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Bose-Einstein condensation and two fluid behavior in ^{4}He

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The basic assumption of this paper is that in the presence of Bose-Einstein condensation (BEC) in liquid ^{4}He, the wave functions of occupied many particle states are the superposition of two components; a phase coherent component, proportional to the ground state wave function and a phase incoherent component. It is shown that this assumption satisfies necessary conditions imposed by the presence of BEC and that the wave functions of the ideal Bose gas and wave functions of the Bijl-Feynman type are of this form. It is shown that this single assumption provides simple microscopic explanations of essentially all the exotic properties of helium II; why BEC implies two fluid behavior, how the superfluid and condensate fractions are linked, how Landau theory is linked to the presence of BEC, why the superfluid exhibits flow without viscosity and macroscopic quantum effects while the normal fluid does not, how the anomalous expansion and loss of spatial order, observed in helium II as it is cooled, is linked to BEC, and how the presence of sharp peaks observed in the dynamic structure factor are linked with BEC. The theory provides predictions that are in quantitative agreement with a wide range of presently unexplained experimental data on helium II.

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I. INTRODUCTION

Liquid helium II (He II) behaves as if it were a mixture of two fluids freely intermingling without any viscous interaction.[1] The superfluid component has weight ρ_S, zero entropy, flows without viscosity and participates in macroscopic quantum effects such as quantized vortices. The normal fluid has weight $\rho_N = 1 - \rho_S$ and does not exhibit these properties. It is widely accepted that two fluid behavior in He II is linked to the presence of Bose-Einstein condensation (BEC) of ^{4}He atoms. In a Bose condensed system, the momentum distribution of the atoms develops a sharp peak, which contains a fraction f (the "condensate fraction") of the total weight (see Fig. 1). As the temperature T is raised f decreases, going to zero at the Bose condensation temperature T_B. Until 1995 BEC had been experimentally observed only in liquid ^{4}He,[2–8] but its observation in dilute trapped gases has strongly stimulated interest in this phenomenon and its potential applications.[9–11]

The experimental and theoretical evidence for the link between BEC and two fluid behavior is very strong. The theoretical basis for this link has been extensively investigated by field theoretical methods,[12,13] which have shown that[14] BEC implies two fluid behavior in weakly interacting Bose gases. Convincing evidence for this link in ^{4}He is provided by neutron scattering measurements[2–5,7,8] and path integral Monte Carlo calculations.[15,16] Both show that BEC appears at the same temperature as a finite superfluid fraction. However, there are still a number of fundamental, unanswered questions about the nature of this link.

For example, Landau theory[17] accurately predicts the properties of the normal fluid in He II. Landau postulated that the normal fluid is a gas of "elementary excitations," of momentum $\hbar q$ and energy $\hbar\omega$. The normal fluid fraction and many other thermodynamic properties of He II can be quite accurately calculated[18] by assuming that these excitations behave as noninteracting Bose particles, with the q,ω relationship obtained from neutron scattering data. However it is not clear how the presence of BEC is connected to the Landau theory. Another unresolved problem is the relationship between the superfluid fraction and the condensate fraction. It is known that the two fractions are not the same. For example, as $T \rightarrow 0$, $\rho_S \rightarrow 1$, whereas neutron scattering measurements[2–4,6,8] and calculations[15] show that in this limit $f \sim 0.07$ in He II. Calculations on weakly interacting Bose gases in the limits $T \rightarrow 0$ (Ref. 19) and $T \rightarrow T_B$ (Ref. 20) also indicate that there is no simple, universal relationship between ρ_S and f.[21] However, it is not clear that the latter conclusion applies away from these limits, or to a strongly interacting system such as He II.

Apart from these questions of principle there are many fundamental experimental properties of He II that have no accepted explanation. For example, He II expands[22] and its spatial order decreases[23] as the temperature is reduced. This uniquely anomalous behavior has been extensively

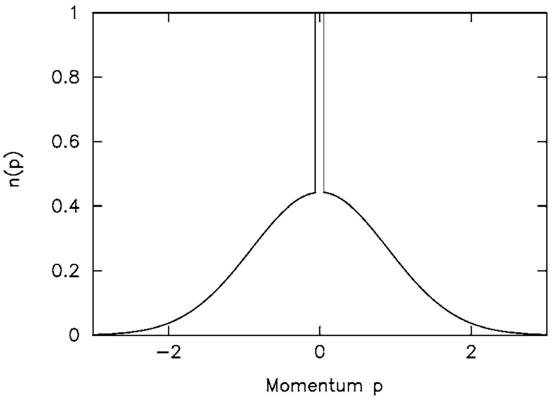

FIG. 1. Schematic illustration of the momentum distribution $n(p)$ in a Bose condensed system. $n(p)$ contains a sharp central peak in the $p=0$ state, with width $\sim\hbar/L$, where $L \sim V^{1/3}$ is the linear dimension of the N particle system. The integrated intensity of the condensate peak comprises a fraction f of the total integrated weight of $n(p)$.

discussed,[24–32] but there is at present no agreement as to why it occurs, or how it is linked to BEC.

Another unique property of He II is that inelastic neutron scattering reveals sharp peaks,[33–35] with essentially zero width[36] at sufficiently low temperatures. There is no consensus as to how the occurrence of these peaks is linked to BEC, or how their intensities vary with temperature. Although recent theory[37,38] suggests that the peak intensities should be proportional to f, there is no general agreement that this is the case, or that the intensities go to zero at the superfluid transition.[39,40]

The present paper is the latest in a series of papers[41–46] in which the basic approach is to investigate the implications of BEC for the properties of the many particle wave functions describing the system. The outline of the paper is as follows. Section II reviews the necessary conditions BEC imposes on the wave functions of many particle states occupied at a given T. The basic assumption of the paper is presented in Sec. III. The remainder of the paper investigates the consequences of this assumption. It is shown in Sec. IV that it implies two fluid behavior. In Sec. V that it implies that the superfluid fraction is proportional to the condensate fraction. It is also shown in this section that the basic assumption provides a simple quantitative explanation of the anomalous thermal expansion and loss of spatial order as He II is cooled. In Sec. VI a physical interpretation is given. In Sec. VII it is shown that the basic assumption provides a simple explanation of the presence and temperature dependence of sharp peaks in the dynamic structure factor of He II and of the link between BEC and Landau theory. Appendix A consider fluctuations in the condensate fraction. In Appendix B it is shown that in the presence of BEC, the ideal Bose gas wave functions and wave functions of the Bijl-Feynman[47,48] type satisfy the basic assumption.

