Theoretical development of the image method for a general magnetic source in the presence of a superconducting sphere or a long superconducting cylinder

Qiong-Gui Lin*

China Center of Advanced Science and Technology (World Laboratory), P.O. Box 8730, Beijing 100080, People's Republic of China and Department of Physics, Sun Yat-Sen University, Guangzhou 510275, People's Republic of China†

(Received 14 January 2006; published 14 July 2006)

For macroscopic problems where the London penetration depth is negligibly small (ideal Meissner state), the image method is developed for a magnetic monopole in the presence of a superconducting sphere (the image is not a single monopole). Then the field for any magnetic source can be obtained by superposition. A magnetic point dipole with arbitrary direction is studied in detail. We also develop the image method for a steady electric current in the presence of the superconducting sphere. Similar problems for a long superconducting cylinder are solved by the same method but the results are rather different. The case of an arbitrary steady electric current in the presence of an infinite superconducting plane is also solved by the image method. In this method the calculation of levitation forces is convenient and the physical picture is simple. The method is also extended to more complicated geometries involving both spherical (or cylindrical) and planar superconducting boundaries.

DOI: 10.1103/PhysRevB.74.024510 PACS number(s): 74.20.De, 41.20.Gz, 74.25.Ha, 74.25.Nf

I. INTRODUCTION

For macroscopic electromagnetic problems of superconductors, the London penetration depth is negligibly small and the superconductors behave as perfect diamagnets (ideal Meissner state). In this limit magnetostatic boundary value problems involving superconductors can be solved by the image method when the boundaries have simple geometric configurations.

The image method is well-known in classical electrodynamics[1] and it has a very long history.[2] This method is well-studied in electrostatics for both conductors and dielectrics.[1] For magnetostatics involving superconductors, however, it is developed only for some special cases. The case of a monopole or a point dipole in the presence of an infinite superconducting plane appears to be well-known.[2] Another case is a magnetic point dipole with radial direction in the presence of a superconducting sphere. The result for the latter case was mentioned in Ref. 2, and a detailed discussion can be found in Ref. 3. The other cases, for example, a magnetic point dipole with transverse direction in the presence of a superconducting sphere, are not well-developed to our knowledge. Some author supposed that the image method is not available for superconductor-magnet systems with curved boundaries, because the boundary condition is of the second type, while for electrostatics it is of the first type.[4] However, in view of the above example of a dipole with radial direction, this is obviously not true.

The purpose of this paper is to develop the image method for rather general magnetic sources in the presence of a superconducting sphere or a long superconducting cylinder, and then extend the method to more complicated geometries involving both spherical (or cylindrical) and planar superconducting boundaries. The contents are arranged as follows.

In Sec. II we first solve the field equation of magnetostatics for a magnetic monopole in the presence of a superconducting sphere. From the result of this simple case, the image method is developed (the image is not a single monopole). Then we can obtain the magnetic scalar potential for a magnetic point dipole with arbitrary direction without solving the field equation. In principle the result for an arbitrary magnetic source can be obtained by superposition. The change of a uniform magnetic field by a superconducting sphere is also solved by the image method. Then we turn to develop the image method for a circular current loop, and extend it to a more general distribution of steady electric current. The levitation force between the superconducting sphere and the magnetic source is calculated in most cases. The physical picture is clear in this method since the force on the source is due to the image. In Sec. III similar problems for a long superconducting cylinder are solved by the same procedure but the results are rather different. Though the image method for an infinite superconducting plane is well-studied, it is mainly developed for sources like a magnetic monopole or dipoles. For an arbitrary steady electric current, the problem seems to be unsolved. This is considered in Sec. IV and the vector potential is obtained again by the image method. In Sec. V we extend the image method to more complicated geometries involving both spherical and planar superconducting boundaries. Similar problems to those considered in Sec. II are solved. In Sec. VI we consider geometries involving both cylindrical and planar superconducting boundaries and give the results for problems similar to those of Sec. III. In Sec. VII we briefly discuss the problem of hollows in superconductors. Section VIII is devoted to a brief summary.

The subject studied here may have interest in several aspects. First, it may have some applications in magnetic force microscopy[5–7] and may be useful to the study of magnetic levitation.[8–16] Indeed, for macroscopic physical problems that can be described by the Maxwell-London equation (which is difficult to solve in many cases), one can treat the superconductor as a perfect diamagnet to give the lowest-order approximation. Second, the image method is one of the fundamental methods in classical electrodynamics, but the development of this method in the presence of superconductors is far from complete, thus it might be of interest to fill the gap. Third, some of the problems studied here are inconvenient when solved by the ordinary method of separation of variables, but in the image method they are rather simple. For example, those with complicated geometries, or even the simple cases of an electric current in the presence of a superconducting sphere or plane. Fourth, some of the results may be of interest in other areas of physics, for example, stationary temperature distribution in heat conduction problems, or fluid mechanics. Throughout this paper we use MKS units.

II. IMAGE METHOD FOR A SUPERCONDUCTING SPHERE

A. Magnetic monopole

First we consider a magnetic monopole in the presence of a superconducting sphere. Though a magnetic monopole has not been observed in experiment and thus is artificial, the result is important for the subsequent problems.

We choose the coordinate system such that the origin coincides with the center of the sphere, and the monopole is located on the positive z axis. In this and some subsequent sections we will use both the rectangular coordinates (x,y,z) and the spherical ones (r,θ,ϕ). The unit vectors of these coordinate systems are denoted by $(\boldsymbol{e}_x,\boldsymbol{e}_y,\boldsymbol{e}_z)$ and $(\boldsymbol{e}_r,\boldsymbol{e}_\theta,\boldsymbol{e}_\phi)$, respectively. When the coordinates of a source or image position is given they are always rectangular ones unless otherwise specified. The radius of the sphere is a. The monopole is located at the position with coordinates $(0,0,d)$ where $d>a$, and its magnetic charge is Q.

Because the superconducting sphere is treated as a perfect diamagnet, the magnetic induction inside the sphere is zero. Thus only the magnetic induction outside the sphere needs to be found. Since there is no free electric current outside the sphere, we can introduce the magnetic scalar potential φ such that the magnetic induction $\boldsymbol{B}=-\mu_0\nabla\varphi$. Because of the rotational symmetry of the system, $\varphi=\varphi(r,\theta)$ is independent of ϕ. It contains two parts:

$$\varphi(r,\theta)=\varphi_Q(r,\theta)+\varphi'(r,\theta), \tag{1}$$

where

$$\varphi_Q(r,\theta)=\frac{Q}{4\pi\sqrt{r^2+d^2-2rd\cos\theta}} \tag{2}$$

is the scalar potential of the monopole, and $\varphi'(r,\theta)$ is the one generated by the induced current on the spherical surface. The latter satisfies the Laplace equation and has the form

$$\varphi'(r,\theta)=\sum_{n=0}^{\infty}\frac{c_n}{r^{n+1}}P_n(\cos\theta), \tag{3}$$

up to a constant which does not contribute to the magnetic induction. Expanding $\varphi_Q(r,\theta)$ in terms of the Legendre polynomials, we have

$$\varphi(r,\theta)=\sum_{n=0}^{\infty}\left(\frac{Q}{4\pi}\frac{r^n}{d^{n+1}}+\frac{c_n}{r^{n+1}}\right)P_n(\cos\theta) \tag{4}$$

near the spherical surface. Since the radial component of the magnetic induction B_r should be continuous at the spherical surface, we have

$$\left.\frac{\partial\varphi}{\partial r}\right|_{r=a}=0. \tag{5}$$

This determines the coefficients c_n in Eq. (3). The result is

$$c_n=\frac{Q}{4\pi}\frac{na^{2n+1}}{(n+1)d^{n+1}},\quad n=0,1,2,\ldots. \tag{6}$$

It is straightforward to verify that Eq. (3) with the coefficients given in Eq. (6) can be recast in the form

$$\varphi'(r,\theta)=\frac{Qa/d}{4\pi\sqrt{r^2+(a^2/d)^2-2r(a^2/d)\cos\theta}}-\int_0^{a^2/d}\frac{(Q/a)du}{4\pi\sqrt{r^2+u^2-2ru\cos\theta}}. \tag{7}$$

This means that for a source monopole which is located at the position $(0,0,d)$ and has charge Q, the image is a point monopole with charge Qa/d located at the ordinary image point $(0,0,a^2/d)$ plus a continuous distribution of magnetic charges on the straight line from the origin to the image point with uniform linear density $-Q/a$. (For convenience the continuous part will be called a line charge occasionally.) Note that the monopole at the image position has the same sign as the source one, and the total magnetic charge inside the sphere vanishes as is expected on the basis of the boundary condition (5). With this result we can deal with more complicated magnetic sources without solving the field equation. Of special interest is a magnetic dipole since this can be realized in practice by a uniformly magnetized sphere. In passing we record the levitation force acted on the monopole

$$\boldsymbol{F}=\frac{\mu_0Q^2a^3}{4\pi d(d^2-a^2)^2}\boldsymbol{e}_z. \tag{8}$$

This can be obtained by $\boldsymbol{F}=Q\boldsymbol{B}'(\boldsymbol{d})$ where $\boldsymbol{B}'(\boldsymbol{r})$ is the magnetic induction of the image charges and $\boldsymbol{d}=d\boldsymbol{e}_z$, or by calculating the force (like the Coulomb one) between the source and the image charges. We also remark that the integral in Eq. (7) can be worked out and the result can be expressed in terms of elementary functions:

$$\varphi'(r,\theta)=\frac{Qa}{4\pi d\sqrt{r^2+(a^2/d)^2-2r(a^2/d)\cos\theta}}-\frac{Q}{4\pi a}\ln\frac{r(1+\cos\theta)}{(r\cos\theta-a^2/d)+\sqrt{r^2+(a^2/d)^2-2r(a^2/d)\cos\theta}}. \tag{7'}$$

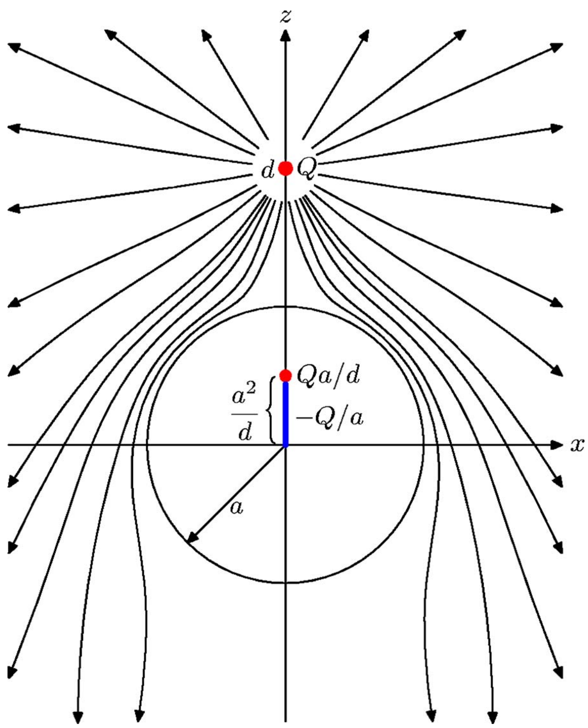

FIG. 1. (Color online) The magnetic induction lines for a magnetic monopole in the presence of a superconducting sphere. The source monopole and the images are also shown, where $-Q/a$ is the linear density of the magnetic line charge.

