# Exact dynamics of a two-qubit system in a spin star environment

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We derive the exact reduced dynamics of a central two-qubit system in a spin star configuration. The exact evolution of the reduced system density matrix is obtained and we compute the limit of an infinite number of environment spins. Initially pure states of the central system evolve into mixed ones and we determine the decoherence-free states of the model. The long-time behavior is studied, partial decoherence is shown to be a result of the coupling of the qubits to the environment, and entanglement evolution of the central system is investigated.

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## I. INTRODUCTION

Multiqubit systems are of great importance in many fields of quantum technology. Experimental and theoretical evidences, accumulated during the last few years, indicate that they exhibit interesting properties that make them central subjects in quantum information processing and quantum computation.[1–3] The inherent dissipation and decoherence phenomena due to the interaction with a surrounding environment with many degrees of freedom, unfortunately, limit their usefulness.

Recently, questions related to entanglement and decoherence of some multiqubit systems have been investigated. Mainly, attention was focused on thermal entanglement, i.e., entanglement induced by the interaction of the multiqubit system with an environment at thermal equilibrium. Usually, these approaches are within the framework of a master equation for the reduced density matrix of the central system and within the Markovian approximation. The main assumption is that the characteristic times of the interacting systems are much longer than those of the environment.[4] The Markovian dynamics is known to be widely applicable in quantum optics and in the study of quantum noise.[5]

Several investigations have shown that dynamics of multiqubit systems shows strong non-Markovian behavior. Therefore, one has to seek new approaches in order to study them. The Ising and transverse Ising model were first applied to the description of the reduced dynamics of one-qubit and two-qubit systems under a symmetry broken environment in thermal equilibrium where phase transitions occur.[6–8] Later, another model was proposed[9] in which the central system is immersed in an environment composed of $N$ spin $\frac{1}{2}$ particles arranged in a star structure. In Ref. 10 the exact solution of the dynamics of one-qubit system in spin star configuration was found assuming a Heisenberg $XY$ interaction. In this model, the spin bath was in an unpolarized infinite temperature state.

The present paper provides an extension of the above model to the dynamics of a two-qubit system coupled to a spin star environment. The model is exactly solvable because of the symmetry of the structure under consideration. As mentioned in Ref. 10, this may represent a method to investigate the validity of approximation techniques and numerical methods applied to the non-Markovian dynamics.

The paper is organized as follows. In Sec. II we give a detailed description of the model. In Sec. III we derive the exact dynamics of the reduced system. In Sec. IV we study the case of an infinite number of environment spins, we determine the correlation functions, and we study the long-time behavior of the density matrix of the central system. We end the paper with a brief conclusion regarding decoherence and evolution of entanglement of the two-qubit system.

## II. THE MODEL

We consider a system of two noninteracting qubits coupled to a set of $N$ independent spin $\frac{1}{2}$ particles (the environment). We restrict ourselves to the case of a spin star configuration. This is a structure in which the two-qubit system is surrounded by the $N$ spin $\frac{1}{2}$ particles located on the surface of a sphere. The central qubits as well as the environment are multipartite systems living in spaces given by twofold and $N$-fold tensor products of the local two-dimensional spin spaces corresponding to the individual particles. From an open quantum system point of view the central system is considered as an open system coupled to an environment with a large number of degrees of freedom.

The nature of the coupling between the qubits and the environment is in general complicated and depends on the details of the interaction. Nevertheless, some symmetry properties characterizing the spin star configuration lead to an enormous simplification of the model. Indeed, under some conditions,[9] the structure in consideration is invariant with respect to the exchange of any two outer spins. Moreover, the spin star configuration is a rotationally invariant system which is the direct result of the isotropy of the environment. More details about SO(3)-invariant spin systems can be found in Ref. 11.

### A. The qubits

Let us first consider the general case of a bipartite system $\mathbf{S}$ composed of two particles with spins $j_1$ and $j_2$. The space $\mathbb{C}^s$ of the composite system is given by the tensor product

 

$$\mathbb{C}^s = \mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}. \tag{1}$$

Here, $d_i = 2j_i + 1$ denotes the dimension of the space $\mathbb{C}^{d_i}$ corresponding to the particle with spin $j_i$. The total angular momentum of the global system is defined by

$$\hat{J} = \hat{J}_1 \otimes \mathbb{I} + \mathbb{I} \otimes \hat{J}_2, \tag{2}$$

where $\hat{J}_1$ and $\hat{J}_2$ are the angular momentum operators of the individual particles and $\mathbb{I}$ denotes the unit matrices on $\mathbb{C}^{d_1}$ and $\mathbb{C}^{d_2}$.

The standard basis in the space $\mathbb{C}^{d_i}$ is composed of the eigenvectors of the operator $\hat{J}_{iz}$ with eigenvalues $m_i = -j_i, -j_i + 1, \ldots, j_i$ with $i = 1, 2$. We denote the vectors of this basis by $|j_i, m_i\rangle$ to stress that on this space $\hat{J}_i^2 = j_i(j_i+1)\mathbb{I}$.

The composite system admits, now, two equivalent orthonormal bases. The first one is formed by the common eigenvectors $|j_1, j_2, m_1, m_2\rangle$ of the set of operators $\{\hat{J}_1^2, \hat{J}_2^2, \hat{J}_{1z}, \hat{J}_{2z}\}$, they are given by the tensor products

$$|j_1, j_2, m_1, m_2\rangle = |j_1, m_1\rangle \otimes |j_2, m_2\rangle. \tag{3}$$

The second one is a standard basis constructed from the simultaneous eigenstates of the square of the total angular momentum operator $\hat{J}^2$ and its projection along the $z$ axis $\hat{J}_z$, namely $\{|j, m\rangle\}$ with $|j_1 - j_2| \leq j \leq j_1 + j_2$ and $-j \leq m \leq j$. As usual, we introduce the lowering and the raising operators $\hat{J}_\pm = \hat{J}_x \pm i\hat{J}_y$. The action of these operators on a vector $|j, m\rangle$ belonging to the standard basis of the total space is given by ($\hbar = 1$)

$$\hat{J}^2|j,m\rangle = j(j+1)|j,m\rangle,$$

$$\hat{J}_z|j,m\rangle = m|j,m\rangle,$$

$$\hat{J}_\pm|j,m\rangle = \sqrt{j(j+1) - m(m\pm 1)}|j,m\pm 1\rangle. \tag{4}$$

In the special case of two spin $\frac{1}{2}$ particles, the total angular momentum $j$ takes either the value one or zero. One possible basis in the four-dimensional space $\mathbb{C}^2 \otimes \mathbb{C}^2$ consists of the state vectors $\{|++\rangle, |+-\rangle, |-+\rangle, |--\rangle\}$ which correspond to the different mutual orientations of the two spin vectors with respect to the $z$ direction. The connection with the standard basis $\{|jm\rangle\}$ of the composite system leads with an appropriate choice of the phase to

$$|1,1\rangle = |++\rangle,$$

$$|1,0\rangle = \frac{1}{\sqrt{2}}(|+-\rangle + |-+\rangle),$$

$$|1,-1\rangle = |--\rangle,$$

$$|0,0\rangle = \frac{1}{\sqrt{2}}(|+-\rangle - |-+\rangle). \tag{5}$$

The picture is equivalent to the decomposition of the two-qubit space $\mathbb{C}^2 \otimes \mathbb{C}^2$ into a direct sum of the spaces $\mathbb{C}$ and $\mathbb{C}^3$ corresponding to spin 0 (antisymmetric vectors) and spin 1 (symmetric vectors), respectively,[12]

$$\mathbb{C}^2 \otimes \mathbb{C}^2 = \mathbb{C} \oplus \mathbb{C}^3. \tag{6}$$

### B. The environment

The above approach can be easily generalized to an arbitrary number of outer spins. In particular, the total angular momentum operator of the spin environment is simply given by the sum of the individual spin $\frac{1}{2}$ operators. The environment space $(\mathbb{C}^2)^{\otimes N}$ is equal to a direct sum of subspaces $\mathbb{C}^{d_j}$ where $0 \leq j \leq \frac{N}{2}$. Due to the different possible orientations of the single spins,[9,13] the angular momentum $j$ will have a degeneracy $\nu(N,j)$. We denote this formally as

$$(\mathbb{C}^2)^{\otimes N} = \bigoplus_{j=0}^{N/2} \nu(N,j)\mathbb{C}^{d_j}. \tag{7}$$