II. NECESSARY PROPERTIES OF THE WAVE FUNCTIONS IN THE PRESENCE OF BEC

The momentum distribution, $n(\vec{p})$, of atoms in a many particle system can be expressed in terms of the wave functions by the following rigorous expressions:

$$n(\vec{p}) = \sum_i \eta_i(T) \int n_{i\vec{S}}(\vec{p}) d\vec{s}, \tag{2.1}$$

where

$$n_{i\vec{S}}(\vec{p}) = \frac{1}{V}\left| \int \Psi_i(\vec{r},s) \exp(i\vec{p}\cdot\vec{r}) d\vec{r} \right|^2. \tag{2.2}$$

The subscript i denotes an N particle eigenstate with energy E_i, $\eta_i(T) \propto \exp(-E_i/T)$ is the probability that the state i is occupied at temperature T, and V is the volume within which the system is enclosed. $\Psi_i(\vec{r},s)$ is the many particle wave function of state i, $\vec{r}$ is a randomly selected particle coordinate (say $\vec{r}_1$—which coordinate is chosen is irrelevant due to the Bose symmetry of the wave function), and $\vec{s} = \vec{r}_2, \vec{r}_3, \ldots, \vec{r}_N$ denotes the coordinates of the other $N-1$ particles. Equations (2.1) and (2.2) follow simply from the standard definition of $n(\vec{p})$ in terms of the one particle density matrix.[49]

In the presence of BEC, a finite fraction $f$of the particles occupies a single momentum state, which for simplicity is taken as $p=0$. Thus $n(\vec{p})$ develops a sharp central peak as illustrated in Fig. 1. The integrated weight of this peak is f and the width of the peak is $\sim\hbar/L$ where $L \sim V^{1/3}$, is the linear dimension of the N particle system. In trapped gases, which may contain only a few thousand atoms, the width of the condensate peak can be significant. However, in liquid ^{4}He, where there are typically $\sim 10^{23}$ atoms, the width of the condensate peak is immeasurably small and the condensate is effectively a δ function in momentum space. Formally it follows from Eqs. (2.1) and (2.2) that the condensate fraction is

$$f(T) = n(0) = \sum_i \eta_i(T) F_i, \tag{2.3}$$

where

$$F_i = \int d\vec{s} \frac{1}{V} \left| \int \Psi_i(\vec{r},s) d\vec{r} \right|^2. \tag{2.4}$$

The wave functions $\Psi_i(\vec{r},s)$ of states occupied at a given T determine the momentum distribution via Eqs. (2.1) and (2.2). The question addressed in this section is as follows: what implications does the presence of a momentum distribution of the form shown in Fig. 1, have for the properties of these wave functions? This question can be simplified by taking account of the generally accepted fact, that in a macroscopic system both thermal and quantum fluctuations in f are negligible. For definiteness, it will be assumed throughout that

$$\frac{\Delta f}{f} \sim \frac{1}{\sqrt{N}}, \tag{2.5}$$

although the results are valid providing that $\Delta f \to 0$ as $N \to \infty$. The sum in Eq. (2.3) represents the thermal average over all states i occupied at a given T. If thermal fluctuations in f are $\sim 1/\sqrt{N}$, it follows that all states that contribute importantly to this sum must give the same value of F_i to within terms $\sim 1/\sqrt{N}$. That this is at a given T, such states must satisfy

$$F_i = f(T) \pm \sim 1/\sqrt{N}. \tag{2.6}$$

Since every state occupied at a given T has the same properties for the purposes of this paper, the subscript i will be dropped to simplify notation. The sum over i can be trivially inserted into the final expressions if required. We consider the contribution of a single "typical" state, selected at random from those occupied at the given T and denote its wave function as $\Psi(\vec{r},\vec{s})$. Thus, for example, Eq. (2.2) is written as

$$n_{\vec{S}}(\vec{p}) = \left| \int \Psi(\vec{r},s) \exp(i\vec{p}\cdot\vec{r}) d\vec{r} \right|^2. \tag{2.7}$$

As T changes and different many particle states are occupied, the properties of $\Psi(\vec{r},s)$ for a typical state and $n_{\vec{S}}(\vec{p})$ will change correspondingly (see Appendix A).

It is clear from the inspection of Eq. (2.2) that when $N=1$, the calculation of $n_{\vec{S}}(\vec{p})$ reduces to the standard single

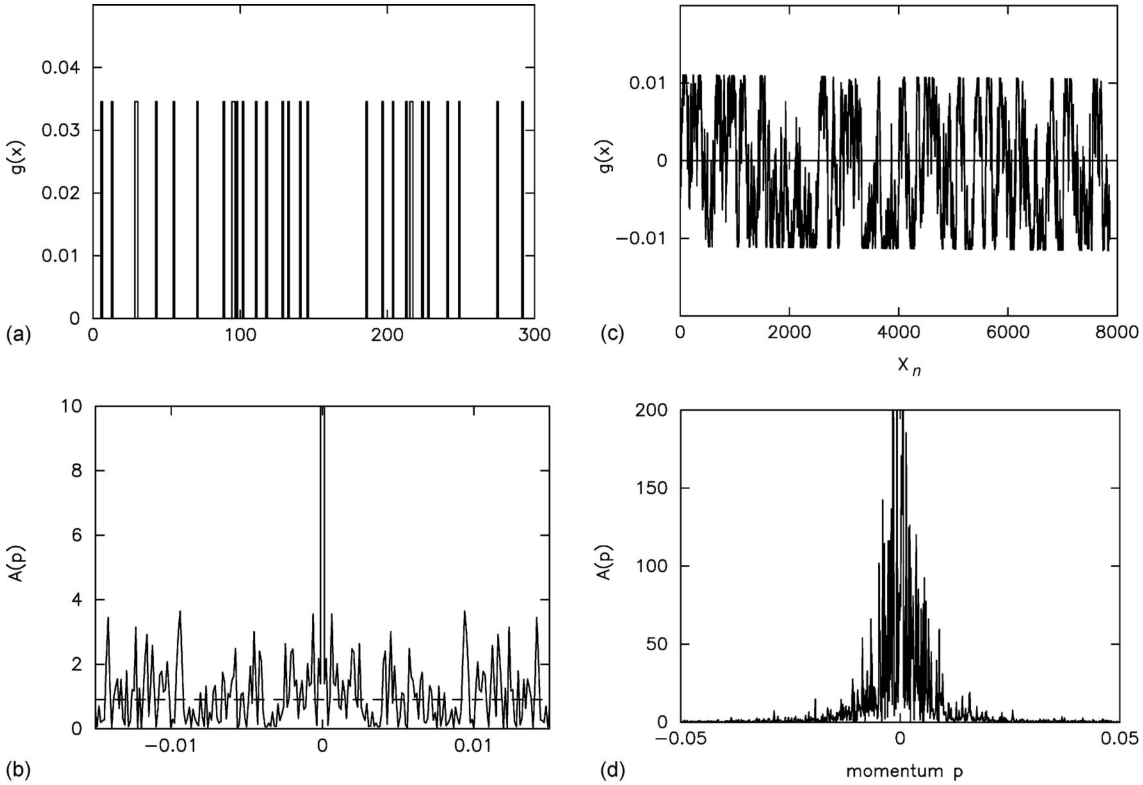

FIG. 2. (a) One-dimensional model wave function $g(x)$, which was generated numerically using a random number generator. 8192 pixels are occupied randomly with probability $f=0.1$ and with the same amplitude and phase. Only the first 300 pixels are shown. (b) $A(p)$, the modulus square of the numerical Fourier transform of $g(x)$. There is a "condensate peak" (which contains a fraction f of the total integrated intensity in this model). (c) The same model with the exception that a random walk of the phase was performed. Pixels separated by greater than the coherence length $r_C \sim 100$ pixels have uncorrelated phases. Only the real part of $g(x)$ is shown; note the different scale to (a). (d) $A(p)$ for the wave function in (c). There is no condensate peak. Instead the distribution is broad with a width $\sim 1/r_C \sim 0.01$. Note that the random variation of $A(p)$ from one momentum state to the next seen in (b) and (d) would be unobservable in a real macroscopic system, since the momentum states would be so closely spaced that it would not be possible to resolve this fine structure. Instead only the mean value, for example, the dashed line in (b), would be observed.

