PHYSICAL REVIEW B **73**, 220502(R) (2006)

Proposal to stabilize and detect half-quantum vortices in strontium ruthenate thin films: Non-Abelian braiding statistics of vortices in a p_x+ip_y superconductor

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(Received 12 April 2006; published 13 June 2006)

We propose a simple way to stabilize half-quantum vortices in superconducting strontium ruthenate, assuming the order parameter is of chiral p_x+ip_y symmetry, as is suggested by recent experiments. The method, first given by Salomaa and Volovik in the context of Helium-3, is very naturally suited for strontium ruthenate, which has a layered, quasi-two-dimensional, perovskite crystal structure. We propose possible experiments to detect their non-Abelian braiding statistics. These experiments are of potential importance for topological quantum computation.

DOI: 10.1103/PhysRevB.73.220502 PACS number(s): 74.20.Rp, 03.67.Lx, 03.67.Mn, 03.67.Pp

The realizable prospect for topological quantum computation using fractional quantum Hall states in two-dimensional (2D) semiconductor structures has been discussed recently in the literature.[1–5] In particular, non-Abelian braiding statistics, manipulated by braiding particles as desired, is the robust quantum mechanical resource underlying quantum computation in these systems. Such non-Abelian quasiparticle statistics arise naturally in certain exotic classes of 2D fractional quantum Hall states (e.g., the so-called Pfaffian quantum Hall state which may exist at the $\sigma_{xy}=\frac{5}{2}e^2/h$ plateau[1–4]), giving rise to the recent excitement about the possibility of quasiparticle manipulation and braiding (and eventually to topological quantum computation) in high-mobility semiconductor heterostructures. Given the great interest in non-Abelian statistics and topological quantum computation, it is relevant to ask whether other physical systems could be studied experimentally to investigate non-Abelian braiding statistics. In this Rapid Communication, we suggest quantum vortex matter in superconducting strontium ruthenate as a prospective candidate for non-Abelian statistics and topological quantum computation. The physical basis for this proposal is the possible p_x+ip_y symmetry of the superconducting order parameter in strontium ruthenate. This order parameter symmetry admits half-quantum vortices, which are analogous to the fractionally charged non-Abelian quasiparticles of the Pfaffian fractional quantum Hall state.

Spin triplet ($\vec{S}=1$), odd parity ($\vec{L}=1$) superfluidity is realized in Helium-3.[6] Recently, a body of evidence, including sensitivity to impurity,[7] nuclear magnetic resonance (NMR) Knight shift,[8] and polarized neutron scattering[9] have strongly indicated that spin-triplet superconductivity is also realized in strontium ruthenate (Sr_2RuO_4).[12] Furthermore, within the superconducting phase of this metallic oxide, time-reversal symmetry breaking has been confirmed by muon spin rotation.[10] The existence of a two-component order parameter is suggested by small angle neutron scattering exploring the flux lattice in the mixed state.[11] Taken together, these experiments point to the conclusion[12,13] that strontium ruthenate, with a highest T_c of about 1.5 K, realizes the simplest unitary T-breaking p-wave state allowed in its tetragonal crystal structure, namely p_x+ip_y, which is analogous to the A phase of He 3. Very recently, phase sensitive superconducting quantum interference device measurements[14] have given almost conclusive evidence of p-wave superconductivity[15] in Sr_2RuO_4. For alternative proposed order parameter structures, see, for instance, Refs. 16 and 17.

For spin-triplet superconductors, the order parameter is a matrix in spin space,[6]

$$\Delta_{\alpha\beta}(\boldsymbol{k}) = \sum_{\mu=1}^{3} d_\mu(\boldsymbol{k})(\sigma_\mu i\sigma_2)_{\alpha\beta}. \tag{1}$$

Here α, β are spin indices, $\boldsymbol{k}$ is the wave vector, $\sigma_{1,2,3}$ are the Pauli matrices, and $\boldsymbol{d}(\boldsymbol{k})$ is a complex three-vector. Orbital p-wave symmetry implies $d_\mu(\boldsymbol{k})=\Sigma_{j=1}^{3}d_{\mu j}\hat{k}_j$. The tensor field $d_{\mu j}$ is the appropriate order parameter. For the He-3 A phase, or the p_x+ip_y superconductor,

$$d_{\mu j} = \Delta_0(T)d_\mu(\hat{m}_j + i\hat{n}_j)e^{i\phi}, \tag{2}$$

where $\Delta_0(T)$ is the temperature-dependent magnitude of the order parameter, $\hat{m}$ and $\hat{n}$ are mutually orthogonal unit vectors in orbital space, and ϕ is the phase angle. Physically, $\hat{d}$ is the unit vector in spin space on which the projection of the Cooper pair spin is zero, and $\hat{l}=\hat{m}\times\hat{n}$ is a preferred direction in the orbital space giving the direction of the Cooper pair angular momentum.

