4 = \int d^3x u(\bar{\psi}\sigma\psi)^2 \quad (56)$$

with $u>0$. This term may be decoupled with a fluctuating $\vec{N}$ field to get a term of the form

$$\frac{(\vec{N})^2}{2u} + i\vec{N} \cdot \bar{\psi}\sigma\psi. \quad (57)$$

Equivalently, the phase diagram and universal aspects of the various phases will be preserved by restricting the $\vec{N}$ vector to have unit magnitude. We thus consider the action

$$S = S_f + S_a, \quad (58)$$

$$S_f = \int d^3x \bar{\psi}[\tau_i(-i\partial_i - a_i) + im\hat{n} \cdot \vec{\sigma}]\psi, \quad (59)$$

$$S_a = \frac{1}{2e^2}(\epsilon_{ijk}\partial_j a_k)^2 \quad (60)$$

with $m>0$. The fields ψ, a, $\hat{n}$ are all to be integrated over. We begin by first doing the quadratic integral over the fermion fields. We assume space-time to be S^3. The resulting fermion determinant has been calculated (in a $1/m$ expansion) by Abanov and Wiegmann[16] and leads to the following remarkable result:

$$S = S_0 + S_j + S_H + S_a, \quad (61)$$

$$S_0 = \int d^3x \frac{1}{g}(\partial_i\hat{n})^2, \quad (62)$$

$$S_j = \int d^3x i J_i a_i, \quad (63)$$

$$S_H = -i\pi H[\hat{n}]. \quad (64)$$

Here $g \sim 1/m$ in S_0. The J_i is the topological current density of the $\hat{n}$ field, i.e., it is the skyrmion current density. Formally,

$$J_i = \frac{1}{8\pi}\epsilon_{ijk}\hat{n} \cdot \partial_j\hat{n} \times \partial_k\hat{n}. \quad (65)$$

The last term of the action involves an interesting topological invariant H corresponding to $\Pi_3(S^2)=Z$ and is known as the Hopf term. H is an integer that distinguishes different space-time configurations of the $\hat{n}$ field. H is conveniently written in terms of a CP^1 spinor z associated with the field $\hat{n}$ (defined through $\hat{n}=z^\dagger\vec{\sigma}z$). From z form, the SU(2) matrix $U=[z -i\sigma^y z^*]$. Then

$$H = \frac{1}{24\pi^2} \int d^3x\epsilon_{ijk}\text{tr}(U^{-1}\partial_i U U^{-1}\partial_j U U^{-1}\partial_k U). \quad (66)$$

In the absence of the coupling to the gauge field, the Hopf term changes the spin and statistics of the skyrmion. Note that the gauge field a_i couples to the skyrmion current density of the $\hat{n}$ vector. Therefore, at long scales the skyrmion density is pinned at zero. Thus the resulting model is the three-dimensional O(3) vector model where skyrmion configurations have been suppressed. This leads to enlargement of the degrees of freedom to a four-component field and the symmetry to O(4). To see this most simply, rewrite the O(3) sigma model above in the CP^1 representation. We have

$$S_0 = \int d^3x \frac{1}{g}|(\partial_i - iA_i)z|^2, \quad (67)$$

$$S_j = \int d^3x \frac{i}{2\pi}\epsilon_{ijk}a_i\partial_j A_k. \quad (68)$$

We may now integrate over the gauge field a. This is conveniently done by choosing a gauge where A_i is transverse (i.e., $\vec{\nabla}\cdot\vec{A}=0$). We get

$$S = S_z + S_H, \quad (69)$$

$$S_z = \int d^3x \frac{1}{g}|(\partial_i - iA_i)z|^2 + \frac{e^2}{8\pi^2}A_i^2. \quad (70)$$

Thus the CP^1 gauge field A_i has been rendered massive due to the coupling to the fluctuating gauge field a. At long distances and low energies, we may thus drop the gauge field altogether. (Strictly speaking, we should just integrate out the gauge field to generate a term that is quartic in the z fields and involves two derivatives—this term is expected to be

irrelevant at the critical point of the resulting theory. A similar result was established in Ref. 19 in the absence of the topological term.) The action S_z then becomes

$$S_z = \int d^3x \frac{1}{g} |\partial_i z|^2, \qquad (71)$$

which has O(4) symmetry as can be made manifest by rewriting in terms of the real and imaginary parts of the two components of z. What happens to the Hopf term? Clearly S_H is invariant both under independent left and right multiplications by constant SU(2) matrices. Thus it too is O(4) invariant. A simple calculation shows that the Hopf invariant H is exactly equal to Q, where Q is the topological index characterizing configurations of an O(4) unit vector field in three dimensions. Thus the action reduces to the O(4) model at $\theta = \pi$.

