Domain growth in the Heisenberg ferromagnet: Effective vector theory of the $S=1/2$ model

V. M. Turkowski,* P. D. Sacramento, and V. R. Vieira
CFIF, Instituto Superior Técnico, Avenida Rovisco Pais, 1049-001 Lisboa, Portugal
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We derive an effective vector theory of the spin $S=1/2$ Heisenberg ferromagnet in an external magnetic field using the Majorana representation for the spin operators and decoupling the interaction term via a Hubbard-Stratonovich transformation. This theory contains both cubic and quartic bosoniclike field terms. We analyze the problem in the Hartree approximation, similarly to the analysis by Boyanovsky *et al.* [Phys. Rev. E **48**, 767 (1993); Phys. Rev. D **48**, 800 (1993)] for the scalar case. The time dependence of the radius of the stable phase domain (bubble) in this bosonic theory is studied in the cases of different dimensionalities and weak magnetic field H. The role of the cubic terms in the process of domain growth is analyzed. It is shown that the field components perpendicular to H acquire a larger amplitude than the component parallel to H, at early times. The domain radius grows as $\sqrt{t}$ for times smaller than the spinodal time or in the case of a very weakly coupled theory. A simple scaling analysis shows that the time dependence changes to $R\sim t$ at long times.

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I. INTRODUCTION

The theoretical description of the time evolution of a nonhomogeneously distributed magnetization is an interesting and important problem of modern theoretical condensed matter physics. In particular, the problem of magnetic bubble formation and growth at low temperatures is of a special interest[1,2] (also see a theoretical paper[3]). Magnetic bubbles are, for instance, domains of reversed magnetization in ferromagnets. Such domains have been observed in thin ferromagnetic films with an uniaxial anisotropy. They arise due to demagnetization effects when an external magnetic field is not large enough. Besides purely academic interest, the solution of this problem is very important from a technological point of view. The domains of reversed magnetization are used for information storage in magnetic recording media. It is important to understand the conditions of the stability of these magnetic domains, and also the temperature and the magnetic field dependencies.

Magnetization domain growth after changing temperature from very high values to values below the critical temperature is another interesting problem, related to the previous one. This phenomenon is the well-known process of spinodal decomposition, when the phase separation in the initially homogeneous system takes place. In this paper we study this problem for the case of the quantum Heisenberg model with ferromagnetic interaction in the presence of an external homogeneous magnetic field. We derive an effective action for the vector field order parameter using the Majorana fermion representation for the spin $S=\frac{1}{2}$ operators[4,5] considering the general situation of a system far from equilibrium. We would like to note that the Majorana fermion representation for the spin $S=\frac{1}{2}$ operators allows one to use a standard Wick theorem for the fermions to calculate quantum statistical averages. Such a theorem does not exist for the spin operators, which makes the results obtained for the case with spin operators less controllable. We approximate the effective action by the field components power expansion up to the fourth order. The presence of the magnetic field results in linear and cubic in field terms in the expression for the free energy. The effective bosonic theory of the ferromagnetic Heisenberg spin $S=1/2$ model was studied in other contexts in Refs. 3 and 6–12, for example (also see reviews in Refs. 13 and 14). In this work we study the unstable domain radius time dependence in this model. We construct the equation of motion for the boson field, which is different from the corresponding classic equation for the scalar ϕ^4 model with a nonconserved order parameter (the Allen-Cahn-type model,[15] or model A, according to the classification given in Ref. 13), which corresponds to the Ising model case.

The case of the model with a nonconserved order parameter and with a zero cubic term has been widely studied in the condensed matter literature in the case of the classical ϕ^4 theory (see, for example, Refs. 15–22) and in the relativistic theory.[23–26] The relativistic $(1+1)$ dimensional model with an additional ϕ^6 term was studied in Ref. 27. The classical vector case was analyzed in Refs. 28 and 29, for example. It is known that the characteristic length grows as $R\sim\sqrt{t}$ in the classical scalar and vector theories (for a review, see Refs. 30 and 31). A similar result was found in the case of a weakly coupled scalar relativistic ϕ^4 theory at short times.[23,24] The principal difference between the relativistic and the classical cases is in the order of the time derivative in the equation for the field. The time derivative is of the first order in the classical Allan-Cahn equation, and it is of the second order in the relativistic case. Nevertheless, this difference does not lead to different time dependencies of the domain size in both theories at short times.

