Description and evolution of anisotropy in superfluid vortex tangles with counterflow and rotation

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We examine several vectorial and tensorial descriptions of the geometry of turbulent vortex tangles. We study the anisotropy in rotating counterflow experiments, in which the geometry of the tangle is especially interesting because of the opposite effects of rotation, which orients the vortices, and counterflow, which randomizes them. We propose to describe the anisotropy and the polarization of the vortex tangle through a tensor, which contains the first and second moments of the distribution of the unit vector $\mathbf{s}'$ locally tangent to the vortex lines. We use an analogy with paramagnetism to estimate the anisotropy, the average polarization, the polarization fluctuations, and the geometrical contribution to the entropy of the tangle in terms of angular velocity and counterflow. We explore the influence of the geometry on the evolution of the vortex line density and propose evolution equations for the geometry of the tangle.

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I. INTRODUCTION

Superfluid turbulence is a fascinating topic, in which quantum effects yield a peculiar behavior at some length scales, whereas at other length scales it exhibits an interesting confluence with classical turbulence.[1–3] Superfluid turbulence is described as a disordered tangle of quantized vortex lines. Here, we will devote our attention to situations combining counterflow and rotation, a currently active topic of investigations.[4–7]

Vortices in ^{4}He (Refs. 1–4) have been much investigated in two physical situations: rotating containers and counterflow experiments (an experimental situation characterized by no matter flow but only heat transport, exceeding a critical heat flux q_c). It is well known that, if helium II is rotated at a constant angular velocity Ω, an ordered array of lines aligned along the rotation axis is created. The vortex array is described by introducing the *line density* L, the average vortex line length per unit volume. In this case, L equals the areal density L_R, which is proportional to the angular velocity Ω of the sample:

$$L = L_R \simeq \frac{2\Omega}{\kappa}, \tag{1.1}$$

where κ is the quantum of vorticity. It is well known too that a disordered tangle of quantized vortex lines is created in the so-called counterflow superfluid turbulence, one of the most typical effects in liquid helium II. When the turbulence is developed, the line density L is proportional to the square of $V=|\langle\mathbf{V}\rangle|$ ($\mathbf{V}=\mathbf{v}_n-\mathbf{v}_s$ being the counterflow velocity and $\mathbf{v}_n$ and $\mathbf{v}_s$ the velocities of the normal and superfluid components; the angular brackets denote the spatial average):

$$L = L_H \simeq \gamma_H^2 \frac{V^2}{\kappa^2}, \tag{1.2}$$

the dimensionless coefficient γ_H being dependent on the helium temperature T.

In usual counterflow experiments it is often assumed that the vortex tangle is practically isotropic, and that the only relevant variable is the vortex line density L. The isotropy hypothesis is no longer tenable in situations involving counterflow and rotation, because the latter tends to orient the vortex lines along the rotation axis. Thus, in the analysis of this situation it is necessary to pay much attention to the geometrical description of the tangle, well beyond the vortex line density. On the other side, also in pure counterflow, the vortex tangle is not completely isotropic, as is shown in some experiments[8] and simulations.[9–12]

Our aim here will be to understand a little better the geometry of the vortex tangle, with special emphasis on its anisotropy, which is given here a tensorial description. In fact, a detailed geometric and statistical description of the vortices has become one of the aims of recent research in this field,[13–15] as the description based only on L—and even a more general one adding to L the tangle polarization—is becoming too restricted to describe the increasing amount of information available on vortex tangles either from experiments or from computer simulations. Furthermore, we will try to relate the geometry to the thermodynamics and dynamics. On the one side, we examine the influence of the anisotropy on the dynamics of the vortex tangle, and the dynamics of the anisotropy itself when the conditions acting on the fluid (the values of the counterflow velocity V or of the angular velocity Ω, for instance) are suddenly changed. On the other side, we consider the geometrical contribution to the entropy of the tangle, which may be of interest to find the chemical potential of the vortex lines, whose gradient would be the thermodynamic force leading to vortex diffusion in inhomogeneous tangles.

As mentioned above, the geometry of the vortex tangle is strongly influenced by the opposite effects of the external rotation, which plays an ordering role on the vortices and tends to align them along it, and of the counterflow, which tends to randomize the directions of vortex segments. The different orientations of vortex segments constituting the

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tangle will have in turn an effect on the dynamics of the tangle itself and of the vortex line density. The understanding of this topic is the subject of both experiments and numerical simulations.

The plan of the paper is the following. In Sec. II we review several proposals for the description of the geometry of vortex tangle and introduce a tensorial description of the tangle, which combines geometrical, dynamical, and thermodynamical meanings. In Sec. III, it is shown how the experimental data on sound propagation allow us to determine the mean parameters characterizing the anisotropy of the tangle. In Sec. IV, we develop with further details an analogy with paramagnetism, previously proposed by some authors,[7] to evaluate from a semimicroscopic basis the parameters characterizing not only the polarization vector, as has been done previously, but also the other anisotropy parameters which appear in our description, and the geometrical contribution to the entropy of the tangle. Section V, finally, is devoted to the study of the dynamics of the geometry of the tangle and, besides, of the influence of the anisotropy of the tangle on the dynamics of the vortex line density L.

II. GEOMETRY OF THE VORTEX TANGLE

The microscopic understanding of the most essential features governing the evolution of quantized vortex filaments in superfluid helium[9–12] has led to an increase of interest in the geometrical aspects of turbulent tangles, which are combined with statistical ingredients and with new topological and geometrical measures of complexity.[13–15] In brief, the vortex lines are described by a function $\mathbf{s}(\xi,t)$, ξ being the arc length measured along the curve of the vortex filament oriented in the direction of the vorticity vector. The derivatives of $\mathbf{s}$ with respect to ξ and t play an essential role in the description of the geometry and dynamics. The most relevant of them are $\mathbf{s}'$, which is the unit vector tangent along the vortex line at a given point, and $\mathbf{s}''$, which is the curvature vector, giving the normal direction to the line (here and in the following a prime denotes differentiation with respect to ξ). The vectorial product $\mathbf{s}'\times\mathbf{s}''$, also called the binormal, is another important vector. All these three vectors and their relative orientations with respect to the counterflow velocity $\mathbf{V}$ play an important role in the description, as they are related to the velocity that a vortex line acquires with respect to the superfluid (besides having, of course, a geometrical meaning too). Indeed, the vortex motion is governed by the equation[1]

$$\mathbf{v}_L = \mathbf{v}_{sl} + \alpha \mathbf{s}' \times (\mathbf{v}_n - \mathbf{v}_{sl}) - \alpha' \mathbf{s}' \times [\mathbf{s}' \times (\mathbf{v}_n - \mathbf{v}_{sl})]. \tag{2.1}$$

Here α and α' are temperature-dependent friction coefficients between the normal fluid and the vortex line, and $\mathbf{v}_{sl} = \mathbf{v}_s + \mathbf{v}_i$ is the local superfluid velocity, the sum of the superfluid velocity $\mathbf{v}_s$ at large distance from any vortex line and of the self-induced velocity $\mathbf{v}_i$, the flow due to any other vortex, including other parts of the same vortex, induced by the curvature of all these lines.

The coefficients α and α' are linked to the Hall-Vinen coefficients B and B' by the relations $\alpha = \frac{\rho_n}{2\rho}B$ and $\alpha' = \frac{\rho_n}{2\rho}B'$.[1] In superfluid ^{4}He, the coefficient B is always positive, while B' is positive only for temperatures T lower than $T_0 = 2.06$ K, where B' is equal to zero. The absolute value of the ratio $\alpha'/\alpha = B'/B$ is always lower than 1, but it is very small only for T near to T_0. In particular, from the data reported in Ref. 1 one deduces that for $T = 1.3$ K, $\alpha'/\alpha = 0.40$, while for $T = 2.170$ K, $\alpha'/\alpha = -0.55$. The coefficient α' is sometimes neglected, but this is correct only very near to $T_0 = 2.06$ K.