Though this result is not instructive, it is useful in some practical calculations, for example, in numerical calculations of the magnetic induction lines, where the form of Eq. (7) is difficult to handle. The magnetic induction lines are illustrated in Fig. 1, where the source and images are also displayed.

B. Magnetic dipole with radial direction

Now we consider a magnetic point dipole with radial direction, which is located at the position $(0,0,d)$ and has the dipole moment $\boldsymbol{m}=m\boldsymbol{e}_z$. This has been studied by solving the field equation,[4] or by the image method[3] where the result is obtained by a guess. (These authors also studied the case when the London penetration depth is finite.) Actually the result of the image method has been mentioned in an appendix of an earlier paper,[2] also without reasoning. Here we will derive the result on the basis of the preceding one.

The above dipole can be theoretically realized by two monopoles, one with charge Q located at $(0,0,d)$, and the other with charge $-Q$ located at $(0,0,d-l)$ where $Q\to\infty$ and $l\to 0$ but $Ql=m$ is fixed. The image of the charge Q is a point monopole with charge Qa/d located at $(0,0,a^2/d)$ plus a uniform line charge with linear density $-Q/a$ distributed on the z axis from the origin to $(0,0,a^2/d)$; the image of the charge $-Q$ is a point monopole with charge $-Qa/(d-l)$ located at $(0,0,a^2/(d-l))$ plus a uniform line charge with linear density Q/a distributed on the z axis from the origin to $(0,0,a^2/(d-l))$. The effect of the two point images is a point dipole with dipole moment $\boldsymbol{m}'=-(a/d)^3\boldsymbol{m}$ plus a monopole with charge $-ma/d^2$, both located at $(0,0,a^2/d)$. The effect of the two line charges is also a monopole located at $(0,0,a^2/d)$ but with charge ma/d^2, which cancels the one just mentioned. Therefore the image is a magnetic point dipole with dipole moment $\boldsymbol{m}'=-(a/d)^3\boldsymbol{m}$ located at the image point $(0,0,a^2/d)$. Then the scalar potential outside the sphere is $\varphi(r,\theta)=\varphi_m(r,\theta)+\varphi'(r,\theta)$, where $\varphi_m(r,\theta)=\boldsymbol{m}\cdot(\boldsymbol{r}-\boldsymbol{d})/4\pi|\boldsymbol{r}-\boldsymbol{d}|^3$ is the potential of the source dipole, and $\varphi'(r,\theta)$ is the one of the image which has a similar form. This is the result previously obtained. It is easy to write down the magnetic induction $\boldsymbol{B}'$ of the image. Fixing the position of the image and placing the source dipole at an arbitrary position $\boldsymbol{r}$, we have

$$\boldsymbol{m}\cdot\boldsymbol{B}'(\boldsymbol{r})=\frac{\mu_0 m^2}{4\pi}\left(\frac{a}{d}\right)^3\frac{x^2+y^2-2(z-a^2/d)^2}{[x^2+y^2+(z-a^2/d)^2]^{5/2}}. \tag{9}$$

The levitation force on the source dipole is

$$\boldsymbol{F}=\nabla(\boldsymbol{m}\cdot\boldsymbol{B}')|_{\boldsymbol{r}=\boldsymbol{d}}=\frac{3\mu_0 m^2}{2\pi}\frac{a^3 d}{(d^2-a^2)^4}\boldsymbol{e}_z\equiv F(d)\boldsymbol{e}_z. \tag{10}$$

The potential energy of the dipole at $\boldsymbol{d}$ is (the reference point is set at infinity)

$$U(d)=\int_d^\infty F(u)du=\frac{\mu_0 m^2}{4\pi}\frac{a^3}{(d^2-a^2)^3}=-\frac{1}{2}\boldsymbol{m}\cdot\boldsymbol{B}'(\boldsymbol{d}). \tag{11}$$

The last equality can be easily verified. Some authors calculate $U(d)=-\frac{1}{2}\boldsymbol{m}\cdot\boldsymbol{B}'(\boldsymbol{d})$ first and then obtain the force by $F(d)=-\partial U(d)/\partial d$. The result is of course the same, but the other components of the force cannot be obtained in this way (though they are zero).

The magnetic induction lines are illustrated in Fig. 2, where the source and image are also displayed. Though the problem has been studied before,[3,4] the figure was not given.

It should be pointed out that the field obtained above is only approximately valid if the source dipole is realized by a uniformly magnetized sphere, since its magnetization may be changed by the interaction with the superconducting sphere. Moreover, the levitation force holds approximately only when the radius of the magnetized sphere is small. These remarks also apply to the similar situations in the following.

C. Magnetic dipole with transverse direction

Next we consider a magnetic point dipole with transverse direction, which is located at the position $(0,0,d)$ and has the dipole moment $\boldsymbol{m}=m(\cos\phi_0,\sin\phi_0,0)$. This has been studied by solving the field equation in both cases when the London penetration depth is finite and negligibly small.[17] The result for the latter case was also published in a separate paper.[18] Unfortunately the result for this case is incorrect because an incorrect boundary condition $\boldsymbol{B}|_{r=a}=0$ instead of

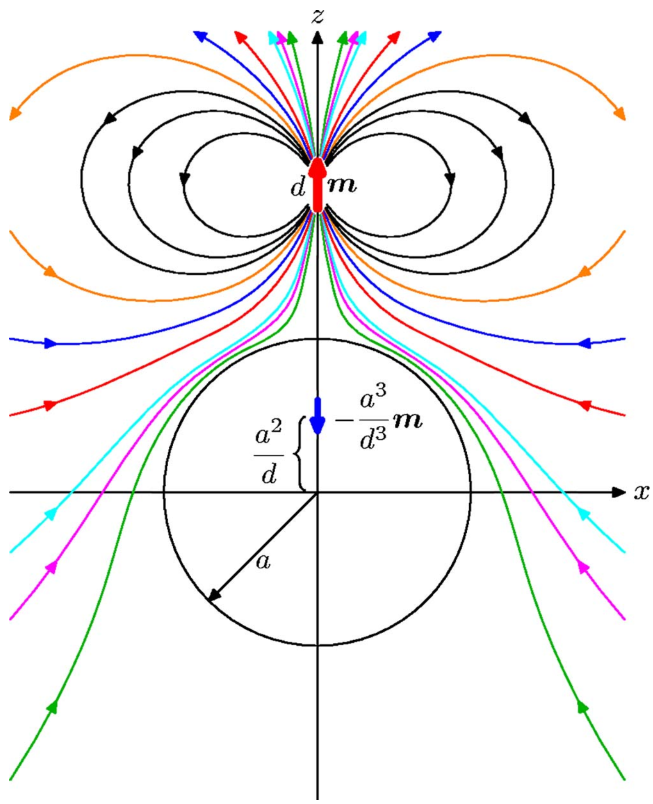

FIG. 2. (Color online) The magnetic induction lines for a radial magnetic dipole in the presence of a superconducting sphere. The source and image dipoles are also shown.

Eq. (5) is used. (The induced field as constructed in these papers does not satisfy the field equation. It is impossible for a solution of the field equation to satisfy their boundary condition.) Here the result is obtained by the image method.

The above dipole can be theoretically realized by two monopoles, one with charge Q located at $(\frac{1}{2}l\cos\phi_0, \frac{1}{2}l\sin\phi_0, d)$, and the other with charge $-Q$ located at $(-\frac{1}{2}l\cos\phi_0, -\frac{1}{2}l\sin\phi_0, d)$ where $Q\to\infty$ and $l\to 0$ but $Ql = m$ is fixed. By similar analysis to that in the last section, it turns out that the image is a magnetic point dipole with dipole moment $\boldsymbol{m}' = (a/d)^3\boldsymbol{m}$ located at the image point $(0,0,a^2/d)$ plus a continuous distribution of point dipoles on the straight line from the origin to the image point, the dipole moment from $(0,0,u)$ to $(0,0,u+du)$ being $-(\boldsymbol{m}/ad)u\,du$. This result is illustrated in Fig. 3 (it is somewhat difficult to draw the magnetic induction lines in this case because the rotational symmetry is absent). It is easy to write down the scalar potential φ_m for the source dipole and φ' for the images:

$$\varphi_m(r,\theta,\phi) = \frac{mr\sin\theta\cos(\phi-\phi_0)}{4\pi(r^2+d^2-2rd\cos\theta)^{3/2}}, \tag{12a}$$

and

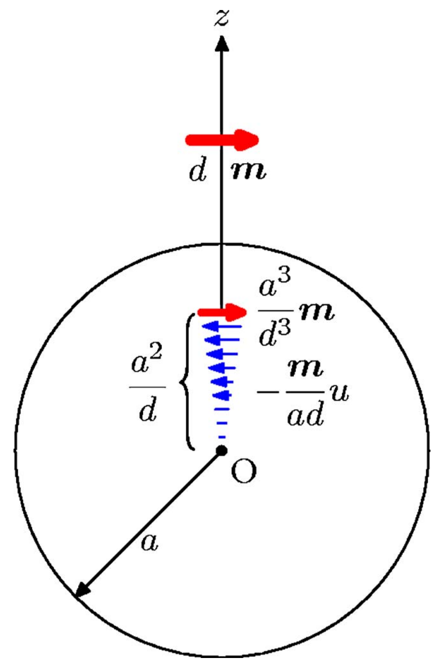

FIG. 3. (Color online) Image method for a transverse magnetic dipole in the presence of a superconducting sphere, where the continuous distribution of point dipoles on the straight line has linear moment density $-(\boldsymbol{m}/ad)u$ at $(0,0,u)$ and $u\in(0,a^2/d)$.