In this paper we show that the transverse part of the effective vector theory of the quantum Heisenberg ferromagnet is equivalent to the relativistic scalar theory in the limit of a very low magnetic field. The magnetic field generates linear and cubic terms in the free energy. These terms were not taken into account in the scalar case. The presence of the cubic term leads to a nontrivial coupling between the transverse and longitudinal field components, which is important for the domain dynamics. We show that the magnetic bubble radius time dependence is $\sqrt{t}$ at times shorter than the spinodal time, similarly to the classical vector and relativistic scalar cases. The scaling analysis in the Hartree approxima-

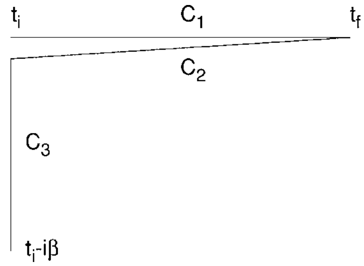

FIG. 1. Integration contour for the time variable. The direction is $t_i \to t_f \to t_i \to t_i - i\beta$.

tion shows that this time dependence changes to linear at long times.

II. EFFECTIVE THEORY

In this section we derive the effective bosoniclike action for the Heisenberg spin Hamiltonian equations (42) and (49). The Hamiltonian for the Heisenberg ferromagnet in an external magnetic field can be written as

$$\hat{H} = -\frac{1}{2}\sum_{ij} J_{ij}\boldsymbol{S}_i\boldsymbol{S}_j - \sum_i \boldsymbol{H}_i\boldsymbol{S}_i, \quad (1)$$

where $J_{ij} > 0$ is the nearest neighbor ferromagnetic coupling, i,j are site coordinates, H is an external and, in general, time-dependent magnetic field. To find the free energy of the system, we derive the expression for the generating function,

$$\mathcal{Z} = \frac{\mathrm{Tr}[\hat{T}e^{-\int_0^\beta \hat{H}\,d\tau}e^{-i\int_{t_f}^{t_i}\hat{H}\,dt}e^{-i\int_{t_i}^{t_f}\hat{H}\,dt}]}{\mathrm{Tr}[\hat{T}e^{-\int_0^\beta \hat{H}\,d\tau}]}, \quad (2)$$

where $\beta = 1/T$ is the inverse temperature of the system. We use the Keldysh formalism[32,33] to study the nonequilibrium properties of the system. In this formalism all time integrals should be performed along the complex three-branch contour C presented in Fig. 1.

It is convenient to represent the spin $S=1/2$ operators in (2) as a vector product of Majorana fermion operators $\boldsymbol{\zeta}_i$ (see, for example, Refs. 4, 5, 7–9, and 12):

$$\boldsymbol{S}_i = -\frac{i}{2}\boldsymbol{\zeta}_i \times \boldsymbol{\zeta}_i. \quad (3)$$

Then, the numerator of the generating function (2) can be written as

$$\int D\boldsymbol{\zeta}\exp\left(-i\int_C dt\left[-\frac{i}{2}\sum_i \boldsymbol{\zeta}_i\frac{d}{dt}\boldsymbol{\zeta}_i - \frac{1}{2}\sum_{ij} J_{ij}\boldsymbol{S}_i\boldsymbol{S}_j - \sum_i \boldsymbol{S}_i\boldsymbol{H}_i\right]\right). \quad (4)$$

To decouple the interaction between the spins in (4) we perform a Hubbard-Stratonovich transformation:

$$\mathcal{Z} = \int \frac{D\boldsymbol{\zeta}\,D\boldsymbol{\Phi}}{\sqrt{\det(2\pi J)}}\exp\left(-i\int_C dt\left[-\frac{i}{2}\sum_i \boldsymbol{\zeta}_i\frac{d}{dt}\boldsymbol{\zeta}_i + \frac{1}{2}\sum_{ij}\boldsymbol{\Phi}_i J_{ij}^{-1}\boldsymbol{\Phi}_j - \sum_i \boldsymbol{\Phi}_i\boldsymbol{S}_i - \sum_i \boldsymbol{H}_i\boldsymbol{S}_i\right]\right). \quad (5)$$

The vector field $\boldsymbol{\Phi}_i(t)$ in (5) is actually an order parameter that characterizes the magnetic properties of the system (see later). With the shift $\boldsymbol{\Phi}_i \to \boldsymbol{\Phi}_i - \boldsymbol{H}_i$ in (5), the partition function can be rewritten in a more convenient form,

$$\mathcal{Z} = \int \frac{D\boldsymbol{\zeta}\,D\boldsymbol{\Phi}}{\sqrt{\det(2\pi J)}}\exp\left(-i\int_C dt\left[-\frac{i}{2}\sum_i \boldsymbol{\zeta}_i\frac{d}{dt}\boldsymbol{\zeta}_i + \frac{1}{2}\sum_{ij}(\boldsymbol{\Phi}_i - \boldsymbol{H}_i)J_{ij}^{-1}(\boldsymbol{\Phi}_j - \boldsymbol{H}_j) - \sum_i \boldsymbol{\Phi}_i\boldsymbol{S}_i\right]\right). \quad (6)$$