particle expression. In a many particle system the calculation of $n_{\vec{S}}(\vec{p})$ involves the additional quantum average over possible $\vec{s}$ in Eq. (2.1). In particular, the condensate fraction is determined by a weighted average over $\vec{s}$ of the quantity (see Appendix A)

$$f(\vec{s}) = \frac{n_{\vec{S}}(0)}{\int n_{\vec{S}}(\vec{p})d\vec{p}}. \tag{2.8}$$

$f(\vec{s})$ is the fraction of the weight of $n_{\vec{S}}(\vec{p})$ in the $p=0$ state. If quantum fluctuations in f are negligible, it follows that (see Appendix A)

$$f(\vec{s}) = f \pm \sim 1/\sqrt{N}. \tag{2.9}$$

Equation (2.9) implies that $n_{\vec{S}}(\vec{p})$ must have the form illustrated in Fig. 1 for essentially *every* $\vec{s}$ that can occur. The question we are addressing thus reduces to the following: what properties must the wave function $\Psi(\vec{r},\vec{s})$ possess, if $n_{\vec{S}}(\vec{p})$, defined in Eq. (2.7), contains a peak in momentum space of width $\sim\hbar/L$ and fractional weight $\sim f$? It has been shown previously[44,45] that there are three conditions that must be satisfied.

(1) $\Psi(\vec{r},s)$ must be a "delocalized" function of $\vec{r}$, that is nonzero over length scales $\sim L$.

A more precise statement of this condition is[45] that $\Psi(\vec{r},s)$ must be a nonzero function of $\vec{r}$ within a total volume $\gtrsim fV$. This condition follows directly from the properties of Fourier transforms and is in fact the standard argument that is used in elementary derivations of the Heisenberg uncertainty principle. The localization in momentum space implied by the presence of a peak in $n_{\vec{S}}(\vec{p})$ of width $\sim\hbar/L$ implies a corresponding delocalization in $\vec{r}$ space.

(2) $\Psi(\vec{r},s)$ must possess "long range phase coherence" in $\vec{r}$ over length scales $\sim L$.

In other words, the phases of $\Psi(\vec{r},s)$ and $\Psi(\vec{r}\,',s)$ are not randomly oriented, even when $|\vec{r}-\vec{r}\,'| \sim L$. The influence of phase incoherence on the momentum distribution is illustrated in Fig. 2. If $\Psi(\vec{r},s)$ has a finite "coherence length" r_C, that is, if the phase of $\Psi(\vec{r},s)$ is correlated with that of $\Psi(\vec{r}\,',s)$ only over distances $|\vec{r}-\vec{r}\,'| \sim r_C$, any peak in $n_{\vec{S}}(\vec{p})$ will have a width of at least $\sim 1/r_C$.

(3) The ground state wave function $\Psi_0(\vec{r},\vec{s})$ must be delocalized and "phase coherent" over length scales $\sim L$. In

other words, the phase of $\Psi_0(\vec{r},\vec{s})$ is precisely determined by the phase of $\Psi_0(\vec{r}',\vec{s})$, even when $|\vec{r}-\vec{r}'|\sim L$. This condition is a consequence of the fundamental result of quantum mechanics[48,50] that the ground state wave function of any Bose system has a constant phase, independent of the particle coordinates $(\vec{r},\vec{s})$. This is not true in Fermi systems and in the view presented here, is the fundamental reason why BEC can occur only in Bose systems.

It follows from condition (3) that as $T\rightarrow 0$, the wave function $\Psi(\vec{r},\vec{s})$ of a typical occupied state must approach delocalization and phase coherence. Moreover it follows from condition (2) that for $0<T<T_B$, $\Psi(\vec{r},\vec{s})$ must retain long range correlations in its phase. Above T_B either condition (1) is not satisfied and $\Psi(\vec{r},\vec{s})$ is localized, or condition (2) is not satisfied and $\Psi(\vec{r},\vec{s})$ is phase incoherent. The obvious question arises as to how the properties of the wave function of a typical occupied state change as the temperature is varied.

III. BASIC ASSUMPTION

The basic assumption of this paper is that for $0<T<T_B$, the wave function $\Psi(\vec{r},\vec{s})$ of a typical occupied state is the superposition of two components

$$\Psi(\vec{r},\vec{s}) = b(\vec{s})\Psi_0(\vec{r},\vec{s}) + \Psi_R(\vec{r},\vec{s}). \tag{3.1}$$

$\Psi_0(\vec{r},\vec{s})$ is the phase coherent ground state, whereas $\Psi_R(\vec{r},\vec{s})$ is phase incoherent in the coordinate $\vec{r}$ and hence does not contribute to the condensate peak. The integrated weight of the ground state component

$$w_C = \int |b(\vec{s})|^2\, d\vec{s} \int |\Psi_0(\vec{r},\vec{s})|^2\, d\vec{r} \tag{3.2}$$

is assumed to decrease smoothly as the temperature is raised, approaching zero for states occupied near the condensation temperature.

The *a priori* justifications for this assumption are as follows:

(1) It satisfies the necessary conditions discussed in the previous section; as $T\rightarrow 0$, $w_C\rightarrow 1$ and occupied states approach the phase coherent ground state. For $0<T<T_B$, $\Psi(\vec{r},\vec{s})$ retains a phase coherent component and hence possesses long-range phase coherence. As $T\rightarrow T_B$, $w_C\rightarrow 0$, and the wave function approaches phase incoherence, with no BEC.

(2) It is shown in Appendix B that the wave functions of the ideal Bose gas are of the form given in Eq. (3.1).

(3) It is also shown in Appendix B that wave functions of the type introduced by Bijl[47] and Feynman[48] are of the form given in Eq. (3.1).

The ideal Bose gas is the only system for which the many particle wave function is known exactly, although it could be argued that this is not a very convincing example, since it describes a system that is Bose condensed but not a superfluid. As is well known, the standard Landau criterion for superfluidity (see Sec. VII) gives a critical velocity of zero for superflow in the ideal Bose gas. This criticism does not apply to wave functions of the Bijl-Feynman type. These accurately predict the phonon-roton curve in He II and hence describe a system that is both Bose condensed and superfluid. Again this does not prove that the wave functions describing He II have the form assumed in Eq. (3.1). For example, wave functions of the Bijl-Feynman type are not energy eigenstates of any known Hamiltonian and Eqs. (2.1) and (2.3) are valid only for energy eigenstates. However, the fact that wave functions of the Bijl-Feynman type accurately describe the excitation spectrum in He II (Refs. 51–56) strongly suggests that they do incorporate the essential physics of the He II wave function.

Ultimately the validity of the basic assumption must be judged on its success in describing the experimental properties of He II. It will be shown in the following sections that this single assumption provides simple microscopic explanations for essentially all of the exotic properties, which make He II such a unique fluid. It also has quantitative consequences that are accurately satisfied by a number of presently unexplained and fundamental experimental properties of He II.