$$\varphi'(r,\theta,\phi) = \frac{ma^3r\sin\theta\cos(\phi-\phi_0)}{4\pi d^3[r^2+(a^2/d)^2-2r(a^2/d)\cos\theta]^{3/2}} - \frac{mr\sin\theta\cos(\phi-\phi_0)}{4\pi ad} \times\int_0^{a^2/d}\frac{u\,du}{(r^2+u^2-2ru\cos\theta)^{3/2}}. \tag{12b}$$

Then the magnetic induction $\boldsymbol{B}'$ of the images can be derived, and the levitation force on the source dipole can be found to be

$$\boldsymbol{F} = \nabla(\boldsymbol{m}\cdot\boldsymbol{B}')|_{\boldsymbol{r}=\boldsymbol{d}} = \frac{3\mu_0 m^2}{4\pi}\left[\frac{a^3 d}{(d^2-a^2)^4} - \frac{a^3(3d^2-a^2)}{6d^3(d^2-a^2)^3}\right]\boldsymbol{e}_z. \tag{13}$$

By solving the field equation the same result can be obtained. The conclusion of Refs. 17 and 18 that the levitation force in this case is half that for a radial dipole is thus incorrect. It holds approximately only when $d-a\ll a$ so that the sphere can be approximately regarded as an infinite plane for which the conclusion holds exactly. The potential energy of the dipole at $\boldsymbol{d}$ can be found to be

$$U(d) = \frac{\mu_0 m^2}{16\pi}\frac{a^3(d^2+a^2)}{d^2(d^2-a^2)^3} = -\frac{1}{2}\boldsymbol{m}\cdot\boldsymbol{B}'(\boldsymbol{d}), \tag{14}$$

where the first equality is obtained by an integral similar to that in Eq. (11) (the calculation is somewhat tedious) and the last one can be verified straightforwardly. In passing we record a simpler form for Eq. (12b):

$$\varphi'(r,\theta,\phi)=\frac{ma^3 r\sin\theta\cos(\phi-\phi_0)}{4\pi d^3[r^2+(a^2/d)^2-2r(a^2/d)\cos\theta]^{3/2}}-\frac{m\cos(\phi-\phi_0)}{4\pi ad\sin\theta}\times\left[1-\frac{r-(a^2/d)\cos\theta}{\sqrt{r^2+(a^2/d)^2-2r(a^2/d)\cos\theta}}\right]. \tag{12b'}$$

The situation is similar to that for Eq. (7), and similar remarks also apply here.

With the results of this and the last section, one can deal with a magnetic dipole with arbitrary direction by superposition.

D. Uniform magnetic field

As another example of the image method we consider the influence of the superconducting sphere on a uniform magnetic field $\boldsymbol{H}_0=H_0\boldsymbol{e}_z$ where H_0 is a constant.

The uniform magnetic field can be realized by two large monopoles at infinity. We put one monopole with charge $-Q$ at $(0,0,L)$ and the other with charge Q at $(0,0,-L)$. The magnetic scalar potential is

$$\varphi_0(r,\theta)=-\frac{Q}{4\pi\sqrt{r^2+L^2-2rL\cos\theta}}+\frac{Q}{4\pi\sqrt{r^2+L^2+2rL\cos\theta}}=-\frac{Q}{2\pi L^2}\sum_{k=0}^{\infty}\frac{r^{2k+1}}{L^{2k}}P_{2k+1}(\cos\theta), \tag{15}$$

where the last expression holds for $r<L$. Let $Q,L\to\infty$ but $Q/2\pi L^2=H_0$ fixed, then only the term with $k=0$ contributes and we have for any finite r

$$\varphi_0(r,\theta)=-H_0 r\cos\theta=-H_0 z,\quad \boldsymbol{H}_0=H_0\boldsymbol{e}_z, \tag{16}$$

that is, a uniform magnetic field.

Now introduce the superconducting sphere, then the field is changed by the induced current on the spherical surface. The image of the charge $-Q$ is a point monopole with charge $-Qa/L$ located at $(0,0,a^2/L)$ plus a uniform line charge with linear density Q/a distributed on the z axis from the origin to $(0,0,a^2/L)$; the image of the charge Q is a point monopole with charge Qa/L located at $(0,0,-a^2/L)$ plus a uniform line charge with linear density $-Q/a$ distributed on the z axis from the origin to $(0,0,-a^2/L)$. In the above limit, the effect of the two point images is a point dipole with dipole moment $\boldsymbol{m}'=-4\pi H_0 a^3\boldsymbol{e}_z$ located at the origin. The effect of the two line charge is also a point dipole with dipole moment $\boldsymbol{m}''=2\pi H_0 a^3\boldsymbol{e}_z$ at the origin. The latter is calculated by $(Q/a\int_0^{a^2/L}u\,du-Q/a\int_{-a^2/L}^{0}u\,du)\boldsymbol{e}_z=(Qa^3/L^2)\boldsymbol{e}_z=2\pi H_0 a^3\boldsymbol{e}_z$, or by writing down the scalar potential and then taking the limit. Therefore the total effect of all the images is a point dipole with dipole moment $\boldsymbol{m}=-2\pi H_0 a^3\boldsymbol{e}_z$ located at the origin. The scalar potential outside the sphere is then

$$\varphi(r,\theta)=-H_0 r\cos\theta-\frac{1}{2}H_0 a^3\frac{\cos\theta}{r^2}. \tag{17}$$

This is of course the same as the result in the textbooks (for example, Ref. 19) obtained by solving the field equation. Compared with the case of a conducting sphere in a uniform electric field (for which the image method can be found in Ref. 1), the difference of the result is caused by two reasons: first, the point image has the same sign with the source monopole; second, there are further contributions from the line image. From the distribution of the images, it is obvious that there is no levitation force on the superconducting sphere in this case.

E. Steady electric current

Consider a circular current loop with electric current I, whose position is described by the spherical coordinates: $r=b$, $\theta=\theta_0$, $0\leqslant\phi<2\pi$, where $b>a$. The vector potential can be found to be

$$\boldsymbol{A}_0(\boldsymbol{r})=\boldsymbol{e}_\phi\frac{\mu_0 Ib\sin\theta_0}{4\pi}\int_0^{2\pi}\frac{\cos\xi d\xi}{\sqrt{r^2+b^2-2rb(\cos\theta_0\cos\theta+\sin\theta_0\sin\theta\cos\xi)}}\equiv A_0(r,\theta)\boldsymbol{e}_\phi. \tag{18}$$

When $\theta_0=\pi/2$ this reduces to the result in the textbooks.[1,19] To include the contribution of the induced current on the spherical surface, we consider a circular current loop inside the sphere as the image of the source current, with the above parameters b, θ_0, and I replaced by b', θ_0', and I', respectively (where $b'<a$). Then we have a similar expression for the vector potential of the image, $\boldsymbol{A}'(\boldsymbol{r})=A'(r,\theta)\boldsymbol{e}_\phi$. The total vector potential is $\boldsymbol{A}(\boldsymbol{r})=A(r,\theta)\boldsymbol{e}_\phi$ where $A(r,\theta)=A_0(r,\theta)+A'(r,\theta)$. Now the boundary condition $B_r|_{r=a}=0$ has the form

$$\left.\frac{\partial}{\partial\theta}[\sin\theta A(r,\theta)]\right|_{r=a}=\frac{d}{d\theta}[\sin\theta A(a,\theta)]=0. \tag{19}$$

This should hold for all θ and thus requires $A(a,\theta)=0$. Therefore the parameters of the image current are found to be

$$\theta_0' = \theta_0, \quad b' = \frac{a^2}{b}, \quad I' = -\frac{b}{a}I. \tag{20}$$

Consequently the field outside the sphere is determined. Note that the coefficient in the last relation is different from the ordinary one in the image method for a conducting sphere in electrostatics. This is because the parameter b appears in the numerator of Eq. (18).

The levitation force on the circular current loop can be found to be

$$\boldsymbol{F} = \boldsymbol{e}_z \mu_0 I^2 a b^3 (b^2 - a^2) \cos\theta_0 \sin^2\theta_0 \int_0^{\pi} \frac{\cos\xi d\xi}{[a^4 + b^4 - 2a^2 b^2 (\cos^2\theta_0 + \sin^2\theta_0 \cos\xi)]^{3/2}}. \tag{21}$$

This can be expressed in terms of elliptic integrals, or expanded as a series involving Legendre polynomials, but the result is not instructive. In the limit $\theta_0 \to 0$ and $I \to \infty$ but $I\pi b^2 \sin^2\theta_0 = m$ is fixed, this reduces to the previous result (10) (with d replaced by b) as expected.

For an arbitrary distribution of source current, it is difficult to work out the field by the image method. However, with the above result, one can work out the vector potential for any source current that can be constructed by superposition of circular current loops whose axes pass through the origin. In particular, for an electric current system with the current density

$$\boldsymbol{J}(\boldsymbol{r}) = J(r,\theta)\boldsymbol{e}_\phi, \quad \boldsymbol{r} \in \Omega, \tag{22}$$

where Ω is the region in which $\boldsymbol{J}$ is nonzero, the vector potential can be explicitly given. In fact, by replacing I by $J(r_0,\theta_0) r_0 dr_0 d\theta_0$ and ξ by ϕ_0 in the above vector potential, and integrating over the source region we have

$$\boldsymbol{A}(\boldsymbol{r}) = A(r,\theta)\boldsymbol{e}_\phi, \tag{23a}$$

where

$$A(r,\theta) = \frac{\mu_0}{4\pi}\left[\int_\Omega \frac{J(r_0,\theta_0)\cos\phi_0 d\boldsymbol{r}_0}{|\bar{\boldsymbol{r}} - \boldsymbol{r}_0|} - \int_\Omega \frac{aJ(r_0,\theta_0)\cos\phi_0 d\boldsymbol{r}_0}{r_0|\bar{\boldsymbol{r}} - (a/r_0)^2\boldsymbol{r}_0|}\right], \tag{23b}$$

where $d\boldsymbol{r}_0 = r_0^2 \sin\theta_0 dr_0 d\theta_0 d\phi_0$, and $\bar{\boldsymbol{r}} = \boldsymbol{r}|_{\phi=0} = r(\sin\theta, 0, \cos\theta)$. The first term in the above equation is generated by the source current and the second by the image one. With this result one can work out the magnetic induction, and calculate the levitation force, etc.