As it follows from this expression, the effective Majorana fermion Hamiltonian for the Heisenberg ferromagnet is

$$\hat{H}_{eff} = -\sum_i \boldsymbol{\phi}_i\boldsymbol{S}_i = \frac{i}{2}\sum_{ilmn}\Phi_i^l\varepsilon^{lmn}\zeta_i^m\zeta_i^n, \quad (7)$$

where l, m, n are vector components. The action in the exponent of (6) is quadratic on the ζ variables. Therefore, it can be formally integrated,

$$\mathcal{Z} = \int \frac{D\boldsymbol{\Phi}}{\sqrt{\det(2\pi J)}}e^{-(i/2)\int_C dt\sum_{ij}(\boldsymbol{\Phi}_i - \boldsymbol{H}_i)J_{ij}^{-1}(\boldsymbol{\Phi}_j - \boldsymbol{H}_j)}\prod_i K(\boldsymbol{\Phi}_i), \quad (8)$$

where

$$K(\boldsymbol{\Phi}_i) = \int D\boldsymbol{\zeta}\exp\left(-\int_C dt\left[\frac{1}{2}\boldsymbol{\zeta}_i\frac{d}{dt}\boldsymbol{\zeta}_i + i\hat{H}_{eff}\right]\right) \quad (9)$$

is a function of the order parameter $\boldsymbol{\Phi}$. The functional $K(\boldsymbol{\Phi}_i)$ is the Helmholtz free energy for a single spin in an external effective field $\boldsymbol{\Phi}_i(t)$.

The integration over the ζ fields in (9) gives

$$\mathcal{Z} = \frac{1}{2^{3N/2}}\int \frac{D\boldsymbol{\Phi}}{\sqrt{\det(2\pi J)}}\exp\left(-\frac{i}{2}\int_C dt\sum_{ij}(\boldsymbol{\Phi}_i - \boldsymbol{H}_i)\times J_{ij}^{-1}(\boldsymbol{\Phi}_j - \boldsymbol{H}_j) + \frac{1}{2}\sum_i \mathrm{Tr}\ln iG^{-1}(\boldsymbol{\Phi}_i)\right), \quad (10)$$

where

$$G_{ij}^{ab}(t,t') = -i\langle \hat{T}\zeta_i^a(t)\zeta_j^b(t')\rangle \quad (11)$$

is the propagator for the ζ field. The effective bosonic action, which corresponds to the partition function (10), has the following form:

$$\beta F = \frac{i}{2}\int_C dt\sum_{i,j}(\boldsymbol{\Phi}_i - \boldsymbol{H}_i)J_{ij}^{-1}(\boldsymbol{\Phi}_j - \boldsymbol{H}_j) - \frac{1}{2}\sum_i \mathrm{Tr}\ln iG_{ii}^{-1}(\boldsymbol{\Phi}_i), \quad (12)$$

where the trace operation assumes the time integration along the contour C, and the summation over the space and spin indexes. The $\mathbf{\Phi}$ fields are not strictly bosonic fields but commutative classical variables.

In general, there is no simple analytical solution for the propagator $G_{ij}^{ab}(t,t')$, since the field $\mathbf{\Phi}_i(t)$ may be both time and space dependent. Therefore, it is very difficult to find the behavior of the system with the free energy (12), especially when some initial nonhomogeneous distribution of the magnetization (or $\mathbf{\Phi}$), like a bubble, for example, takes place. It is convenient to obtain a simpler bosonic theory that can describe the behavior of the system. We shall study the behavior of a small magnetic domain at short to intermediate times. In this case it is possible to approximate the free energy of the system by the power expansion in the deviations of the field $\boldsymbol{\phi}=\mathbf{\Phi}-\mathbf{\Phi}^{(0)}$ up to some finite order in the effective action (12), $\mathbf{\Phi}^{(0)}$ is the saddle point equilibrium solution (homogeneous high temperature solution). Usually, it is enough to consider the fourth order contribution to get the main physical effects in the system.

Minimization of the thermodynamic potential with respect to $\mathbf{\Phi}$ gives the saddle-point equation for the order parameter:

$$\mathbf{\Phi}_i^{(0)}(t)=\boldsymbol{H}_i(t)+J_{ij}\boldsymbol{M}_j^{(0)}(t), \tag{13}$$

where the mean-field magnetization $M_j^{l(0)}(t)$ is

$$M_j^{l(0)}(t)=\frac{\delta}{i\delta\Phi_j^l(t)}\frac{1}{2}\,\mathrm{Tr}\ln iG^{-1}(\mathbf{\Phi}_j^{(0)})=\frac{1}{2}\varepsilon^{lsr}G_{jj}^{sr}(t,t). \tag{14}$$

Taking the time limit properly, one gets

$$M_j^{l(0)}(t)=\frac{1}{4}\varepsilon^{lsr}[G_{jj}^{(0)sr}(t^+,t)+G_{jj}^{(0)sr}(t^-,t)]. \tag{15}$$