IV. TWO FLUID BEHAVIOR

For a wave function of the form in Eq. (3.1), since only the phase coherent component contributes to the condensate peak, Eqs. (2.4), (2.6), and (3.1) imply that to within $\sim 1/\sqrt{N}$,

$$f = \int |b(\vec{s})|^2 d\vec{s} \left| \frac{1}{V} \int \Psi_0(\vec{r},\vec{s}) d\vec{r} \right|^2. \tag{4.1}$$

If quantum fluctuations in f are $\sim 1/\sqrt{N}$, it follows from (4.1) by similar reasoning to that used in the derivation of Eq. (2.9) that (see Appendix A)

$$|b(\vec{s}|^2 = w_C \pm \sim 1/\sqrt{N}, \tag{4.2}$$

where w_C is defined in Eq. (3.2). Equations (3.1) and (4.2) imply that

$$\int |\Psi(\vec{r},\vec{s})|^2\, d\vec{r} = w_C \int |\Psi_0(\vec{r},\vec{s})|^2\, d\vec{r} + \int |\Psi_R(\vec{r},\vec{s})|^2\, d\vec{r} + X(\vec{s}) \pm \sim 1/\sqrt{N}. \tag{4.3}$$

The "cross terms" in Eq. (4.3) are of the form

$$X(\vec{s}) = \int \Psi_0(\vec{r},\vec{s})\Psi_R(\vec{r},\vec{s}) d\vec{r} + \text{c.c.}, \tag{4.4}$$

where c.c. denotes the complex conjugate. Since the phase of $\Psi_0(\vec{r},\vec{s})$ is constant, the phase of the integrand in Eq. (4.4) varies randomly with $\vec{r}$ over length scales $\vec{r}_C$, where $\vec{r}_C$ is the coherence length of $\Psi_R(\vec{r},\vec{s})$. This implies that $X(\vec{s})$ has an average value of zero and amplitude $\sim 1/\sqrt{N}$ compared to the other terms in Eq. (4.3). Hence $X(\vec{s})$ simply adds to the fluctuations and Eq. (4.3) reduces to

$$\int |\Psi(\vec{r},\vec{s})|^2\, d\vec{r} = w_C \int |\Psi_0(\vec{r},\vec{s})|^2\, d\vec{r} + \int |\Psi_R(\vec{r},\vec{s})|^2\, d\vec{r} \pm \sim 1/\sqrt{N}. \tag{4.5}$$

It follows from Eqs. (4.5) and (2.7), by application of Parseval's theorem for Fourier transforms, that, to within terms $\sim 1/\sqrt{N}$, $n_{\vec{S}}(\vec{p})$ and hence $n(\vec{p})$ splits into two independent components

$$n(\vec{p}) = w_C n_0(\vec{p}) + w_R n_R(\vec{p}), \tag{4.6}$$

where w_R is defined as

$$w_R = 1 - w_C. \tag{4.7}$$

Since the macroscopic fluid flow is proportional to the first moment of $n(\vec{p})$, Eq. (4.6) also implies that the flow can be split into the sum of two independent components. It can further be shown, by similar arguments to those used in the derivation of (4.5), that, to within terms $\sim 1/\sqrt{N}$, the total fluid energy splits into two independent components

$$E = w_C E_0 + w_R E_R. \tag{4.8}$$

Similarly the entropy, $S=(\partial E/\partial T)_V$, splits into two independent components. Furthermore, the entropy of the ground state component $(\partial E_0/\partial T)_V$ is zero. Clearly the two components have identical properties to the superfluid and normal fluid components of the standard two fluid model. Thus the phase coherent component can be identified with the superfluid and the phase incoherent component with the normal fluid

$$\rho_S = w_C, \quad \rho_N = w_R. \tag{4.9}$$

Further evidence for this identification is provided by the wave functions discussed in Appendix B. For both the ideal Bose gas and wave functions of the Bijl-Feynman type, the weight w_R is determined by the number of excitations, as is the case for ρ_N in Landau theory.

V. MEASURABLE CONSEQUENCES

The most fundamental consequence of the basic assumption of Sec. III is that the condensate fraction should be proportional to the superfluid fraction. Only the phase coherent component contributes to f. This component has weight ρ_S and makes an identical contribution to that from the ground state. Hence it follows that

$$f = \rho_S f_0. \tag{5.1}$$

Equation (5.1) is accurate to $\sim 1/\sqrt{N}$ and follows formally from Eqs. (4.1), (4.2), and (4.9), as is shown in Appendix A. Note that Eq. (5.1) determines f only to within a proportionality constant f_0, the condensate fraction for the exact ground state. Figure 3 shows neutron scattering measurements of f/f_0 as a function of T. Also shown is the superfluid fraction, ρ_S. It can be seen that Eq. (5.1) is accurately obeyed in He II. Only "state of the art" measurements are shown, although other measurements[2–4] and calculations[15] are also consistent with Eq. (5.1).

In a macroscopic system Eq. (5.1) should be highly accurate and more accurate calculations or measurements of $f(T)$ in He II would be of great interest to test its validity. It should be noted that Eq. (5.1) is not valid as $T \to T_B$, where the assumption that fluctuations in f are negligible breaks down. It should also be noted that w_R is not well defined by Eq. (4.7) in the limit $T\to 0$ and hence cannot be compared with calculations of[19] ρ_N in this limit. Furthermore the cross terms $X(\vec{s}) \sim 1/\sqrt{N}$ in Eq. (4.3) are significant in the critical regions, again implying that w_C is not well defined in the limit $T \to T_B$, and w_R is not well defined in the limit $T\to 0$.

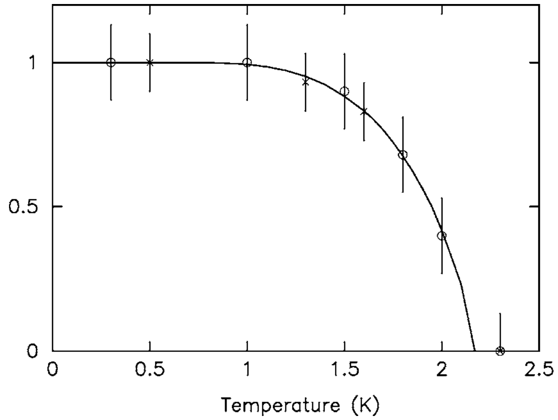

FIG. 3. The crosses (Ref. 7) and circles (Ref. 6) are measurements of $f(T)/f(0)$. The solid line shows the superfluid fraction as a function of temperature (Ref. 18).

The basic assumption of Sec. III has much wider consequences than just the prediction of two fluid behavior. It implies that all physical properties of the many particle system split into two independent components to within terms $\sim 1/\sqrt{N}$. For example, Eq. (4.8) implies that in a macroscopic system, the superfluid and the normal fluid behave as two independent thermodynamic systems. In particular, it follows from (4.8) that the pressure exerted by the fluid, $P=-(\partial E/\partial V)_S$, splits into two independent contributions

$$P = \rho_S P_0 + \rho_N P_R. \tag{5.2}$$

It has been shown previously[46] that with the additional assumption that the normal fluid pressure P_R is identical to the pressure at T_B, Eq. (5.2) supplies quantitative agreement with the observed T dependence of the fluid density in He II. The pressure is the sum of two partial pressures, with $P_0 > P_R$. Thus, as the superfluid fraction increases, P also increases, and at constant external pressure, He II will undergo an anomalous thermal expansion as T is reduced. The additional pressure in the superfluid has a purely quantum origin due to the presence of the condensate. It is known from experiments[57] that f_0 decreases with decreasing volume. The corresponding increase in the kinetic energy of atoms adds an extra contribution to the derivative $P_0=-(\partial E_0/\partial V)_S$. Since the normal fluid does not contribute to f, P_R does not contain this contribution.