III. IMAGE METHOD FOR A LONG SUPERCONDUCTING CYLINDER

A. Magnetic monopole

The case of magnetic sources in the presence of a long superconducting cylinder seems to be less studied. As before, we first consider a long magnetic line charge parallel to the axis of the cylinder. For convenience we also call the line charge a monopole since it is a point charge from a two-dimensional viewpoint.

We choose the coordinate system such that the z axis coincides with the axis of the cylinder. In all subsequent discussions of this section everything is independent of the coordinate z, so we need only the rectangular coordinates (x,y) and the polar ones (ρ,ϕ) on the xy plane (but the unit vector $\boldsymbol{e}_z$ will be used occasionally). The unit vectors in the polar coordinate system are denoted by $(\boldsymbol{e}_\rho, \boldsymbol{e}_\phi)$. The position vector on the xy plane is denoted by $\boldsymbol{\rho}$. As before, source and image positions are always given in rectangular coordinates. The radius of the cylinder is a. The line charge passes through the xy plane at the point $(d,0)$ where $d > a$ [equivalently we will say that the monopole is located at $(d,0)$], and the linear density of the magnetic line charge is Q (also called the charge of the monopole). The scalar potential outside the cylinder contains two parts:

$$\varphi(\rho,\phi) = \varphi_Q(\rho,\phi) + \varphi'(\rho,\phi), \tag{24}$$

where

$$\varphi_Q(\rho,\phi) = -\frac{Q}{2\pi}\ln|\boldsymbol{\rho} - \boldsymbol{d}| = -\frac{Q}{2\pi}\ln\sqrt{\rho^2 + d^2 - 2\rho d\cos\phi}, \tag{25}$$

where $\boldsymbol{d} = d\boldsymbol{e}_x$ is the scalar potential of the monopole, and $\varphi'(\rho,\phi)$ is the one generated by the induced current on the cylindrical surface. The latter satisfies the Laplace equation and has the form

$$\varphi'(\rho,\phi) = a_0 \ln\rho + \sum_{n=1}^{\infty} a_n \rho^{-n}\cos n\phi, \tag{26}$$

up to a constant which does not contribute to the magnetic induction. The geometric symmetry has been taken into account so that sine terms do not appear in the above series. Expanding $\varphi_Q(\rho,\phi)$ as a Fourier series near $\rho = a$ and using the boundary condition

$$\left.\frac{\partial\varphi}{\partial\rho}\right|_{\rho=a} = 0, \tag{27}$$

we obtain the coefficients a_n in Eq. (26). The result is

$$a_0 = 0, \quad a_n = \frac{Q}{2\pi}\frac{1}{n}\left(\frac{a^2}{d}\right)^n, \quad n = 1,2,\ldots . \tag{28}$$

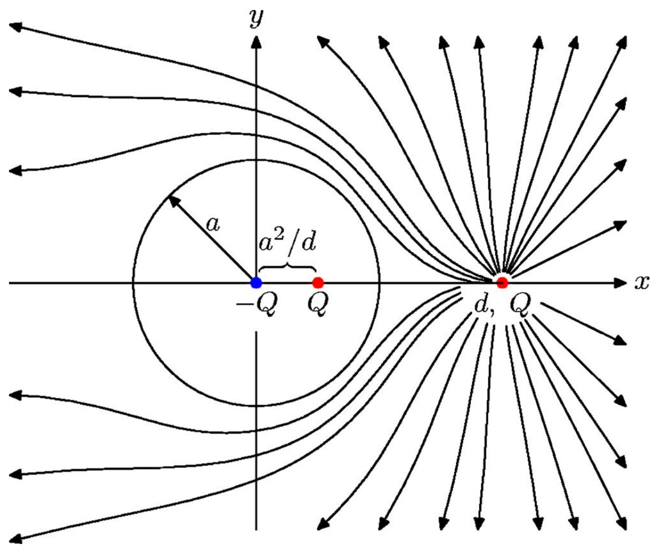

FIG. 4. (Color online) The magnetic induction lines for a two-dimensional magnetic monopole (a long magnetic line charge) in the presence of a long superconducting cylinder. The source and image monopoles are also shown.

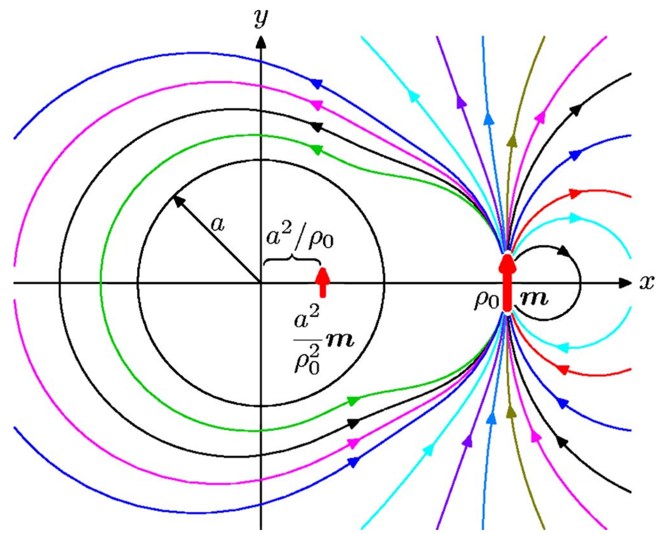

FIG. 5. (Color online) The magnetic induction lines for a two-dimensional magnetic dipole (continuous distribution of point dipoles on a long straight line) with transverse direction in the presence of a long superconducting cylinder. The source and image dipoles are also shown.

It is straightforward to verify that Eq. (26) with the coefficients given in Eq. (28) can be recast in the form

$$\varphi'(\rho,\phi) = \frac{Q}{2\pi}\ln\rho - \frac{Q}{2\pi}\ln\left|\boldsymbol{\rho} - \frac{a^2}{d}\boldsymbol{e}_x\right|. \tag{29}$$

This means that for a source monopole with charge Q located at $(d,0)$, the image contains two monopoles: one with charge Q located at the ordinary image point $(a^2/d,0)$, and the other with charge $-Q$ located at the origin. Again note that the monopole at the image point has the same sign as the source one, and the total magnetic charge inside the cylinder vanishes as expected. This result is rather different from that in Sec. II for a superconducting sphere. The levitation force acted on the monopole (actually it is the linear force density on the line charge) is

$$\boldsymbol{F} = \frac{\mu_0 Q^2 a^2}{2\pi d(d^2-a^2)}\boldsymbol{e}_x. \tag{30}$$

This can be obtained by $\boldsymbol{F}=Q\boldsymbol{B}'(\boldsymbol{d})$ where $\boldsymbol{B}'(\boldsymbol{\rho})$ is the magnetic induction of the image charges.

The magnetic induction lines are illustrated in Fig. 4, where the source and images are also displayed. The field lines are similar to those in Fig. 1, but note that in the three-dimensional case the field lines in the whole space are obtained by rotating Fig. 1 about the z axis, while for the current case they are obtained by translating Fig. 4 along the z axis.

B. Magnetic dipole

Now we consider a two-dimensional magnetic dipole with dipole moment $\boldsymbol{m}$ located at $\boldsymbol{\rho}_0$ (where $\rho_0 > a$). This is of physical interest since it can be realized by a long cylinder uniformly magnetized in the transverse direction. Theoretically the dipole can be realized by two monopoles, one with charge $-Q$ located at $\boldsymbol{\rho}_0$ and the other with charge Q located at $\boldsymbol{\rho}_0+\boldsymbol{l}$, where $\boldsymbol{l}\to 0$ and $Q\to\infty$ but $Q\boldsymbol{l}=\boldsymbol{m}$ is fixed. The image of the monopole $-Q$ at $\boldsymbol{\rho}_0$ is a monopole with charge $-Q$ at $(a^2/\rho_0^2)\boldsymbol{\rho}_0$ and one with charge Q at the origin; the image of the monopole Q at $\boldsymbol{\rho}_0+\boldsymbol{l}$ is a monopole with charge Q at $(a^2/|\boldsymbol{\rho}_0+\boldsymbol{l}|^2)(\boldsymbol{\rho}_0+\boldsymbol{l})$ and one with charge $-Q$ at the origin. The two monopoles at the origin cancel. Therefore the image is a magnetic dipole at $(a^2/\rho_0^2)\boldsymbol{\rho}_0$, with dipole moment

$$\boldsymbol{m}' = \frac{a^2}{\rho_0^2}\left[\boldsymbol{m} - \frac{2(\boldsymbol{\rho}_0\cdot\boldsymbol{m})\boldsymbol{\rho}_0}{\rho_0^2}\right]. \tag{31}$$

For a source dipole with radial direction, $\boldsymbol{m}=m\boldsymbol{\rho}_0/\rho_0$ (where m may be either positive or negative), we have $\boldsymbol{m}'=-(a^2/\rho_0^2)\boldsymbol{m}$. On the other hand, for one with transverse direction, $\boldsymbol{m}\cdot\boldsymbol{\rho}_0=0$, and we have $\boldsymbol{m}'=(a^2/\rho_0^2)\boldsymbol{m}$. For the radial case the result is similar to the corresponding one in Sec. II while for the transverse case it is rather different.

The magnetic induction lines for the radial case are similar to those in Fig. 2, but notice the remarks for Fig. 4. For the transverse case they are illustrated in Fig. 5, where the source and image dipoles are also displayed.

