Strict site-occupation constraint in two-dimensional Heisenberg models and dynamical mass generation in QED_3 at finite temperature

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We study the effect of site occupation in two-dimensional quantum spin systems at finite temperature in a π-flux state description at the mean-field level. We impose each lattice site to be occupied by a single $SU(2)$ spin. This is realized by means of a specific prescription. We consider the low-energy Hamiltonian which is mapped into a QED_3 Lagrangian of spinons. We compare the dynamically generated mass to the one obtained by means of an average site occupation constraint.

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I. INTRODUCTION

Quantum electrodynamics $QED_{(2+1)}$ is a common framework aimed to describe strongly correlated systems such as quantum spin systems in 1 time and 2 space dimensions, as well as related specific phenomena like high-T_c superconductivity[1–4]. A gauge field formulation of antiferromagnetic Heisenberg models in $d=2$ dimensions leads to a QED_3 action for spinons, see Ghaemi and Senthil[2] and Morinari.[3] This description raises the problem of the mean-field solution and the correlated question of the confinement of test charges which leads to the impossibility to determine the quantum fluctuation contributions through a loop expansion in this approach.[5–7]

We consider here the π-flux state approach introduced by Affleck and Marston.[8,9] The occupation of sites of the system by a single particle is generally introduced by means of a Lagrange multiplier procedure.[10,11] In the present work we implement a strict site occupation. It can be constructed by means of constraints imposed through a specific projection operator which introduces an imaginary chemical potential. This has been proposed by Popov and Fedotov[12] for $SU(2)$ spins and generalized by Kiselev *et al.*[13] to $SU(N)$ semifermionic Hamiltonians. It is our aim in the present work to confront the outcome of the two approaches.

Here we concentrate on the behavior of the spinon mass which is generated dynamically by an $U(1)$ gauge field. Appelquist *et al.*[14,15] showed that at zero temperature the originally massless fermion can acquire a dynamically generated mass when the number N of fermion flavors is lower than the critical value $N_c=32/\pi^2$. Later Maris[16] confirmed the existence of a critical value $N_c \simeq 3.3$ below which the dynamical mass can be generated. Since we consider only spin-1/2 systems, $N=2$ and hence $N<N_c$.

At finite temperature Dorey and Mavromatos[17] and Lee[18] showed that the dynamically generated mass vanishes at a temperature T larger than the critical one T_c.

We shall show below that the imaginary chemical potential introduced by Popov and Fedotov[12] modifies noticeably the effective potential between two charged particles and doubles the dynamical mass transition temperature, in agreement with former work at the same mean-field level.[19]

The outline of the paper is the following. In Sec. II we recall the projection procedure introduced by Popov and Fedotov (PFP) leading to a rigorous constraint on the lattice site occupation. In Sec. III we derive the Lagrangian which couples a spinon field to a $U(1)$ gauge field. Section IV is devoted to the comparison of the effective potential constructed with and without strict occupation constraint. In Sec. V we present the calculation of the mass term using the Schwinger-Dyson equation of the spinon.

II. SITE OCCUPATION CONSTRAINT FOR QUANTUM SPIN SYSTEMS AT FINITE TEMPERATURE

Heisenberg quantum spin Hamiltonians of the type

$$H = \frac{1}{2}\sum_{i,j} J_{ij}\vec{S}_i\vec{S}_j, \tag{1}$$

with $\{J_{ij}\}>0$ can be projected onto Fock space by means of the transformation

$$S_i^+ = f_{i,\uparrow}^\dagger f_{i,\downarrow},$$
$$S_i^- = f_{i,\downarrow}^\dagger f_{i,\uparrow},$$
$$S_i^z = \tfrac{1}{2}(f_{i,\uparrow}^\dagger f_{i,\uparrow} - f_{i,\downarrow}^\dagger f_{i,\downarrow}), \tag{2}$$

where $\{f_{i,\sigma}^\dagger, f_{i,\sigma}\}$ are anticommuting fermion operators which create and annihilate spinon with $\sigma=\pm 1/2$.

This transformation is not bijective because the dimensionality of Fock space is larger than the dimensionality of the space in which the spin operators $\{\vec{S}_i\}$ are acting. Indeed, in Fock space, each site i can be occupied by 0, 1, or 2 fermions corresponding to the states $|0,0\rangle$, $|1,0\rangle$, $|0,1\rangle$, $|1,1\rangle$ where $|0,0\rangle$ is the particle vacuum, $|1,0\rangle=|+1/2\rangle$, $|0,1\rangle=|-1/2\rangle$ and $|1,1\rangle=|+1/2,-1/2\rangle$ in terms of spin 1/2 projections. Since one wants to keep states with one fermion per site the states $|0,0\rangle$ and $|1,1\rangle$ have to be eliminated. This can be performed on the partition function for a system at inverse temperature β

$$\mathcal{Z} = \mathrm{Tr}[e^{-\beta H}]$$

where the trace is taken over the whole Fock space by the introduction of a projection operator

$$\mathcal{Z} = \mathrm{Tr}[e^{-\beta(H-\mu N)}], \tag{3}$$

where N is the particle number operator and $\mu = i\pi/2\beta$ an imaginary chemical potential.[12] Indeed, the presence of the states $|0,0\rangle_i$ and $|1,1\rangle_i$ on site i leads in Z to phase contributions which eliminate each other

$$e^{i*0} + e^{i*\pi} = 0,$$

and hence the contributions of these spurious states are canceled as a whole. The application of the Hamiltonian defined in terms of the fermion operators introduced in (2) on the states with fermion number 0 and 2 leads indeed to a cancellation of their contributions.

This result is valid at zero temperature since the partition function $\mathcal{Z}$ can be written as $\mathcal{Z} = \mathrm{Tr}[e^{-\beta H} e^{i(\pi/2)N}]$ which shows that there is a smooth connection from finite to zero temperature.