The function $G_{ij}^{mn}(t_1,t_2)$ is local in space and it satisfies the following equation in the Cartesian basis:

$$\left(i\frac{d}{dt_1}\delta^{pm}-i\varepsilon^{pml}\Phi_i^{(0)l}(t_1)\right)G_{ij}^{mn}(t_1,t_2)=\delta^{pn}\delta(t_1-t_2). \tag{16}$$

This Green's function can be found analytically on the mean-field level in the case of an homogeneous time-dependent field $\mathbf{\Phi}^{(0)}(t)=[0,0,\Phi^{(0)}(t)]$ in the cylindrical basis $G_{ii}^{(0)\alpha\beta}(t_1,t_2)=-i\langle\hat{T}[\xi_i^{\alpha}(t_1)\xi_i^{\beta}(t_2)]\rangle$ with the coordinates $\xi_i^{\pm}(t)=1/\sqrt{2}[\xi_i^x(t)]\pm i\xi_i^y(t)$, $\xi_i^0(t)=\xi_i^z(t)$, but we do not present the general expression here. In the case the system is homogeneously magnetized and $\Phi^{(0)}$ is time independent, the Green's function is diagonal in the spin and space indexes and it satisfies the following equation:

$$\left(i\frac{d}{dt}-\alpha\Phi^{(0)}\right)G_{\alpha}(t-t')=\delta(t-t'). \tag{17}$$

The components of the Green's function are

$$G_{jj}^{(0)\alpha}(t_1,t_2)=G_{jj}^{\alpha>}(t_1,t_2)\theta_C(t_1-t_2)+G_{jj}^{\alpha<}(t_1,t_2)\theta_C(t_2-t_1), \tag{18}$$

$$G_{jj}^{\alpha>}(t_1,t_2)=-\frac{i}{\exp(-\beta\alpha\Phi_j^{(0)})+1}e^{-i\alpha\Phi_j^{(0)}(t_1-t_2)}, \tag{19}$$

$$G_{jj}^{\alpha<}(t_1,t_2)=\frac{i}{1+\exp(\beta\alpha\Phi_j^{(0)})}e^{-i\alpha\Phi_j^{(0)}(t_1-t_2)}. \tag{20}$$

The zeroth-order Green's function in the Cartesian basis has the following structure:

$$\hat{G}_{jj}^{(0)sr}(t_1,t_2)=\begin{pmatrix} G_{jj}^{(0)xx}(t_1,t_2) & G_{jj}^{(0)xy}(t_1,t_2) & 0\\ -G_{jj}^{(0)xy}(t_1,t_2) & G_{jj}^{(0)xx}(t_1,t_2) & 0\\ 0 & 0 & G_{jj}^{(0)zz}(t_1,t_2)\end{pmatrix}. \tag{21}$$

The Green's function components in the Cartesian basis are connected with the Green's function components in a cylindrical basis as:

$$G_{jj}^{(0)xx}(t_1,t_2)=\frac{1}{2}[G_{jj}^{(0)+}(t_1,t_2)+G_{jj}^{(0)-}(t_1,t_2)], \tag{22}$$

$$G_{jj}^{(0)xy}(t_1,t_2)=-\frac{1}{2i}[G_{jj}^{(0)+}(t_1,t_2)-G_{jj}^{(0)-}(t_1,t_2)], \tag{23}$$

$$G_{jj}^{(0)zz}(t_1,t_2)=G_{jj}^{(0)0}(t_1,t_2). \tag{24}$$

Therefore, Eq. (15) in this case has the following form:

$$M_i^{(0)}=\frac{1}{2}\tanh\left(\frac{\Phi_i^{(0)}}{2T}\right). \tag{25}$$

The system of equations (13) and (25) allows us to find the equilibrium magnetization $M^{(0)}$ and the order parameter $\Phi^{(0)}$ of the system in an external magnetic field H and at temperature T. The mean-field critical temperature of the system is $T_c=dJ/2$ $(H=0)$, where d is the dimensionality of the system. We shall consider below the situation when the temperature of the system is switched from a high value $T_i\gg T_c$ to some low value $T<T_c$ at time $t=0$. It is important that the mean-field solution $\Phi^{(0)}$ is time independent when the external field is constant in time. It follows from (13) and (25) that the mean field solution is $\mathbf{\Phi}^{(0)}=\boldsymbol{H}$, when the initial temperature is very high. The time dependence of the initial bubble will be described by a fluctuation field $\boldsymbol{\phi}=\mathbf{\Phi}-\boldsymbol{H}$, which is the deviation of the field from the mean-field high T solution $\mathbf{\Phi}^{(0)}=\boldsymbol{H}$.