The degeneracy $\nu(N,j)$ is given by[13]

$$\nu(j,N) = \binom{N}{N/2 - j} - \binom{N}{N/2 - j - 1} \quad \text{with} \binom{N}{-1} = 0. \tag{8}$$

Obviously, the following equality holds:

$$\sum_{j=0}^{N/2} \nu(j,N)(2j+1) = 2^N. \tag{9}$$

### C. The Hamiltonian

We assume that the two qubits do not interact with each other. Moreover, we will neglect any kind of interactions between the constituents of the environment, the main contribution to the total Hamiltonian comes from the interaction between the central qubits and the environment. The strength of the interaction is supposed to be the same for any two interacting particles, this insures the symmetry with respect to permutations of the outer spins. The qubits are coupled to the environment via an Heisenberg $XY$ interactions whose Hamiltonian is given by

$$H = \alpha[(\sigma_+^1 + \sigma_+^2) \otimes J_- + (\sigma_-^1 + \sigma_-^2) \otimes J_+], \tag{10}$$

where $\alpha$ denotes the strength of the interaction. In this expression, $\sigma_1$ and $\sigma_2$ are Pauli matrices associated to each of the central qubits and $J_\pm$ denote the raising and lowering operators of the environment which consists of $N$ spin 1/2 particles.

Consequently, the action of $H$ on any state vector of the form $|00\rangle \otimes |\Phi_B\rangle$ always gives a vanishing result. Taking into account this fact and the symmetry of the problem, it is sufficient to consider only the space $\mathbb{C}^3 \otimes (\mathbb{C}^2)^{\otimes N}$. The subspace $\mathbb{C}^3$ is spanned by the vectors $|1,-1\rangle$, $|1,0\rangle$, and $|1,1\rangle$. In this basis the lowering and raising operators admit the following representation:

$$\sigma_+ = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad \sigma_- = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}. \tag{11}$$

Therefore, the Hamiltonian $H$ can be written as

$$H = \alpha \begin{pmatrix} 0 & J_+ & 0 \\ J_- & 0 & J_+ \\ 0 & J_- & 0 \end{pmatrix}. \tag{12}$$

One can easily prove by induction that powers of $H$ are given by

$$H^{2n} = \alpha^{2n} \begin{pmatrix} J_+K^{n-1}J_- & 0 & J_+K^{n-1}J_+ \\ 0 & K^n & 0 \\ J_-K^{n-1}J_- & 0 & J_-K^{n-1}J_+ \end{pmatrix}, \tag{13}$$

$$H^{2n+1} = \alpha^{2n+1} \begin{pmatrix} 0 & J_+K^n & 0 \\ K^nJ_- & 0 & K^nJ_+ \\ 0 & J_-K^n & 0 \end{pmatrix}. \tag{14}$$

Here $K$ denotes the anticommutator of the operators $J_+$ and $J_-$, that is

$$K = J_+J_- + J_-J_+ = 2(J^2 - J_z^2). \tag{15}$$

However, $K$ is diagonal in the standard basis of $(\mathbb{C}^2)^{\otimes N}$ with eigenvalues $2[j(j+1)-m^2]$ and satisfies the following commutation relations:

$$[K,J^2] = [K,J_z] = 0,$$

$$[K,J_-J_+] = [K,J_+J_-] = 0. \tag{16}$$

Equations (13) and (14) allow us to explicitly write out any function of the Hamiltonian restricted to $\mathbb{C}^3 \otimes (\mathbb{C}^2)^{\otimes N}$. In particular the explicit form of the time evolution operator $U(t)=\exp(-iHt)$ reads

$$U(t) = \begin{pmatrix} 1 + J_+\dfrac{\cos(\alpha t\sqrt{K})-1}{K}J_- & -iJ_+\dfrac{\sin(\alpha t\sqrt{K})}{\sqrt{K}} & J_+\dfrac{\cos(\alpha t\sqrt{K})-1}{K}J_+ \\ -i\dfrac{\sin(\alpha t\sqrt{K})}{\sqrt{K}}J_- & \cos(\alpha t\sqrt{K}) & -i\dfrac{\sin(\alpha t\sqrt{K})}{\sqrt{K}}J_+ \\ J_-\dfrac{\cos(\alpha t\sqrt{K})-1}{K}J_- & -iJ_-\dfrac{\sin(\alpha t\sqrt{K})}{\sqrt{K}} & 1 + J_-\dfrac{\cos(\alpha t\sqrt{K})-1}{K}J_+ \end{pmatrix}. \tag{17}$$

## III. EXACT REDUCED DYNAMICS

The state of the composite system is completely characterized by the total density matrix $\rho(t)$ whose evolution in time is given by

$$\rho(t) = U(t)\rho(0)U^\dagger(t). \tag{18}$$

Here $U(t)$ is the time evolution operator and $\rho(0)$ denotes the initial density matrix in the space $\mathbb{C}^3 \otimes (\mathbb{C}^2)^{\otimes N}$. For time-independent Hamiltonians, the operator $U(t)$ takes the simple form

$$U(t) = \exp(-iHt) \tag{19}$$

and we could use the expression (17).

Alternatively, one can use the Liouville superoperator $\mathcal{L}$ to describe the evolution of the total density matrix $\rho(t)$[4]

$$\mathcal{L}\rho(t) = -i[H,\rho(t)]. \tag{20}$$

This leads to the von Neumann differential equation,

$$\frac{d}{dt}\rho(t) = \mathcal{L}\rho(t), \tag{21}$$

whose integral form is

$$\rho(t) = \exp(\mathcal{L}t)\rho(0). \tag{22}$$

Tracing over the environment degrees of freedom in the space $(\mathbb{C}^2)^{\otimes N}$, enables us to determine the dynamics of the reduced system density matrix, that is

$$\rho_S(t) = \mathrm{tr}_B\{\rho(t)\}. \tag{23}$$

We have used the letters $B$ and $S$ to denote the environment (bath) and the qubits (system). Both descriptions of the dynamics are of course completely equivalent. The difference just consists in a regrouping of terms. We only use the Liouville operator to obtain a more concise description of the dynamics in (34) and (35).

### A. Initial conditions

We assume that the initial condition factorizes into the uncorrelated tensor product state

$$\rho(0) = \rho_S(0) \otimes \rho_B(0), \tag{24}$$

where $\rho_S(0)$ and $\rho_B(0)$ are the initial density matrices describing the local state of the qubits and the environment, respectively. The matrices $\rho_S(0)$ and $\rho_B(0)$ are self-adjoint, positive and have trace one.