For topological classification of vortices,[18] note that the full symmetry group $G=\mathrm{SO}(3)\times\mathrm{SO}(3)\times\mathrm{U}(1)$, implying rotational symmetries in the spin and the orbital spaces, and the global gauge symmetry. The p_x+ip_y state has residual symmetry $H=\mathrm{U}(1)\times\mathrm{U}(1)\times Z_2$, where the first U(1) is for spin rotation symmetry about the vector $\hat{d}$, the second U(1) is the combined gauge-orbital symmetry, and the Z_2, most important for our purposes, signifies a discrete combined gauge-spin rotation symmetry[20] of the order parameter, $(\hat{d},\phi)\to(-\hat{d},\phi+\pi)$. The first homotopy group of the degeneracy space R then corresponds to $\pi_1(R)=\pi_1(G/H)=\pi_1[(S^2\times\mathrm{SO}_3)/Z_2]=Z_4$, the cyclic group with four elements. Here S^2 denotes the surface of a sphere. The vortices, then, correspond to only four classes, with vorticities $N=-\frac{1}{2},0,\frac{1}{2},1=-1$. The surprising fact is the existence of $N=\pm\frac{1}{2}$, which correspond to a half-flux quantum per vortex. The half-quantum vortex is crucial for non-Abelian statistics in this system.

The simplest representation of the class $N=\pm\frac{1}{2}$ is given by

$$\hat{d}=\hat{x}\cos(\theta/2)+\hat{y}\sin(\theta/2), \quad \phi=\pm\theta/2,$$

$$\hat{l}=\text{const.} \tag{3}$$

Here, it has been assumed, without any loss of generality, that $\hat{d}$ lies in the x-y plane and θ is the polar angle with the vortex core as the origin. The flux enclosed by a half-vortex is half the flux quantum, $\frac{1}{2}\phi_0$, where $\phi_0=hc/2e$. The existence of the half-vortex is possible because $\hat{d}$ can change sign upon circulating the core, and the phase angle ϕ can simultaneously change by π to keep the order parameter, Eq. (2), unchanged.

However, in real systems, half-vortices are energetically costlier than vortices with $N=1,2$ (a vortex with $N=2$ is continuously deformable to $N=0$). This is because of small, but nonzero, spin-orbit or dipole energy, $E_{\text{so}}=-\Omega_{\text{so}}(\hat{d}\hat{l})^2$, which binds $\hat{l}$ and $\hat{d}$ parallel, and so penalizes spatial modulation of $\hat{d}$ around the vortex core. The texture for a half-quantum vortex requires $\hat{d}$ and $\hat{l}$ to be nonparallel over a region of macroscopic size. Because of E_{so}, this region forms a domain wall terminated by the vortex, expending energy proportional to the volume of the domain wall. Another way of looking at this is to note that in the case of the dipole-locking of $\hat{d}$ and $\hat{l}$, the extra Z_2 symmetry of H is absent. The first homotopy group of R is then $\pi_1(S^2\times \text{SO}_3)=Z_2$. There are only two classes of vortices, the uniform class, $N=0$ (a vortex with $N=2$ remains continuously deformable to this class), and the class with vorticity $N=1$. Half-quantum vortices are no longer possible. Hence, to stabilize the half-quantum vortices, the spin-orbit energy must be neutralized.

A simple way to neutralize the spin-orbit energy, first proposed in the context of He-3,[21] is very naturally suited to strontium ruthenate. In the He-3 A phase, one has to use an artificial parallel plate geometry so that the direction of the Cooper pair angular momentum, $\hat{l}$, is constrained to be along the plate normals, $\hat{n}$. In the presence of a sufficiently strong magnetic field $\hat{H}\|\hat{n}$, the Cooper pair spins will align themselves with $\hat{H}$. The vector $\hat{d}$ is then in the plane of the parallel plates, expending the maximum spin-orbit energy. However, this energy cost is a constant for *all* types of vortices. $\hat{d}$ is also free to rotate in the plane to $-\hat{d}$ spending no additional spin-orbit energy. In this way one can avoid the large domain wall energy cost for the half-quantum vortices, making them competitive with other topological excitations.