The magnetic scalar potential of the source dipole is

$$\varphi_m(\boldsymbol{\rho}) = \frac{\boldsymbol{m}\cdot(\boldsymbol{\rho}-\boldsymbol{\rho}_0)}{2\pi|\boldsymbol{\rho}-\boldsymbol{\rho}_0|^2}, \tag{32a}$$

and the magnetic induction is

$$\boldsymbol{B}_m(\boldsymbol{\rho}) = \frac{\mu_0}{2\pi}\frac{2[\boldsymbol{m}\cdot(\boldsymbol{\rho}-\boldsymbol{\rho}_0)](\boldsymbol{\rho}-\boldsymbol{\rho}_0) - |\boldsymbol{\rho}-\boldsymbol{\rho}_0|^2\boldsymbol{m}}{|\boldsymbol{\rho}-\boldsymbol{\rho}_0|^4}. \tag{32b}$$

For the image dipole we have similar expressions. The levitation force can be found to be

$$\boldsymbol{F}(\boldsymbol{\rho}_0) = \nabla(\boldsymbol{m}\cdot\boldsymbol{B}')|_{\boldsymbol{\rho}=\boldsymbol{\rho}_0} = \frac{\mu_0 m^2}{\pi}\frac{a^2\boldsymbol{\rho}_0}{(\rho_0^2-a^2)^3}, \qquad (33)$$

where $m=|\boldsymbol{m}|$ and $\boldsymbol{B}'(\boldsymbol{\rho})$ is the magnetic induction of the image dipole. The calculation would be easier if one set $\boldsymbol{\rho}_0 = \rho_0\boldsymbol{e}_x$ and $\boldsymbol{m}=m_1\boldsymbol{e}_x+m_2\boldsymbol{e}_y$, which yields $\boldsymbol{m}'=(a^2/\rho_0^2)(-m_1\boldsymbol{e}_x + m_2\boldsymbol{e}_y)$. The levitation force does not depend on the direction of the dipole. This is also different from the result in Sec. II.

The potential energy of the dipole can be easily found to be

$$U(\boldsymbol{\rho}_0) = \frac{\mu_0 m^2}{4\pi}\frac{a^2}{(\rho_0^2-a^2)^2} = -\frac{1}{2}\boldsymbol{m}\cdot\boldsymbol{B}'(\boldsymbol{\rho}_0). \qquad (34)$$

The first equality holds because $-\nabla_{\boldsymbol{\rho}_0}U(\boldsymbol{\rho}_0)=\boldsymbol{F}(\boldsymbol{\rho}_0)$, while the second can be verified straightforwardly.

C. Uniform magnetic field

As in Sec. II we consider the influence of the superconducting cylinder on a uniform magnetic field $\boldsymbol{H}_0=H_0\boldsymbol{e}_x$ where H_0 is a constant.

The uniform magnetic field can be realized by two large monopoles at infinity. We put one monopole with charge $-Q$ at $(L,0)$ and the other with charge Q at $(-L,0)$. The magnetic scalar potential is

$$\begin{aligned}\varphi_0(\rho,\phi) &= \frac{Q}{2\pi}\ln\sqrt{\rho^2+L^2-2\rho L\cos\phi} \\ &\quad - \frac{Q}{2\pi}\ln\sqrt{\rho^2+L^2+2\rho L\cos\phi} \\ &= -\frac{Q}{\pi L}\sum_{k=0}^{\infty}\frac{\rho^{2k+1}}{(2k+1)L^{2k}}\cos(2k+1)\phi, \qquad (35)\end{aligned}$$

where the last expression holds for $\rho<L$. Let $Q,L\to\infty$ but $Q/\pi L=H_0$ fixed, then only the term with $k=0$ contributes and we have for any finite ρ

$$\varphi_0(\rho,\phi) = -H_0\rho\cos\phi = -H_0 x, \quad \boldsymbol{H}_0 = H_0\boldsymbol{e}_x, \qquad (36)$$

that is, a uniform magnetic field.

Now introduce the superconducting cylinder, then the field is changed by the induced current on the cylindrical surface. The image of the charge $-Q$ is two point monopoles, one with charge $-Q$ located at $(a^2/L,0)$ and the other with charge Q at the origin; the image of the charge Q is also two point monopoles, one with charge Q located at $(-a^2/L,0)$ and the other with charge $-Q$ at the origin. The two monopoles at the origin cancel. The effect of the other two monopoles is in the above limit a point dipole with dipole moment $\boldsymbol{m}=-2\pi H_0 a^2\boldsymbol{e}_x$, located at the origin. The scalar potential outside the cylinder is then

$$\varphi(\rho,\phi) = -H_0\rho\cos\phi - H_0 a^2\frac{\cos\phi}{\rho}. \qquad (37)$$

Compared with the case of a conducting cylinder in a uniform electric field, there is a sign difference in the second term of the above result because the monopole at the image point has the same sign as the source one. From the distribution of the images, it is obvious that there is no levitation force on the superconducting cylinder.

D. Steady electric current

First we consider an electric current of the form $\boldsymbol{J}(\rho) = J(\rho)\boldsymbol{e}_\phi$ outside the cylinder. More specifically, we consider the case with $J(\rho)=K\delta(\rho-d)$. This can be realized by a long solenoid of radius d (where $d>a$ as before) and with electric current K per unit length. The magnetic induction generated by this current is $\boldsymbol{B}=\mu_0 K\vartheta(d-\rho)\boldsymbol{e}_z$ where ϑ is the step function. To cancel this field inside the cylinder, there should be an induced current $\boldsymbol{J}'(\rho)=J'(\rho)\boldsymbol{e}_\phi$ with $J'(\rho)=-K\delta(\rho-a)$ on the surface of the cylinder. However, this current has no effect on the field outside the cylinder. Therefore the field outside the cylinder is the one generated by the source current, and no image is needed. This is also true for a more general $J(\rho)$.

Next we consider a long straight wire parallel to the z axis carrying an electric current I. It passes through the xy plane at the point $\boldsymbol{\rho}_0$ (where $\rho_0>a$ as before). The current density is $\boldsymbol{J}(\boldsymbol{\rho})=I\delta(\boldsymbol{\rho}-\boldsymbol{\rho}_0)\boldsymbol{e}_z$. If $\boldsymbol{\rho}_0=0$, the magnetic induction is well-known

$$\boldsymbol{B}_0(\boldsymbol{\rho}) = \frac{\mu_0 I}{2\pi\rho}\boldsymbol{e}_\phi = \boldsymbol{e}_z\times\left(\frac{\mu_0 I}{2\pi}\frac{\boldsymbol{\rho}}{\rho^2}\right) = \boldsymbol{e}_z\times\nabla\left(\frac{\mu_0 I}{2\pi}\ln\rho\right).$$

Thus the magnetic induction for the above current is

$$\boldsymbol{B}_0(\boldsymbol{\rho}) = \boldsymbol{e}_z\times\nabla\left(\frac{\mu_0 I}{2\pi}\ln|\boldsymbol{\rho}-\boldsymbol{\rho}_0|\right). \qquad (38)$$

Now we consider an image current inside the cylinder with the parameters I and $\boldsymbol{\rho}_0$ replaced by I' and $\boldsymbol{\rho}_0'$ (where $\rho_0' < a$), respectively. We have a similar expression for the magnetic induction $\boldsymbol{B}'(\boldsymbol{\rho})$ of the image current. The total magnetic induction is

$$\begin{aligned}\boldsymbol{B}(\boldsymbol{\rho}) &= \boldsymbol{e}_z\times\nabla\left(\frac{\mu_0 I}{2\pi}\ln|\boldsymbol{\rho}-\boldsymbol{\rho}_0| + \frac{\mu_0 I'}{2\pi}\ln|\boldsymbol{\rho}-\boldsymbol{\rho}_0'|\right) \\ &\equiv \boldsymbol{e}_z\times\nabla\Phi(\boldsymbol{\rho}). \qquad (39)\end{aligned}$$

To satisfy the boundary condition $B_\rho|_{\rho=a}=0$, we should have $\partial_\phi\Phi(\rho,\phi)|_{\rho=a}=\partial_\phi\Phi(a,\phi)=0$. This requires that $\Phi(a,\phi)$ is a constant. Since $\Phi(\boldsymbol{\rho})$ has the form of the scalar potential for two electric point charges in two dimensions, from the image method in electrostatics, we know that the boundary condition is satisfied if

$$\boldsymbol{\rho}_0' = \frac{a^2}{\rho_0^2}\boldsymbol{\rho}_0, \quad I' = -I. \qquad (40)$$

In order to generate the same field as the induced supercurrent, we still need another image current I along the z axis. Note that we have implicitly assumed that the superconducting cylinder is insulated to external current sources so that the total supercurrent in the z direction should be zero.

For a more general current distribution outside the cylinder

$$\boldsymbol{J}(\boldsymbol{\rho}) = J(\boldsymbol{\rho})\boldsymbol{e}_z, \quad \boldsymbol{\rho} \in \Omega, \tag{41}$$

where Ω is the region on the xy plane where $J(\boldsymbol{\rho})$ is nonzero, the result may be obtained from the above one by replacing I by $J(\boldsymbol{\rho}_0)d\boldsymbol{\rho}_0$, and integrating over the source region:

$$\boldsymbol{B}(\boldsymbol{\rho}) = \frac{\mu_0}{2\pi}\boldsymbol{e}_z \times \nabla\left[\int_\Omega J(\boldsymbol{\rho}_0)\ln|\boldsymbol{\rho}-\boldsymbol{\rho}_0|d\boldsymbol{\rho}_0 - \int_\Omega J(\boldsymbol{\rho}_0)\ln\left|\boldsymbol{\rho}-\frac{a^2}{\rho_0^2}\boldsymbol{\rho}_0\right|d\boldsymbol{\rho}_0 + \ln\rho\int_\Omega J(\boldsymbol{\rho}_0)d\boldsymbol{\rho}_0\right], \tag{42}$$

where the first term is the contribution of the source current and the subsequent terms the images.