Any state vector of the qubits can be written as

$$|\psi\rangle = \beta|--\rangle + \gamma_+|+-\rangle + \gamma_-|-+\rangle + \delta|++\rangle, \qquad (25)$$

where $\beta$, $\gamma_\pm$, and $\delta$ are complex numbers satisfying $|\beta|^2 + |\gamma_+|^2 + |\gamma_-|^2 + |\delta|^2 = 1$. Using the relations (5) it is possible to rewrite $|\psi\rangle$ in the standard basis of $\mathbb{C}\oplus\mathbb{C}^3$ as

$$|\psi\rangle = \beta|1,-1\rangle + \gamma|1,0\rangle + \delta|1,1\rangle + \gamma'|0,0\rangle, \qquad (26)$$

where $\gamma=(\gamma_+ + \gamma_-)/\sqrt{2}$ and $\gamma'=(\gamma_+ - \gamma_-)/\sqrt{2}$. Thus the initial density matrix corresponding to the pure state vector $|\psi\rangle$ reads as follows:

$$\rho_S(0) = \begin{pmatrix} |\beta|^2 & \beta\gamma^* & \beta\delta^* & \beta\gamma'^* \\ \gamma\beta^* & |\gamma|^2 & \gamma\delta^* & \gamma\gamma'^* \\ \delta\beta^* & \delta\gamma^* & |\delta|^2 & \delta\gamma'^* \\ \gamma'\beta^* & \gamma'\gamma^* & \gamma'\delta^* & |\gamma'|^2 \end{pmatrix}. \qquad (27)$$

Here $z^*$ denotes the complex conjugate of $z$.

Once again, because of the symmetry of the problem and the degeneracy of the antisymmetric state vector $|0,0\rangle$, our task is reduced to study the dynamics of a spin-one particle in the space $\mathbb{C}^3$. Without loss of generality, we represent the initial reduced system density matrix restricted to this subspace by

$$\rho_S(0) = \begin{pmatrix} \rho_{11}^0 & \rho_{12}^0 & \rho_{13}^0 \\ \rho_{12}^{0*} & \rho_{22}^0 & \rho_{23}^0 \\ \rho_{13}^{0*} & \rho_{23}^{0*} & \rho_{33}^0 \end{pmatrix}. \qquad (28)$$

Obviously, one has to keep in mind that the actual normalization condition for the initial density matrix of the qubits reads $\Sigma_{i=1}^4 \rho_{ii}^0 = 1$, where $\rho_{44}^0 = |\gamma'|^2$. Although, our attention is focused on the subspace $\mathbb{C}^3$, we will investigate in parallel the evolution in time of the remaining density matrix elements.

Let us now take a look at the initial condition of the environment. It is well known that the density matrix characterizing an unpolarized bath in thermal equilibrium at temperature $T$ is given by $\rho_B(0) = (\mathrm{e}^{-H_B/k_BT})/Z$ where $H_B$ is the Hamiltonian of the environment, $k_B$ is the Boltzmann constant and $Z = \mathrm{tr}_B \mathrm{e}^{-H_B/k_BT}$ is the partition function. In our model, we assume that the environment is initially in a state of infinite temperature with a corresponding density matrix

$$\rho_B(0) = 2^{-N}\mathbb{I}_B, \qquad (29)$$

where $\mathbb{I}_B$ denotes the unity operator in the environment space.

### B. Reduced system dynamics

The time-dependent reduced density matrix is obtained by taking the partial trace over the environment degrees of freedom

$$\rho_S(t) = \mathrm{tr}_B\{\exp(-iHt)\rho_S(0)\otimes\rho_B(0)\exp(iHt)\} \qquad (30)$$

$$= \mathrm{tr}_B\{\exp(\mathcal{L}t)\rho_S(0)\otimes\rho_B(0)\}. \qquad (31)$$

Expanding the exponential function (31) in a Taylor series gives

$$\rho_S(t) = \sum_{k=1}^{\infty} \frac{t^k}{k!}\,\mathrm{tr}_B\{\mathcal{L}^k\rho_S(0)\otimes 2^{-N}\mathbb{I}_B\}. \qquad (32)$$

In the above equation, powers of the Liouville operator appear. In order to evaluate them we expand the unitary evolution operators in (30) to obtain

$$\mathcal{L}^n\rho = i^n\sum_{\ell=0}^{n}(-1)^\ell\binom{\ell}{n}H^\ell\rho H^{n-l}. \qquad (33)$$

For odd $n$'s, one always gets an extra lowering or raising operator under the trace, thus

$$\mathrm{tr}_B\{\mathcal{L}^{2n+1}\rho_S(0)\otimes 2^{-N}\mathbb{I}_B\} = 0. \qquad (34)$$

In fact, this holds for any number of central spins interacting with the environment in a similar way as in (10).

With the help of the trace properties one can find that for nonzero $n$

$$\mathrm{tr}_B\{\mathcal{L}^{2n}\rho_S(0)\otimes 2^{-N}\mathbb{I}_B\} = \sum_{k=1}^{n-1}\binom{2n}{2k}\mathcal{S}_{2k}^{2n} - \sum_{k=0}^{n-1}\binom{2n}{2k+1}\mathcal{S}_{2k+1}^{2n} + \mathcal{F}^{2n}, \qquad (35)$$

where

$$\mathcal{S}_{2k}^{2n} = (-\alpha)^n\begin{pmatrix} (\rho_{11}^0-\rho_{33}^0)R_{n-2}+\rho_{33}P_n & \rho_{12}^0 Q_k^n & \rho_{13}^0 O_k^n \\ \rho_{12}^{0*}Q_{n-k}^n & \rho_{22}^0 F_n & \rho_{23}^0 Q_{k+1}^n \\ \rho_{13}^{0*}O_{n-k}^n & \rho_{23}^{0*}Q_{n-k+1}^n & -(\rho_{11}^0-\rho_{33}^0)R_{n-2}+\rho_{33}^0 F_n+(\rho_{11}^0-2\rho_{33}^0)P_n \end{pmatrix}, \qquad (36)$$

$$\mathcal{S}_{2k+1}^{2n} = (-\alpha)^n\begin{pmatrix} \rho_{22}^0 P_n & \rho_{23}^0 Q_{k+1}^n & 0 \\ \rho_{23}^{0*}Q_{n-k}^n & (\rho_{11}^0-\rho_{33}^0)P_n+\rho_{33}^0 F_n & \rho_{12}^0 Q_{k+1}^n \\ 0 & \rho_{12}^{0*}Q_{n-k}^n & \rho_{22}^0(F_n-P_n) \end{pmatrix}, \qquad (37)$$

and

$$\mathcal{F}^{2n} = (-\alpha)^n \begin{pmatrix} 2\rho_{11}^0 P_n & \rho_{12}^0 (F_n + P_n) & \rho_{13}^0 F_n \\ \rho_{12}^{0*}(F_n + P_n) & 2\rho_{22}^0 F_n & \rho_{23}^0 (2F_n - P_n) \\ \rho_{13}^0 F_n & \rho_{23}^{0*}(2F_n - P_n) & 2\rho_{33}^0 (F_n - P_n) \end{pmatrix}. \tag{38}$$

Here we have introduced the environment correlation functions

$$R_n = 2^{-N}\,\mathrm{tr}_B\{(J_-J_+)^2 K^{n-2}\}, \tag{39}$$

$$Q_k^n = 2^{-N}\,\mathrm{tr}_B\{J_+ K^{k-1} J_- K^{n-k}\}, \tag{40}$$

$$O_k^n = 2^{-N}\,\mathrm{tr}_B\{J_+ J_+ K^{k-1} J_- J_- K^{n-k-1}\}, \tag{41}$$

$$P_n = 2^{-N}\,\mathrm{tr}_B\{J_- J_+ K^{n-1}\}, \tag{42}$$

$$F_n = 2^{-N}\,\mathrm{tr}_B\{K^n\}. \tag{43}$$

Notice that the above correlation functions are independent for small $N$, they were obtained with the help of the commutation relations (16) where we could derive simple expressions relating them. Nevertheless, the number of independent functions still remains large since the operator $K$ does not commute with any polynomial of the lowering and raising operators $J_\pm$. Besides this fact, one can see that there exists some similarity among these correlation functions as it is the case between $R_n$ and $O_k^n$ on one hand and $P_n$ and $Q_k^n$ on the other hand.