Another simple consequence is that the static structure factor $S(\vec{q})$, and hence the pair correlation function $g(r)$, split into two independent contributions from the normal fluid and superfluid. $S(\vec{q})$ can be expressed in terms of the wave functions as[43]

$$S(\vec{q}) = \sum_i \eta_i(T) \int |\Psi_i(\vec{r},\vec{s})|^2 \frac{1}{N} \left| \sum_n \exp(i\vec{q}\cdot\vec{r}_n) \right|^2 d\vec{r}_N. \tag{5.3}$$

By similar arguments to those used in the derivation of Eq. (4.5), the contribution of a single state to $S(\vec{q})$ is (dropping the subscript i)

$$\begin{aligned} S(\vec{q}) &= \rho_S \int |\Psi_0(\vec{r},\vec{s})|^2 \frac{1}{N} \left| \sum_n \exp(i\vec{q}\cdot\vec{r}_n) \right|^2 d\vec{r}\, d\vec{s} \\ &\quad + \int |\Psi_R(\vec{r},\vec{s})|^2 \frac{1}{N} \left| \sum_n \exp(i\vec{q}\cdot\vec{r}_n) \right|^2 d\vec{r}\, d\vec{s} \\ &= \rho_S S_0(q) + \rho_N S_R(q), \end{aligned} \tag{5.4}$$

where Eq. (5.4) defines $S_0(\vec{q})$ and $S_R(\vec{q})$. It has been shown previously[44–46] that, with the assumption that the contribution $S_R(\vec{q})$ from the incoherent component is identical to $S(\vec{q})$ at the condensation temperature, Eq. (5.4) gives quantitative agreement with available measurements of $S(\vec{q})$ in He II. The uniquely anomalous loss of spatial correlations in He II as it is cooled can be attributed to the lower spatial order of the superfluid compared with the normal fluid. As T is lowered and ρ_S increases, measured spatial order therefore decreases. The physical reasons for the lower spatial order of the superfluid in He II will be discussed in the following section.

VI. PHYSICAL INTERPRETATION

The relevant physics of the wave function of He II can be understood by considering the simple model of the ground state introduced by Feynman[48] and used by Penrose and Onsager[58] in the first accurate calculation of the condensate fraction in He II

$$\Psi_0(\vec{r},\vec{s}) = 0, \quad \text{if } |\vec{r}_n - \vec{r}_m| < a,$$

$$\Psi_0(\vec{r},\vec{s}) = C, \text{ otherwise}. \tag{6.1}$$

a is the hard core diameter of a helium atom, C is a normalization constant, and $\vec{r}_n$, $\vec{r}_m$ denote any two particle coordinates. In the Feynman model $\Psi_0(\vec{r},\vec{s})$ is nonzero only for $\vec{r}$, where it would be possible to insert the center of an impenetrable sphere, given $N-1$ impenetrable spheres already present at coordinates $\vec{s}=\vec{r}_2,\vec{r}_3,\ldots,\vec{r}_N$. $\Psi_0(\vec{r},\vec{s})$ is zero for any point $\vec{r}$ within a distance a of one of the other atoms at $\vec{s}$ and has the same amplitude and phase at all other points, as illustrated in Fig. 4. Helium atoms have a hard-core interaction, hence one would expect that the wave functions in He II will be similar to that shown in Fig. 4. The main differences are that $\Psi_0(\vec{r},\vec{s})$ in He II will have an amplitude that falls smoothly to zero as $\vec{r}$ approaches the hard core exclusion region of another atom and that wave functions of states occupied at finite T will have a phase that varies with $\vec{r}$.

The spatial ordering of the many particle system is determined by the probability $|\Psi(\vec{r},\vec{s})|^2$ of different particle configurations $\vec{r},\vec{s}=\vec{r}_1,\vec{r}_2,\ldots,\vec{r}_N$, as Eq. (5.3) demonstrates. For a wave function of the form in Eq. (3.1), it follows from Eq. (5.4) that the spatial order of the superfluid is determined by $|\Psi_0(\vec{r},\vec{s})|^2$, while that of the normal fluid is determined by $|\Psi_R(\vec{r},\vec{s})|^2$. As discussed in Sec. V, measurements of $S(\vec{q})$ imply that the superfluid is more disordered than the normal fluid. Thus $|\Psi_0(\vec{r},\vec{s})|^2$ and $|\Psi_R(\vec{r},\vec{s})|^2$ must have significant amplitude for different particle configurations $\vec{r},\vec{s}$. The first condition for BEC in Sec. II, implies that for $\vec{s}$, where $\Psi_0(\vec{r},\vec{s})$ has significant amplitude, the "spaces" (see Fig. 4) in the liquid structure must comprise at least $f_0 \sim 7\%$ of the total fluid volume in He II. Thus the structure of the ground state must be relatively open, with many spaces as illustrated in Fig. 5(a). There is no such necessary condition on $\Psi_R(\vec{r},\vec{s})$, which makes no contribution to the condensate. Configurations of particles $(\vec{r},\vec{s})$ for which $\Psi_R(\vec{r},\vec{s})$ has significant amplitude, could define a structure with few spaces [Fig. 5(b)].

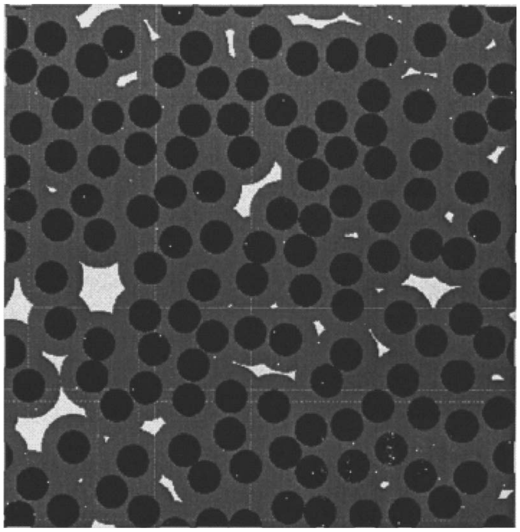

FIG. 4. A schematic illustration of the wave function $\Psi(\vec{r},\vec{s})$ in He II. The black circles denote $N-1$ atoms centered at positions $\vec{s}=\vec{r}_2,\vec{r}_3,\ldots,\vec{r}_N$. The wave function is nonzero only for $\vec{r}$, where it would be possible to insert place the center of an atom without hard-core overlap with the $N-1$ atoms already present. In liquid He II at $T=0$, these "spaces" (the white regions of the figure) must comprise at least ~7% of the total fluid volume.