In the local induction approximation, the self-induced velocity $\mathbf{v}_i$ is approximated by[1]

$$\mathbf{v}_i^{(loc)} = \tilde{\beta}[\mathbf{s}' \times \mathbf{s}'']_{s=s_0} \quad \text{with } \tilde{\beta} = \frac{\kappa}{4\pi} \ln\left(\frac{\tilde{c}}{a_0 L^{1/2}}\right), \tag{2.2}$$

with $\tilde{c}$ a constant of the order of unity and a_0 the dimension of the vortex core, of the order of 1 Å. The intensity of $\mathbf{v}_i$ is $|\mathbf{v}_i| = \tilde{\beta}/R$, with R the curvature radius of the vortex line, and $\tilde{\beta} \approx \kappa$ results. The self-induced velocity is zero if the vortices are straight lines. The coefficient $\tilde{\beta}$ is linked to the internal energy per unit length of the vortex line (the tension of the vortex line), by the relation $\epsilon_V = \rho_s \kappa \tilde{\beta}$.

In most studies of counterflow superfluid turbulence, the vortex tangle is supposed homogeneous and isotropic. This isotropy refers essentially to the angular distribution of tangent vectors $\mathbf{s}'$ with respect to the counterflow velocity over the whole ensemble of vortex lines. The experimental information is obtained by means of propagation of second sound across the tangle in several directions. Indeed, second sound is much more attenuated when it propagates normally to vortices than when it propagates along them. The experimental observations, however, confirm the isotropy hypothesis only in a first rough approximation.[1,8] Further, numerical simulations show that the vortices in the tangle tend to move outward in a plane orthogonal to $\mathbf{V}$, thus leading to anisotropy in the vortex line distribution with vortices concentrated in planes orthogonal to $\mathbf{V}$. Consequently, also in pure counterflow, it is important to study the anisotropy of the tangle.

A. General considerations

To describe the information on the distribution of $\mathbf{s}'$, one may write three tensors with $\mathbf{s}'$, namely,

$$\langle \mathbf{s}'\mathbf{s}' \rangle, \quad \langle \mathbf{U} - \mathbf{s}'\mathbf{s}' \rangle, \quad \langle \mathbf{W} \cdot \mathbf{s}' \rangle, \tag{2.3}$$

where $\mathbf{s}'\mathbf{s}'$ is the dyadic product and $\mathbf{U}$ is the unit matrix; using for clarity index notation, one has

$$[\mathbf{s}'\mathbf{s}']_{ij} = s_i' s_j', \quad [\mathbf{U} - \mathbf{s}'\mathbf{s}']_{ij} = \delta_{ij} - s_i' s_j', \tag{2.4}$$

with $i,j = 1,2,3$ and $\mathbf{s}' = (s_i')$. The first and second tensors in (2.3) are closely related to each other but it is useful to separate them, whereas $\mathbf{W}$ is the Ricci tensor, a completely antisymmetric third-order tensor which makes the third matrix of (2.3) an antisymmetric matrix; in index and matrix notation, one has

$$[\mathbf{W}\cdot\mathbf{s}']_{ij}=W_{ijk}s'_k,\quad \mathbf{W}\cdot\mathbf{s}'=\begin{pmatrix}0 & s'_3 & -s'_2\\ -s'_3 & 0 & s'_1\\ s'_2 & -s'_1 & 0\end{pmatrix}. \tag{2.5}$$

The angular brackets in (2.3) denote the average along the total length of the vortices in the sample volume Λ; for instance,

$$\langle\mathbf{s}'\mathbf{s}'\rangle=\frac{1}{\Lambda L}\int \mathbf{s}'\mathbf{s}'d\xi,\quad \langle\mathbf{W}\cdot\mathbf{s}'\rangle=\frac{1}{\Lambda L}\int \mathbf{W}\cdot\mathbf{s}'d\xi. \tag{2.6}$$

One could go to more detailed statistical specifications, as for instance to the probability regarding not only the spatial orientation of $\mathbf{s}'$, but also the number of closed vortex loops according to their respective lengths, which have led to interesting statistical proposals,[13] which go beyond the aims of the present paper.

Note that $\mathbf{s}'\mathbf{s}'$ and $\mathbf{U}-\mathbf{s}'\mathbf{s}'$ project any vector in the direction parallel to $\mathbf{s}'$ or orthogonal to $\mathbf{s}'$, respectively. Consider, for instance, the counterflow velocity $\mathbf{V}$: then one has

$$\mathbf{s}'\mathbf{s}'\cdot\mathbf{V}=\mathbf{s}'(\mathbf{s}'\cdot\mathbf{V}), \tag{2.7}$$

$$(\mathbf{U}-\mathbf{s}'\mathbf{s}')\cdot\mathbf{V}=\mathbf{V}-\mathbf{s}'(\mathbf{s}'\cdot\mathbf{V})=-\mathbf{s}'\times[\mathbf{s}'\times\mathbf{V}]; \tag{2.8}$$

finally

$$\mathbf{W}\cdot\mathbf{s}'\cdot\mathbf{V}=-\mathbf{s}'\times\mathbf{V}. \tag{2.9}$$

It is convenient to separate the contributions (2.7) and (2.8) because it is known that the flow of normal component only produces a force on locally straight vortex lines perpendicular to it. Indeed, the force per unit length of vortex on locally straight vortex lines is given by

$$\mathbf{f}=-\alpha\rho_s\kappa\mathbf{s}'\times[\mathbf{s}'\times\mathbf{V}]-\alpha'\rho_s\kappa\mathbf{s}'\times\mathbf{V}, \tag{2.10}$$

whereas, when the curvature of the lines cannot be neglected, this force depends also on the self-induced velocity $\mathbf{v}_i$ which has the direction of the binormal vector [see Eq. (2.2)], and it is[1–3]

$$\mathbf{f}=-\alpha\rho_s\kappa\mathbf{s}'\times[\mathbf{s}'\times(\mathbf{V}-\mathbf{v}_i)]-\alpha'\rho_s\kappa\mathbf{s}'\times(\mathbf{V}-\mathbf{v}_i). \tag{2.11}$$

Therefore, the force acting on vortex lines is related to $\langle(\mathbf{U}-\mathbf{s}'\mathbf{s}')\cdot(\mathbf{V}-\mathbf{v}_i)\rangle$ and $\langle\mathbf{W}\cdot\mathbf{s}'\cdot(\mathbf{V}-\mathbf{v}_i)\rangle$.

We will consider that $\mathbf{V}$ has a constant direction. This is usually assumed, but it is not completely clear, because it is natural to expect that, locally, the interaction with tangle may deviate $\mathbf{V}$ from the direction it has outside the tangle. Under the assumption, however, that $\mathbf{V}$ is indeed constant, we will have

$$\mathbf{V}\cdot\langle\mathbf{U}-\mathbf{s}'\mathbf{s}'\rangle\cdot\mathbf{V}=V^2-\langle(\mathbf{V}\cdot\mathbf{s}')^2\rangle=V^2[1-\langle\Psi^2\rangle], \tag{2.12}$$

where $\Psi=\cos\theta$ is the polarization of $\mathbf{s}'$ along the direction of $\mathbf{V}$. In a completely isotropic tangle $\langle\Psi^2\rangle=\frac{1}{3}$ and one finds $\mathbf{V}\cdot\langle\mathbf{U}-\mathbf{s}'\mathbf{s}'\rangle\cdot\mathbf{V}=(2/3)V^2$. Indeed, for a completely isotropic tangle, since $\mathbf{s}'$ is normalized to 1, the tensor $\langle\mathbf{U}-\mathbf{s}'\mathbf{s}'\rangle$ reduces to

$$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}-\begin{pmatrix}1/3&0&0\\0&1/3&0\\0&0&1/3\end{pmatrix}=\frac{2}{3}\mathbf{U}. \tag{2.13}$$

We have written explicitly all the components to achieve maximum clarity. Thus, to describe the anisotropy of the tangle, one could use the tensor $\langle\mathbf{U}-\mathbf{s}'\mathbf{s}'\rangle$ or, for the sake of simplicity in the equations to be written below, the tensor[16]

$$\mathbf{\Pi}^s\equiv\frac{3}{2}\langle\mathbf{U}-\mathbf{s}'\mathbf{s}'\rangle, \tag{2.14}$$

which reduces to $\mathbf{U}$ for an isotropic distribution of $\mathbf{s}'$.