By substitution into Eq. (32) one finds the explicit form of the various matrix elements of $\rho_S(t)$ (see Appendix B). One can check that the diagonal elements are given by

$$\rho_{11}(t) = \rho_{11}^0(1 + 2g(t)) + (\rho_{11}^0 - \rho_{33}^0) f(t) + \rho_{33}^0 e(t) + \rho_{22}^0 h(t), \tag{44}$$

$$\rho_{22}(t) = \rho_{22}^0 + (\rho_{11}^0 - \rho_{33}^0) h(t) + (\rho_{33}^0 - \rho_{22}^0)\,\ell(t), \tag{45}$$

and

$$\rho_{33}(t) = \rho_{33}^0(1 - 2g(t)) - (\rho_{11}^0 - \rho_{33}^0) f(t) - \rho_{22}^0 h(t) + (\rho_{11}^0 - 2\rho_{33}^0) e(t) + (\rho_{22}^0 - \rho_{33}^0)\,\ell(t). \tag{46}$$

The off-diagonal elements read

$$\rho_{12}(t) = \rho_{12}^0[\tilde{\ell}(t) + \tilde{e}_1(t)] + \rho_{23}^0 \tilde{h}(t), \tag{47}$$

$$\rho_{13}(t) = \rho_{13}^0[\tilde{\ell}(t) + \tilde{f}(t)], \tag{48}$$

$$\rho_{23}(t) = \rho_{23}^0[\tilde{\ell}(t) + \tilde{e}_2(t)] + \rho_{12}^0 \tilde{h}(t), \tag{49}$$

$$\rho_{21}(t) = \rho_{12}^*(t), \tag{50}$$

$$\rho_{31}(t) = \rho_{13}^*(t), \tag{51}$$

$$\rho_{32}(t) = \rho_{23}^*(t). \tag{52}$$

Here we have introduced the functions

$$f(t) = 2^{-N}\,\mathrm{tr}_B\left\{ J_- J_+ \frac{\cos(\alpha t\sqrt{K}) - 1}{K} \right\}^2, \tag{53}$$

$$g(t) = 2^{-N}\,\mathrm{tr}_B\left\{ J_- J_+ \frac{\cos(\alpha t\sqrt{K}) - 1}{K} \right\}, \tag{54}$$

$$h(t) = 2^{-N}\,\mathrm{tr}_B\left\{ J_- J_+ \frac{\sin^2(\alpha t\sqrt{K})}{K} \right\}, \tag{55}$$

$$e(t) = 2^{-N}\,\mathrm{tr}_B\left\{ J_- J_+ \frac{(\cos(\alpha t\sqrt{K}) - 1)^2}{K} \right\}, \tag{56}$$

$$\ell(t) = 2^{-N}\,\mathrm{tr}_B\{\sin^2(\alpha t\sqrt{K})\}, \tag{57}$$

$$\tilde{\ell}(t) = 2^{-N}\,\mathrm{tr}_B\{\cos(\alpha t\sqrt{K})\}. \tag{58}$$

The remaining functions are quite different in their analytical form from those listed above. They are given explicitly by

$$\tilde{f}(t) = 2^{-N}\,\mathrm{tr}_B\left\{ J_-^2 \frac{\cos(\alpha t\sqrt{K}) - 1}{K} J_+^2 \frac{\cos(\alpha t\sqrt{K}) - 1}{K} \right\}, \tag{59}$$

$$\tilde{e}_1(t) = 2^{-N}\,\mathrm{tr}_B\left\{ J_+ \frac{\cos(\alpha t\sqrt{K}) - 1}{K} J_- \cos(\alpha t\sqrt{K}) \right\}, \tag{60}$$

$$\tilde{e}_2(t) = 2^{-N}\,\mathrm{tr}_B\left\{ J_- \frac{\cos(\alpha t\sqrt{K}) - 1}{K} J_+ \cos(\alpha t\sqrt{K}) \right\}, \tag{61}$$

$$\tilde{h}(t) = 2^{-N}\,\mathrm{tr}_B\left\{ J_+ \frac{\sin(\alpha t\sqrt{K})}{\sqrt{K}} J_- \frac{\sin(\alpha t\sqrt{K})}{\sqrt{K}} \right\}. \tag{62}$$

One has to be careful when dividing by the operator $K$ since its eigenvalue corresponding to $j=0$ vanishes. To overcome this difficulty, it is sufficient to write the quantity under the trace sign in the normal order, that is to first apply the lowering operator $J_-$ on the state $|0,0\rangle$ which leads obviously to zero.

In fact, the function $e(t)$ can be expressed in terms of $g(t)$ and $h(t)$. We will leave it in this form in order to maintain its symmetry with the functions $\tilde{e}_1(t)$ and $\tilde{e}_2(t)$. Altogether, we need a set of nine real-valued functions to describe the reduced system dynamics in $\mathbb{C}^3$. In the special case of one-qubit dynamics[10] the number of independent functions is significantly reduced to two because of the rotational invariance of the star configuration.

When the conditions $\rho_{11}^0=\rho_{33}^0$ and $\rho_{22}^0\neq 0$ are satisfied [one can, e.g., set $\beta=\delta$ in Eq. (26)], the diagonal elements take the relatively simple form

$$\frac{\rho_{11}(t)}{\rho_{11}^0} = 1 + (\xi - 1)h(t), \tag{63}$$

$$\frac{\rho_{22}(t)}{\rho_{22}^0} = 1 + \left(\frac{1-\xi}{\xi}\right)\ell(t), \tag{64}$$

$$\frac{\rho_{33}(t)}{\rho_{33}^0} = 1 + (\xi - 1)(\ell(t) - h(t)), \tag{65}$$

where the parameter $\xi$ is given by $\rho_{22}^0/\rho_{11}^0$.

It is not difficult to check that the solutions (44)–(46) as well as (63)–(65) ensure that the trace is preserved, that is $\Sigma_{i=1}^3\rho_{ii}(t)=\Sigma_{i=1}^3\rho_{ii}^0$. This actually results from the fact that the time evolution operator $U(t)$ is unitary and hence trace preserving. It is worth noting that the density matrix element $\rho_{44}$ does not evolve in time; the time evolution operator is reduced to 1 in the space $\mathbb{C}$. This is due to the symmetry of the Hamiltonian $H$. The subspace $\mathbb{C}$ is said to be decoherence free which was expected because of the degeneracy in energy of the antisymmetric state vector $|00\rangle$. Moreover, the density matrix elements $\rho_{i4}, i=1,2,3$ evolve according to

$$\rho_{i4}(t) = 2^{-N}\sum_{k=1}^{3}\mathrm{tr}_B\{U_{ik}(t)\}\rho_{k4}^0, \tag{66}$$

since $U_{i4}(t)$ is equal to $\delta_{i4}$. The last relation shows that the off-diagonal elements behave like the components of a three-dimensional state vector. Taking into account the fact that the partial trace of any off-diagonal element of $U(t)$ is zero, it is not difficult to find that

$$\begin{pmatrix}\rho_{14}(t)\\ \rho_{24}(t)\\ \rho_{34}(t)\end{pmatrix} = \begin{pmatrix}\rho_{14}^0(1+g(t))\\ \rho_{24}^0\widetilde{\ell}(t)\\ \rho_{34}^0(1+g(t))\end{pmatrix}. \tag{67}$$