To study the time dependence of the field $\boldsymbol{\phi}$, we shall evaluate an approximate expression for the free energy (12) in powers of this field (up to the fourth order). For this purpose, it is convenient to rewrite the last term in (12) in the following form:

$$\begin{aligned}\frac{1}{2}\,\mathrm{Tr}\ln iG^{-1}(\mathbf{\Phi})&=\frac{1}{2}\,\mathrm{Tr}\ln iG^{(0)-1}(\boldsymbol{H})+\frac{1}{2}\,\mathrm{Tr}\ln\{1+G^{(0)}(\boldsymbol{H})\\&\quad\times[G^{-1}(\mathbf{\Phi})-G^{(0)-1}(\boldsymbol{H})]\}\\&=\frac{1}{2}\,\mathrm{Tr}\ln iG^{(0)-1}(\boldsymbol{H})-\frac{1}{2}\,\mathrm{Tr}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\\&\quad\times\{G^{(0)}(\boldsymbol{H})[G^{-1}(\mathbf{\Phi})-G^{(0)-1}(\boldsymbol{H})]\}^n.\end{aligned} \tag{26}$$

The following expression for the Green's function can be used:

$$G_{ij}^{-1mn}(t_1,t_2) = \left(i\frac{d}{dt_1}\delta^{nn}\delta_{ij} - i\varepsilon^{mnl}\delta_{ij}\Phi_i^l(t_1) \right)\delta(t_1-t_2). \tag{27}$$

Therefore,

$$[G^{-1}(\mathbf{\Phi}) - G^{(0)-1}(\boldsymbol{H})]_{ij}^{lm}(t_1,t_2) = -i\varepsilon^{lmn}\ \delta_{ij}\phi_i^n(t_1)\delta(t_1-t_2). \tag{28}$$

The system of equations (12), (26), and (79) defines the power expansion of the effective bosonic theory for the Heisenberg model. The free energy has the following form in the ϕ^4 approximation:

$$\begin{aligned} F &= \frac{i}{2}\int_C dt \sum_{i,j} (\mathbf{\Phi}_i - \boldsymbol{H}_i)J_{ij}^{-1}(\mathbf{\Phi}_j - \boldsymbol{H}_j) - \frac{1}{2}\sum_i \mathrm{Tr}\ln iG_{ii}^{-1}(\mathbf{\Phi}_i) \\ &\simeq -\frac{1}{2}\sum_i \mathrm{Tr}\ln iG_{ii}^{-1}(\boldsymbol{H}_i) - i\int_C dt_1\, j_i^l(t_1)\phi_i^l(t_1) + \frac{i}{2}\int_C dt_1 \int_C dt_2\ \phi_i^l(t_1)[J_{ij}^{-1}\delta^{lm}\delta(t_1-t_2) - i\ \delta_{i,j}\bar{\chi}_{ii}^{lm}(t_1,t_2)]\phi_j^m(t_2) \\ &\quad + \frac{1}{3!}\int_C dt_1\int_C dt_2 \int_C dt_3 C_i^{lmn}(t_1,t_2,t_3)\phi_i^l(t_1)\phi_i^m(t_2)\phi_i^n(t_3) \\ &\quad + \frac{1}{4!}\int_C dt_1\int_C dt_2\int_C dt_3\int_C dt_4\ \lambda_i^{lmnr}(t_1,t_2,t_3,t_4)\phi_i^l(t_1)\phi_i^m(t_2)\phi_i^n(t_3)\phi_i^r(t_4), \end{aligned} \tag{29}$$

where

$$j_i^l(t_1) = M_i^{l(0)}(t_1), \tag{30}$$

$$\bar{\chi}_{ii}^{lm}(t_1,t_2) = -\frac{1}{2}\varepsilon^{ll_1l_2}\varepsilon^{mm_1m_2}G_{ii}^{(0)l_2m_1}(t_1,t_2)G_{ii}^{(0)m_2l_1}(t_2,t_1), \tag{31}$$

$$C_i^{lmn}(t_1,t_2,t_3) = -i\varepsilon^{ll_1l_2}\varepsilon^{mm_1m_2}\varepsilon^{nn_1n_2}G_{ii}^{(0)l_2m_1}(t_1,t_2)G_{ii}^{(0)m_2n_1}(t_2,t_3)G_{ii}^{(0)n_2l_1}(t_3,t_1), \tag{32}$$

$$\lambda_i^{lmnr}(t_1,t_2,t_3,t_4) = 3\varepsilon^{ll_1l_2}\varepsilon^{mm_1m_2}\varepsilon^{nn_1n_2}\varepsilon^{rr_1r_2}G_{ii}^{(0)l_2m_1}(t_1,t_2)G_{ii}^{(0)m_2n_1}(t_2,t_3)G_{ii}^{(0)n_2r_1}(t_3,t_4)G_{ii}^{(0)r_2l_1}(t_4,t_1). \tag{33}$$

The function $\bar{\chi}_{ij}^{lm}(t_1,t_2)$ is in fact the zeroth-order local magnetic susceptibility of the system, calculated as $\bar{\chi}_{ij}^{lm}(t_1,t_2) = \delta M_i^{(0)l}(t_1)/i\ \delta H_j^{(0)m}(t_2)$. Note that there are linear and cubic terms. The linear term is present since this is not a saddle-point expansion at each temperature but an expansion around the high-temperature mean-field solution. Obviously, this ϕ^4 approximation in (29) is valid only when the field is small. This is always correct at short times. It is also correct at intermediate and long times when the external field is weak or/and the temperature is high enough. Since the last case is the case under consideration, we can safely consider the free energy in the fourth order in the perturbation theory.