The common alternative approximate projection procedure would be to introduce a chemical potential in terms of *real* Lagrange multipliers $\{\lambda_i\}$

$$\mathcal{Z} = \mathrm{Tr}\left[e^{-\beta H} \prod_i \int d\lambda_i e^{\lambda_i(n_i - 1)} \right],$$

where n_i is the particle number operator on site i and the $\{\lambda_i\}$ are fixed by means of a saddle point procedure.

III. SPIN STATE MEAN-FIELD ANSATZ IN TWO-DIMENSIONAL

In two-dimensional (2D) space the Heisenberg Hamiltonian given by Eq. (1) can be written in terms of composite nonlocal operators $\{\mathcal{D}_{ij}\}$ ("diffusons")[10] defined as

$$\mathcal{D}_{ij} = f^{\dagger}_{i,\uparrow} f_{j,\uparrow} + f^{\dagger}_{i,\downarrow} f_{j,\downarrow}.$$

If the coupling strengths are fixed as

$$J_{ij} = J \sum_{\vec{\eta}} \delta(\vec{r}_i - \vec{r}_j \pm \vec{\eta}),$$

where $\vec{\eta}$ is a lattice vector $\{a_1, a_2\}$ in the $\vec{Ox}$ and $\vec{Oy}$ directions the Hamiltonian takes the form

$$H = -J \sum_{\langle ij \rangle} \left(\frac{1}{2} \mathcal{D}^{\dagger}_{ij} \mathcal{D}_{ij} - \frac{n_i}{2} + \frac{n_i n_j}{4} \right), \tag{4}$$

where i and j are nearest neighbor sites.

The number operator products $\{n_i n_j\}$ in Eq. (4) are quartic in terms of creation and annihilation operators in Fock space. In principle the formal treatment of these terms requires the introduction of a mean field procedure. One can however show that the presence of this term has no influence on the results obtained from the partition function. Indeed these terms lead to a constant quantity under the exact site-occupation constraint and hence are of no importance for the physics described by the Hamiltonian (4). As a consequence we leave it out from the beginning as well as the contribution corresponding to the $\{n_i\}$ terms.

A. Exact occupation procedure

Starting with the Hamiltonian

$$H = -\frac{J}{2} \sum_{\langle ij \rangle} \mathcal{D}^{\dagger}_{ij} \mathcal{D}_{ij} - \mu N, \tag{5}$$

the partition function Z can be written in the form

$$\mathcal{Z} = \int \prod_{i,\sigma} \mathcal{D}(\{\xi^*_{i,\sigma}, \xi_{i,\sigma}\}) e^{-A(\{\xi^*_{i,\sigma}, \xi_{i,\sigma}\})},$$

where the $\{\xi^*_{i,\sigma}, \xi_{i,\sigma}\}$ are Grassmann variables corresponding to the operators $\{f^{\dagger}_{i\sigma}, f_{i\sigma}\}$ defined above. They depend on the imaginary time τ in the interval $[0, \beta]$. In the continuum limit the action A is given by

$$A(\{\xi^*_{i,\sigma}, \xi_{i,\sigma}\}) = \int_0^{\beta} d\tau \Bigg(\sum_{i,\sigma} \xi^*_{i,\sigma}(\tau) \partial_{\tau} \xi_{i,\sigma}(\tau) + \mathcal{H}(\{\xi^*_{i,\sigma}(\tau), \xi_{i,\sigma}(\tau)\}) \Bigg),$$

where

$$\mathcal{H}(\tau) = H(\tau) - \mu N(\tau), \tag{6}$$

and $N(\tau)$ is the total particle number operator. A Hubbard-Stratonovich transformation on the corresponding functional integral partition function in which the action contains the occupation number operator as seen in Eq. (6) eliminates the quartic contributions generated by Eq. (2) and introduces the mean fields $\{\Delta_{ij}\}$. The Hamiltonian takes then the form

$$\mathcal{H} = \frac{2}{|J|} \sum_{\langle ij \rangle} \bar{\Delta}_{ij} \Delta_{ij} + \sum_{\langle ij \rangle} [\bar{\Delta}_{ij} \mathcal{D}_{ij} + \Delta_{ij} \mathcal{D}^{\dagger}_{ij}] - \mu N. \tag{7}$$

The fields $\{\Delta_{ij}\}$ and their complex conjugates $\bar{\Delta}_{ij}$ can be decomposed into a mean-field contribution and a fluctuation term

$$\Delta_{ij} = \Delta^{mf}_{ij} + \delta\Delta_{ij}.$$

The field Δ^{mf}_{ij} can be chosen as a complex quantity $\Delta^{mf}_{ij} = |\Delta^{mf}_{ij}| e^{i\phi^{mf}_{ij}}$.

The phase ϕ^{mf}_{ij} is fixed in the following way. Consider a square plaquette $\Box \equiv (\vec{i}, \vec{i}+\vec{e}_x, \vec{i}+\vec{e}_y, \vec{i}+\vec{e}_x+\vec{e}_y)$ where $\vec{e}_x$ and $\vec{e}_y$ are the unit vectors along the directions $\vec{Ox}$ and $\vec{Oy}$ starting from site $\vec{i}$ on the lattice. On this plaquette we define

$$\phi = \prod_{(ij) \in \Box} \phi^{mf}_{ij},$$

which is taken to be constant. If the gauge phases ϕ^{mf}_{ij} fluctuate in such a way that ϕ stays constant the average of Δ^{mf}_{ij} will be equal to zero in agreement with Elitzur's theorem.[20]

Mean-field Hamiltonians (7) can be defined arbitrarily by choosing different ϕ^{mf}_{ij} configurations over the plaquette. However these choices may not show the same symmetries of the mean-field Hamiltonian, the flux through the plaquette may break the $SU(2)$ invariance down to $U(1)$ or Z_2 gauge structure.[21] We would like to keep the original spin $SU(2)$ invariance of (1) intact in the mean-field Hamiltonian (7). In order to guarantee the $SU(2)$ symmetry of the mean-field Hamiltonian along the plaquette we follow[21,8,9,11,22] and introduce the configuration