Note that in the O(4) symmetry broken phase there are three gapless linear dispersing modes. In the gauge theory description, this corresponds to broken chiral symmetry with $\langle \hat{n} \rangle \neq 0$. Here again there are three gapless linear dispersing modes—two are spin waves in $\hat{n}$ while the third is simply the gapless photon.

A somewhat similar possible duality between massless $N=2$ QED$_3$ and the usual critical O(4) model was conjectured recently in Ref. 12 using very different arguments. Our derivation shows that such a duality indeed exists but necessarily includes the topological term. Does the topological term make any difference to the properties of the model? We turn to this question in the following subsection.

Isotropic O(4) model with a θ term

What may we say about the properties of the *isotropic* [i.e., O(4) symmetric] model at $\theta = \pi$ from the analysis in this paper? First we expect that a phase with broken O(4) symmetry is stable for weak coupling. The θ term presumably plays a very little role at low energies in this phase as it only affects topological configurations that cost large energy. At strong coupling, paramagnetic phases where O(4) symmetry is preserved are presumably possible. The arguments in previous sections show that a trivial featureless paramagnet cannot exist. To see this, first assume that such a paramagnetic phase can indeed exist. Then turn on a small anisotropy that breaks the O(4) symmetry to O(3)×Z_2. Such an anisotropy will have no effect on the ground state of such a trivial paramagnet—but the arguments of Sec. II show that such trivial paramagnetic ground states do not exist in the anisotropic model. Thus such states are forbidden for the isotropic model as well. What are the possibilities then? Gapped phases can exist if the ground state has topological order. Gapless paramagnetic states are also not precluded by these arguments.

Similar and more interesting conclusions may also be drawn by considering weak O(2)×O(2) anisotropy [instead of O(3)×Z_2]. First note that with this anisotropy the model has an extra Z_2 symmetry that interchanges the two O(2) symmetries. It is clear that this is exactly equivalent to the Motrunich-Vishwanath self-duality symmetry[15] of the equivalent easy plane NCCP^1 model. We can now discuss the

phase diagram of the $\theta = \pi$ O(4) model with weak O(2)×O(2) anisotropy in terms of the known phases and phase transitions of models with the same field content as the easy plane NCCP^1 model. First consider the O(4) ordered phase. With the O(2)×O(2) anisotropy, this becomes the location of a first-order "spin flop" transition between phases that separately break either of the two distinct O(2) symmetries. What about phases that preserve O(2)×O(2) symmetry? Here there are two possibilities. First the self-dual second-order deconfined critical line of the easy plane NCCP^1 appears as a paramagnetic *phase* of the sigma model. Thus this is a concrete example of the possibility of a gapless strong-coupling paramagnetic phase at $\theta = \pi$ in the O(4) model [albeit with some O(2)×O(2) anisotropy]. It is conceivable that the self-dual critical manifold of the easy plane NCCP^1 admits a special multicritical point with higher O(4) symmetry. The existence of such a fixed point would imply the possibility of a gapless paramagnetic phase at $\theta = \pi$ with full O(4) symmetry. The other possibility that preserves full O(2)×O(2) symmetry is a gapped Z_2 topologically ordered paramagnet that coresponds to the Z_2 spin liquid allowed in the lattice easy plane spin-1/2 antiferromagnet. This phase should persist with full O(4) symmetry as well.

In view of the above, it seems clear that the transitions out of the ordered phases with O(2)×O(2) anisotropy would be in a different universality class from those in a model without the θ term. With full O(4) symmetry, it thus seems rather likely that at $\theta = \pi$ the transition is in a different universality class from that at $\theta = 0$.

V. DISCUSSION AND PROSPECTS

In this paper, we have provided a number of examples illustrating how the physics of two-dimensional spin-1/2 quantum magnets can in principle be described in "super-spin" sigma models that describe the slow competing orders. As such, these are as close as one might get to a Landau-Ginzburg description of these competing orders. An important feature of these sigma models is the presence of topological terms that reflect the underlying quantum nature of the spins. From the conventional point of view, it is these topological terms that complicate a direct application of naive Landau-Ginzburg-Wilson (LGW) thinking to quantum magnetism. Thus they are at the root of the failure of the LGW paradigm in describing various quantum phases and phase transitions.