Spaces in a structure imply a loss of spatial correlations—for example, the development of vacancies in a crystal structure implies that the amplitudes of Bragg peaks in $S(\vec{q})$ are reduced, whereas the diffuse intensity between Bragg peaks increases. Spaces in a liquid structure will similarly reduce the oscillations observed in $S(\vec{q})$ and hence the measured spatial order. It has been shown previously[44–46] by the use of lattice models, that the assumption that the ground state contains sufficient spaces in its structure to give the observed value of $f_0 \sim 7\%$, while the normal fluid contains negligible spaces, is quantitatively consistent with the observed T dependence of $S(\vec{q})$ in He II. Furthermore, this is true only at the known hard-core diameter and packing density in He II. Changing the hard-core diameter a by only 10% is sufficient to destroy the agreement with experiment. This provides strong circumstantial evidence that this explanation is correct.

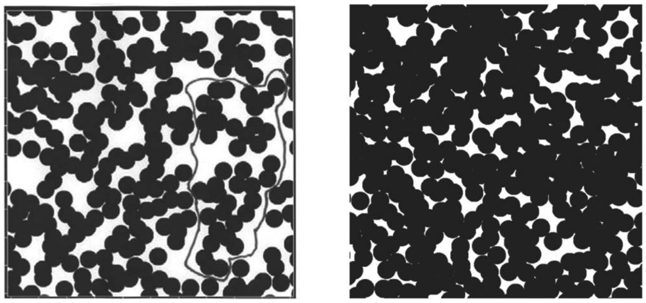

FIG. 5. The left hand figure illustrates a typical configuration $\vec{s}$ of $N-1$ particles, for which $\Psi_0(\vec{r},\vec{s})$ has significant amplitude. The structure is open with many spaces. Macroscopic loops over $\vec{r}$ where $\Psi_0(\vec{r},\vec{s})\neq 0$ are possible, as illustrated, leading to quantized vortices. The right hand figure illustrates a typical configuration $\vec{s}$ for which $\Psi_R(\vec{r},\vec{s})$ has significant amplitude. The structure has fewer spaces and regions where $\Psi_R(\vec{r},\vec{s})\neq 0$ are not connected. Macroscopic loops such as those shown in the left hand figure are not possible. The $\sim 1\%$ lower density of the normal fluid, compared with the superfluid, has been exaggerated for purposes of illustration. In fact, calculations based on lattice models (Ref. 45) suggest that the extra spaces within the superfluid are produced essentially by a rearrangement of atoms at almost constant fluid density.

As discussed in Sec. III, the ground state of any Bose condensed system must be phase coherent over length scales $\sim L$. Phase coherence in the wave function is generally associated with connectivity—that is, it is possible to follow a path over which $\Psi_0(\vec{r},\vec{s})$ is always nonzero, between any two phase coherent points $\vec{r}$ and $\vec{r}\,'$. This is plausible in He II at the known values of f_0, packing density, and hard-core radius.[45] Assuming that this is the case, then for $\vec{s}$, where $\Psi_0(\vec{r},\vec{s})$ has significant amplitude, it is possible to perform integrations over macroscopic loops in $\vec{r}$, such as that illustrated in Fig. 5(a). It is easily shown[45] that this implies that the circulation must bequantized over macroscopic length scales, leading to macroscopic quantized vortices and other macroscopic quantum effects in the superfluid. In contrast, the phase incoherence of $\Psi_R(\vec{r},\vec{s})$ implies no connectivity over macroscopic length scales. Hence macroscopic loops such as those shown in Fig. 5(a) are not possible and the normal fluid cannot contribute to quantized vortices or other macroscopic quantum effects. The normal fluid properties are a phase incoherent sum of contributions from localized regions of size $\sim r_C$, where r_C is the phase coherence length of $\Psi_R(\vec{r},\vec{s})$. For the wave functions discussed in Appendix B, $r_C\sim\lambda_T$, where λ_T is the well-known "thermal wavelength," $\sim 5-10$ Å for $T>1$ K in He II.

VII. BEC AND LANDAU THEORY

The fundamental premise of the Landau theory of superfluidity is that excitations exist with a well-defined relationship between their momentum $\hbar q$ and energy $\hbar\omega$. An equivalent statement is that the intensity in the dynamic structure factor $S(q,\omega)$ follows a well-defined line in (q,ω) space, with a narrow width in both q and ω. The rigorous expression for $S(q,\omega)$ in terms of the wave functions is[43]

$$S(\vec{q},\omega)=\frac{1}{N}\sum_i \eta_i(T)\sum_f |\tilde{A}_{if}(\vec{q})|^2\delta(\hbar\omega+E_i-E_f), \tag{7.1}$$

where the "scattering amplitude" $\tilde{A}_{if}(\vec{q})$ for a transition between states i and f is

$$\tilde{A}_{if}(\vec{q})=\int \Psi_i^*(\vec{r}_1,\vec{s})\sum_n \exp(i\vec{q}\cdot\vec{r}_n)\Psi_f(\vec{r}_1,\vec{s})\,d\vec{r}_1\,d\vec{s}. \tag{7.2}$$

The initial and final state wave functions are $\Psi_i(\vec{r},\vec{s})$ and $\Psi_f(\vec{r},\vec{s})$, respectively, with corresponding energies E_i and E_f. The δ function expresses conservation of energy. Using the Bose symmetry of the wave function and denoting $\vec{r}_1$ as $\vec{r}$ and $\vec{r}_2,\vec{r}_3..\vec{r}_N$ as $\vec{s}$, it follows from (7.2) that[43]

$$\tilde{A}_{if}(\vec{q})=N\int \Psi_i^*(\vec{r},\vec{s})\Psi_f(\vec{r},\vec{s})\exp(i\vec{q}\cdot\vec{r})d\vec{s}\,d\vec{r}. \tag{7.3}$$

The influence of the phase coherence properties of $\Psi_i(\vec{r},\vec{s})$ on $S(q,\omega)$ can be understood by first considering the case when $\Psi_i(\vec{r},\vec{s})$ is phase incoherent in $\vec{r}$, with coherence length r_C. This implies that the function

$$g(\vec{r})=N\int \Psi_i^*(\vec{r},\vec{s})\Psi_f(\vec{r},\vec{s})d\vec{s} \tag{7.4}$$

is also phase incoherent with coherence length r_C. Then for similar reasons that the function $g(x)$ shown in Fig. 2 gives no sharp peaks in p space, any peak in

$$|\tilde{A}_{if}(\vec{q})|^2=\left|\int g(\vec{r})\exp(i\vec{q}\cdot\vec{r})d\vec{r}\right|^2 \tag{7.5}$$

must have a width of at least $\Delta q\sim 1/r_C$. Thus the normal fluid, where $r_C\sim 10$ Å, makes a diffuse contribution to $S(q,\omega)$, with minimum width $\Delta q\sim 1$ Å^{-1}, at constant ω. In contrast, the superfluid, for which $r_C\sim L$, can contribute to peaks with minimum width $\sim 1/L$. It also follows, by similar arguments to those used in Sec. IV that the superfluid and normal fluid contribute separately to $S(q,\omega)$.[59] Thus $S(q,\omega)$ will be the sum of two components with quite different properties: a diffuse component of minimum width $\Delta q\sim 1/r_C\sim 1$ Å^{-1} and integrated weight ρ_N and a sharp component of weight ρ_S. This is what is observed in neutron scattering measurements of $S(q,\omega)$ in He II. The measured intensity contains sharp and diffuse components and the intensity of the sharp component is proportional to ρ_S to within an experimental error of $\sim 2\%$.[33,35]