The levitation force (per unit length) on the source current is

$$\begin{aligned}\boldsymbol{F} &= \int_\Omega J(\boldsymbol{\rho})\boldsymbol{e}_z \times \boldsymbol{B}'(\boldsymbol{\rho})d\boldsymbol{\rho} \\ &= \frac{\mu_0}{2\pi}\int_\Omega J(\boldsymbol{\rho})J(\boldsymbol{\rho}')\frac{\boldsymbol{\rho}-(a^2/\rho'^2)\boldsymbol{\rho}'}{|\boldsymbol{\rho}-(a^2/\rho'^2)\boldsymbol{\rho}'|^2}d\boldsymbol{\rho}d\boldsymbol{\rho}' \\ &\quad - \frac{\mu_0}{2\pi}\int_\Omega J(\boldsymbol{\rho})d\boldsymbol{\rho}\int_\Omega \frac{\boldsymbol{\rho}'J(\boldsymbol{\rho}')}{\boldsymbol{\rho}'^2}d\boldsymbol{\rho}'.\end{aligned} \tag{43}$$

For the simple case of a long straight wire considered above, $J(\boldsymbol{\rho})=I\delta(\boldsymbol{\rho}-\boldsymbol{\rho}_0)$, the above result reduces to

$$\boldsymbol{F} = \frac{\mu_0 I^2}{2\pi}\frac{a^2\boldsymbol{\rho}_0}{\rho_0^2(\rho_0^2-a^2)}. \tag{44}$$

IV. IMAGE METHOD FOR A SUPERCONDUCTING PLANE

The various magnetostatic boundary value problems involving a superconducting plane have been studied by many authors, for example, Refs. 20–24. The image method for this case is simple and appears to be well-known.[2] We first review the result for a monopole and a dipole, which will be used in subsequent sections. Then we derive the result for an arbitrary electric current.

Assume that the superconductor occupies the half space $z<0$, and the half space $z>0$ is vacuum. The boundary condition in this case is $B_z|_{z=0}=0$. For a monopole with charge Q, a longitudinal dipole with moment $\boldsymbol{m}=m\boldsymbol{e}_z$ or a transverse one with $\boldsymbol{m}=m_1\boldsymbol{e}_x+m_2\boldsymbol{e}_y$, located at $(0,0,d)$, the image is a monopole with charge Q, a dipole with moment $\boldsymbol{m}'=-\boldsymbol{m}$ or one with $\boldsymbol{m}'=\boldsymbol{m}$, respectively, located at the image point $(0,0,-d)$. The levitation force on the transverse dipole is half that on the longitudinal one.

Now consider an electric current with an arbitrary current density

$$\boldsymbol{J}(\boldsymbol{r}) = J_1(\boldsymbol{r})\boldsymbol{e}_x + J_2(\boldsymbol{r})\boldsymbol{e}_y + J_3(\boldsymbol{r})\boldsymbol{e}_z, \tag{45}$$

distributed in a region Ω in the upper half space. We consider an image current in the image region Ω' (obtained from the region Ω by a reflection about the xy plane) with current density

$$\boldsymbol{J}'(\boldsymbol{r}) = -J_1(\bar{\boldsymbol{r}})\boldsymbol{e}_x - J_2(\bar{\boldsymbol{r}})\boldsymbol{e}_y + J_3(\bar{\boldsymbol{r}})\boldsymbol{e}_z, \quad \boldsymbol{r} \in \Omega', \tag{46}$$

where $\bar{\boldsymbol{r}}=(x,y,-z)$ if $\boldsymbol{r}=(x,y,z)$. The vector potential in the upper half space is then given by

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\int_\Omega \frac{\boldsymbol{J}(\boldsymbol{r}_0)}{|\boldsymbol{r}-\boldsymbol{r}_0|}d\boldsymbol{r}_0 + \frac{\mu_0}{4\pi}\int_\Omega \frac{-J_1(\boldsymbol{r}_0)\boldsymbol{e}_x - J_2(\boldsymbol{r}_0)\boldsymbol{e}_y + J_3(\boldsymbol{r}_0)\boldsymbol{e}_z}{|\boldsymbol{r}-\bar{\boldsymbol{r}}_0|}d\boldsymbol{r}_0, \tag{47}$$

where in the second term we have made a replacement of integral variable $z_0 \to -z_0$ so that the second integral is also over the region Ω. It is straightforward to verify that this satisfies all field equations $\nabla\cdot\boldsymbol{A}=0$ and $\nabla^2\boldsymbol{A}=-\mu_0\boldsymbol{J}$, and the boundary condition. Therefore it is a correct solution in the upper half space.

Of special interest is a circular current loop studied in Sec. II, which carries current I and has the position described by the spherical coordinates: $r=b$, $\theta=\theta_0$, $0\leqslant\phi<2\pi$ (where now $\theta_0<\pi/2$). From Eq. (46), we know that the image is also a circular current loop, which carries current $-I$ and has the position described by the spherical coordinates: $r=b$, $\theta=\pi-\theta_0$, $0\leqslant\phi<2\pi$. The levitation force on the source current loop can be found to be

$$\boldsymbol{F} = \boldsymbol{e}_z\frac{\mu_0 I^2\cos\theta_0\sin^2\theta_0}{\sqrt{2}}\int_0^\pi \frac{\cos\xi d\xi}{(1+\cos^2\theta_0-\sin^2\theta_0\cos\xi)^{3/2}}. \tag{48}$$

This can be expressed in terms of elliptic integrals, but the result is not instructive. In the limit $\theta_0\to 0$ and $I\to\infty$ but $I\pi b^2\sin^2\theta_0=m$ is fixed, this reduces to the result for a longitudinal dipole as expected.

V. MORE COMPLICATED BOUNDARY IN THREE DIMENSIONS

In this section we consider more complicated geometries involving both spherical and planar superconducting boundaries. We mainly deal with the case where the superconductor occupies the lower half space $z<0$ plus the upper half of the sphere $r<a$. Other cases will be discussed briefly. It is inconvenient to solve such boundary value problems by the ordinary method of separation of variables in mathematical physics. However, in the image method they are simple. The boundary condition is that the normal component of the magnetic induction should vanish at all boundaries.

First of all consider a monopole with charge Q located at $\boldsymbol{r}_1$ whose spherical coordinates are (r_0,θ_0,ϕ_0) (where $\theta_0<\pi/2$). According to the results of the preceding sections, the image contains five parts: three monopoles, one with charge Q located at $\boldsymbol{r}_2$ with spherical coordinates $(r_0,\pi-\theta_0,\phi_0)$, the other two both with charge Qa/r_0 located at $\boldsymbol{r}_1'=(a^2/r_0^2)\boldsymbol{r}_1$ and $\mathbf{r}_2'=(a^2/r_0^2)\boldsymbol{r}_2$; and two continuous distributions of magnetic charges on straight lines, one from the origin to $\boldsymbol{r}_1'$ and the other from the origin to $\boldsymbol{r}_2'$, both with linear density $-Q/a$. The magnetic scalar potential in the vacuum region satisfying the boundary condition is

$$\varphi(\boldsymbol{r}) = \frac{Q}{4\pi|\boldsymbol{r}-\boldsymbol{r}_1|} + \frac{Q}{4\pi|\boldsymbol{r}-\boldsymbol{r}_2|} + \frac{Qa}{4\pi r_0|\boldsymbol{r}-(a^2/r_0^2)\boldsymbol{r}_1|} + \frac{Qa}{4\pi r_0|\boldsymbol{r}-(a^2/r_0^2)\boldsymbol{r}_2|} - \int_0^{a^2/r_0} \frac{Q du}{4\pi a|\boldsymbol{r}-u\boldsymbol{e}_1|} - \int_0^{a^2/r_0} \frac{Q du}{4\pi a|\boldsymbol{r}-u\boldsymbol{e}_2|}, \tag{49}$$

where $\boldsymbol{e}_1=\boldsymbol{r}_1/r_0$ and $\boldsymbol{e}_2=\boldsymbol{r}_2/r_0$ are unit vectors in the directions of $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$, respectively.

Now consider magnetic dipoles. First we consider a point dipole with radial direction. It is located at $\boldsymbol{r}_1$ and has the dipole moment $\boldsymbol{m}_1=m\boldsymbol{e}_1$. The image contains three point dipoles, the first with moment $\boldsymbol{m}_2=m\boldsymbol{e}_2$ located at $\boldsymbol{r}_2$, the second with moment $\boldsymbol{m}_1'=-(a/r_0)^3\boldsymbol{m}_1$ located at $\boldsymbol{r}_1'$, and the third with moment $\boldsymbol{m}_2'=-(a/r_0)^3\boldsymbol{m}_2$ located at $\boldsymbol{r}_2'$. The scalar potential is then

$$\varphi_1(\boldsymbol{r}) = \frac{m(\boldsymbol{r}\cdot\boldsymbol{e}_1-r_0)}{4\pi|\boldsymbol{r}-\boldsymbol{r}_1|^3} + \frac{m(\boldsymbol{r}\cdot\boldsymbol{e}_2-r_0)}{4\pi|\boldsymbol{r}-\boldsymbol{r}_2|^3} - \frac{ma^3(\boldsymbol{r}\cdot\boldsymbol{e}_1-a^2/r_0)}{4\pi r_0^3|\boldsymbol{r}-(a^2/r_0^2)\boldsymbol{r}_1|^3} - \frac{ma^3(\boldsymbol{r}\cdot\boldsymbol{e}_2-a^2/r_0)}{4\pi r_0^3|\boldsymbol{r}-(a^2/r_0^2)\boldsymbol{r}_2|^3}. \tag{50}$$