Notice that the set of functions (53)–(62) can be rewritten in the standard basis of the environment space $(\mathbb{C}^2)^{\otimes N}$. For example, we can write the functions $f(t)$ and $\widetilde{e}_1(t)$ as

$$f(t) = \sum_{j,m}\nu(j,N)\left\{\frac{\chi(j,m)}{\omega(j,m)}\cos(\alpha t\sqrt{\omega(j,m)})\right\}^2 \tag{68}$$

and

$$\widetilde{e}_1(t) = \sum_{j,m}\nu(j,N)\frac{\chi(j,m)}{\omega(j,m-1)}\cos[\alpha t\sqrt{\omega(j,m-1)}] \times \cos[\alpha t\sqrt{\omega(j,m)}], \tag{69}$$

where the quantities $\chi(j,m)$ and $\omega(j,m)$ are the eigenvalues of the operators $J_-J_+$ and $K$, respectively,

$$\chi(j,m) = j(j+1) - m(m-1), \tag{70}$$

$$\omega(j,m) = 2(j(j+1) - m^2). \tag{71}$$

Taking the trace over the environment yields a superposition of weighted periodic functions with different frequencies. Roughly speaking, this means that the time-dependent density matrix elements evolve anharmonically starting from their initial values.

## IV. THE LIMIT OF LARGE NUMBER OF BATH SPINS

In this section we will investigate the behavior of the solution found previously when the number of the environment spins becomes very large, that is the limit $N\to\infty$.

To this end, let us anticipate and say that in the limit of large number of degrees of freedom, the environment has the tendency to behave as a classical system. Consequently, one can expect that the various operators related to the environment do commute at least for the case where the total angular momentum $j$ is very large compared to the quantum number $m$. As we will see, this will enable us to determine the long-time behavior of the reduced system density matrix.

### A. Environment correlation functions

The trace operation over the environment degrees of freedom can be carried out by writing the lowering and raising operators in the standard basis of the environment space $\{\otimes_{i=1}^N|s^i\rangle\}$, namely,

$$J_\pm = \sum_{i=1}^{N}\sigma_\pm^i, \tag{72}$$

where $\sigma_\pm^i|s^i\rangle=\pm s^i|s^i\rangle$. With help of the formula

$$K^n = \sum_{\ell=0}^{n}\binom{n}{\ell}(J_+J_-)^{n-\ell}(J_-J_+)^\ell, \tag{73}$$

the problem is reduced to the calculation of terms having the following general structure:

$$\mathcal{A}_n = \mathrm{tr}_B\left\{\prod_{i_1,i_2}^{n}J_{\kappa_{i_1}}J_{\kappa_{i_2}}\right\} = \mathrm{tr}_B\left\{\prod_{i_1,i_2}^{n}\sum_{j_1,j_2}^{N}\sigma_{\kappa_{i_1}}^{j_1}\sigma_{\kappa_{i_2}}^{j_2}\right\}, \tag{74}$$

where the index $\kappa$ indicates the nature of the operator, raising or lowering. The main restriction here is that the lowering and raising operators $J_-$ and $J_+$ must appear the same number of times under the trace in order to insure that the result is not zero. In general, $\mathcal{A}_n$ leads to a polynomial of order $n$ in the environment spins number $N$. The main contribution to such quantities comes from terms having the maximum number of indices labeling the operators $\sigma_{\kappa_i}$. This is due to the fact that these terms are characterized by the largest combinatorial weight and hence yield the largest exponent in $N$.

It is shown in Ref. 10 that

$$\mathrm{tr}_B\{(J_+J_-)^n\} \sim \mathrm{tr}_B\{(J_+J_-)^{n-\ell}(J_-J_+)^\ell\} \approx \frac{2^N N^n n!}{2^n}. \tag{75}$$

With the help of the last relation, it is easy to compute the environment correlation functions for the two-qubit case. For example, we have for $R_n$

$$R_n = 2^{-N}\mathrm{tr}_B\left\{\sum_{\ell=0}^{n-2}\binom{n-2}{\ell}(J_+J_-)^{n-\ell-2}(J_-J_+)^{2+\ell}\right\}$$
$$\approx \sum_{\ell=0}^{n-2}\binom{n-2}{\ell}\frac{N^n n!}{2^n}, \tag{76}$$

and thus

$$R_n \approx \frac{N^n n!}{4}. \tag{77}$$

Similarly, we find as $N \to \infty$ for the remaining correlation functions

$$O_k^n \sim R_n \approx \frac{N^n n!}{4}, \tag{78}$$

$$Q_k^n \sim P_n \approx \frac{N^n n!}{2}, \tag{79}$$

$$F_n \approx N^n n\,! \,. \tag{80}$$

The above method does not apply for correlation functions where at least one of the upper or lower indices is zero. In these cases the operator $K$ appears in the denominator of the correlation functions and hence the expansion (74) is no longer applicable. One alternative way to determine them is by writing the trace in the eigenbasis of $J_z$ and $J^2$. We will not present all the results here but just give one example since the method is the same and can be applied to the other correlation functions (see Appendix A). The computation yields

$$R_0 = \frac{1}{4} + \Omega_N, \tag{81}$$

where

$$\Omega_N = 2^{-N}\sum_{j,m}\nu(j,N)\frac{m^2}{4[j(j+1)-m^2]^2}. \tag{82}$$

Here we have used the property

$$\sum_{m=-j}^{j}\frac{m}{j(j+1)-m^2} = 0. \tag{83}$$

The quantity $\Omega_N$ is very small compared to 1 and can be neglected. Under this assumption both methods lead to the same result, this is actually the same thing as assuming that $K=2J_-J_+$. Thus the environment operators behave as if they commute when $N$ tends to infinity, a result which confirms the statement we gave in the beginning of this section.

### B. Time evolution

The dynamics of the reduced system can easily be determined in the limit $N \to \infty$ by properly rescaling the coupling constant $\alpha$. The substitution of the correlation functions (78)–(80) into Eqs. (36)–(38) yields

$$\mathrm{tr}_B\{\mathcal{L}^{2n}\rho_S(0)\otimes 2^{-N}\mathbb{I}_B\} = \{2^{n-1}A + C\}\frac{(-N)^n n!}{4}, \tag{84}$$

where the matrices $A$ and $C$ are given by

$$A = \begin{pmatrix} \rho_{11}^0 + \rho_{33}^0 - 2\rho_{22}^0 & 2(\rho_{12}^0 - \rho_{23}^0) & \rho_{13}^0 \\ 2(\rho_{12}^{0*} - \rho_{23}^{0*}) & 4\rho_{22}^0 - 2(\rho_{11}^0 + \rho_{33}^0) & 2(\rho_{23}^0 - \rho_{12}^0) \\ \rho_{13}^{0*} & 2(\rho_{23}^{0*} - \rho_{12}^{0*}) & \rho_{11}^0 + \rho_{33}^0 - 2\rho_{22}^0 \end{pmatrix} \tag{85}$$

and

$$C = \begin{pmatrix} 2(\rho_{11}^0 - \rho_{33}^0) & 2\rho_{12}^0 & 2\rho_{13}^0 \\ 2\rho_{12}^{0*} & 0 & 2\rho_{23}^0 \\ 2\rho_{12}^{0*} & 2\rho_{23}^{0*} & 2(\rho_{33}^0 - \rho_{11}^0) \end{pmatrix}. \tag{86}$$