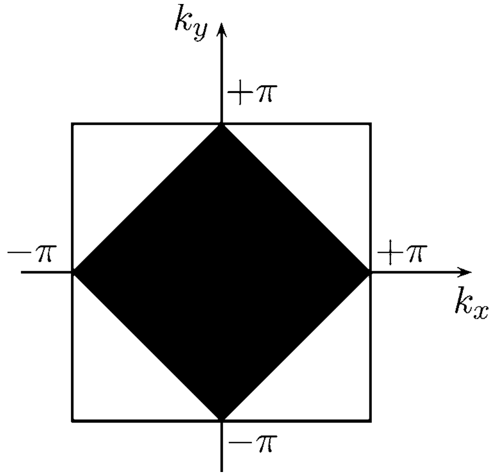

FIG. 1. The two-dimensional spin Brillouin Zone (black area) and the lattice Brillouin Zone (whole square).

$$\phi_{ij}^{mf} = \begin{cases} \frac{\pi}{4}(-1)^i, & \text{if } \vec{r}_j = \vec{r}_i + \vec{e}_x \\ -\frac{\pi}{4}(-1)^i, & \text{if } \vec{r}_j = \vec{r}_i + \vec{e}_y, \end{cases}$$

where $\vec{e}_x$ and $\vec{e}_y$ join the site i to its nearest neighbors j. Then the total flux through the fundamental plaquette is such that $\phi = \pi$ which guarantees that the $SU(2)$ symmetry of the plaquette is respected.[23]

At the mean-field level the partition function reads

$$\mathcal{Z}_{mf} = e^{-\beta(\mathcal{H}_{mf} - \mu N)},$$

where

$$\mathcal{H}_{mf} = \frac{2}{|J|}\sum_{\langle ij \rangle} \bar{\Delta}_{ij}^{mf}\Delta_{ij}^{mf} + \sum_{\langle ij \rangle} [\bar{\Delta}_{ij}^{mf}\mathcal{D}_{ij} + \Delta_{ij}^{mf}\mathcal{D}_{ij}^{\dagger}] - \mu N, \quad (8)$$

as read immediately from Eq. (7).

After a Fourier transformation the Hamiltonian (8) takes the form

$$\mathcal{H}_{mf} = \mathcal{N}z\frac{\Delta^2}{|J|} + \sum_{\vec{k} \in SBZ} \sum_{\sigma} (f_{\vec{k},\sigma}^{\dagger} f_{\vec{k}+\vec{\pi},\sigma}^{\dagger})[\tilde{H}]\begin{pmatrix} f_{\vec{k},\sigma} \\ f_{\vec{k}+\vec{\pi},\sigma} \end{pmatrix}, \quad (9)$$

with

$$[\tilde{H}] = \begin{bmatrix} -\mu + \Delta \cos\frac{\pi}{4} z\gamma_{k_x,k_y} & -i\Delta \sin\frac{\pi}{4} z\gamma_{k_x,k_y+\pi} \\ +i\Delta \sin\frac{\pi}{4} z\gamma_{k_x,k_y+\pi} & -\mu - \Delta \cos\frac{\pi}{4} z\gamma_{k_x,k_y} \end{bmatrix},$$

where $\Delta \equiv |\Delta^{mf}|$. The spin brillouin zone (SBZ) covers half of the Brillouin zone (see Fig. 1) which originates from the bipartite structure of the antiferromagnetic spin lattice and describes the Fourier space of the sublattice spin excitations. The $\gamma_{\vec{k}}$'s are defined by

$$\gamma_{\vec{k}} = \frac{1}{z}\sum_{\vec{\eta}} e^{i\vec{k}\cdot\vec{\eta}} = \frac{1}{2}(\cos k_x a_1 + \cos k_y a_2).$$

B. The π-flux Dirac action

As already shown in earlier work by Ghaemi and Senthil[2] and Morinari[3] the spin liquid Hamiltonian (9) for systems at low energy can be described by four-component Dirac spinons in the continuum limit. The Dirac action of spin liquid in (2+1) dimensions is derived in Appendix A. In Euclidean space this action reads

$$S_E = \int_0^{\beta} d\tau \int d^2\vec{r} \sum_{\sigma} \bar{\psi}_{\vec{r}\sigma}[\gamma^0(\partial_\tau - \mu) + \tilde{\Delta}\gamma^k\partial_k]\psi_{\vec{r}\sigma}, \quad (10)$$

where $\tilde{\Delta} = 2\Delta \cos\frac{\pi}{4}$ is the "light velocity," and $\{\gamma^\mu\}$ are the Dirac γ matrices in (2+1) dimensions. Spinons move in a "gravitational" field and the metric can be handled in a Minkowskian (or Euclidean) metric[24] assuming $\tilde{\Delta} = 1$ without altering the physics of the problem.

Since the Heisenberg Hamiltonian (1) is gauge invariant in the transformation $\psi \to e^{ig\theta}\psi$ the Dirac action has to be written in the form

$$S_E = \int_0^{\beta} \int d^2\vec{r} \left\{ -\frac{1}{2}a_\mu[(\Box\,\delta^{\mu\nu} + (1-\lambda)\partial^\mu\partial^\nu)]a_\nu + \sum_{\sigma} \bar{\psi}_{\vec{r}\sigma}[\gamma_\mu(\partial_\mu - iga_\mu)]\psi_{\vec{r}\sigma} \right\}. \quad (11)$$

Here g is the coupling strength between the gauge field $a_\mu = \partial_\mu\theta$ and the Dirac spinons ψ. In (11) the first term corresponds to the "Maxwell" term $-\frac{1}{4}f_{\mu\nu}f^{\mu\nu}$ of the gauge field a_μ where $f^{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu$, λ is the parameter of the Faddeev-Popov gauge fixing term[25] $-\lambda(\partial^\mu a_\mu)^2$ and $\delta^{\mu\nu}$ the Kronecker δ. $\Box = \partial_\tau^2 + \vec{\nabla}^2$ is the Laplacian in Euclidean space time. This form of the action originates from a shift of the imaginary time derivation $\partial_\tau \to \partial_\tau + \mu$ and leads to a new definition of the Matsubara frequencies only for the fermion fields ψ[12] which read then

$$\tilde{\omega}_{F,n} = \omega_{F,n} - \mu/i = \frac{2\pi}{\beta}(n + 1/4).$$

This modification will induce substantial consequences as it will be shown in the following.