Second we consider a point dipole with one of the two independent transverse directions. It is located at $\boldsymbol{r}_1$ and has the dipole moment $\boldsymbol{m}_1=m\boldsymbol{e}_0$, where $\boldsymbol{e}_0=-\sin\phi_0\boldsymbol{e}_x+\cos\phi_0\boldsymbol{e}_y$ is the unit vector $\boldsymbol{e}_\phi$ at $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$. The image contains five parts: three point dipoles, the first with moment $\boldsymbol{m}_2=\boldsymbol{m}_1$ located at $\boldsymbol{r}_2$, the second with moment $\boldsymbol{m}_1'=(a/r_0)^3\boldsymbol{m}_1$ located at $\boldsymbol{r}_1'$, and the third with moment $\boldsymbol{m}_2'=(a/r_0)^3\boldsymbol{m}_2=(a/r_0)^3\boldsymbol{m}_1$ located at $\boldsymbol{r}_2'$; and two continuous distributions of point dipoles on straight lines, one from the origin to $\boldsymbol{r}_1'$, the dipole moment from $u\boldsymbol{e}_1$ to $(u+du)\boldsymbol{e}_1$ being $-(\boldsymbol{m}/ar_0)udu$, and the other from the origin to $\boldsymbol{r}_2'$, the dipole moment from $u\boldsymbol{e}_2$ to $(u+du)\boldsymbol{e}_2$ being also $-(\boldsymbol{m}/ar_0)udu$. The scalar potential is then

$$\varphi_2(\boldsymbol{r}) = \frac{m\boldsymbol{r}\cdot\boldsymbol{e}_0}{4\pi|\boldsymbol{r}-\boldsymbol{r}_1|^3} + \frac{m\boldsymbol{r}\cdot\boldsymbol{e}_0}{4\pi|\boldsymbol{r}-\boldsymbol{r}_2|^3} + \frac{ma^3\boldsymbol{r}\cdot\boldsymbol{e}_0}{4\pi r_0^3|\boldsymbol{r}-(a^2/r_0^2)\boldsymbol{r}_1|^3} + \frac{ma^3\boldsymbol{r}\cdot\boldsymbol{e}_0}{4\pi r_0^3|\boldsymbol{r}-(a^2/r_0^2)\boldsymbol{r}_2|^3} - \int_0^{a^2/r_0} \frac{m\boldsymbol{r}\cdot\boldsymbol{e}_0}{4\pi a r_0|\boldsymbol{r}-u\boldsymbol{e}_1|^3}udu - \int_0^{a^2/r_0} \frac{m\boldsymbol{r}\cdot\boldsymbol{e}_0}{4\pi a r_0|\boldsymbol{r}-u\boldsymbol{e}_2|^3}udu. \tag{51}$$

Third we consider a point dipole with another independent transverse direction. It is located at $\boldsymbol{r}_1$ and has the dipole moment $\boldsymbol{m}_1=m\boldsymbol{e}_0\times\boldsymbol{e}_1$, where $\boldsymbol{e}_0\times\boldsymbol{e}_1$ is the unit vector $\boldsymbol{e}_\theta$ at $\boldsymbol{r}_1$. The image again contains five parts: three point dipoles, the first with moment $\boldsymbol{m}_2=-m\boldsymbol{e}_0\times\boldsymbol{e}_2$ located at $\boldsymbol{r}_2$, the second with moment $\boldsymbol{m}_1'=(a/r_0)^3\boldsymbol{m}_1$ located at $\boldsymbol{r}_1'$, and the third with moment $\boldsymbol{m}_2'=(a/r_0)^3\boldsymbol{m}_2$ located at $\boldsymbol{r}_2'$; and two continuous distributions of point dipoles on straight lines, one from the origin to $\boldsymbol{r}_1'$, the dipole moment from $u\boldsymbol{e}_1$ to $(u+du)\boldsymbol{e}_1$ being $-(\boldsymbol{m}_1/ar_0)udu$, and the other from the origin to $\boldsymbol{r}_2'$, the dipole moment from $u\boldsymbol{e}_2$ to $(u+du)\boldsymbol{e}_2$ being $-(\boldsymbol{m}_2/ar_0)udu$. The scalar potential is then

$$\varphi_3(\boldsymbol{r}) = \frac{m\boldsymbol{r}\cdot(\boldsymbol{e}_0\times\boldsymbol{e}_1)}{4\pi|\boldsymbol{r}-\boldsymbol{r}_1|^3} - \frac{m\boldsymbol{r}\cdot(\boldsymbol{e}_0\times\boldsymbol{e}_2)}{4\pi|\boldsymbol{r}-\boldsymbol{r}_2|^3} + \frac{ma^3\boldsymbol{r}\cdot(\boldsymbol{e}_0\times\boldsymbol{e}_1)}{4\pi r_0^3|\boldsymbol{r}-(a^2/r_0^2)\boldsymbol{r}_1|^3} - \frac{ma^3\boldsymbol{r}\cdot(\boldsymbol{e}_0\times\boldsymbol{e}_2)}{4\pi r_0^3|\boldsymbol{r}-(a^2/r_0^2)\boldsymbol{r}_2|^3} - \int_0^{a^2/r_0} \frac{m\boldsymbol{r}\cdot(\boldsymbol{e}_0\times\boldsymbol{e}_1)}{4\pi a r_0|\boldsymbol{r}-u\boldsymbol{e}_1|^3}udu + \int_0^{a^2/r_0} \frac{m\boldsymbol{r}\cdot(\boldsymbol{e}_0\times\boldsymbol{e}_2)}{4\pi a r_0|\boldsymbol{r}-u\boldsymbol{e}_2|^3}udu. \tag{52}$$

With the above results one can deal with a point dipole with an arbitrary direction located at an arbitrary position, and calculate the magnetic induction, the levitation force, etc. More complicated sources can be treated as superposition of monopoles in principle.

For a uniform magnetic field $\boldsymbol{H}_0=H_0\boldsymbol{e}_x$, it is easy to find the result in the presence of the superconductor by similar analysis to that in Sec. II. The scalar potential is

$$\varphi(\boldsymbol{r}) = -H_0x - \frac{1}{2}H_0a^3\frac{x}{r^3} = -H_0r\sin\theta\cos\phi - \frac{1}{2}H_0a^3\frac{\sin\theta\cos\phi}{r^2}. \tag{53}$$

For a uniform magnetic field in the z direction, however, the image method fails even when the hemisphere is absent (only the superconducting plane is present), and we still do not know how to solve this boundary value problem. Note that the boundary conditions at infinity are rather different in the two cases. In the case just solved, the condition is $\lim_{r\to\infty}\boldsymbol{H}\to H_0\boldsymbol{e}_x$ in all directions (in the unoccupied region of course). On the other hand, in the case with a uniform magnetic field in the z direction, the well posed boundary condition should be $\lim_{z\to+\infty}\boldsymbol{H}\to H_0\boldsymbol{e}_z$, since the z component of the magnetic field should be zero when $\theta=\pi/2$, no matter how large r is.

The final example for the present geometry is a circular current loop studied in Sec. II. For a source current with position parameters b, θ_0 and electric current I (where $b>a$ and $\theta_0<\pi/2$), we have three image currents, the first with parameters b, $\pi-\theta_0$, and $-I$, the second with a^2/b, θ_0, and $-Ib/a$, and the third with a^2/b, $\pi-\theta_0$, and Ib/a. We denote the function $A_0(r,\theta)$ defined in Eq. (18) by a new notation $A_0(r,\theta;b,\theta_0,I)$ which explicitly displays the parameters. Then the vector potential in the vacuum region is $\boldsymbol{A}(\boldsymbol{r})=A(r,\theta)\boldsymbol{e}_\phi$ where

$$A(r,\theta) = A_0(r,\theta;b,\theta_0,I) + A_0(r,\theta;b,\pi-\theta_0,-I) + A_0(r,\theta;a^2/b,\theta_0,-Ib/a) + A_0(r,\theta;a^2/b,\pi-\theta_0,Ib/a). \tag{54}$$

One can also consider a more general current with the current density (22), and write down the vector potential in the vacuum region as above. We will not go into further details.

To conclude this section we briefly discuss some other geometries involving planar or spherical boundary. One typical case is the region between the two intersecting superconducting planes. The image method is applicable if the angle between them is $2\pi/n$ where $n>1$ is a natural number. The simplest case with $n=2$ involves only one superconducting plane. Another simple case with $n=4$ is a 1/4 space (the superconductor occupies 3/4 space) which may be characterized by $x>0$ and $y>0$. In this case a monopole has three images and the magnetic scalar potential contains four terms, so does a point dipole. If the part of the sphere $r<a$ between the two planes is also occupied by the superconductor, the image method still works. For example, in the case $n=4$ the vacuum region is described by $x>0$, $y>0$, and $r>a$. In this case a monopole has 11 images (seven point monopoles and four line charges) and the magnetic scalar potential contains 12 terms. The situation is similar for a point dipole with transverse direction. However, a point dipole with radial direction has seven images, all being point dipoles, and the magnetic scalar potential contains eight terms. A more complicated case may be the first octant with the corresponding 1/8 sphere excluded. This region is described by $x>0$, $y>0$, $z>0$, and $r>a$. In this case a monopole has 23 images (15 point monopoles and 8 line charges) and the scalar potential contains 24 terms. Finally, one can consider two parallel superconducting planes. In this case, a monopole has an infinite number of images (all have the same sign as the source one) and the magnetic scalar potential is an infinite series.

VI. MORE COMPLICATED BOUNDARY IN TWO DIMENSIONS

In this section we consider similar problems to those in Sec. V. For convenience we use two-dimensional language in the following. Since the results in two dimensions are similar to but simpler than those in three dimensions, in most cases we will give the results directly. We mainly deal with the case where the superconductor occupies the lower half plane $y<0$ plus the upper half of the circular disk $\rho<a$. Other cases will be discussed briefly.