An apparently different way to characterize the anisotropy of the tangle has been used by Schwarz,[1,12] who introduced the two scalar quantities $I_\parallel$ and $I_\perp$:

$$I_\parallel=\frac{1}{\Lambda L}\int[1-(\mathbf{s}'\cdot\hat{\mathbf{r}}_\parallel)^2]d\xi, \tag{2.15}$$

$$I_\perp=\frac{1}{\Lambda L}\int[1-(\mathbf{s}'\cdot\hat{\mathbf{r}}_\perp)^2]d\xi, \tag{2.16}$$

where $\hat{\mathbf{r}}_\parallel$ and $\hat{\mathbf{r}}_\perp$ are the unit vectors for directions parallel and perpendicular to the $\mathbf{V}$ direction. The functions $I_\parallel$ and $I_\perp$ are strictly related to the tensor $\langle\mathbf{U}-\mathbf{s}'\mathbf{s}'\rangle$. In fact, if one chooses a frame with the first axis in the direction of the counterflow $\mathbf{V}$ and assumes symmetry in the y,z plane, one obtains

$$\langle\mathbf{U}-\mathbf{s}'\mathbf{s}'\rangle=\begin{pmatrix}I_\parallel&0&0\\0&I_\perp&0\\0&0&I_\perp\end{pmatrix}. \tag{2.17}$$

Our description of the anisotropy, which uses the tensor $\mathbf{\Pi}^s$, is equivalent to the Schwarz description, which uses the two scalar quantities $I_\parallel$ and $I_\perp$, when the tangle is axially symmetric, as in counterflow in a cylindrical channel. But our tensor $\mathbf{\Pi}^s$ is also able to describe the geometry of the tangle in the absence of axial symmetry, as happens, for example, in channels with high aspect ratio (in these situations, in fact, the presence of the walls may produce anisotropy in the vortex tangle) or in simultaneous presence of counterflow and rotation, when the angular velocity $\mathbf{\Omega}$ of the sample is not collinear with the counterflow velocity $\mathbf{V}$. Thus our tensor $\mathbf{\Pi}^s$, which is able to describe also situations without symmetries, provides an elementary unified description of some aspects of the global geometry of vortex tangle.

Another measure of anisotropy is the so-called polarity, which is defined as the average projection of $\mathbf{s}'$ along some given direction, for instance, it refers to $\mathbf{s}'\cdot\hat{\mathbf{V}}$ (where $\hat{\mathbf{V}}=\mathbf{V}/|\mathbf{V}|$) if one takes the polarity along the direction of $\mathbf{V}$. If one chooses the direction of $\mathbf{\Omega}$ parallel to $\mathbf{V}$, as is found in most experimental situations studied up to now, and one takes this direction as first axis, one has

$$p_x = \langle s'_x \rangle = \frac{1}{\Lambda L} \int \hat{\mathbf{x}} \cdot \mathbf{s}' d\xi, \tag{2.18}$$

and it is $\langle s'_x \rangle = 1$ in pure rotation and $\langle s'_x \rangle = 0$ in an isotropic tangle. We can also introduce a vector $\mathbf{p}$ characterizing the polarity, whose components in an orthogonal frame are

$$\mathbf{p} = \langle \mathbf{s}' \rangle = (\langle s'_x \rangle, \langle s'_y \rangle, \langle s'_z \rangle). \tag{2.19}$$

In homogeneous counterflow superfluid turbulence, which deals with vortices far from the boundaries of the sample, the vortex tangle consists mainly of closed loops, in such a way that the integrals in (2.18) and (2.19) vanish. However, in the presence of rotation, there appear some vortex lines along the axis of rotation, which terminate on the upper and lower boundaries of the container. Then, in the presence of rotation it is logical to expect some degree of polarization; see, for example, simulations by Barenghi *et al.*[17] and by Finne *et al.*,[18] which show arrangements of vortex lines in turbulent helium in the rotating frame. Furthermore, even in the absence of rotation, some vortex lines may begin and end at the boundaries, thus providing some contribution to the anisotropy, which is expected to be small when the container is wide enough; this may happen, for instance, in the regime of low-density lines (the so-called TI turbulent regime), in which could appear small localized arrays of quantized vortices locally polarized.[19]

Finally, other quantities characterizing the anisotropy of the vortex tangle are the vectors

$$\mathbf{I} = \frac{\langle \mathbf{s}' \times \mathbf{s}'' \rangle}{\langle |s''| \rangle}, \quad \mathbf{J} = \frac{\langle \mathbf{s}'' \rangle}{\langle |s''| \rangle}; \tag{2.20}$$

the vector $\mathbf{I}$, introduced by Schwarz, describes the information on the distribution of $\mathbf{s}' \times \mathbf{s}''$, linked to the mean direction of the self-induced velocity $\mathbf{v}_i$. The second vector $\mathbf{J}$, which Schwarz does not consider, was introduced in Ref. 20, where a new derivation of Vinen's equation,[21] valid for anisotropic tangles, has been shown.

B. Influence of counterflow and rotation

A situation of special interest concerning the anisotropy is found in superfluid turbulence under the simultaneous influence of counterflow velocity $\mathbf{V}$ and rotation with angular speed $\mathbf{\Omega}$. As has been said, counterflow velocity $\mathbf{V}$ tends to produce a disordered tangle, whereas $\mathbf{\Omega}$ produces an ordered array of vortex lines parallel to rotation axis.

As in pure counterflow, to describe the information on the distribution of $\mathbf{s}'$, we will use a tensor $\mathbf{\Pi}$, which now will be supposed nonsymmetric and whose symmetric part is furnished by (2.14). We propose to describe the anisotropy of the tangle using the tensor

$$\mathbf{\Pi} = \mathbf{\Pi}^s + \mathbf{\Pi}^a \equiv \frac{3}{2}\left[\langle \mathbf{U} - \mathbf{s}'\mathbf{s}' \rangle + \frac{\alpha'}{\alpha} \langle \mathbf{W} \cdot \mathbf{s}' \rangle \right], \tag{2.21}$$

where $\mathbf{\Pi}^s$, the symmetric part of tensor $\mathbf{\Pi}$, is defined in Eq. (2.14), while $\mathbf{\Pi}^a$, the antisymmetric part of $\mathbf{\Pi}$, is linked to the polarity $\mathbf{p}$. In this way, the tensor $\mathbf{\Pi}$ contains the main quantities characterizing the anisotropy of the vortex tangle, in the distribution of $\mathbf{s}'$, namely, the first and second moments of $s'_x = \cos\theta$ (θ being the angle between $\mathbf{s}'$ and the first axis). This choice is related to the tensor $\mathbf{P}_\omega$ introduced in Ref. 22 and has some similarities with the geometrical description of polymer solutions or of nematic liquid crystals or dumbbell-like molecules in condensed matter.[23–25]

In a completely isotropic vortex tangle tensor $\mathbf{\Pi}^a$ is zero. Indeed, in this case, it is directly assumed that $\langle \mathbf{s}' \rangle = 0$, i.e., that, on the average, there are as many fragments of vortex array along one direction as opposite to this direction. In the presence of rotation only, the vortices are parallel and oriented along the rotation axis; therefore $\mathbf{s}' = \hat{\mathbf{\Omega}}$, and the tensor $\mathbf{\Pi}$ takes the form:[16]

$$\mathbf{\Pi} = \mathbf{\Pi}_R = \frac{3}{2}\left[\left(\mathbf{U} - \hat{\mathbf{\Omega}}\hat{\mathbf{\Omega}} \right) + \frac{\alpha'}{\alpha} \mathbf{W} \cdot \hat{\mathbf{\Omega}} \right]. \tag{2.22}$$