Inserting Eq. (84) into Eq. (32) yields a power series with terms of the general form $[(\alpha t)^2 N]^k$. It is then natural to rescale the coupling constant by setting

$$\alpha \to \frac{\alpha}{\sqrt{N}}. \tag{87}$$

It is shown in Appendix B that in the limit $N \to \infty$ the functions (53)–(62) become

$$f(t) = \tilde{f}(t) = \frac{1}{4}\zeta(2t) - \zeta(t), \tag{88}$$

$$g(t) = \zeta(t), \tag{89}$$

$$h(t) = \tilde{h}(t) = -\frac{1}{2}\zeta(2t), \tag{90}$$

$$\ell(t) = -\zeta(2t), \tag{91}$$

$$\tilde{\ell}(t) = 1 + 2\zeta(t), \tag{92}$$

$$e(t) = \frac{1}{2}\zeta(2t) - 2\zeta(t), \tag{93}$$

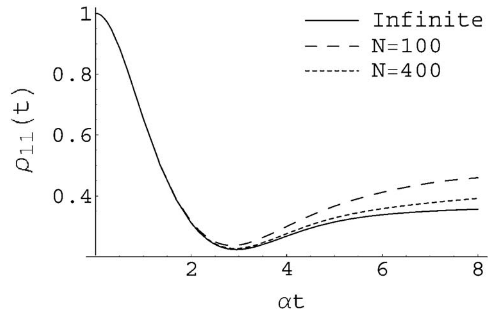


FIG. 1. The time evolution of the density matrix element $\rho_{11}$. Initial state of the two-qubit system is the pure state $|--\rangle$. The figure shows the plots obtained for $N=100$, $N=400$, and the limit $N\to\infty$.

$$\tilde{e}_{1,2}(t) = \frac{1}{2}\zeta(2t) - \zeta(t), \tag{94}$$

where

$$\zeta(t) = -\frac{\alpha t}{2} D_+\left(-\frac{\alpha t}{2}\right). \tag{95}$$

Here $D_+(x)$ denotes the Dawson function, also called Dawson's integral which arises from the calculation of the Voigt spectral lines shape.[14] It is given by

$$D_+(x) = e^{-x^2}\int_0^x e^{t^2}dt. \tag{96}$$

Dawson's function is related to the imaginary error function erfi$(x)$ by[15]

$$D_+(x) = \frac{\sqrt{\pi}}{2} e^{-x^2}\,\mathrm{erfi}(x). \tag{97}$$

As opposed to the ordinary error function, the imaginary one is unbounded. It is given by the following series expansion:

$$\mathrm{erfi}(x) = \frac{2}{\sqrt{\pi}}\sum_{k=0}^{\infty}\frac{x^{2k+1}}{k\,!\,(2k+1)}. \tag{98}$$

It is then sufficient to substitute the above functions into the set of Eqs. (44)–(49) to get the new form of the density matrix elements.

Fortunately, the function $\zeta(t)$ is bounded and admits a limit when $t$ tends to infinity. In order to determine this limit let us stress that the $J_\pm/\sqrt{N}$ are well behave fluctuation operators with respect to the tracial state on the bath. From a mathematical point of view, the above statement means that $J_+/\sqrt{N}$ converges to a complex random variable $z$ with probability density function

$$z \to \frac{2}{\pi} e^{-2|z|^2}. \tag{99}$$

The explicit form of the functions (53)–(62) shows that it is sufficient to calculate the expectation value of the function $\cos(\beta|z|)$ where $\beta \in \mathbb{R}$, namely,

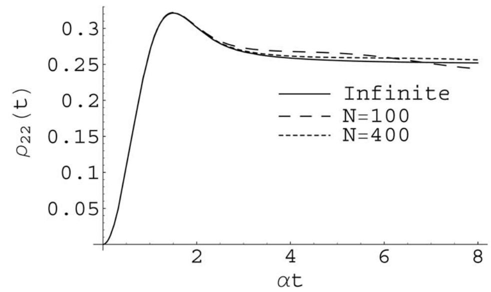


FIG. 2. The evolution in time of the density matrix element $\rho_{22}$ as a function of time. The initial condition of the two-qubit system is the pure state $|--\rangle$. The figure shows the plots obtained for $N=100$, $N=400$, and the limit $N\to\infty$.

$$G(\beta) = \frac{2}{\pi}\int_{\mathbb{C}} dz dz^* e^{-2|z|^2}\cos(\beta|z|). \tag{100}$$

In order to obtain the asymptotic state we need to take the limit $\beta\to\infty$, but this is straightforward by the Riemann-Lebesgue lemma and so we simply obtain

$$\lim_{\beta\to\infty} G(\beta) = 0. \tag{101}$$

It is easy to see that $G(\alpha t) \equiv \tilde{\ell}(t)$, it follows that the limit of the function $\xi(t)$ is equal to $-\frac{1}{2}$. Moreover, we can show that the following relation holds for any value of the nonzero real parameter $\theta$:

$$\lim_{t\to\infty}\zeta(\theta t) = -\frac{1}{2}. \tag{102}$$

Therefore, as $t\to\infty$

$$f(t), \tilde{f}(t) \to \frac{3}{8}, \tag{103}$$

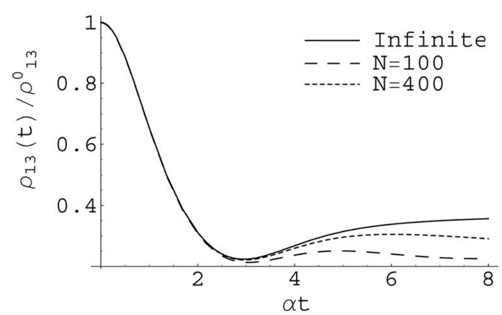


FIG. 3. The evolution in time of the density matrix element $\rho_{13}$. The initial condition of the two-qubit system is the entangled state $\frac{1}{\sqrt{2}}(|++\rangle+|--\rangle)$. The figure shows the plots obtained for $N=100$, $N=400$, and the limit $N\to\infty$.

$$g(t) \to -\frac{1}{2}, \tag{104}$$

$$h(t), \tilde{h}(t) \to \frac{1}{4}, \tag{105}$$

$$\ell(t) \to \frac{1}{2}, \tag{106}$$

$$\tilde{\ell}(t) \to 0, \tag{107}$$

$$e(t) \to \frac{3}{4}, \tag{108}$$

$$\tilde{e}_{1,2}(t) \to \frac{1}{4}. \tag{109}$$

Consequently, the long-time limit of the reduced system dynamics yields the following density matrix in $\mathbb{C} \oplus \mathbb{C}^3$:

$$\rho_S^\infty = \begin{pmatrix} \frac{3}{8}\left(\rho_{11}^0 + \rho_{33}^0 + \frac{2}{3}\rho_{22}^0\right) & \frac{1}{4}\rho_{12}^0 + \frac{1}{4}\rho_{23}^0 & \frac{3}{8}\rho_{13}^0 & \frac{1}{2}\rho_{14}^0 \\ \frac{1}{4}\rho_{12}^{0*} + \frac{1}{4}\rho_{23}^{0*} & \frac{1}{4}(\rho_{11}^0 + \rho_{33}^0 + 2\rho_{22}^0) & \frac{1}{4}\rho_{23}^0 + \frac{1}{4}\rho_{12}^0 & 0 \\ \frac{3}{8}\rho_{13}^{0*} & \frac{1}{4}\rho_{23}^{0*} + \frac{1}{4}\rho_{12}^{0*} & \frac{3}{8}\left(\rho_{11}^0 + \rho_{33}^0 + \frac{2}{3}\rho_{22}^0\right) & \frac{1}{2}\rho_{34}^0 \\ \frac{1}{2}\rho_{14}^{0*} & 0 & \frac{1}{2}\rho_{34}^{0*} & \rho_{44}^0 \end{pmatrix}. \tag{110}$$

In Figs. 1 and 2, we have drawn the variation of the diagonal elements $\rho_{11}(t)$ and $\rho_{22}(t)$, respectively, for the pure initial state $|--\rangle$. The graphs were obtained for $N=100$, $N=400$ and the limit $N\to\infty$. The evolution in time of the off-diagonal element $\rho_{13}(t)$ corresponding to the maximally entangled state $\frac{1}{\sqrt{2}}(|++\rangle+|--\rangle)$ is given in Fig. 3. The plots show that the solution corresponding to infinite number of environment spins $N\to\infty$ is almost identical to the exact solutions up to a value of time given by $\alpha t \approx 3$ then the curves start to diverge.