However, this restrictive geometry is completely natural for strontium ruthenate. It is a metallic oxide with a highly 2D, layered, tetragonal crystal structure. In fact, the crystal structure of Sr_2RuO_4 is almost identical to that of the perovskite $(La,Sr)_2CuO_4$ high temperature superconductor.[12] Two-dimensional, correlated RuO_2 layers are separated by an intermediate strontium reservoir. Electronic conduction is mediated by d orbitals of ruthenium ions in 2D planes. This highly planar structure is also evident by the superconducting state properties such as the upper critical field H_{c2}, which is about 20 times smaller when applied parallel to the c axis than when applied parallel to the ab plane.

Sr_2RuO_4 thus naturally realizes a p_x+ip_y superconductor in a parallel plate geometry. It is believed that $\hat{l}$ is in the c direction, and that $\hat{d}$ is pinned to $\hat{l}$ due to spin-orbit energy, although conclusive proof is lacking.[13] If this is indeed the case, the most direct way to stabilize half-quantum vortices is to simply apply a magnetic field, strong enough to overcome the spin-orbit energy, parallel to the c axis. In the case of He-3, Ref. 21 claimed that at low enough temperatures $T\ll T_c$, vortices with $N=\frac{1}{2}$ are energetically slightly more favorable than those with $N=1,2$. In the absence of a definitive proof of even the symmetry of the order parameter being (p_x+ip_y)-type and lack of knowledge of other system parameters, such microscopic energy comparisons for Sr_2RuO_4 are premature. However, half-quantum vortices are expected to be at least competitive in energy and may even be more favorable in the presence of magnetic fields simply because of the smaller vorticity (see also Ref. 19).

The required magnetic field H can be estimated from the spin-orbit or dipole coupling constant Ω_{so}. Unfortunately, we are unaware of any reliable theoretical estimate of this quantity. In the He-3 A phase, the required H was estimated to be greater than ~25 G.[21] In Sr_2RuO_4, a recent experimental estimate of the spin-orbit decoupling field is ~250 G.[22] We suggest applying a magnetic field of this order of magnitude in order to decouple d from l. Note that this can be done without destroying superconductivity since the upper critical field in the c direction is still larger: $H_{c2\|c}\sim 750$ G.[13]

Once stabilized, the half-quantum vortices may be detectable by exploiting their statistics under an interchange of positions. In the following, we recapitulate the arguments leading to their non-Abelian braiding statistics.[23–25] An interpretation of the half-quantum vortex can be given, when the subsystems of up and down spins are uncoupled, in terms of a single-quantum vortex for only one of the spins and zero vorticity for the other spin component.[24,26] This can be seen by explicitly writing out the components of the order parameter matrix, Eq. (1). Ignoring, for the moment, the orbital dependencies of all quantities, we get, $\Delta_{\uparrow\uparrow}=\Delta_0\exp(i\phi)\times(-d_x+id_y)$, $\Delta_{\downarrow\downarrow}=\Delta_0\exp(i\phi)(d_x+id_y)$, and $\Delta_{\uparrow\downarrow}=\Delta_{\downarrow\uparrow}=\Delta_0\exp(i\phi)d_z$. Using the form of $\hat{d}$ and ϕ in Eq. (3), with $\phi=\theta/2$ (positive half-quantum vortex), it is easy to see that $\Delta_{\uparrow\uparrow}=-\Delta_0$, $\Delta_{\downarrow\downarrow}=\Delta_0\exp(i\theta)$, and $\Delta_{\uparrow\downarrow}=\Delta_{\downarrow\uparrow}=0$. Because the phase angle $\phi=\theta$ changes by 2π upon circulating the core, the down spin pairs see a full single-quantum vortex, while the up spin pairs see no vortex at all. Because of this property, the Bogoliubov–de Gennes equations separate into two pieces: the part for the down spin component is identical to that for an ordinary single-quantum vortex in a p-wave superconductor,[27] albeit for spinless electrons, while the part for the up spin component is devoid of any vortex. As in Ref. 27, the low-energy spectrum in the vortex core for the down spins is given by $E_n=n\omega_0$, where $\omega_0=\Delta_0^2/\epsilon_F$ with ϵ_F the

Fermi energy, and n an integer. For $n=0$, there is a zero energy state in the vortex core.