IV. THE "PHOTON" PROPAGATOR AT FINITE TEMPERATURE

Integrating over the fermion fields ψ leads to a pure gauge Lagrangian $\mathcal{L}_a = \frac{1}{2}a_\mu\Delta_{\mu\nu}^{-1}a_\nu$ where $\Delta_{\mu\nu}$ is the dressed photon propagator from which we shall extract an effective interaction potential $V(R)$ between two test particles and extract a dynamically generated fermion mass.

The finite temperature photon propagator in Euclidean space (imaginary time formulation) verifies the Dyson equation

$$\Delta_{\mu\nu}^{-1} = \Delta_{\mu\nu}^{(0)-1} + \Pi_{\mu\nu}. \quad (12)$$

The detailed calculation of the polarization function $\Pi_{\mu\nu}$ is given in Appendix B.

FIG. 2. The dressed photon propagator. Wavy lines correspond to the photon and solid loops to the fermion insertions.

Since the system is at finite temperature and "relativistic" covariance should be kept the polarization function may be put in the form[26]

$$\Pi_{\mu\nu} = \Pi_A A_{\mu\nu} + \Pi_B B_{\mu\nu},$$

where Π_A and Π_B are related to $\tilde{\Pi}_k$ by $\Pi_A = \tilde{\Pi}_1 + \tilde{\Pi}_2$ and $\Pi_B = \tilde{\Pi}_3$. The expressions of $\tilde{\Pi}_1$, $\tilde{\Pi}_2$, and $\tilde{\Pi}_3$ are explicitly worked out in Appendix B. $A_{\mu\nu}$ and $B_{\mu\nu}$ generate an orthogonal tensor basis transversal to the photon momentum q^μ

$$A_{\mu\nu} = \tilde{\eta}_{\mu\nu} - \frac{\tilde{q}_\mu \tilde{q}_\nu}{\tilde{q}^2},$$

$$B_{\mu\nu} = \frac{q^2}{\tilde{q}^2} \bar{u}_\mu \bar{u}_\nu,$$

$$A_{\mu\nu} + B_{\mu\nu} = \delta_{\mu\nu} - \frac{q_\mu q\nu}{q^2},$$

with $\tilde{\eta}_{\mu\nu} = \delta_{\mu\nu} - u_\mu u_\nu$, $\bar{u}_\mu = u_\mu - \frac{(q.u)}{q^2} q_\mu$, and $\tilde{q}_\mu = q_\mu - (qu) u_\mu$. Here $u_\mu = (1,0,0)$ is the three vector of the thermal bath.

The dressed photon propagator $\Delta_{\mu\nu}$ is obtained by the summation of the geometric series shown in Fig. 2 and reads

$$\Delta_{\mu\nu} = \frac{A_{\mu\nu}}{q^2 + \tilde{\Pi}_1 + \tilde{\Pi}_2} + \frac{B_{\mu\nu}}{q^2 + \tilde{\Pi}_3} - (1 - 1/\lambda) \frac{q_\mu q_\nu}{(q^2)^2}. \quad (13)$$

A. Effective potential between test particles

The effective static potential between two test particles of opposite charges g at distance R is given by

$$V(R) = -g^2 \int_0^\beta d\tau \Delta_{00}(\tau, R) = -g^2 \frac{1}{2\pi} \int \frac{d^2\vec{q}}{(2\pi)^2} \Delta_{00}(q^0 = 0, \vec{q}) e^{i\vec{q}\cdot\vec{R}} = -\frac{g^2}{2\pi} \int_0^\infty dq q J_0(qR) \frac{1}{q^2 + \tilde{\Pi}_3(m=0)},$$

where $J_0(qR)$ is the zero order Bessel function. The polarization contribution $\tilde{\Pi}_3(q^0 = 0, \vec{q})$ is equal to $\frac{\alpha}{\pi\beta} \int_0^1 dx \log 2(\cosh \beta q \sqrt{x(1-x)})$ when taking the PFP imaginary chemical potential into account. This has to be compared to the expression $\frac{2\alpha}{\pi\beta} \int_0^1 dx \log 2[\cosh \frac{\beta}{2} q \sqrt{x(1-x)}]$ when the Lagrange multiplier method for which $\lambda = 0$ is used.[17]

For small momentum $q \to 0$, $\tilde{\Pi}_3(m=0)$ can be identified as a mass term $[M_0^{PF}(\beta)]^2$. For $R \gg (M_0^{PF})^{-1}$ the effective potential reads

$$V(R,\beta) \simeq -\frac{g^2}{2\pi} \int_0^\infty dq \frac{q J_0(qR)}{q^2 + M_0^{PF2}} = -\frac{\alpha}{N} \sqrt{\frac{1}{8\pi R M_0^{PF}}} e^{-M_0^{PF} R},$$

where $N = 2$ since we consider only $S = 1/2$ spins.