### C. Decoherence and entanglement

From formula (110) we see that the off-diagonal elements show partial decoherence. Indeed, the ratio between the asymptotic and the initial values of the density matrix element $\rho_{13}$ is equal to $\frac{3}{8}$. The contribution to the final result of the two other off-diagonal elements, $\rho_{12}$ and $\rho_{23}$, is symmetrically shared by their original values with the same weight, namely $\frac{1}{4}$. This can be seen, for example, in the case where the initial condition is the separable state $\frac{1}{\sqrt{2}}|-\rangle(|+\rangle+|-\rangle)$ or $\frac{1}{\sqrt{2}}|+\rangle(|+\rangle+|-\rangle)$. In particular, if the condition $\rho_{12}^0=\rho_{23}^0$ is satisfied, both matrix elements relax and assume half of their initial value. Similarly, the off-diagonal elements $\rho_{14}$ and $\rho_{13}$ evolve asymptotically to half their original values whereas the element $\rho_{24}$ relaxes and tends to zero.

A first look at the explicit form of the diagonal elements of the density matrix reveals that they only depend on the corresponding initial ones. Let us choose $\xi\to 1$ in relations (63)–(65) and assume that the remaining off-diagonal elements vanish. The resulting density matrix corresponds to the diagonal initial state $\frac{1}{3}(|1,1\rangle\langle 1,1| + |1,0\rangle\langle 1,0| + |1,-1\rangle\langle 1,-1|)$. It is not a hard task to see that this state does not change in time. Consequently, the two qubits do not feel the presence of the environment. The same result holds for the entangled antisymmetric state $|00\rangle$ which belongs to the decoherence-free subspace $\mathbb{C}$.

Because of the coupling between the central system and the environment, entanglement between the two qubits may appear. Assume, for instance, that the two-qubit system was initially in a pure state, $|--\rangle$ or $|++\rangle$, for example. This corresponds to the preparation of a spin-one particle in the pure states $|1,-1\rangle$ and $|1,1\rangle$, respectively. Once the interaction is switched on, the system evolves into a mixed state.

The case where the initial condition is one of the maximally entangled states $\frac{1}{\sqrt{2}}(|+-\rangle \pm |-+\rangle)$, is quite special. Indeed, the latter are regarded as pure states for the composite system, they generally evolve into mixed states when exposed to the environment. One then asks whether the evolving state is entangled or separable.

In order to quantify the amount of entanglement created between the two qubits, we shall use the concurrence, $C(\rho)$, as a measure of entanglement for mixed states. The numerical values of the concurrence range from 0 for separable states to 1 for maximally entangled states. According to Refs. 16 and 17, $C(\rho)$ is defined as follows:

$$C(\rho) = \max\{0, \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4}\}. \tag{111}$$

$\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 \geqslant \lambda_4$ are the eigenvalues of the operator $\rho(V\otimes V)\rho(V\otimes V)$ where $V$ is a linear skew-adjoint operator in $\mathbb{C}\oplus\mathbb{C}^3$ such that $VV=-\mathbb{I}$. In our case

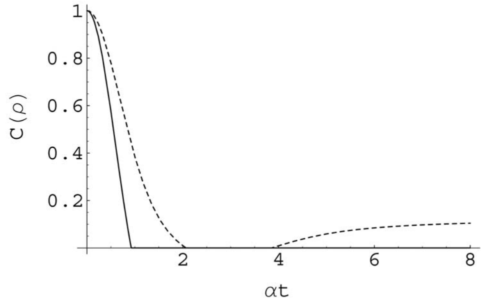


FIG. 4. Concurrence as a function of time for initial states $\frac{1}{\sqrt{2}}(|+-\rangle+|-+\rangle)$ (solid curve) and $\frac{1}{\sqrt{2}}(|--\rangle+|++\rangle)$ (dashed curve).

$$V\otimes V=\begin{pmatrix}0 & 0 & 1 & 0\\ 0 & -1 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}. \tag{112}$$

In the following we pick out some typical results.

(i) The concurrence corresponding to the initial separable state $|--\rangle$ is equal to

$$C(\rho)=\max\{0,-\rho_{22}(t)\}=0. \tag{113}$$

However, the two-qubit state maintains its separability during time which means that no entanglement will be produced by the interaction with the environment. For the same reason, the initial state $|++\rangle$ evolves into a separable state too. In fact, the latter result is also true for the general case of pure separable states when one of the qubits is in the state $|-\rangle$ (or $|+\rangle$) and the other one is at an angle, say $\theta$, from the first qubit.

(ii) If the initial state is the maximally entangled state $|\Psi\rangle=\frac{1}{\sqrt{2}}(|+-\rangle+|-+\rangle)$, then the concurrence takes the form

$$C(\rho)=\max\{0,\rho_{22}(t)-2\sqrt{\rho_{11}(t)\rho_{33}(t)}\}. \tag{114}$$

The time behavior of $C(\rho)$ is shown in the plot of Fig. 4 where one can see that it quickly decreases and vanishes after a relatively short time. The two-qubit state is completely disentangled whence the asymptotic state becomes separable. Consequently, the coupling between the central system, initially in the maximally entangled state $|\Psi\rangle$, and the spin environment causes the qubits to lose entanglement.

(iii) Let us now consider the maximally entangled state $|\Phi\rangle=\frac{1}{\sqrt{2}}(|--\rangle+|++\rangle)$. In this case the concurrence reads

$$C(\rho)=\max\{0,2\rho_{13}(t)-\rho_{22}(t)\}. \tag{115}$$

The entanglement dynamics in this case is significantly different from the one corresponding to $|\Psi\rangle$. Indeed, the entanglement here decays from its maximum value, one, and vanishes within a certain interval of the time, then starts to increase and tends asymptotically to $C^{\infty}(\rho)=\frac{1}{8}$ as shown in Fig. 4. Hence, the state loses its entanglement for a short period of time in which the state is separable, entanglement between the qubits will appear again while the asymptotic state is partially entangled. Thus the effect of the environment is to decrease the amount of entanglement of the initial state.

The above state is a special case of the so-called Werner states; the general form of the density matrix corresponding to these states is given by

$$\rho^0=\frac{1}{4}(1-p)\mathbb{I}_4+p|\Phi\rangle\langle\Phi| \tag{116}$$

with $0\leqslant p\leqslant 1$. One can show that the asymptotic density matrix is

$$\rho^{\infty}=\begin{pmatrix}\frac{2+p}{8} & 0 & \frac{3p}{16} & 0\\ 0 & \frac{1}{4} & 0 & 0\\ \frac{3p}{16} & 0 & \frac{2+p}{8} & 0\\ 0 & 0 & 0 & \frac{1-p}{4}\end{pmatrix} \tag{117}$$

and has the concurrence

$$C(\rho^{\infty})=\max\left\{0,\frac{5p-4}{8}\right\}. \tag{118}$$

This implies that the stationary state of the two-qubit system is entangled if $p>\frac{4}{5}$. When the last condition is satisfied the concurrence behaves in the same manner as the one associated with $|\Phi\rangle$, i.e., decreases from its initial maximum value, vanishes for certain interval of time to increase asymptotically to $C(\rho^{\infty})$. Once again, we find that the two-qubit state becomes partially entangled.