It is interesting to note that the Bogoliubov quasiparticles corresponding to the levels E_n are linear superpositions of particle and hole operators from the *same* species, namely, down spin electrons, $\gamma_i^\dagger = u\psi_{i\downarrow}^\dagger + v\psi_{i\downarrow}$, where $\psi_{i\downarrow}$ is the electron annihilation field for down spins at the location of the ith vortex. They satisfy $\gamma_i^\dagger(E_n)=\gamma_i(-E_n)$. The zero energy state, then, is a Majorana fermion state, satisfying $\gamma_i^\dagger(0) = \gamma_i(0)$. Each half-quantum vortex i has a zero energy Majorana fermion γ_i in its core. In the case of $2m$ such spatially separated vortices, the $2m$ Majorana operators can be pairwise combined to produce m complex fermion operators, each of which can be occupied or unoccupied in the ground state. The ground state, thus, is 2^m-fold degenerate. Further, using the property that γ_i's carry odd charge with respect to the gauge field of a single-quantum vortex, that is, $\gamma_i \to -\gamma_i$ for a phase change of 2π, it follows that,[24] upon interchange of two neighboring half-quantum vortices γ_i, γ_{i+1},[28]

$$\gamma_i \to \gamma_{i+1},$$

$$\gamma_{i+1} \to -\gamma_i,$$

$$\gamma_j \to \gamma_j, \quad j \neq i,\ j \neq i+1. \tag{4}$$

These topological properties are identical to those of the Pfaffian quantum Hall state, which is a candidate description of the observed fractional quantum Hall plateau with $\sigma_{xy} = \frac{5}{2}e^2/h$ in two-dimensional electron gases.[1–4] This state is distinguished by the non-Abelian braiding statistics of its quasiparticles, which are precisely (4). This state has recently been the subject of much interest as a platform for fault-tolerant quantum computation.[1] Recent papers,[2,3] elaborating on an earlier proposed experiment,[29] explain how non-Abelian statistics could be directly detected through quantum interference measurements. In this paper, we suggest that the same underlying physics could arise in Sr_2RuO_4. Following Ref. 1, we note that the two possible states of a pair of vortices in a 2D p_x+ip_y superconductor (corresponding to the presence or absence of a neutral fermion) can be used as the two states of a qubit. When the two vortices are far apart, no local measurement can distinguish the two states of the qubit, so they form a decoherence-free subspace. However, there are important differences between Sr_2RuO_4 and 2D electron gas that lead to practical differences, which we discuss below.

Vortices in Sr_2RuO_4 are neutral, unlike quasiparticles in a quantum Hall liquid. Therefore, a direct quantum interference measurement of the quasiparticle transport as proposed in Ref. 1 would not apply to Sr_2RuO_4. However, the Josephson relation tells us that the motion of a vortex across a line connecting points 1 and 2 causes a phase slip, thereby inducing a voltage drop between 1 and 2 according to

$$V_x = \frac{h}{e} j_y^{\text{vortex}}. \tag{5}$$

When vortices become depinned, either as a result of nonequilibrium effects or thermal fluctuations, there is dissipation.

A second important difference is that, even in a thin film, there are many 2D layers in a Sr_2RuO_4 crystal. Hence, a vortex that runs through N layers actually has N Majorana zero modes associated with it. Consequently, a pair of vortices has N qubits associated with it. When two vortices are braided, their topological interactions, as summarized in (4), occur independently in each layer. In other words, the Majorana modes in layer α only affect other Majorana modes in layer α, not those in layer β.

Another important difference is that vortices are potentially heavy objects, so it is less clear that one can observe the coherent quantum motion of vortices. However, vortices are expected to become light as the Mott transition is approached. While the normal state of Sr_2RuO_4 appears to be a Fermi liquid at low temperatures, transport at temperatures above 30 K is anomalous.[13] If this is an indication of proximity to a zero-temperature Mott transition (note that proximity to a quantum phase transition is revealed only at a relatively high temperature, where the system crosses over to a quantum critical regime), then it could imply that vortices are light. In general, we expect the vortex motion to be coherent in the absence of appreciable quasiparticle excitations. Having very low temperature would help prevent decoherence by quasiparticles and by phonons. At magnetic fields of approximately 250 G, the intervortex spacing $a \sim \sqrt{\phi_0/B} \sim 10\,000$ Å is significantly larger than the vortex size, ~ 660 Å, so that we expect individual vortices to move.