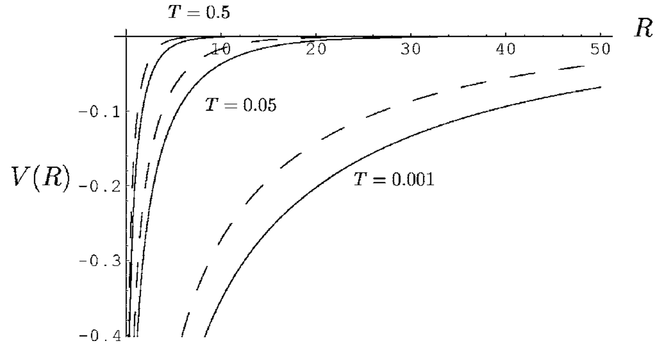

FIG. 3. Effective static potential with (full line) and without (dashed line) the Popov-Fedotov imaginary chemical potential for the temperature $T = \{0.001, 0.05, 0.5\}$.

Figure 3 shows the effective potential between two opposite test charges at distance $R \gg (M_0^{PF})^{-1}$. The screening effect is smaller when the imaginary chemical potential μ is implemented rather than the Lagrange multiplier λ. By inspection one sees that $(M_0^{PF})^{-1} = \sqrt{2}(M_0^{\lambda=0})^{-1}$. At zero temperature and at large distances the well known logarithmic potential is screened by the vacuum polarization and reduces to a Coulombic form independently of the PFP. Temperature adds screening effects and leads to the exponential decay of the effective potential at large distances.[17]

V. DYNAMICAL MASS GENERATION

We show now how the PFP doubles the "chiral" restoring transition temperature of the dynamical mass generation. The Schwinger-Dyson equation for the spinon propagator at finite temperature reads

$$G^{-1}(k) = G^{(0)-1}(k) - \frac{g}{\beta} \sum_{\tilde{\omega}_{F,n}} \int \frac{d^2\vec{P}}{(2\pi)^2} \gamma_\mu G(p) \Delta_{\mu\nu}(k-p) \Gamma_\nu, \quad (14)$$

where $p = (p_0 = \tilde{\omega}_{F,n}, \vec{P})$, G is the spinon propagator, Γ_ν the spinon-"photon" vertex which will be approximated here by its bare value $g\gamma_\nu$ and $\Delta_{\mu\nu}$ is the dressed photon propagator (13). The second term in (14) is the fermion self-energy Σ, $(G^{-1} = G^{(0)-1} - \Sigma)$. Performing the trace over the γ matrices in Eq. (14) leads to a self-consistent equation for the self-energy

$$\Sigma(k) = \frac{g^2}{\beta} \sum_{\tilde{\omega}_{F,n}} \int \frac{d^2\vec{P}}{(2\pi)^2} \Delta_{\mu\mu}(k-p) \frac{\Sigma(p)}{p^2 + \Sigma(p)^2}. \quad (15)$$

In the low energy and momentum limit $m(\beta) = \Sigma(k) \simeq \Sigma(0)$ Eq. (15) simplifies to

$$1 = \frac{g^2}{\beta} \sum_{\tilde{\omega}_{F,n}} \int \frac{d^2\vec{P}}{(2\pi)^2} \Delta_{\mu\mu}(-p) \frac{1}{p^2 + m(\beta)^2}. \quad (16)$$

If the main contribution comes from the longitudinal part $\Delta_{00}(0, -\vec{P})$ of the photon propagator (16) goes over to

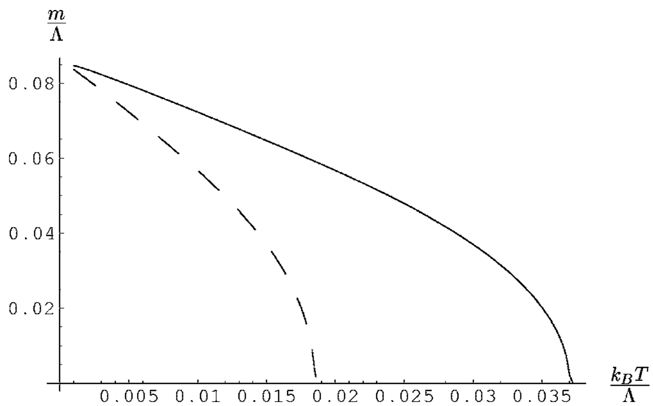

FIG. 4. Temperature dependence of the dynamical mass generated with (full line) and without (dashed line) the use of the Popov-Fedotov procedure.

$$1 = \frac{g^2}{\beta} \sum_{\tilde{\omega}_{F,n}} \int \frac{d^2\vec{P}}{(2\pi)^2} \left(\frac{1}{\vec{P}^2 + \tilde{\Pi}_3(m=0)} \frac{1}{\tilde{\omega}_{F,n}^2 + \vec{P}^2 + m(\beta)^2} \right). \quad (17)$$

Performing the summation over the fermion Matsubara frequencies $\tilde{\omega}_{F,n}$ the self-consistent equation takes the form

$$1 = \frac{\alpha}{4\pi N} \int_0^{\Lambda} dP \left(\frac{P \tanh \beta \sqrt{\vec{P}^2 + m(\beta)^2}}{[\vec{P}^2 + \tilde{\Pi}_3(m=0)] \sqrt{\vec{P}^2 + m(\beta)^2}} \right). \quad (18)$$

Equation (18) can be solved numerically with a cutoff Λ fixed at ∞ in an analytical calculation. By inspection of Eq. (18) and the corresponding result obtained by Dorey and Mavromatos[17] and Lee[18] one sees that the imaginary chemical potential used which fixes rigorously one spin per lattice site of the original Hamiltonian (1) doubles the transition temperature. This result is coherent with the results obtained elsewhere[19] where spinons are massless.