We shall derive the effective theory, local in time and continuous in space, in the following way. First, we pass to the continuous version of the hopping operator in the quadratic term in (29):

$$J_{ij}^{-1} \to \frac{1}{2\,dJ}\left(1 - \frac{\boldsymbol{\nabla}^2}{2d}a^2\right); \tag{34}$$

where a is the lattice constant. In order to get the term local in time we make a Fourier transform of the susceptibility with respect to the relative time coordinate $t=t_1-t_2$. We keep the lowest terms in the frequency ω and then substitute $\omega \to id/dt$. The susceptibility does not depend on the center of mass time coordinate $(t_1+t_2)/2$ and, therefore, it can be integrated out along the contour. This gives just a factor $-i\beta$. The expression for the quadratic part of the free energy is in this case

$$\begin{aligned} F_2 = iT\int_c dt \int d\mathbf{x}\Bigg[&\frac{1}{2}\boldsymbol{\phi}_\perp(\mathbf{x},t)\left(\gamma_1\frac{d^2}{dt^2} + (\widetilde{m}^{xx})^2\right)\boldsymbol{\phi}_\perp(\mathbf{x},t) \\ &+ \frac{1}{2}(\widetilde{m}^{zz})^2\phi^{z2}(\mathbf{x},t) + \frac{1}{2}\left(\phi^x(\mathbf{x},t)\gamma_2\frac{d}{dt}\phi^y(\mathbf{x},t)\right. \\ &\left. - \phi^y(\mathbf{x},t)\gamma_2\frac{d}{dt}\phi^x(\mathbf{x},t)\right) + \frac{a^2}{16\,dT_c}\sum_{l=x,y,z}[\boldsymbol{\nabla}\phi^l(\mathbf{x},t)]^2\Bigg], \end{aligned} \tag{35}$$

where $\boldsymbol{\phi}_\perp = (\phi^x, \phi^y, 0)$ is the planar field component,

$$\gamma_1 = (1/2H^3)\tanh(\beta H/2), \tag{36}$$

$$\gamma_2 = (1/2H^2)\tanh(\beta H/2), \quad (37)$$

and the effective mass squares are

$$(\tilde{m}^{xx})^2 = \frac{1}{2\,dJ} - \frac{1}{2H}\tanh\left(\frac{\beta H}{2}\right), \quad (38)$$

$$(\tilde{m}^{zz})^2 = \frac{1}{2\,dJ} - \frac{\beta}{4\cosh^2(\beta H/2)}. \quad (39)$$

The effective mass squares are negative in the ordered phase. The detailed derivation of (35) is presented in Appendix A. Let us note that only the planar components have time derivatives. These correspond to the presence of the planar component ferromagnetic wave modes in the system. The z component has no time derivatives since $\chi_{ii}^{zz}(t_1,t_2)$ is time independent, and the mixed xz and yz components are equal to 0 (see Appendix A).[34] It is interesting to note also that the terms linear in frequency in the effective action, which corresponds to Eq. (35), have a form different from the one obtained in the case of the Hubbard model.[35] In the last case the lowest frequency contribution is proportional to ω divided by the product of the Fermi velocity and the momentum modulus $v_F|\mathbf{q}|$, which means that "…the decay mechanism for the paramagnon excitations is Landau damping lifetime of the free particle-hole pair with the total momentum $|\mathbf{q}|$ is $(v_F|\mathbf{q}|)^{-1}$…."[35] In our case the standard magnon dispersion relation $\omega \sim \mathbf{k}^2$ can be obtained from the expression for the RPA susceptibility. This expression can be evaluated starting from the free energy (29) (for details, see Ref. 9). In this paper we consider also the terms second order in frequency in the effective action. As it will be shown below, the terms $\sim\omega^2$ are important in the case when one of the planar components of the field (ϕ_x or ϕ_y) is space and time independent. In this case, the bubble radius has an unusual time dependence at long times (see Sec. III B).