Under the simultaneous influence of counterflow velocity $\mathbf{V}$ and rotation with angular speed $\mathbf{\Omega}$, rotation produces an ordered array of vortex lines parallel to rotation axis, whereas counterflow velocity tends to produce a disordered (approximatively isotropic) tangle. In this way the total ensemble of vortex lines will be a superposition of both contributions:

$$\begin{aligned} \mathbf{\Pi}^s &= (1-b)\mathbf{\Pi}^s_H + b\mathbf{\Pi}^s_R, \\ \mathbf{\Pi}^a &= (1-c)\mathbf{\Pi}^a_H + c\mathbf{\Pi}^a_R. \end{aligned} \tag{2.23}$$

If $\mathbf{V}$ is parallel to $\mathbf{\Omega}$, as it happens in most experiments and simulations, one will have $\mathbf{V} \cdot \mathbf{\Pi}^s \cdot \mathbf{V} = (1-b)V^2$. In Eq. (2.23) b and c are parameters between 0 and 1 related to the anisotropy of vortex lines, describing the relative weight of the array of vortex lines parallel to $\mathbf{\Omega}$ and the disordered tangle of counterflow: when $b = c = 0$ we recover an approximatively isotropic tangle and when $b = c = 1$ the ordered array (total anisotropy). In general situations these coefficient depend on Ω, V, and the angle between $\mathbf{V}$ and $\mathbf{\Omega}$.

Here we examine explicitly two simplified situations. In the first one $\mathbf{V}$ is parallel to $\mathbf{\Omega}$ and there is cylindrical symmetry with respect to the rotation axis; in the second one, $\mathbf{\Omega}$ and $\mathbf{V}$ are orthogonal each other, and no symmetries are present in the vortex tangle.

We consider first the case in which $\mathbf{V}$ is parallel to $\mathbf{\Omega}$, as happens in some experiments and simulations.[26–29] In this case, in both tensors $\mathbf{\Pi}_H$ and $\mathbf{\Pi}_R$ it is possible to suppose isotropy in planes orthogonal to the rotation axis; choosing $\mathbf{V}$ and $\mathbf{\Omega}$ in the direction of the x axis, one has

$$\mathbf{\Pi}^s_H = \frac{3}{2}\begin{pmatrix} 2a & 0 & 0 \\ 0 & 1-a & 0 \\ 0 & 0 & 1-a \end{pmatrix}, \quad \mathbf{\Pi}^s_R = \frac{3}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{2.24}$$

Here a is a parameter characterizing the anisotropy of the tangle in pure counterflow; comparing (2.24) with (2.17) we see that in this situation

$$I_{\|} = 2a, \quad I_{\perp} = 1 - a. \tag{2.25}$$

In most studies of counterflow superfluid turbulence, the vortex tangle is supposed isotropic. This corresponds to choosing $a=1/3$, but this hypothesis is confirmed by experimental observations only in a first rough approximation. This coefficient a has been determined through accurate measurements of second-sound attenuation propagating in both directions parallel and orthogonal to $\mathbf{V}$.[8] Summarizing, for the symmetric part of tensor $\mathbf{\Pi}$, from (2.23) and (2.24) we get

$$\mathbf{\Pi}^s = \frac{3}{2}\langle \mathbf{U} - \mathbf{s}'\mathbf{s}'\rangle = \frac{3}{2}\begin{pmatrix} 2a(1-b) & 0 & 0 \\ 0 & 1-a(1-b) & 0 \\ 0 & 0 & 1-a(1-b) \end{pmatrix}. \tag{2.26}$$

Therefore, from Eq. (2.26) we obtain

$$\langle s_x'^2\rangle = 1 - 2a(1-b), \quad \langle s_y'^2\rangle = \langle s_z'^2\rangle = a(1-b). \tag{2.27}$$

Comparing (2.26) with (2.17), we see that

$$I_{\|} = 2a(1-b), \quad I_{\perp} = 1 - a(1-b). \tag{2.28}$$

Once the coefficient a is known, from experiments in pure counterflow,[8] from (2.27) we can obtain the dependence of b from the second moments of $\mathbf{s}'$.

In the second situation, when the counterflow velocity $\mathbf{V}$ is orthogonal to the angular velocity $\mathbf{\Omega}$, choosing $\mathbf{\Omega}$ in the direction of the x axis and $\mathbf{V}$ in that of the z axis, one has

$$\mathbf{\Pi}_H^s = \frac{3}{2}\begin{pmatrix} 1-a & 0 & 0 \\ 0 & 1-a & 0 \\ 0 & 0 & 2a \end{pmatrix}, \tag{2.29}$$

and, for the symmetric part of tensor $\mathbf{\Pi}$, we get

$$\mathbf{\Pi}^s = (1-b)\mathbf{\Pi}_H^s + b\mathbf{\Pi}_R^s = \frac{3}{2}\begin{pmatrix} (1-a)(1-b) & 0 & 0 \\ 0 & (1-a)(1-b)+b & 0 \\ 0 & 0 & 2a(1-b)+b \end{pmatrix}. \tag{2.30}$$

As a consequence,

$$\langle s_x'^2\rangle = a(1-b) + b, \quad \langle s_y'^2\rangle = a(1-b),$$
$$\langle s_z'^2\rangle = (1-2a)(1-b). \tag{2.31}$$

If we make the rough approximation that the tangle produced by the counterflow is isotropic in the distribution of $\mathbf{s}'$ one has $a=1/3$ and $\mathbf{\Pi}_H^s = \mathbf{U}$. In this case, we obtain, in both situations, for the symmetric part of the tensor $\mathbf{\Pi}$ the following expression:

$$\mathbf{\Pi}^s = \frac{3}{2}\begin{pmatrix} 1-\langle s_x'^2\rangle & 0 & 0 \\ 0 & 1-\langle s_y'^2\rangle & 0 \\ 0 & 0 & 1-\langle s_z'^2\rangle \end{pmatrix} = \begin{pmatrix} 1-b & 0 & 0 \\ 0 & 1+b/2 & 0 \\ 0 & 0 & 1+b/2 \end{pmatrix}, \tag{2.32}$$

and

$$\langle s_x'^2\rangle = \frac{1+2b}{3}, \quad \langle s_y'^2\rangle = \langle s_z'^2\rangle = \frac{1-b}{3}; \tag{2.33}$$

in this simplified case, comparing with Eq. (2.17), we see that b is linked to the two scalar quantities $I_{\|}$ and $I_{\perp}$ in Eqs. (2.15) and (2.16), by the relation $b=1-(3/2)I_{\|}=3I_{\perp}-2$.

The antisymmetric part of tensor $\mathbf{\Pi}$ furnishes the polarity of the tangle; in fact, since $\mathbf{W}$ is a constant tensor, one can write, from Eqs. (2.21) and (2.23),

$$\mathbf{\Pi}^a \equiv \frac{3}{2}\frac{\alpha'}{\alpha}\langle \mathbf{W}\cdot\mathbf{s}'\rangle = (1-c)\mathbf{\Pi}_{\mathrm{H}}^a + c\mathbf{\Pi}_{\mathrm{R}}^a, \tag{2.34}$$

from which we deduce

$$\langle s_x'\rangle_{HR} = (1-c)\langle s_x'\rangle_H + c\langle s_x'\rangle_R = (1-c)\langle s_x'\rangle_H + c, \tag{2.35}$$

for high values of the counterflow, we can neglect the influence of pinned vortices, and we get

$$\langle s_x'\rangle_{HR} = c. \tag{2.36}$$

In the presence of rotation only, $\mathbf{s}'$ is parallel to the angular velocity $\mathbf{\Omega}$. In this case, all the vortices are parallel and oriented along the rotation axis and $\langle s_x'\rangle=1$, $\langle s_x'^2\rangle=1$, $b=1$, and from Eq. (2.35) we get $c=1$.