In terms of Landau theory, this implies that the momenta of Landau excitations in the normal fluid are uncertain to within $\sim\hbar/r_C$, whereas excitations in the superfluid have momentum that can be defined precisely, to within $\sim\hbar/L$. It follows that the basic premise of Landau theory—that excitations with a well-defined q,ω relationship exist—can be satisfied only in the presence of BEC and the associated phase coherent component. The uncertainty in the momentum of normal fluid excitations also provides a simple explanation of why the normal fluid is viscous. The Landau criterion for superflow[17] is that the flow velocity must be less than the critical velocity given by

$$v_C = (\omega/q)_{\min}.$$

$(\omega/q)_{\min}$ is the minimum value of the ratio of the energy and momentum of any process by which the fluid can exchange energy and momentum, for example, with the walls of a tube along which the liquid is flowing. The uncertainty in the momentum of normal fluid excitations implies that excitations of arbitrarily low energy have a finite probability of exchanging significant momentum $\sim \hbar / r_C$. Hence the normal fluid has a critical velocity ~ 0. In contrast, the well-defined momentum of excitations in the superfluid implies that the superfluid has a nonzero critical velocity, which as is well known is given by the ratio ω/q at the roton minimum. Thus the superfluid can undergo nonviscous flow, while the normal fluid cannot.

VIII. SUMMARY

The basic assumption of this paper is that in a Bose condensed system at a given temperature, many particle wave functions of occupied states are the superposition of two components. One component is phase coherent over macroscopic length scales and identical to the ground state wave function. The other component has phase coherence only over length scales comparable to the thermal wavelength. It has been shown that this assumption satisfies necessary conditions that must be obeyed by the wave functions of any Bose condensed system. It is shown in Appendix B that wave functions of the ideal Bose gas and wave functions of the Bijl-Feynman type are of this form. This single assumption, together with the assumption that quantum and thermal fluctuations in the condensate fraction are negligible, is sufficient to explain essentially all the features that make He II such a unique liquid. It provides a simple microscopic explanation of why two fluid behavior occurs, why BEC is necessary for the validity of Landau theory, and why the superfluid undergoes macroscopic quantum effects and nonviscous flow, while the normal fluid does not. It also provides explanations of a number of fundamental and presently unexplained experimental properties of He II. It implies that the superfluid fraction is proportional to the condensate fraction, in accurate agreement with experiment, provides quantitative explanations of the anomalous expansion and loss of spatial correlations in He II as it is cooled, and of the presence and temperature dependence of sharp peaks in the dynamic structure factor of He II.

APPENDIX A

1. Thermal fluctuations in *f*

The sum over states *i*of energy E_i in Eq. (2.3) can be converted in the usual way to an integral over the density of states $g(E)$

$$\sum_i \eta_i(T) = \int g(E)\eta(E)dE. \tag{A1}$$

$g(E)$ increases very rapidly with energy E, whereas $\eta(E)$ falls very rapidly. At a given T, the product of these two functions is a sharply peaked function of E that is centered at $E(T)$. The size of thermal fluctuations in the macroscopic energy E is determined by the width of this band, which, according to standard arguments, is $\sim E(T)/\sqrt{N}$.[60] Thus all states i within the band have energy

$$E_i = E(T) \pm \sim 1/\sqrt{N}. \tag{A2}$$

The assumption that thermal fluctuations in f are $\sim 1/\sqrt{N}$ implies that all states within the occupied band have wave functions $\Psi_i(\vec{r},\vec{s})$ that give the same value of F_i, to within terms $\sim 1/\sqrt{N}$, where F_i is defined in Eq. (2.4). The T dependence of f and of the corresponding nature of wave functions $\Psi_i(\vec{r},\vec{s})$ of occupied states, is due to the fact that, as T changes, the band moves to a different energy $E(T)$. Thus different many particle states are occupied and these states give a different value of $F_i \sim f(T)$.

2. Quantum fluctuations in *f*

We consider a randomly chosen state in the occupied band and drop the subscript i to simplify notation. It follows from Eqs. (2.7) and (2.8) that

$$f(\vec{s}) = \frac{\frac{1}{V}\left|\int \Psi(\vec{r},\vec{s})d\vec{r}\right|^2}{\int |\Psi(\vec{r},\vec{s})|^2 \, d\vec{r}}, \tag{A3}$$

where Parseval's theorem has been used to convert the denominator of Eq. (2.8) to an integral over $\vec{r}$. Defining

$$P(\vec{s}) = \int |\Psi(\vec{r},\vec{s})|^2 \, d\vec{r}, \tag{A4}$$

it then follows from Eqs. (2.4), (A3), and (A4) that (dropping the subscript i)

$$F = \int f(\vec{s})P(\vec{s})d\vec{s}. \tag{A5}$$

According to the standard physical interpretation of the wave function, $|\Psi(\vec{r},\vec{s})|^2$ is the probability that a particular configuration $\vec{r},\vec{s}$ of particles will occur in the state under consideration. It therefore follows from (A4) that $P(\vec{s})$ is the probability that a particular $\vec{s}$ will occur. Thus Eq. (A5) defines F as the average of the quantity $f(\vec{s})$, over the probability distribution for $\vec{s}$. The mean square quantum fluctuation in F therefore can be defined as

$$[\Delta f]^2 = \int P(\vec{s})[f(\vec{s}) - F]^2 \, d\vec{s}. \tag{A6}$$

If $\Delta f \sim 1/\sqrt{N}$, it follows from (A6) that

$$f(\vec{s}) = F \pm \sim 1/\sqrt{N} \qquad \text{(A7)}$$

for all $\vec{s}$ for which $P(\vec{s}) \neq 0$. Equation (2.9) follows trivially from Eqs. (A7) and (2.6).

The physical origin of the small quantum fluctuations in f is the delocalization of the wave function in the presence of BEC. Since BEC implies that $\Psi(\vec{r},\vec{s})$ must be a nonzero function of $\vec{r}$ over a volume $\sim fV$ [see condition (1a) of Sec. III], the integrals in Eq. (A3) are averages over a macroscopic volume. The microscopic details of the structure defined by $\vec{s}$ become irrelevant in these macroscopic averages, provided that $L = V^{1/3}$ is much greater than the interatomic separation.[61] In contrast, fluctuations in f are not negligible if $\Psi(\vec{r},\vec{s})$ is a localized function of $\vec{r}$ or if $\Psi(\vec{r},\vec{s})$ is phase incoherent. In the latter case the integral over $\Psi(\vec{r},\vec{s})$ in the numerator of Eq. (A9) is the sum of $\sim N$ randomly phased contributions. In this case $f(\vec{s}) \sim 1/\sqrt{N}$, with fluctuations comparable to its amplitude.

It follows from Eq. (4.1) that

$$f = \int |b(\vec{s})|^2 f_0(\vec{s}) P_0(\vec{s}) d\vec{s}, \qquad \text{(A8)}$$

where $f_0(s)$ and $P_0(\vec{s})$ are the ground state values of the quantities defined in Eqs. (A3) and (A4). It follows by the same argument used to derive Eq. (A7) that if fluctuations in f are $\sim 1/\sqrt{N}$,

$$|b(\vec{s})|^2 f_0(\vec{s}) = f \pm \sim 1/\sqrt{N}. \qquad \text{(A9)}$$

It also follows from (A7) that

$$f_0(s) = f_0 \pm \sim 1/\sqrt{N}. \qquad \text{(A10)}$$

It follows from Eqs. (A9) and (A10) that

$$|b(\vec{s})|^2 = \frac{f}{f_0} \pm \sim \frac{1}{\sqrt{N}}. \qquad \text{(A11)}$$

Equation (5.1) follows from Eqs. (A11), (4.2), and (4.9).