For a monopole with charge Q located at $\boldsymbol{\rho}_1$ with polar coordinates (ρ_0,ϕ_0), we have four images: three monopoles, all with charge Q, located at $\boldsymbol{\rho}_2$ with polar coordinates $(\rho_0, -\phi_0)$, $\boldsymbol{\rho}_1'=(a^2/\rho_0^2)\boldsymbol{\rho}_1$ and $\boldsymbol{\rho}_2'=(a^2/\rho_0^2)\boldsymbol{\rho}_2$; and another monopole with charge $-2Q$ located at the origin. The scalar potential is

$$\varphi(\boldsymbol{\rho}) = -\frac{Q}{2\pi}\left[\ln|\boldsymbol{\rho}-\boldsymbol{\rho}_1| + \ln|\boldsymbol{\rho}-\boldsymbol{\rho}_2| + \ln\left|\boldsymbol{\rho}-\frac{a^2}{\rho_0^2}\boldsymbol{\rho}_1\right| + \ln\left|\boldsymbol{\rho}-\frac{a^2}{\rho_0^2}\boldsymbol{\rho}_2\right|\right] + \frac{Q}{\pi}\ln\rho. \tag{55}$$

For a point dipole located at $\boldsymbol{\rho}_1$ with dipole moment $\boldsymbol{m}=m\boldsymbol{e}_1$ where $\boldsymbol{e}_1=\boldsymbol{\rho}_1/\rho_0$ (this is one with the radial direction), the scalar potential is

$$\varphi(\boldsymbol{\rho}) = \frac{m(\boldsymbol{\rho}\cdot\boldsymbol{e}_1-\rho_0)}{4\pi|\boldsymbol{\rho}-\boldsymbol{\rho}_1|^2} + \frac{m(\boldsymbol{\rho}\cdot\boldsymbol{e}_2-\rho_0)}{4\pi|\boldsymbol{\rho}-\boldsymbol{\rho}_2|^2} - \frac{ma^2(\boldsymbol{\rho}\cdot\boldsymbol{e}_1-a^2/\rho_0)}{4\pi\rho_0^2|\boldsymbol{\rho}-(a^2/\rho_0^2)\boldsymbol{\rho}_1|^2} - \frac{ma^2(\boldsymbol{\rho}\cdot\boldsymbol{e}_2-a^2/\rho_0)}{4\pi\rho_0^2|\boldsymbol{\rho}-(a^2/\rho_0^2)\boldsymbol{\rho}_2|^2}, \tag{56}$$

where $\boldsymbol{e}_2=\boldsymbol{\rho}_2/\rho_0$.

For a point dipole located at $\boldsymbol{\rho}_1$ with dipole moment $\boldsymbol{m}=m\boldsymbol{e}_z\times\boldsymbol{e}_1$ (this is one with transverse direction), the scalar potential is

$$\varphi(\boldsymbol{\rho}) = \frac{m\boldsymbol{\rho}\cdot(\boldsymbol{e}_z\times\boldsymbol{e}_1)}{4\pi|\boldsymbol{\rho}-\boldsymbol{\rho}_1|^2} - \frac{m\boldsymbol{\rho}\cdot(\boldsymbol{e}_z\times\boldsymbol{e}_2)}{4\pi|\boldsymbol{\rho}-\boldsymbol{\rho}_2|^2} + \frac{ma^2\boldsymbol{\rho}\cdot(\boldsymbol{e}_z\times\boldsymbol{e}_1)}{4\pi\rho_0^2|\boldsymbol{\rho}-(a^2/\rho_0^2)\boldsymbol{\rho}_1|^2} - \frac{ma^2\boldsymbol{\rho}\cdot(\boldsymbol{e}_z\times\boldsymbol{e}_2)}{4\pi\rho_0^2|\boldsymbol{\rho}-(a^2/\rho_0^2)\boldsymbol{\rho}_2|^2}. \tag{57}$$

For a uniform magnetic field $\boldsymbol{H}_0=H_0\boldsymbol{e}_x$, in the presence of the superconductor, the result is still given by Eq. (37). For one in the y direction, however, the image method fails, and the problem seems difficult.

Then consider an electric current I (along the z direction) located at $\boldsymbol{\rho}_1$, according to the result of Secs. III and IV, the magnetic induction is given by

$$\boldsymbol{B}(\boldsymbol{\rho}) = \frac{\mu_0 I}{2\pi}\boldsymbol{e}_z\times\left[\frac{\boldsymbol{\rho}-\boldsymbol{\rho}_1}{|\boldsymbol{\rho}-\boldsymbol{\rho}_1|^2} - \frac{\boldsymbol{\rho}-\boldsymbol{\rho}_2}{|\boldsymbol{\rho}-\boldsymbol{\rho}_2|^2} - \frac{\boldsymbol{\rho}-(a^2/\rho_0^2)\boldsymbol{\rho}_1}{|\boldsymbol{\rho}-(a^2/\rho_0^2)\boldsymbol{\rho}_1|^2} + \frac{\boldsymbol{\rho}-(a^2/\rho_0^2)\boldsymbol{\rho}_2}{|\boldsymbol{\rho}-(a^2/\rho_0^2)\boldsymbol{\rho}_2|^2}\right]. \tag{58}$$

One can consider the region between two intersecting rays. This is essentially the region between two intersecting planes, except that in the two-dimensional case any source is an infinitely long one along the z direction. Results for this geometry are similar to those in the last section.

Another geometry is the above region between two rays but with the part of the circular disk in that region excluded. In fact, the situation considered in the main body of this section belongs to this category. A slightly more complicated case is the first quadrant with the 1/4 circular disk in it excluded. In this case a monopole with charge Q has eight images, of which seven have charge Q located at their respective image positions and one with charge $-4Q$ located at the origin. We will not go into further details.

VII. HOLLOWS IN SUPERCONDUCTORS

Consider a spherical hollow in a superconductor. For a monopole, the boundary condition that the normal component of the magnetic induction vanishes on the surface of the hollow cannot be satisfied. Thus the problem has no solution. When the monopole is placed at the center of the spherical hollow, the physical picture is very clear: any induced current on the surface of the hollow cannot generate a magnetic induction (outside the hollow, or inside the superconductor) that will cancel the one produced by the monopole, since the latter is radially symmetric. Actually the conclusion holds for a hollow of any shape, since it is well-known in mathematical physics that the Green function of the Poisson equation with the second-type boundary condition does not exist (here we remark that for the region outside a sphere as considered in Sec. II, we have another boundary condition that the magnetic scalar potential is nonsingular at infinity, so the situation is different). For a dipole with radial direction, it is easy to find the solution on the basis of the result in Sec. II. For one with transverse direction, however, there is no ready solution. Though the problem can be studied by solving the field equation, it is not in the scope of our discussion.

Then consider a long cylindrical hollow in the superconductor. For a monopole (a line charge), there is no solution as mentioned above. For a dipole of either radial or transverse direction, it is easy to find the solution on the basis of the results in Sec. III.

One can still consider an electric current in the spherical or cylindrical hollow, and find the result by the image method without difficulty. We will not go into further details.

VIII. SUMMARY

In this paper we first solve the field equation of magnetostatics for a magnetic monopole in the presence of a superconducting sphere in the ideal Meissner state. The result in the form of an infinite series involving the Legendre polynomials is recast in a finite form, from which we obtain the images of the monopole inside the sphere which generates the induced field. With this result we develop the image method for a point dipole with arbitrary direction. In principle the result for an arbitrary magnetic source can be obtained by superposition. Then we develop the image method for a circular current loop whose axis passes through the center of the superconducting sphere. This is used to obtain the result for a more general current distribution. A parallel procedure is carried out for the case of a long superconducting cylinder. The results for this case are rather different from those for a superconducting sphere. The vector potential for an arbitrary electric current in the presence of an infinite superconducting plane is also obtained by the image method. The levitation force between the superconductor and the magnetic source is calculated in most cases. In this method it is the force acted on the source by the image, thus the physical picture is simple. More complicated geometries involving both spherical and planar superconducting boundaries are studied in detail. Results for geometries involving both cylindrical and planar boundaries are also given. We also briefly discuss hollows in superconductors. Except for the two elementary cases involving a monopole, all results are obtained without solving the field equation.

ACKNOWLEDGMENT

This work was supported by the National Natural Science Foundation of the People's Republic of China (Grant number 10275098).

*Electronic address: linqionggui@tsinghua.org.cn
†Mailing address.

[1] J. D. Jackson, *Classical Electrodynamics*, 3rd ed. (Wiley, New York, 1998).
[2] W. M. Saslow, Am. J. Phys. **59**, 16 (1991).
[3] M. W. Coffey, J. Supercond. **13**, 381 (2000).
[4] Z. J. Yang, Solid State Commun. **107**, 745 (1998).
[5] Y. Martin and H. K. Wickramasinghe, Appl. Phys. Lett. **50**, 1455 (1987).
[6] J. J. Sáenz, N. García, P. Grütter, E. Meyer, H. Heinzelmann, R. Wiesendanger, L. Rosenthaler, H. R. Hidber, and H.-J. Güntherodt, J. Appl. Phys. **62**, 4293 (1987).
[7] D. Rugar, H. J. Mamin, P. Güthner, S. E. Lambert, J. E. Stern, I. McFadyen, and T. Yogi, J. Appl. Phys. **68**, 1169 (1990).
[8] E. H. Brandt, Appl. Phys. Lett. **53**, 1554 (1988); Science **243**, 349 (1989); Am. J. Phys. **58**, 43 (1990).
[9] F. Hellman, E. M. Gyorgy, D. W. Johnson, Jr., H. M. O'Bryan, and R. C. Sherwood, J. Appl. Phys. **63**, 447 (1988).
[10] Z. J. Yang, T. H. Johansen, H. Bratsberg, G. Helgesen, and A. T. Skjeltorp, Physica C **160**, 461 (1989).
[11] L. C. Davis, J. Appl. Phys. **67**, 2631 (1990).
[12] R. J. Adler and W. W. Anderson, J. Appl. Phys. **68**, 695 (1990).
[13] T. Torng and Q. Y. Chen, J. Appl. Phys. **73**, 1198 (1992).
[14] M. J. Qin, G. Li, H. K. Liu, S. X. Dou, and E. H. Brandt, Phys. Rev. B **66**, 024516 (2002).
[15] A. Sanchez and C. Navau, Phys. Rev. B **64**, 214506 (2001); C. Navau and A. Sanchez, *ibid.* **64**, 214507 (2001).
[16] C. Navau and A. Sanchez, Phys. Rev. B **58**, 963 (1998).
[17] M. W. Coffey, Phys. Rev. B **65**, 214524 (2002).
[18] M. W. Coffey, J. Supercond. **15**, 257 (2002).
[19] S.-H. Guo, *Electrodynamics*, 2nd ed. (Higher Education Press, Beijing, 1997) (in Chinese).
[20] J. Pearl, J. Appl. Phys. **37**, 4139 (1966).
[21] Z. J. Yang, Jpn. J. Appl. Phys., Part 2 **31**, L477 (1992); **31**, L938 (1992); Physica C **234**, 263 (1994).
[22] M. W. Coffey, Phys. Rev. B **52**, R9851 (1995).
[23] H. J. Reittu and R. Laiho, Supercond. Sci. Technol. **5**, 448 (1992).
[24] M. W. Coffey and E. T. Phipps, Phys. Rev. B **53**, 389 (1996).