Second, we make the time average of the other nonlocal in time terms in the following way:

$$\int_C dt_1 \int_C dt_2 \int_C dt_3 C_i^{lmn}(t_1,t_2,t_3)\phi_i^l(t_1)\phi_i^m(t_2)\phi_i^n(t_3) \simeq \int_C dt\, \tilde{C}_i^{lmn}(t)\phi_i^l(t)\phi_i^m(t)\phi_i^n(t), \quad (40)$$

where

$$\tilde{C}_i^{lmn}(t) \simeq \int_C dt_1 \int_C dt_2\, C_i^{lmn}(t_1,t_2,t). \quad (41)$$

The same procedure is applied to the j and λ terms. This local in time approximation is valid in the nonequilibrium case when the fields rapidly change with time, so the equal time products make the main contribution to the integral. It is also a good approximation at very long times when the system approaches equilibrium and the fields almost do not depend on time. The short and the long time cases are the most interesting to us; therefore we use this approximation. However, the study of the intermediate time regime requires a more accurate approach.

The expression for the free energy is then reduced to

$$F = iT\int_c dt \int d\mathbf{x}\Bigg[-\tilde{j}^z\phi^z(\mathbf{x},t) + \frac{1}{2}\boldsymbol{\phi}_\perp(\mathbf{x},t)\left(\gamma_1\frac{d^2}{dt^2} + (\tilde{m}^{xx})^2\right)\boldsymbol{\phi}_\perp(\mathbf{x},t) + \frac{1}{2}(\tilde{m}^{zz})^2\phi^{z2}(\mathbf{x},t) + \frac{1}{2}\left(\phi^x(\mathbf{x},t)\gamma_2\frac{d}{dt}\phi^y(\mathbf{x},t) - \phi^y(\mathbf{x},t)\gamma_2\frac{d}{dt}\phi^x(\mathbf{x},t)\right) + \frac{a^2}{16\,dT_c}\sum_{l=x,y,z}[\boldsymbol{\nabla}\phi^l(\mathbf{x},t)]^2 + \frac{1}{3!}\tilde{C}^{xxz}\boldsymbol{\phi}_\perp^2(\mathbf{x},t)\phi^z(\mathbf{x},t) + \frac{1}{3!}\tilde{C}^{zzz}\phi^{z3}(\mathbf{x},t) + \frac{1}{4!}\tilde{\lambda}^{xxxx}\boldsymbol{\phi}_\perp^4(\mathbf{x},t) + \frac{1}{4!}\tilde{\lambda}^{xxzz}\boldsymbol{\phi}_\perp^2(\mathbf{x},t)\phi^{z2}(\mathbf{x},t) + \frac{1}{4!}\tilde{\lambda}^{zzzz}\phi^{z4}(\mathbf{x},t)\Bigg]. \quad (42)$$

The expressions for the independent parameters are

$$\tilde{j}^z = \frac{1}{2}\tanh\left(\frac{\beta H}{2}\right), \quad (43)$$

$$\tilde{C}^{xxz} = -\frac{3\beta}{4H\cosh^2(\beta H/2)} + \frac{3}{2H^2}\tanh\left(\frac{\beta H}{2}\right), \quad (44)$$

$$\tilde{C}^{zzz} = \frac{\beta^2\tanh(\beta H/2)}{4\cosh^2(\beta H/2)}, \quad (45)$$

$$\tilde{\lambda}^{xxxx} = \frac{3}{4H^3\cosh^2(\beta H/2)}[\sinh(\beta H) - \beta H], \quad (46)$$

$$\tilde{\lambda}^{xxzz} = \frac{6}{H^3}\left[\frac{\beta^2H^2\tanh(\beta H/2)}{4\cosh^2(\beta H/2)} + \frac{\beta H}{2\cosh^2(\beta H/2)} - \tanh\left(\frac{\beta H}{2}\right)\right], \quad (47)$$

$$\tilde{\lambda}^{zzzz} = \frac{\beta^3}{4\cosh^2(\beta H/2)}\left(\frac{3}{2\cosh^2(\beta H/2)} - 1\right). \quad (48)$$

All the coefficients are space independent, since the magnetic field is homogeneous in space. It follows from expressions (46)–(48) for the free energy coefficients that the ϕ^4 approximation we use is valid only at some values of the ratio H/T. Namely, in order to have a well-defined theory, the coefficients $\tilde{\lambda}^{xxxx}$ and $\tilde{\lambda}^{zzzz}$ must be positive. Otherwise, the global minimum of the free energy will take place at $|\boldsymbol{\phi}|=\infty$. The coefficient λ^{xxxx} is always positive and the coefficient λ^{zzzz} becomes negative, when $\cosh(H/2T) > \sqrt{3/2}$, or at $H < 2a\cosh(\sqrt{3/2})T \simeq 1.316T$. Therefore, only the fields $H < 1.316T$ have a physical sense. The fact that the ratio $H/(2T)$ is smaller than 1 allows one to make an expansion of the coefficients in powers of $\beta H/2$ (see later).