The sigma model formulations provide a potential alternate to the gauge theoretic descriptions that have been used thus far to describe the non-LGW physics. However, as things stand we know even less about how to handle the effects of topological terms than we know about gauge theories. So it is not clear at present how useful the sigma model will be.

Finally, our results raise the possibility of such a sigma model description for stable algebraic spin liquids in two dimensions (of which the dRVB state popular in cuprate physics may be a possible example). These also have slow power-law correlations in a number of physical observables. The results on $N=2$ massless QED$_3$ provide some positive

hints. Stronger evidence is the sigma model description of the deconfined Neel-VBS critical point. These may be thought of as a special kind of algebraic spin liquid that has one relevant perturbation. In the gauge theory description, they have gapless bosonic matter fields coupled to a noncompact $U(1)$ gauge field. As these theories seem to have sigma model descriptions, perhaps their fermionic cousins do as well. Perhaps such descriptions might even be useful.

ACKNOWLEDGMENTS

APPENDIX: DUALITY OF THE $\theta = \pi$ O(2)×O(2) MODEL TO NCCP[1]

Proceeding as usual, we solve the current conservation condition Eq. (53) by writing

$$\vec{j}_2 = \vec{\nabla} \times \vec{A} \tag{A1}$$

with $\vec{A}$ an integer living on the links $\langle ij \rangle$. We have for the action

$$S = S_{t1} + S_A + S_{aA} + S_a, \tag{A2}$$

$$S_A = u \sum_P (\vec{\nabla} \times \vec{A})^2, \tag{A3}$$

$$S_{aA} = \sum_{IJ} \vec{a}_2 \cdot \vec{\nabla} \times \vec{A} \tag{A4}$$

with S_{t1} and S_a as before. This must be supplemented with the mutual statistics condition

$$\prod_P \sigma_{ij}(-1)^{A_{ij}} = 1. \tag{A5}$$

To handle this, we write

$$A_{ij} = 2A'_{ij} + s_{ij} \tag{A6}$$

with $s = 0, 1$ and A' an integer. Then we have

$$\prod_P \sigma_{ij}(-1)^{s_{ij}} = 1. \tag{A7}$$

This is solved by

$$\sigma_{ij}(-1)^{s_{ij}} = \alpha_i \alpha_j \tag{A8}$$

with $\alpha_i = \pm 1$. The integer condition on A' can be implemented (softly) with a term

$$-t_c \cos(2\pi A'_{ij}) = -t_c \cos[\pi(A_{ij} - s_{ij})] \tag{A9}$$

$$= -t_c \sigma_{ij} \alpha_i \alpha_j \cos(\pi A_{ij}). \tag{A10}$$

We now separate out the longitudinal part of $\vec{A}$,

$$\vec{A} = \vec{A}^T + \frac{1}{\pi}\vec{\nabla}\theta_c. \tag{A11}$$

Collecting all the pieces of the action together and integrating out the gauge field $\vec{a}_2$, we see that $\vec{A}^T$ is massive. We will therefore simply drop it from now on. We are then left with

$$S = S_{t1} + S_{tc} + S_{a1}, \tag{A12}$$

$$S_{tc} = -t_c \sum_{ij} \sigma_{ij} \cos(\vec{\nabla}\theta_c), \tag{A13}$$

$$S_{a1} = K \sum_P (\vec{\nabla} \times \vec{a}_1)^2. \tag{A14}$$

In writing the term S_{tc}, we have absorbed the α_i into the θ_{ci}. The sum over the Z_2 gauge field σ_{ij} may now be straightforwardly performed. It generates a number of terms of which the most important have the structure

$$-\kappa[\cos(\vec{\nabla}\theta_+ - a_1) + \cos(\vec{\nabla}\theta_- + a_1)], \tag{A15}$$

where $\theta_\pm = \theta_c \pm \phi_1$. Together with the term S_{a1}, this is precisely the action for the easy plane NCCP[1] theory.

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[20] For the π flux state, this involves coupling the fermionic spinons to a gapless SU(2) gauge field

[21] We will ignore the Z_4 anisotropy on the VBS order parameter for the present discussion. It can be reinstated without any major changes if necessary and in any case has been argued to be irrelevant at the deconfined critical fixed point.