Since the mass can be identified with a superconducting gap one can evaluate the parameter $r = \frac{2m(0)}{k_B T_c}$ where $m(0)$ is the mass at zero temperature and T_c the transition temperature for which the mass becomes zero. Dorey and Mavromatos[17] obtained $r \simeq 10$ and Lee[18] computed the mass by taking into account the frequency dependence and obtained $r \simeq 6$. We have shown above that the imaginary chemical potential doubles the transition temperature so that the parameter r is $\simeq 4.8$ for $\alpha/\Lambda = \infty$ to compare with the result of Dorey and Mavromatos and $r \simeq 3$ to compare with Lee's result. Recall that the BCS parameter r is roughly equal to 3.5 and the YBaCuO parameter $r \simeq 8$ as given by the experiment.[27]

The dynamical mass generation can be identified as the spontaneous breaking of $SU(2)$ spin symmetry.[28] In this case the mass term can be interpreted as a Néel-like order parameter. The present results are in agreement with previous work.[19]

VI. CONCLUSION

We mapped a Heisenberg 2D Hamiltonian describing an antiferromagnetic quantum spin system into a $QED_{(2+1)}$ Lagrangian coupling a Dirac spinon field with a $U(1)$ gauge field. In this framework we showed that the implementation of the constraint which fixes rigorously the site occupation in a quantum spin system described by a 2D Heisenberg model leads to a substantial quantitative modification of the transition temperature at which the dynamically generated mass vanishes in the $QED_{(2+1)}$ description. It modifies consequently the effective static potential which acts between two test particles of opposite charges.

The imaginary chemical potential[12] reduces the screening of this static potential between test fermions when compared to the potential obtained from standard $QED_{(2+1)}$ calculations by Dorey and Mavromatos[17] who implicitly used a Lagrange multiplier procedure in order to fix the number of particles per lattice site[11,29] since $\lambda = 0$ at the mean-field level.

We showed that the transition temperature to "chiral" symmetry restoration corresponding to the vanishing of the spinon mass $m(\beta)$ is doubled by the introduction of the Popov-Fedotov imaginary chemical potential. The trend is consistent with earlier results concerning the value of T_c.[19] It reduces sizably the parameter $r = \frac{2m(0)}{k_B T_c}$ determined by Dorey and Mavromatos[17] and Lee.[18]

Since only gauge configurations of the flux states belonging to the Z_2 symmetry ($\pm\pi$) are allowed. As shown by Marston,[23] the flux through the plaquette is restricted to $\phi = \pm\pi$. In order to remove "forbidden" $U(1)$ gauge configuration (which are all configurations for which $\phi \neq \pm\pi$) of the antiferromagnet Heisenberg model a Chern-Simons term should be naturally included in the QED_3 action and fix the total flux through a plaquette. When the magnetic flux through a plaquette is fixed the system becomes 2π invariant in the gauge field a_μ and instantons appear in the system. This is the case when the present noncompact formulation of QED_3 is replaced by its correct compact version[30] and leads to fundamental and open problems.[6,7]

It is our next aim to implement a Chern-Simons term[31] in a system constrained by a rigorous site occupation.

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APPENDIX A: DERIVATION OF THE EUCLIDEAN QED ACTION IN (2+1) DIMENSIONS

At low energy near the two independent points $\vec{k} = \left(\pm\frac{\pi}{2}, \frac{\pi}{2}\right) + \vec{k}$ of the spin Brillouin zone (see Fig. 1) the Hamiltonian (9) can be rewritten in the form

$$H = \sum_{\vec{k} \in SBZ} \sum_{\sigma} (f^{\dagger}_{1,\vec{k},\sigma} f^{\dagger}_{1,\vec{k}+\vec{\pi},\sigma} f^{\dagger}_{2,\vec{k},\sigma} f^{\dagger}_{2,\vec{k}+\vec{\pi},\sigma}) \times \Bigg\{ -\mu\mathbb{I} + \sqrt{2}\Delta\left[-k_x\begin{pmatrix}\tau_3 & 0\\ 0 & \tau_3\end{pmatrix} - k_y\mathbb{I}\right] + \sqrt{2}\Delta\left[-k_x\begin{pmatrix}\tau_2 & 0\\ 0 & -\tau_2\end{pmatrix} + k_y i\mathbb{I}\right]\Bigg\}\begin{pmatrix} f_{1,\vec{k},\sigma}\\ f_{1,\vec{k}+\vec{\pi},\sigma}\\ f_{2,\vec{k},\sigma}\\ f_{2,\vec{k}+\vec{\pi},\sigma}\end{pmatrix},$$

with $\vec{\pi}=(\pi,\pi)$ the Brillouin vector. τ_1, τ_2, and τ_3 are Pauli matrices

$$\tau_1 = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \quad \tau_2 = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}, \quad \tau_3 = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}.$$

$f^{\dagger}_{1,\vec{k},\sigma}$ and $f_{1,\vec{k},\sigma}$ ($f^{\dagger}_{2,\vec{k},\sigma}$ and $f_{2,\vec{k},\sigma}$) are fermion creation and annihilation operators near the point $\left(\frac{\pi}{2},\frac{\pi}{2}\right)$ $\left(\left(-\frac{\pi}{2},\frac{\pi}{2}\right)\right)$.

Rotating the operators

$$f_{\vec{k}} = \tfrac{1}{\sqrt{2}}(f_{a,\vec{k}} + f_{b,\vec{k}}),$$

$$f_{\vec{k}+\vec{\pi}} = \tfrac{1}{\sqrt{2}}(f_{a,\vec{k}} - f_{b,\vec{k}}),$$

leads to

$$H = \sum_{\vec{k} \in SBZ} \sum_{\sigma} \psi^{\dagger}_{\vec{k}\sigma}\left[-\mu\mathbb{I} + \tilde{\Delta}k_+\begin{pmatrix}\tau_1 & 0\\ 0 & \tau_2\end{pmatrix} - \tilde{\Delta}k_-\begin{pmatrix}\tau_2 & 0\\ 0 & \tau_1\end{pmatrix}\right]\psi_{\vec{k}\sigma},$$

where $k_+=k_x+k_y$ and $k_-=k_x-k_y$, $\tilde{\Delta}=2\Delta\cos\frac{\pi}{4}$ and

$$\psi_{\vec{k}\sigma} = \begin{pmatrix} f_{1a,\vec{k}\sigma}\\ f_{1b,\vec{k}\sigma}\\ f_{2a\vec{k}\sigma}\\ f_{2b\vec{k}\sigma}\end{pmatrix}.$$