Tsubota *et al.*[7,27–29] have studied what happens to vortices in the simultaneous presence of counterflow and rotation. They have obtained from numerical simulations the polarization $p_x=\langle s_x'\rangle$ as a function of V and Ω. As expected, the polarization decreases with V at constant Ω and increases with Ω at constant V, which shows the competition between these two effects. However, in our proposal (2.21) for the description of anisotropy, also the second moments of $\mathbf{s}'$ appear, besides the first moments studied by Tsubota *et al.* Further, as we will show in the following section, the knowledge of b is important, because this coefficient appears in the expression for the attenuation of second sound, which is the most usual method to determine the value of the vortex line density L. Therefore, to know the effective value of L we need to know the value of b. In the following section, we will show how the anisotropy coefficients introduced in this section modify the expression of the attenuation of second sound, while in Sec. IV, the dependence of b and c on Ω and V will be determined on a microscopic basis.

III. SECOND-SOUND PROPAGATION RESULTS AS A MEASURE OF ANISOTROPY OF THE TANGLE

The interaction between second sound and vortex lines is a classical topic of research, and provides the basis to obtain the values of the vortex line density L, from the attenuation of low-amplitude second sound. However, in the usual counterflow analyses, the assumption of isotropy of the tangle is *a priori* imposed on the analysis, in such a way that the usual results cannot be automatically applied to obtain information on the anisotropy. In experiments in the simultaneous presence of counterflow and rotation the anisotropy of the tangle cannot be neglected. We briefly deal with this topic, for the sake of completeness.

For the sake of simplicity, we will assume that the contribution to the vortex tangle is reduced to mutual friction between the normal component and static vortex tangle structure. In fact, for a more detailed analysis, the tangle is not static and its back reaction on the applied flow corresponding to the second sound should be considered (see, for instance, Refs. 30–32). However, here we only pretend to show that even in the simplest situation the anisotropy will have some influence on the dispersion equation for second sound. We defer to a future work a more detailed incorporation of the dynamic back reaction of the tangle, which will need additional equations to those in (3.1), describing the mutual interaction of the tangle and the second sound.

To study the second-sound propagation in superfluid counterflow turbulence in a rotating container one may use the standard two-fluid model, using ρ_n, ρ_s, and $\mathbf{v}_n-\mathbf{v}_s$ as variables, or a one-fluid model, using ρ, T, and $\mathbf{q}$ as variables. Both models yield essentially equivalent results for second sound although there are some interesting differences in the prediction of fourth sound in porous media.[33,34]

We start our analysis by writing the field equations in the framework of the one-fluid model of liquid helium II,[22,35] deduced from extended irreversible thermodynamics,[36] because in this framework the tensor $\mathbf{\Pi}$ characterizing the anisotropy of the vortex tangle was first introduced. Then, we rewrite the equations obtained in the more familiar variables of the two-fluid model. As we will see, in the description of counterflow superfluid turbulence in rotating containers, the tensor $\mathbf{\Pi}$ plays an explicit role.

The fundamental fields of the one-fluid model of liquid helium II are density ρ, velocity $\mathbf{v}$, absolute temperature T, and heat flux $\mathbf{q}$.[22,35] Neglecting bulk and shear forces and thermal dilatation (which in helium II are very small), the linearized system of field equations, in a noninertial frame, rotating at uniform velocity $\mathbf{\Omega}$, in absence of external forces, is:[22]

$$\frac{\partial \rho}{\partial t} + \rho \nabla \cdot \mathbf{v} = 0, \tag{3.1a}$$

$$\rho \frac{\partial \mathbf{v}}{\partial t} + \nabla p + \rho(\mathbf{i}^0 + 2\Omega \times \mathbf{v}) = 0, \tag{3.1b}$$

$$\frac{\partial T}{\partial t} + \frac{1}{\rho c_V} \nabla \cdot \mathbf{q} = 0, \tag{3.1c}$$

$$\frac{\partial \mathbf{q}}{\partial t} + \zeta \nabla T + 2\mathbf{\Omega} \times \mathbf{q} = -\mathbf{P}_\omega \cdot \mathbf{q}. \tag{3.1d}$$

In (3.1) p is the thermostatic pressure, c_V the specific heat, ζ a positive coefficient linked to the second-sound velocity, and $\rho(\mathbf{i}^0+2\mathbf{\Omega}\times\mathbf{v})$ the inertial force. Convective terms have been neglected owing to linearization. The effect of vortices is described by incorporating the source term

$$\boldsymbol{\sigma}_\omega = -\mathbf{P}_\omega \cdot \mathbf{q} = -\frac{1}{3}\kappa B L \mathbf{\Pi} \cdot \mathbf{q} \tag{3.2}$$

into the evolution equation of heat flux, with $\mathbf{\Pi}$ given by Eqs. (2.21) and B the dissipative Hall-Vinen coefficient.[1]

In Ref. 22 two situations were considered: second-sound propagation in a rotating frame and second-sound propagation in a cylindrical tube in the presence of stationary thermal counterflow. In the first case, the vortices are parallel and oriented along the rotation axis; therefore $\mathbf{s}'=\hat{\mathbf{\Omega}}=\frac{\mathbf{\Omega}}{|\Omega|}$ and the corresponding production term becomes

$$\boldsymbol{\sigma}_\omega^R = -\frac{1}{3}\kappa B L_R \mathbf{\Pi}_R \cdot \mathbf{q} = \frac{1}{2}\kappa B L_R\left(\hat{\mathbf{\Omega}} \times \hat{\mathbf{\Omega}} \times \mathbf{q} + \frac{\alpha'}{\alpha}\hat{\mathbf{\Omega}} \times \mathbf{q}\right). \tag{3.3}$$

In most experiments in counterflow superfluid turbulence, the vortex arrangement is described as an isotropic tangle. Therefore, in this approximation, the production term is expressed by

$$\boldsymbol{\sigma}_\omega^H = -\frac{1}{3}\kappa B L_H \mathbf{\Pi}_H \cdot \mathbf{q} = -\frac{1}{3}\kappa B L_H \mathbf{q}. \tag{3.4}$$

Here we consider the combined situation in which vortex lines are introduced by both rotation and counterflow along rotation axis and without assuming *a priori* that the tangle is isotropic in absence of rotation. Assuming a heat flux along x direction and isotropy in the transversal yz plane, the tensor $\mathbf{\Pi}$ has the explicit expressions (2.26) and (2.34). Then, the vortex tangle is described by four dynamical quantities, namely, $L=L(T,\Omega,V^2)$, $a=a(T,V^2)$, $b=b(T,\Omega,V^2)$, and $c=c(T,\Omega,V^2)$.

In Refs. 22 and 35 it has been shown that, performing the change of variables

$$\mathbf{q} = \rho_s T s \mathbf{V}, \quad \mathbf{v} = \frac{\rho_s}{\rho}\mathbf{v}_s + \frac{\rho_n}{\rho}\mathbf{v}_n, \tag{3.5}$$

with s the entropy of helium II, and putting

$$\zeta = \rho \frac{\rho_s}{\rho_n} T s^2, \tag{3.6}$$

the linearized equations of the one-fluid model can be identified with those of the two-fluid nondissipative model. If we perform in the field equation (3.1d) the change of variables (3.5) and (3.6), to the first-order approximation with respect to the relative velocity $\mathbf{V}$ and the derivatives of the field variables, we obtain

$$\frac{\partial \mathbf{V}}{\partial t} + \frac{\rho s}{\rho_n}\nabla T + 2\mathbf{\Omega} \times \mathbf{V} = -\frac{1}{3}\kappa B L \mathbf{\Pi} \cdot \mathbf{V}. \tag{3.7}$$

In this equation appears the tensor $\mathbf{\Pi}$. When $\mathbf{V}=0$ (i.e., $\mathbf{q}=0$), the result is $\mathbf{\Pi}=\mathbf{\Pi}_R$, $L=2\Omega/\kappa$, and (3.7) reduces to the well-known Hall-Vinen equation; when $\Omega=0$, $\mathbf{\Pi}=\mathbf{\Pi}_H$, $L=\gamma_H^2 V^2/\kappa^2$, and under the additional hypothesis of isotropy in the vortex tangle, (3.7) reduces to the Gorter-Mellink equation. In combined rotation and counterflow $\mathbf{\Pi}$ is expressed by (2.21), which in our case ($\mathbf{v}$ and $\mathbf{\Omega}$ collinear) leads to Eqs. (2.26) and (2.34)