Keeping all of these caveats in mind, we turn to experiments that could observe the topological properties of vortices in Sr_2RuO_4 thereby, in principle, taking the first steps towards topologically protected qubits. Our goal is to observe the two topologically degenerate states of a vortex.

Recently, Stone and Chung[30] emphasized that when two vortices are brought together, the difference between the two states of a qubit is now manifest: there either is or is not a neutral fermion in the core of the combined vortex. There is an energy cost in the latter case. Since we expect neither an excess of charge nor spin in the vortex core, we can look for subtle differences in the charge distribution or try to observe the energy difference. The idea is the following. While the total charge is the same whether or not the neutral fermion is present, the local charge distribution will be different in the two cases. Thus, if one had a spatial resolution smaller than the size of a vortex core ~ 660 Å, one could try to look with a scanning tunneling microscope for the particular charge density profiles associated with the presence or absence of a neutral fermion. In principle, gravitational effects can be used to detect this energy difference, but the required sensitivity may be very difficult to achieve.

An alternate strategy is to observe the braiding statistics of vortices through their effect on quantum interference. Consider the experimental setup in Fig. 1. We assume that a thin film has a hole bored in it so that it has an annular topology. Flux Φ is threaded through the hole, so that there are N_v vortices in the hole. A magnetic field $H > H_{c1}$ is applied to the rest of the film so that vortices penetrate the rest of the film as well. A current J is driven in the y direction. Vortex motion in the x direction generates a voltage drop in the y direction, as in (5). If the motion of a vortex is coherent, then the two trajectories in the figure will contribute to the resistivity according to Ref. 29:

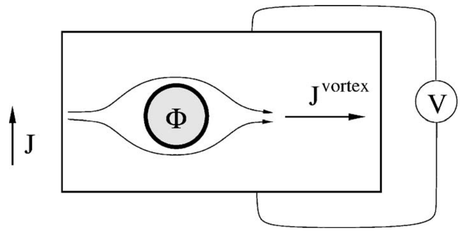

FIG. 1. In the experimental geometry depicted above, a thin film of Sr_2RuO_4 has a central hole with a flux Φ through it. When an electrical current flows in the y direction, depinned vortices flow in the x direction. This generates a voltage drop in the x direction. If the motion of a vortex is coherent, then there will be interference between trajectories that go around either side of the hole. The nature of this interference will depend strongly on whether the number of half-quantum vortices in the hole is odd or even.

$$R_{yy} \propto |t_1|^2 + |t_2|^2 + 2\,\mathrm{Re}\{t_1 t_2^* \langle \Psi | B | \Psi \rangle\}, \tag{6}$$

where t_1, t_2 are the amplitudes for the two trajectories, B is the braiding operator for a vortex to go around the vortices in the hole, and $|\Psi\rangle$ is the combined state of the moving vortex and the vortices in the hole. When the number of vortices in the hole is even, $\langle \Psi | B | \Psi \rangle$ is a phase $e^{i\theta}$. However, when the number of vortices in the hole is odd, $\langle \Psi | B | \Psi \rangle = 0$.[2,3] If we can vary t_1 and t_2, then we can distinguish these two cases. It is not clear how to do this, but there are several possibilities. (1) If we gate the Sr_2RuO_4 thin film along one of the possible trajectories and apply a gate voltage, this will suppress the electron density along this trajectory. This, in turn, will change the amplitude for this trajectory. (2) If pressure is applied to the part of the film where one of the trajectories is located, the lattice constant will be different there, presumably translating into a different amplitude for this trajectory. (3) Applying a temperature gradient will cause the superfluid density to be different along the two different trajectories which, again, would lead to a variation in their respective amplitudes. However, there is a caveat to this final option: raising the temperature may be dangerous since we want the vortex motion to be coherent. One could also try to observe interference in the Nernst effect in the geometry of Fig. 1, but this is difficult experimentally.

In summary, we propose an experimentally feasible technique for stabilizing half-quantum vortices in Sr_2RuO_4, assuming the system to be a $p_x + ip_y$ superconductor. If stabilized, such half-quantum vortices should exhibit non-Abelian braiding statistics and, potentially, lead to topological quantum computation.

We have been supported by the ARO-ARDA under Grant No. W911NF-04-1-0236. S.D.S. and S.T. have also been supported by the LPS-NSA. C.N. has also been supported by the NSF under Grant No. DMR-0411800.

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