(iv) Because of the symmetry, the concurrence corresponding to the initial states $\frac{1}{2}(|+\rangle\pm|-\rangle)(|+\rangle\pm|-\rangle)$ and $\frac{1}{2}(|+\rangle\pm|-\rangle)(|+\rangle\mp|-\rangle)$ is identically zero. The corresponding asymptotic states are always separable.

## V. CONCLUSION

In this paper, we have studied the dynamics of a two-qubit system in a spin star configuration. The Hamiltonian we chose describes a Heisenberg *XY* interaction. We obtained the exact analytical solution for the time evolution of the reduced system density matrix. This model can also describe the dynamics of a spin-one particle coupled to an environment. It may be used to test validity of numerical approximation techniques.

The solution which we have obtained simplifies in the limit of a large number of environmental spins. The limit is carried out by rescaling the coupling constant $\alpha$. The long-time behavior of the density matrix reveals that some of the off-diagonal elements show partial decoherence. The pure entangled state $\frac{1}{\sqrt{2}}(|+-\rangle-|-+\rangle)$ of the two qubits is found to be decoherence-free, the mixed state $\frac{1}{3}(|1,-1\rangle\langle 1,-1|+|1,0\rangle\langle 1,0|+|1,1\rangle\langle 1,1|)$ written in the standard basis of a spin-one particle does not evolve in time. In these cases the central system does not feel the presence of the environment.

Any pure state of the two-qubit system evolves into a mixed state. It turns out that the environment has no effect on the separability of pure separable states. On the contrary, it has the tendency to decrease the degree of entanglement of initially entangled states of the two-qubit system. This can be understood from the high symmetry of the *XY* interaction.

Many scenarios are possible regarding the extension of the model. One first step may consist of adding a suitable term to the interaction Hamiltonian and investigate the production of entanglement between the two qubits. Recently the dynamics of three qubits in a symmetry broken fermionic environment has been exactly solved.[8] This could be also investigated within the framework of the Heisenberg interaction and may be extended to more qubit cases. Because of the high symmetry of the Hamiltonian, we expect that some structure will appear when the number of central spins increases.[11] It will also be of interest to investigate the dynamics for environments that are in coherent or squeezed states.

## ACKNOWLEDGMENTS

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## APPENDIX A: CORRELATION FUNCTIONS

This Appendix is devoted to the derivation of the explicit form of some correlation functions in which the operator $K$ appears with a negative power. The point here is to write the trace over the environment in the joint standard basis of $J^2$ and $J_z$, this gives

$$\mathrm{tr}_B\left(\frac{J_-J_+}{K}\right)^2 = \sum_{j,m} \nu(j,N)\frac{[j(j+1)-m(m+1)]^2}{4[j(j+1)-m^2]^2}. \quad \text{(A1)}$$

This equation can be rewritten as

$$\mathrm{tr}_B\left(\frac{J_-J_+}{K}\right)^2 = \frac{1}{4}\sum_{j,m} \nu(j,N)\left\{1+\frac{m^2}{(j(j+1)-m^2)^2}\right\}, \quad \text{(A2)}$$

where we have used the fact that

$$\sum_{j,m} \nu(j,N)\frac{m}{j(j+1)-m^2} = 0. \quad \text{(A3)}$$

Taking into account the relation $\Sigma_{j,m}\nu(j,N)=2^N$, we find

$$R_0 = \frac{1}{4}+\Omega_N \quad \text{and} \quad \Omega_N = 2^{-N}\sum_{j,m}\frac{m^2}{4(j(j+1)-m^2)^2}. \quad \text{(A4)}$$

Similarly, we have

$$\mathrm{tr}_B\left\{\frac{(J_-J_+)^2}{K}\right\} = \frac{1}{2}\sum_{j,m}\nu(j,N)\left\{j(j+1)-m^2+\frac{m^2}{(j(j+1)-m^2)}\right\}. \quad \text{(A5)}$$

With the help of Eq. (A3), we find

$$\mathrm{tr}_B\left\{\frac{(J_-J_+)^2}{K}\right\} = \frac{1}{2}\,\mathrm{tr}_B\{J_-J_+\} + \frac{1}{2}\sum_{j,m}\nu(j,N)\frac{m^2}{j(j+1)-m^2}. \quad \text{(A6)}$$

Then

$$R_1 = \frac{N}{2}+\Gamma_N$$

and

$$\Gamma_N = 2^{-N}\frac{1}{2}\sum_{j,m}\nu(j,N)\frac{m^2}{j(j+1)-m^2}. \quad \text{(A7)}$$

With the same method one can find that

$$Q_0^0 = \frac{1}{4}-\Omega_N, \quad Q_0^1 = \frac{N}{2}-\Gamma_N \quad \text{and} \quad P_0 = \frac{1}{2}. \quad \text{(A8)}$$

One can check that for sufficient large values of $N$

$$\Omega_N \ll 1 \quad \text{and} \quad \Gamma_N \ll N. \quad \text{(A9)}$$

## APPENDIX B: A USEFUL DERIVATION

In this Appendix we show how to find the explicit form of the functions (53)–(62) when $N\to\infty$. We just consider $f(t)$, the other functions can be determined with a similar procedure. We have from Eq. (53)

$$f(t) = 2^{-N}\,\mathrm{tr}_B\left\{\left(J_-J_+\frac{\cos(\alpha t\sqrt{K})}{K}\right)^2 - 2\left(\frac{J_-J_+}{K}\right)^2\cos(\alpha t\sqrt{K}) + \left(\frac{J_-J_+}{K}\right)^2\right\}. \quad \text{(B1)}$$

The first term in the right-hand side of the above equation can be written as

$$2^{-N}\,\mathrm{tr}_B\left\{\left(\frac{J_-J_+}{K}\right)^2\left[\frac{1}{2}+\sum_{n=0}^{\infty}(-1)^n\frac{(2\alpha t)^{2n}}{2(2n)!}K^n\right]\right\}=R_0-R_1(\alpha t)^2+\sum_{n=2}^{\infty}(-1)^n\frac{(2\alpha t)^{2n}}{2(2n)!}R_n=\frac{1}{4}+\Omega_N-\Gamma_N(\alpha t)^2+\frac{1}{4}\sum_{n=1}^{\infty}(-1)^n n\,!\,\frac{(2\alpha t\sqrt{N})^{2n}}{2(2n)!}. \quad \text{(B2)}$$

Similarly, we find

$$2^{-N}\,\mathrm{tr}_B\left\{\left(\frac{J_-J_+}{K}\right)^2\cos(\alpha t\sqrt{K})\right\}=\frac{1}{4}+\Omega_N-\frac{1}{2}\Gamma_N(\alpha t)^2+\frac{1}{2}\sum_{n=1}^{\infty}(-1)^n n\,!\,\frac{(\alpha t\sqrt{N})^{2n}}{2(2n)!}. \quad \text{(B3)}$$

It is then sufficient to rescale the coupling constant and use the definition of the function $\zeta(t)$ in Eqs. (95) and (98) to get the final form (88) of the function $f(t)$.


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