The equation for the velocity of the superfluid component can be obtained by multiplying Eq. (3.7) by $-\rho_n/\rho$ and adding it to the balance equation for the velocity (3.1b). Recalling that one can write

$$\mathbf{v}_s = \mathbf{v} - \frac{\rho_n}{\rho}\mathbf{V}, \tag{3.8}$$

we find

$$\frac{\partial \mathbf{v}_s}{\partial t} - s\,\nabla T + \frac{1}{\rho}\nabla p + (\mathbf{i}^0 + 2\mathbf{\Omega}\times\mathbf{v}_s) = \frac{1}{3}\frac{\rho_n}{\rho}\kappa BL\mathbf{\Pi}\cdot\mathbf{V}. \tag{3.9}$$

By virtue of the equation $d\mu=(1/\rho)dp-s\,dT$, which relates the chemical potential $\mu=\epsilon-Ts+(p/\rho)$ to the equilibrium variables,[35] the field equation for the superfluid velocity takes the form

$$\rho_s\frac{\partial \mathbf{v}_s}{\partial t} + \rho_s\nabla\mu + \rho_s(\mathbf{i}^0 + 2\mathbf{\Omega}\times\mathbf{v}_s) = \frac{1}{3}\frac{\rho_n\rho_s}{\rho}\kappa BL\mathbf{\Pi}\cdot\mathbf{V}. \tag{3.10}$$

Observe that in the literature the form $-(\rho_n\rho_s/3\rho)\kappa BL\mathbf{\Pi}\cdot\mathbf{V}$ for the source term in Eq. (3.10) is not found; instead, one uses the particular form (3.3), relating the source term directly to the angular rotation $\mathbf{\Omega}$ of the container, or the form (3.4) relating the source term to the intensity of the thermal counterflow. The form proposed here is able to recover very directly the above results and can be used in combined situations too.

We consider the propagation of second-sound harmonic plane waves. Let $\mathbf{n}$ be the unit vector in the direction of wave propagation. We assume heat flux and rotation parallel to the x axis. In the case $\mathbf{n}\parallel\hat{\mathbf{x}}$, it is seen that when the temperature wave is propagated along the direction of the rotation axis, it suffers an attenuation expressed as

$$k_s^{\parallel} \simeq \frac{1}{2u_2}\kappa BLa(1-b). \tag{3.11}$$

In particular, if $b=1$ (rotation) one obtains $k_s^{\parallel}=0$; if $b=0$ (counterflow) the result is $k_s^{\parallel}=\gamma\kappa La/u_2$. In the case $\mathbf{n}\perp\hat{\mathbf{x}}$, under the hypothesis of small dissipation (L and k_s small), one sees that, when the temperature wave is propagated orthogonal to the rotation axis, it is

$$k_s^{\perp} \simeq \frac{1}{4u_2}\kappa BL[1-a(1-b)]. \tag{3.12}$$

In particular, if $b=1$ (rotation), $k_s^{\perp}=\kappa BL/4u_2$; if $b=0$ (counterflow) $k_s^{\perp}=\kappa BL(1-a)/4u_2$. In Ref. 37 a more detailed calculation has been made, in which also the influence of the antisymmetric nondissipative part of the tensor $\mathbf{\Pi}$ on wave propagation is taken into account.[37]

Observe that, because we have supposed isotropy in the yz plane, we can use the expression (2.17) for $\mathbf{\Pi}$. In the case $\mathbf{n}\parallel\hat{\mathbf{x}}$, (3.11) becomes $k_s^{\parallel}=\frac{1}{4u_2}\kappa BLI_{\parallel}$, while from Eq. (3.12), in the case $\mathbf{n}\perp\hat{\mathbf{x}}$, one can write $k_s^{\perp}=\frac{1}{4u_2}\kappa BLI_{\perp}$. These two latter equations are formally identical to the expressions for the attenuation of the second sound found in only counterflow, when the wave is propagated parallel or orthogonal to the axis of rotation;[1,16] their physical meaning is, however, different, because in this case $I_{\parallel}$ and $I_{\perp}$ are obtained by (2.28) and not by (2.25).

Note that in the usual analyses studying the vortex tangle in pure counterflow, b is directly set equal to zero and one may use Eqs. (3.11) and (3.12) to obtain L and a. However, in general circumstances b is not known *a priori*; in this a way Eqs. (3.11) and (3.12) contain the three unknowns L, a, and b. To our knowledge, there are no direct measures of the anisotropy of the tangle in counterflow-rotational superfluid turbulence.

In the following section, using an analogy with paramagnetism, the dependence of b and c on Ω and V will be determined on a microscopic basis.

IV. MICROSCOPIC EVALUATION OF THE ANISOTROPY

Up to now, our description has been purely macroscopic; thus, some parameters of the tensor $\mathbf{\Pi}$, as for instance the anisotropy coefficient b introduced in (2.23), or the tangle polarization $\mathbf{p}=\langle\mathbf{s}'\rangle$ given in (2.19), which will be in general functions of $\mathbf{\Omega}$ and $\mathbf{V}$, cannot be explicitly determined. We will consider here only the case in which $\mathbf{V}$ is parallel to the rotation axis, which we choose as the x axis. In this case the anisotropy coefficient b is expressed by Eq. (2.27) and the polarity p_x by Eq. (2.34).

To discuss them in some detail, we will use an analogy with paramagnetism proposed by several authors[4,5,7] to describe the competition between the ordering influence of rotation and the disordering influence of counterflow. Previous authors[7] have studied only the polarity, related to $\langle s_x'\rangle=\langle\cos\theta\rangle$ (with θ the angle between $\mathbf{s}'$ and the x axis); we want also to obtain the anisotropy factor, related to $\langle s_x'^2\rangle=\langle\cos^2\theta\rangle$, and the entropy of the tangle. We will consider, as in Refs. 4, 5, 7, and 27–29 that $\mathbf{\Omega}$ plays on the vortex segments an orienting influence—arising from the friction force—analogous to that of an external magnetic field $\mathbf{H}$ on a magnetic dipole $\vec{\mu}$. Indeed, the orienting influence of rotation could be phenomenologically described by a term proportional to $\mathbf{\Omega}\cdot\vec{\omega}$ ($\vec{\omega}$ being the local vorticity), analogous to the magnetic interaction $-\mathbf{H}\cdot\vec{\mu}$. In a magnetic system, the dipoles are disordered because of the thermal motion, described by k_BT; the vortices in the tangle are instead disordered by the counterflow $\mathbf{V}$, through instabilities and vortex reconnections which randomize the orientation of the vortices.

Since all relative orientations of the vortex segments with respect to $\mathbf{V}$ are possible, we will consider the Langevin model of paramagnetism, by following Tsubota *et al.*[7] (in our paper[5] we proposed a simpler analogy based on an Ising model of spin-1/2 particles, with only two possible orientations, but the Langevin model is indeed more realistic). In the Langevin model of paramagnetism,[38] the partition function is

$$Z(T,H) = \int e^{x\cos\theta}\sin\theta\, d\theta\, d\phi = 4\pi\frac{\sinh x}{x}, \quad (4.1)$$

with $x=\mu H/k_B T$. From here, the average value of the magnetization and its second moments are easily obtained as

$$p_x = \langle\cos\theta\rangle = \frac{\partial \ln Z}{\partial x} = \coth x - \frac{1}{x} \quad (4.2)$$

and

$$\langle(\Delta p_x)^2\rangle = \langle\cos^2\theta\rangle - \langle\cos\theta\rangle^2 = \frac{\partial^2 \ln Z}{\partial x^2} = -\frac{1}{\sinh^2 x} + \frac{1}{x^2}. \quad (4.3)$$

Note that Eq. (4.3) yields the second moment of the fluctuations in the polarization of the tangle Δp_x, which in principle are also accessible to measurement and which may provide valuable statistical information on the tangle.