APPENDIX B

1. The ideal Bose gas

The N particle wave function of the ideal Bose gas is[62]

$$\Phi(\vec{r}_1,\vec{s}) = \frac{1}{\sqrt{N_P V^N}} \sum_P \exp(ik_0 \cdot \vec{r}_1)\exp(ik_1 \cdot \vec{r}_2) \cdots \exp(ik_J \cdot \vec{r}_N). \qquad \text{(B1)}$$

$\vec{r}_2, \ldots, \vec{r}_N$ is denoted as $\vec{s}$ and the sum is over the N_P permutations of the N particle coordinates in the occupied states $\vec{k}_j$. $\Phi(\vec{r}_1,\vec{s})$ can be rearranged into the form

$$\Phi(\vec{r}_1,\vec{s}) = \sum_j \alpha_j(\vec{s}) \exp(i\vec{k}_j \cdot \vec{r}_1), \qquad \text{(B2)}$$

where $\alpha_i(\vec{s})\exp(i\vec{k}_j \cdot \vec{r}_1)$ is the sum of the terms in (B1) containing the factor $\exp(i\vec{k}_j \cdot \vec{r}_1)$. This sum can be divided into two parts. Denoting $\vec{r}_1$ as $\vec{r}$,

$$\Phi(\vec{r},\vec{s}) = \alpha_0(\vec{s}) + \Phi_R(\vec{r},\vec{s}) \qquad \text{(B3)}$$

$\alpha_0(\vec{s})$ is the sum of all terms containing the factor $\exp(i\vec{k}_0 \cdot \vec{r}_1)$, where $\vec{k}_0 = 0$. Since the phase of $\alpha_0(\vec{s})$ is the same for all $\vec{r}$, this part is phase coherent in $\vec{r}$. The second part,

$$\Phi_R(\vec{r},\vec{s}) = \sum_{j \neq 0} \alpha_j(\vec{s}) \exp(i\vec{k}_j \cdot \vec{r}_1), \qquad \text{(B4)}$$

has a phase that varies randomly with $\vec{r}$, over distances $r_C \sim 1/\Delta k$, where Δk is the range of values of $\vec{k}_j$ included in (B4). Hence $\Phi_R(\vec{r},\vec{s})$ is phase incoherent. Since $\Delta k \sim (2MkT)^{1/2}$ in the ideal Bose gas, $r_C \sim \lambda_T$, where λ_T is the well-known "thermal wavelength."

The weight of $\Phi_R(\vec{r},\vec{s})$ is determined by the number of particles excited from the condensate state $\vec{k}_0 = 0$. When N particles are excited, $\Phi_C(\vec{r},\vec{s}) = 0$ and $\Phi(\vec{r},\vec{s})$ is phase incoherent, whereas with no particles excited $\Phi_R(\vec{r},\vec{s}) = 0$ and $\Phi(\vec{r},\vec{s})$ reduces to the phase coherent ground state. The phase coherence properties of $\Phi(\vec{r},\vec{s})$ vary smoothly between these two limiting behaviors as $T$is varied and different numbers of particles are excited. Thus the wave functions of the ideal Bose gas have the properties assumed in Sec. III.

2. Wave functions of the Bijl-Feynman type

The Bijl-Feynman form for wave functions of excited states of liquid ^{4}He is[63]

$$\Psi(\vec{r},\vec{s}) = \Theta(\vec{r},\vec{s}) \times \Psi_0(\vec{r},\vec{s}), \qquad \text{(B5)}$$

where $\Psi_0(\vec{r},\vec{s})$ is the ground state wave function. The standard form for $\Theta(\vec{r},\vec{s})$ is[52,55,63] (denoting $\vec{r}_2, \ldots, \vec{r}_N$ as $\vec{s}$)

$$\Theta(\vec{r}_1,\vec{s}) = C \prod_{j=1} \left(\sum_{n=1}^{N} \exp(ik_j \cdot r_n) \right)^{n_j}. \qquad \text{(B6)}$$

C is a normalization factor, n_j is the number of phonon-roton excitations with wave vector $\vec{k}_j$, and the total number of excitations is $M = \Sigma n_j$.

$\Theta(\vec{r},\vec{s})$ can be expanded as the sum of N^M terms, each containing M factors of the form $\exp(ik_j \cdot r_n)$. Writing Eq. (B6) in the form

$$\Theta(\vec{r}_1,\vec{s}) = C \prod_{n=2} \left(\exp(ik_j \cdot r_1) + \sum_{n=2}^{N} \exp(ik_j \cdot r_n) \right)^{n_j}, \qquad \text{(B7)}$$

it can be seen that $(N-1)^M$ of these terms do not contain $\vec{r}_1$. Thus $\Theta(\vec{r},\vec{s})$ can be divided into two parts, as was the case in the ideal Bose gas

$$\Theta(\vec{r},\vec{s}) = b(\vec{s}) + \Theta_R(\vec{r},\vec{s}). \qquad \text{(B8)}$$

$b(\vec{s})$ is the sum of the $(N-1)^M$ terms not containing $\vec{r}_1 (= \vec{r})$, and hence is phase coherent in $\vec{r}$. The function $\Theta_R(\vec{r},\vec{s})$ is the sum of all terms containing $\vec{r}$, and has a phase that varies randomly over distances $\sim r_C \sim \hbar/\Delta k$. In this case Δk is the range of wave vectors of $\vec{k}_j$ of phonon-roton excitations existing at the given temperature. As in the ideal Bose gas, $r_C \sim \lambda_T \sim 5-10$ Å. Since the phase of $\Psi_0(\vec{r},\vec{s})$ is constant, the phase of $\Psi(\vec{r},\vec{s})$ defined in Eq. (B5) is the same as the phase of $\Theta_R(\vec{r},\vec{s})$. Hence $\Psi(\vec{r},\vec{s})$ also can be divided into phase coherent and phased incoherent components, as assumed in Sec. III. The fraction of terms containing $\vec{r}_1$ and that therefore contribute to $\Theta_R(\vec{r},\vec{s})$ is

$$1-\frac{(N-1)^M}{N^M} \sim \frac{M}{N} \tag{B9}$$

in the limit $N \rightarrow \infty$. Thus the weight of the phase coherent component increases smoothly as temperature is raised and more excitations are created, as is the case for the normal fluid in Landau theory. The wave function is completely phase incoherent and BEC ceases to exist, when $\sim N$ excitations are present.

It is worth noting that in two-dimensional (2D) systems long-range phase coherence in the wave function is present only in the ground state, since at any finite temperature there is no BEC.[64] In the ideal 2D Bose gas, there are N particles excited from the condensed state at finite $T$and the phase coherent component therefore has zero weight. Similarly 2D wave functions of the Bijl-Feynman type must contain $\sim N$ excitations at finite T. In both cases the coherence length decreases smoothly from $\sim L$ as T is raised, whereas in three dimensions the wave function retains a component with coherence length $\sim L$ at all temperatures below the condensation temperature.

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