In the Euclidean metric the action reads

$$S_E = \int_0^{\tau} d\tau \sum_{\vec{k}\in SBZ}\sum_{\sigma} \psi^{\dagger}_{\vec{k}\sigma}\begin{pmatrix}\tau_3 & 0\\ 0 & \tau_3\end{pmatrix}\left[(\partial_\tau - \mu)\begin{pmatrix}\tau_3 & 0\\ 0 & \tau_3\end{pmatrix} + i\tilde{\Delta}k_+\begin{pmatrix}\tau_2 & 0\\ 0 & -\tau_1\end{pmatrix} + i\tilde{\Delta}k_-\begin{pmatrix}\tau_1 & 0\\ 0 & -\tau_2\end{pmatrix}\right]\psi_{\vec{k}\sigma}.$$

Through the unitary transformation

$$\psi_{\vec{k}\sigma} \to \begin{pmatrix}1 & 0\\ 0 & e^{i(\pi/4)\tau_3}\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -\tau_1\end{pmatrix}\psi_{\vec{k}\sigma}$$

and writing $k_+=k_2$ and $k_-=k_1$

$$S_E = \int_0^{\beta} d\tau \sum_{\vec{k}\in SBZ}\sum_{\sigma} \bar{\psi}_{\vec{k}\sigma}[\gamma^0(\partial_\tau - \mu) + \tilde{\Delta}ik_1\gamma^1 + \tilde{\Delta}ik_2\gamma^2]\psi_{\vec{k}\sigma},$$

where $\bar{\psi}=\psi^{\dagger}\gamma^0$ and the γ matrices are defined as

$$\gamma^0 = \begin{pmatrix}\tau_3 & 0\\ 0 & -\tau_3\end{pmatrix}, \quad \gamma^1 = \begin{pmatrix}\tau_1 & 0\\ 0 & -\tau_1\end{pmatrix},$$

$$\gamma^2 = \begin{pmatrix}\tau_2 & 0\\ 0 & -\tau_2\end{pmatrix}.$$

Using the inverse Fourier transform $\psi_{\vec{k}\sigma}=\int d^2\vec{r}\psi_{\vec{r}\sigma}e^{i\vec{k}\cdot\vec{r}}$ the Euclidean action finally reads

$$S_E = \int_0^{\beta} d\tau \int d^2\vec{r}\sum_{\sigma} \bar{\psi}_{\vec{r}\sigma}[\gamma^0(\partial_\tau - \mu) + \tilde{\Delta}\gamma^k\partial_k]\psi_{\vec{r}\sigma}.$$

With a "light velocity" $v_\mu=(1,\tilde{\Delta},\tilde{\Delta})$. The covariant derivative which takes v_μ into account[24] reads

$$D_\mu = \partial_\mu + \tfrac{1}{8}\omega_{\alpha,ab}[\gamma^a,\gamma^b],$$

where $\omega_{\alpha,ab}=e^{\nu}_a(\partial_\alpha e_{\nu b} - \Gamma^{\gamma}_{\alpha\mu}e_{\gamma b})$, e^{μ}_a are the *vierbein*[32] for which the metric is defined as $g^{\mu\nu}=\eta^{mn}e^{\mu}_m e^{\nu}_n = v^{\mu}\delta^{\mu\nu}$ with $\eta^{00}=-1$, $\eta^{ij}=\delta^{ij}$, and Γ is the Christoffel symbols. Since $\tilde{\Delta}$ is constant we see clearly that the *vierbein* are also constant, $\omega_{\alpha,ab}=0$ in a dilated flat space-time with the Euclidean metric $g_{\mu\nu}=v_\mu\delta_{\mu\nu}$.

APPENDIX B: DERIVATION OF THE PHOTON POLARIZATION FUNCTION AT FINITE TEMPERATURE

The Fourier transformation of the spinon action given by Eq. (11) reads

$$S_E[\psi] = \sum_{\sigma}\sum_{\tilde{\omega}_{F,1},\tilde{\omega}_{F,2}} \int \frac{d^2\vec{k}_1}{(2\pi)^2}\int \frac{d^2\vec{k}_2}{(2\pi)^2}\bar{\psi}_\sigma(k_1)\left[\frac{i\gamma^\mu k_\mu}{(2\pi)^2\beta}\times\delta(k_1 - k_2) - \frac{ig\gamma^\mu a_\mu(k_1-k_2)}{(2\pi)^2\beta}\right)^2\psi_\sigma(k_2),$$

with $k=(\tilde{\omega}_F \equiv \frac{2\pi}{\beta}(n+1/4),\vec{k})$. Integrating over the fermion field ψ and keeping the second order in the gauge field leads to the effective gauge action

$$S_{eff}[a] = \tfrac{1}{2}\,\mathrm{Tr}[G_F ig\gamma^\mu a_\mu]^2$$

with $\mathrm{Tr}=\sum_{\omega'_F}\int\frac{d^2\vec{k}'}{(2\pi)^2}\sum_{\omega''_F}\int\frac{d^2\vec{k}''}{(2\pi)^2}\mathrm{tr}$. The trace tr extends over the γ matrix space, and $G_F^{-1}(k_1-k_2)=i\frac{\gamma^\mu k_\mu}{(2\pi)^2\beta}\delta(k_1-k_2)$. The pure gauge action comes as

$$S_{eff}[a] = -g^2\frac{1}{2\beta}\sum_{\sigma}\sum_{\omega_{F,1}}\int\frac{d^2\vec{k}_1}{(2\pi)^2}\frac{1}{\beta}\sum_{\omega''_F}\int\frac{d^2\vec{k}''}{(2\pi)^2}\times\mathrm{tr}\left[\frac{\gamma^\rho k_{1,\rho}}{k_1^2}\gamma^\mu a_\mu(k_1-k'')\frac{\gamma^\eta k''_\eta}{k''^2}\gamma^\nu a_\nu(-(k_1-k''))\right].$$