Up to now, only the average value (4.2) of $\cos\theta$ has been considered, but in (2.21) the average value of $\cos^2\theta$ is needed; therefore, it is useful to push the paramagnetic analysis to obtain also $\langle\cos^2\theta\rangle$, which is, from Eqs. (4.2) and (4.3),

$$\langle\cos^2\theta\rangle = 1 + \frac{2}{x}\left(\frac{1}{x} - \coth x\right). \quad (4.4)$$

The situation we are considering is very far from equilibrium, and therefore the partition function (4.1) and its consequences (4.2)–(4.4) cannot be safely extrapolated to it. Anyway, since we are lacking for the moment other information, the distribution function related to (4.1) seems a valuable starting point to set our analysis on microscopic grounds. Another criticism that could be addressed to the analogy with paramagnetism is that it explicitly ignores the interaction among vortices, in the same way as in paramagnetic models the interaction among microscopic magnetic moments is neglected. However, it could be argued, in favor of this analogy, that it may also be interpreted as a mean-field model—as in the Weiss mean-field model of ferromagnetism,[38] where every vortex experiences a local rotation that is the external rotation of the container plus the average local rotations induced by the neighboring vortices. Finally, it could be said that a distribution function formally analogous to the canonical one could be obtained from information theory under the restriction of an average value of the total polarity, related to the average value of $\cos\theta$. Here, being aware of the limits of validity of the paramagnetic analogy, we will explore anyway its results as far as possible. This will allow us to gain a further microscopic understanding of the model, and it will reveal whether the paramagnetic model is reasonable enough, or, alternatively, it will show what are the main shortcomings—something that will be illuminating too.

In our case the ratio x between ordering and disordering factors will be taken as proportional to the ratio of the dimensionless combinations $\Omega/\kappa L$ and $V/\kappa L^{1/2}$, combined in such a way that L disappears, because we are studying local disorder which does not depend on L, at least in the case when the interaction between vortices is neglected. This yields

$$x = \tilde{a}\frac{\Omega/\kappa L}{V^2/\kappa^2 L} = \tilde{a}\frac{\Omega\kappa}{V^2}, \quad (4.5)$$

with $\tilde{a}$ a numerical constant. An analogous combination has been used by Tsubota *et al.*,[7] who have used Eq. (4.2) to describe the polarization of the vortex under rotation. They have taken for x

$$x \propto \frac{2\Omega/\kappa}{\gamma_H^2 V^2/\kappa^2} = \frac{L_R}{L_H}, \quad (4.6)$$

where L_R and L_H are, respectively, the vortex line densities corresponding to the presence of pure rotation or pure counterflow. Here, γ_H is the dimensionless parameter relating L_H and V [see Eq. (1.2)]. Thus, it follows that Eqs. (4.5) and (4.6) correspond essentially to the same combination. Note that the term V^2 in the denominator of Eq. (4.5) could be considered in some way as an effective nonequilibrium temperature, different from the purely thermodynamic temperature, characterizing the fluctuations in the orientation of the vortex segments. This is an open topic for further discussion.[39]

The results (4.2) and (4.4) allow us to express the tensor $\mathbf{\Pi}$ as function of Ω and V^2, using Eq. (4.5). Indeed, the coefficients a and b describing the anisotropy in (2.21) may be evaluated according to (2.27) and b turns out to be, using Eq. (4.4),

$$b = 1 - \frac{1-\langle\cos^2\theta\rangle}{2a} = 1 + \frac{3V^2}{\tilde{a}\Omega\kappa}\left(\frac{V^2}{\tilde{a}\Omega\kappa} - \coth\frac{\tilde{a}\Omega\kappa}{V^2}\right). \quad (4.7)$$

If we suppose $a=1/3$ (total isotropy of the tangle in pure counterflow) Eq. (4.7) simplifies as

$$b = \frac{3\langle\cos^2\theta\rangle - 1}{2} = 1 + \frac{3V^2}{\tilde{a}\Omega\kappa}\left(\frac{V^2}{\tilde{a}\Omega\kappa} - \coth\frac{\tilde{a}\Omega\kappa}{V^2}\right). \quad (4.8)$$

The coefficient c in (2.21) is, in accord with (2.36),

$$c = \langle s'_x\rangle = \langle\cos\theta\rangle = \coth\frac{\tilde{a}\Omega\kappa}{V^2} - \frac{V^2}{\tilde{a}\Omega\kappa}. \quad (4.9)$$

The limiting values of b are $b=1$ for high values of Ω, which means that in this case all the vortices are oriented along $\mathbf{\Omega}$, and $b=0$ when $\mathbf{\Omega}$ tends to zero (or when V becomes very high), meaning that the tangle is completely disordered. The limiting values of c are $c=1$ for high values of $\mathbf{\Omega}$, and $c=0$ when $\mathbf{\Omega}$ tends to zero (or when V becomes very high). We cannot determine the value of $\tilde{a}$ from first principles, but we may take for it the value empirically found by Tsubota *et al.*[7] by studying the polarization of the vortex, which is $\tilde{a}=22/\gamma_H^2$. Here, the coefficient γ_H depends on temperature; for instance, its value is 9.8×10^{-4} at $T=1.65$ K.

It is also worthwhile to use the paramagnetic analogy to evaluate not only $\langle\cos\theta\rangle$ and $\langle\cos^2\theta\rangle$, but also the entropy due to the geometry of the tangle. A topic that arises in the context of the geometry of the tangle is its contribution to entropy; this has an obvious interest when one wants to study diffusion of vortices in inhomogeneous tangles. If the tangle is everywhere isotropic (locally isotropic) the only difference from point to point will be the value of L, and the diffusion flux $\mathbf{J}$ of vortex lines will be given by

$$\mathbf{J} = -\beta'\kappa\nabla L, \tag{4.10}$$

where β' is a dimensionless constant and κ the quantum of vorticity, which has the same dimensions as a diffusion constant, namely, $(\text{length})^2/(\text{time})$. For instance, numerical simulations by Tsubota *et al.* have shown that at zero temperature the dimensionless constant β' takes the value 0.1.[40] In fact, from a thermodynamic perspective, it would be more rigorous to use, instead of Eq. (4.10)

$$\mathbf{J} = -\beta''\kappa\nabla\mu(L,\mathbf{\Pi}), \tag{4.11}$$

where $\mu(L,\mathbf{\Pi},\mathbf{p})$ is the chemical potential of the vortices, i.e.,

$$\mu = \mu(L,\langle\cos\theta\rangle,\langle\cos^2\theta\rangle) = \frac{\partial}{\partial L}[\epsilon(L) - Ts(L,\langle\cos\theta\rangle,\langle\cos^2\theta\rangle)] \tag{4.12}$$

or

$$\mu = \mu(L,\mathbf{\Pi}) = \frac{\partial}{\partial L}[\epsilon(L) - Ts(L,\mathbf{\Pi})], \tag{4.13}$$

$\epsilon(L)$ and $s(L,\mathbf{\Pi})$ being the internal energy and the entropy of the vortices per unit volume. The contribution $\epsilon(L)$ is well known; it is given by $\epsilon_V = \partial\epsilon(L)/\partial L = \rho_s\kappa\tilde{\beta}$, with $\tilde{\beta}$ expressed as in Eq. (2.2).

When the geometrical complexity of the tangle differs from point to point, the entropy will also change, being higher where the anisotropy is lower. In general terms, this geometric entropy would be given by

$$S_{tangle} = -k_B L\int \Psi(\mathbf{n})\ln\Psi(\mathbf{n})d\mathbf{n}, \tag{4.14}$$

with $\Psi(\mathbf{n})$ being the orientational distribution function of vortex segments. Here, we will use for (4.14) the entropy due to the orientational disorder of magnetic moments, namely, $S = \partial(k_B T\ln Z)/\partial T$ which in our case leads to

$$S_{tangle}(x) = k_B L\left(\ln Z - x\frac{\partial\ln Z}{\partial x}\right), \tag{4.15}$$

with Z given by (4.1), i.e.,

$$\frac{S_{tangle}(\Omega,V)}{Lk_B} = 1 + \ln 4\pi - \frac{\tilde{a}\Omega\kappa}{V^2}\coth\frac{\tilde{a}\Omega\kappa}{V^2} + \ln\left(\frac{V^2}{\tilde{a}\Omega\kappa}\sinh\frac{\tilde{a}\Omega\kappa}{V^2}\right). \tag{4.16}$$

Recall that expression (4.15) corresponds to a classical system, and that it is not good enough to describe systems at low temperatures (where the number of available microstates is small), where a quantum description is needed. In (4.16), this is the case when Ω is very high and all the vortices are oriented along it, in which situation the entropy should be zero. Thus, expression (4.15) would give the entropy only for relatively small values of $\Omega\kappa/V^2$. The knowledge of the entropy of the vortex tangle could be of interest in future thermodynamic or informational analysis concerning transport of vortex lines or the most probable geometry of the tangle under given external conditions.

V. GEOMETRY AND EVOLUTION

In this section we will propose evolution equations for the geometry of the tangle, i.e., for the anisotropy tensor (2.21), and for the influence of the geometry of the tangle on the evolution of vortex line density. These are tentative phenomenological proposals, waiting for a more fundamental microscopic approach.

In previous sections, we have described the geometry of the tangle by means of tensor $\mathbf{\Pi}$. Now, we would like to describe the evolution of $\mathbf{\Pi}$ following the modification of $\mathbf{V}$ and $\mathbf{\Omega}$. Assume that, under some given values of $\mathbf{V}$ and $\mathbf{\Omega}$, the tensor $\mathbf{\Pi}$ takes asymptotically a given form $\mathbf{\Pi}_0$. For instance, in Eq. (2.23) one would have

$$\mathbf{\Pi}_0^s = (1-b_0)\mathbf{\Pi}_H^s + \frac{3}{2}b_0\left(\mathbf{U} - \hat{\mathbf{\Omega}}\hat{\mathbf{\Omega}}\right),$$

$$\mathbf{\Pi}_0^a = \frac{\alpha'}{\alpha}\mathbf{W}\cdot\mathbf{p}_0 = \frac{\alpha'}{\alpha}c_0\mathbf{W}\cdot\hat{\mathbf{\Omega}}, \tag{5.1}$$

with b_0 and c_0 as given, for instance, by the expressions (4.7) and (4.9) found in Sec. IV. Now, assume a sudden change in V and/or Ω. The tensor $\mathbf{\Pi}$ will tend to the new steady-state value. We tentatively suggest for $\mathbf{\Pi}^s$ and $\mathbf{\Pi}^a$ relaxational evolution equations (which will be valid if the changes in V and Ω are not too high) of the form

$$\tau_s\frac{d\mathbf{\Pi}^s}{dt} + \mathbf{\Pi}^s = \mathbf{\Pi}_0^s(\text{final}), \tag{5.2}$$

$$\tau_a\frac{d\mathbf{\Pi}^a}{dt} + \mathbf{\Pi}^a = \mathbf{\Pi}_0^a(\text{final}). \tag{5.3}$$

These equations will describe the evolution of $\mathbf{\Pi}$ toward the final steady state. The relaxation times τ_s and τ_a will depend, in principle, on L. Then, we assume, for instance, τ_s and τ_a proportional to L^{-1} because the relaxation times will be shorter for higher values of L. Dimensional analysis yields

$$\tau_s = \frac{a_s}{\kappa L}, \quad \tau_a = \frac{a_a}{\kappa L}, \tag{5.4}$$

with a_s and a_a numerical constants.

If we apply this idea to the situation of coupled rotation and counterflow, i.e., to tensor (2.23), we simply obtain for the evolution of $\mathbf{\Pi}^s$ [Eq. (2.26)]

$$\tau_s \frac{da}{dt} + a = a_0(\text{final}), \tag{5.5}$$

$$\tau_s \frac{db}{dt} + b = b_0(\text{final}), \tag{5.6}$$

while, for the evolution of the antisymmetric tensor $\mathbf{\Pi}^a$ [Eq. (2.34)], we get

$$\tau_a \frac{dc}{dt} + c = c_0(\text{final}). \tag{5.7}$$

In Ref. 37 it is analytically shown that, in the presence of eddies in a normal fluid, a net polarization of the tangle in the direction of the normal fluid rotation is created, and the polarization of the entire tangle is also numerically computed. Numerical simulations of the evolution of the polarization of the tangle in counterflow in rotating containers are also reported in Ref. 28. These numerical computations show a behavior that is in qualitative accord with Eq. (5.7), and suggest that the coefficients in (5.4) are given by $3/\alpha$, with α the friction coefficient in (2.1). To our knowledge, numerical simulations for the evolution of the second moments of $\mathbf{s}'$ do not exist. We hope that this work will stimulate more experiments and simulations.

For the sake of completeness, we want to mention here other evolution equations found in previous works, which take into account the anisotropy of the tangle. To describe the dynamics of the anisotropy, in Ref. 41 Lipniacki derives an equation for $\mathbf{I}$ which turns out to be

$$\tau \frac{d\mathbf{I}}{dt} + \mathbf{I} = \mathbf{I}_0(\text{final}), \tag{5.8}$$

where the steady-state value of $\mathbf{I}$ is given by

$$\mathbf{I}_0 = \frac{2}{3}[\mathbf{V}(1 - \mathbf{II}) + \mathbf{I} \times (\mathbf{V} \times \mathbf{I})], \tag{5.9}$$

and the relaxation time τ is given by

$$\tau = \frac{2}{\pi(1+\alpha_{Lip}^2)^{1/2}\beta L} \approx \frac{1}{\kappa L}. \tag{5.10}$$

Note that this expression for the relaxation time is also inversely proportional to L, as was assumed in (5.4).

VI. CONCLUDING REMARKS

We have proposed a description of the geometry and dynamics of vortex tangle as related to possible anisotropy in the directional distribution of the tangent to the lines $\mathbf{s}'$. To describe this anisotropy we have used a tensor $\mathbf{\Pi}$, proposed in Eq. (2.21), which projects $\mathbf{V}$ on the direction perpendicular to the vortex lines. This projection is linked to the force of the counterflow on the vortex tangle which, when multiplied by $\mathbf{V}$ itself, yields the average power supplied to the counterflow by the tangle.[20]

The geometry of the vortex tangle deserves more attention, especially in order to understand in detail the interplay between rotation and counterflow, which have opposite influences on the orientation of the vortex segments. In turn, this orientation has an influence on the dynamics, in such a way that understanding this geometry is a key ingredient in the analysis of rotating counterflow.

Note that we have taken into account the orientation of vortex lines and the total length of vortex lines per unit volume, but we have not taken into consideration the length distribution of vortex lines with different lengths; in fact, the vortex lines are either closed on themselves or attached by their ends to the walls of the container, and when they break they immediately reconnect with themselves or with other vortex lines, but this does not preclude that they may have different lengths. This would be an interesting topic of analysis; indeed, despite experimental information on this feature is not yet available, the increasing abundance of simulations of the tangle may clarify its role for the understanding of the dynamics, and interesting proposals on this direction have been carried out.

Another aspect of interest in the analysis of the geometry of the tangle is in attempting to understand the thermodynamics of turbulent superfluids. Indeed, the tangle will have two kind of contributions to the free energy: on the one side, the rotational energy of the vortices, and, on the other side, the entropy related to the orientational disorder of the vortex lines. Based on a paramagnetic analogy, we have computed in Sec. IV, not only the polarity of the vortex tangle, related to $\langle \cos\theta \rangle$, but also the second moment of the polarity fluctuations (4.3), the anisotropy factor b (2.22), related to $\langle \cos^2\theta \rangle$, and the geometrical entropy of the tangle (4.15). We have shown its relevance in computing the chemical potential of vortex lines and, therefore, its possible influence on vortex transport in inhomogeneous tangles. It would be interesting to explore whether the analogy with the Langevin paramagnetic model does indeed offer a consistent quantitative model not only for polarization, but also for anisotropy and entropy; in fact, an approach similar to this paramagnetic analogy could be obtained in a more direct way from the perspective of maximum entropy formalism, by imposing conditions on L and $\langle \cos\theta \rangle$. We leave this for future research, for which more detailed data on anisotropy would be convenient.

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