With the change of variables $k_1-k''=q$ and $k_1=k$

$$S_{eff} = -\frac{g^2}{2\beta}\sum_{\omega_B}\int\frac{d^2\vec{q}}{(2\pi)^2}a_\mu(-q)\Pi^{\mu\nu}(q)a_\nu(q),$$

where $q=\left(\omega_B=\frac{2\pi}{\beta}m,\vec{q}\right)$ and the polarization function is given by

$$\Pi^{\mu\nu}(q) = \frac{g^2}{\beta}\sum_{\omega_F,\sigma}\int \frac{d^2\vec{k}}{(2\pi)^2}\mathrm{tr}\left[\frac{\gamma^\rho k_\rho}{k^2}\gamma^\mu\gamma^\eta\frac{(k_\eta+q_\eta)}{(k+q)}\gamma^\nu\right].$$

Then using the Feynmann identity

$$\frac{1}{ab} = \int_0^1 dx \frac{1}{[ax+(1-x)b]^2}.$$

$\Pi^{\mu\nu}$ can be rewritten as

$$\Pi^{\mu\nu}(q) = \frac{g^2}{\beta}\sum_\sigma\sum_{\omega_F}\int\frac{d^2\vec{k}}{(2\pi)^2} \times\mathrm{tr}[\gamma^\rho\gamma^\mu\gamma^\eta\gamma^\nu].\int_0^1 dx\frac{k_\rho(k_\eta+q_\eta)}{[(k+q)^2x+(1-x)k^2]^2}.$$

By means of a change of variables $k\to k'-xq$ and using the identity

$$\mathrm{tr}[\gamma^\rho\gamma^\mu\gamma^\eta\gamma^\nu] = 4[\delta_{\rho\mu}\delta_{\eta\nu}-\delta_{\rho\eta}\delta_{\mu\nu}+\delta_{\rho\nu}\delta_{\mu\eta}],$$

one obtains

$$\Pi^{\mu\nu}(q) = 4\alpha\int_0^1 dx\frac{1}{\beta}\sum_{\omega_F'}\int\frac{d^2\vec{k}'}{(2\pi)^2}\{[2k'_\mu k'_\nu+(1-2x)(k'_\mu q_\nu + q_\mu k'_\nu)-x(1-x)2q_\mu q_\nu \quad - \quad \delta_{\mu\nu}\sum_\eta(k'^2_\eta+(1-2x)k'_\eta q_\eta - x(1-x)q_\eta^2)]/[k'^2+x(1-x)q^2]^2,$$

where $\alpha = g^2\sum_{\sigma=1}^{N=2}1$. Following Dorey and Mavromatos,[17] Lee,[18] Aitchison *et al.*[33] and Gradshteyn[34] we define

$$S_1 = \sum_{n=-\infty}^{\infty}\frac{1}{[k'^2+x(1-x)q^2]} = \frac{\beta^2}{4\pi Y}\left[\frac{\sinh(2\pi Y)}{\cosh(2\pi Y)-\cos(2\pi X)}\right],$$

$$S_2 = \sum_{n=-\infty}^{\infty}\frac{1}{[k'^2+x(1-x)q^2]^2} = -\frac{\beta^2}{8\pi^2}\frac{1}{Y}\frac{\partial S_1}{\partial Y},$$

$$S^* = \sum_{n=-\infty}^{\infty}\frac{\omega_F'}{[k'^2+x(1-x)q^2]^2} = -\frac{\beta}{4\pi}\frac{\partial S_1}{\partial X},$$

with $X = xm+1/4$ and $Y=\frac{\beta}{2\pi}\sqrt{\vec{k}'^2+x(1-x)q^2}$. The polarization can be expressed in terms of these sums and reads

$$\Pi^{00} = \frac{\alpha}{\beta}\int_0^1 dx\int\frac{d^2\vec{k}'}{(2\pi)^2}\{S_1 - 2[\vec{k}'^2+x(1-x)q_0^2]S_2 + (1-2x)q_0 S^*\},$$

for the temporal component and

$$\Pi^{ij} = \frac{\alpha}{\beta}\int_0^1 dx\int\frac{d^2\vec{k}'}{(2\pi)^2}[2x(1-x)(q^2\delta_{ij}-q_iq_j)S_2 - (1-2x)q_0\delta_{ij}S^*],$$

for the spatial components.

Integrating over the fermion momentum $\vec{k}'$ one gets

$$\Pi^{00} = \tilde{\Pi}_3 - \frac{q_0^2}{q^2}\tilde{\Pi}_1 - \tilde{\Pi}_2,$$

$$\Pi^{ij} = \tilde{\Pi}_1\left(\delta_{ij}-\frac{q_iq_j}{q^2}\right)+\tilde{\Pi}_2\delta_{ij},$$

where

$$\tilde{\Pi}_1 = \frac{\alpha q}{\pi}\int_0^1 dx\sqrt{x(1-x)}\frac{\sinh\beta q\sqrt{x(1-x)}}{D(X,Y)},$$

$$\tilde{\Pi}_2 = \frac{\alpha m}{\beta}\int_0^1 dx(1-2x)\frac{\cos 2\pi xm}{D(X,Y)},$$

$$\tilde{\Pi}_3 = \frac{\alpha}{\pi\beta}\int_0^1 dx \log 2D(X,Y),$$

and $D(X,Y)=\cosh[\beta q\sqrt{x(1-x)}]+\sin(2\pi xm)$.

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