The expression for the free energy (42) can be rewritten in the cylindric representation $\boldsymbol{\phi}=[\phi_\perp\cos(\varphi), \phi_\perp\sin(\varphi), \phi^z]$ as

$$
\begin{aligned}
F = iT\int_c dt \int d\mathbf{x}\Bigg[&-\tilde{j}^z\phi^z(\mathbf{x},t) \\
&+ \frac{1}{2}\boldsymbol{\phi}_\perp(\mathbf{x},t)\left(\gamma_1\frac{d^2}{dt^2} + (\tilde{m}^{xx})^2\right)\boldsymbol{\phi}_\perp(\mathbf{x},t) + \frac{1}{2}(\tilde{m}^{zz})^2\phi^{z2}(\mathbf{x},t) \\
&+ \frac{1}{2}\gamma_2\phi_\perp^2(\mathbf{x},t)\frac{d\varphi(\mathbf{x},t)}{dt} + \frac{a^2}{16\,dT_c}\{[\nabla\phi_\perp(\mathbf{x},t)]^2 + \phi_\perp^2(\mathbf{x},t) \\
&\times[\nabla\varphi(\mathbf{x},t)]^2 + [\nabla\phi^z(\mathbf{x},t)]^2\} + \frac{1}{3!}\tilde{C}^{xxz}\boldsymbol{\phi}_\perp^2(\mathbf{x},t)\phi^z(\mathbf{x},t) \\
&+ \frac{1}{3!}\tilde{C}^{zzz}\phi^{z3}(\mathbf{x},t) + \frac{1}{4!}\tilde{\lambda}^{xxxx}\boldsymbol{\phi}_\perp^4(\mathbf{x},t) \\
&+ \frac{1}{4!}\tilde{\lambda}^{xxzz}\boldsymbol{\phi}_\perp^2(\mathbf{x},t)\phi^{z2}(\mathbf{x},t) + \frac{1}{4!}\tilde{\lambda}^{zzzz}\phi^{z4}(\mathbf{x},t)\Bigg]. \qquad (49)
\end{aligned}
$$

The first term in the second line of (49) is the energy of the precession of the field around the z axis. Contrary to the classical case,[16,31] the free energy contains the second time derivative term for the field amplitude. This form of the free energy is more adequate in the case of the ferromagnet (see, for example, Ref. 36).

III. QUANTUM SPINODAL DECOMPOSITION

To study the spinodal decomposition in the system, it is convenient to analyze the structure factor $S^{ll}(\mathbf{r},t) = \langle\phi^l(\mathbf{r},t)\phi^l(0,t)\rangle$, or its Fourier transform $S^{ll}(\mathbf{k},t) = \int d^d\mathbf{r}\exp(-i\mathbf{kr})S^{ll}(\mathbf{r},t) = \langle\phi_\mathbf{k}^l(t)\phi_{-\mathbf{k}}^l(t)\rangle$.[23] To find the structure factor $S^{ll}(\mathbf{k},t)$, one needs to find the solution of the equation for $\boldsymbol{\phi}(\mathbf{x},t)$, which can be found by minimizing the free energy (42) and (49) with respect to the field $\boldsymbol{\phi}$ components. The equations have the simplest form in the cylindric representation of the field:

$$
\begin{aligned}
\Bigg(&\gamma_2\frac{d^2}{dt^2} + (\tilde{m}^{xx})^2 - \frac{a^2}{8dT_c}\nabla^2 + \gamma_1\frac{d\varphi}{dt} + \frac{a^2}{8dT_c}[\nabla\varphi(\mathbf{x},t)]^2 \\
&+ \frac{\tilde{C}^{xxz}}{3}\phi^z(\mathbf{x},t) + \frac{\tilde{\lambda}^{xxxx}}{3!}\phi_\perp^2(\mathbf{x},t) + \frac{\tilde{\lambda}^{xxzz}}{12}\phi^{z2}(\mathbf{x},t)\Bigg)\phi_\perp(\mathbf{x},t) \\
&= 0, \qquad (50)
\end{aligned}
$$

$$
\gamma_2\frac{d}{dt}[\phi_\perp^2(\mathbf{x},t)] - \frac{a^2}{8\,dT_c}\nabla^2[\phi_\perp^2(\mathbf{x},t)] = 0, \qquad (51)
$$

$$
\begin{aligned}
\Bigg(&(\tilde{m}^{zz})^2 - \frac{a^2}{8dT_c}\nabla^2 + \frac{\tilde{C}^{zzz}}{2}\phi^z(\mathbf{x},t) + \frac{\tilde{\lambda}^{xxzz}}{12}\phi_\perp^2(\mathbf{x},t) \\
&+ \frac{\tilde{\lambda}^{zzzz}}{3!}\phi^{z2}(t)(\mathbf{x},t)\Bigg)\phi^z(\mathbf{x},t) = \tilde{j}^z - \frac{\tilde{C}^{xxz}}{3!}\phi_\perp^2(\mathbf{x},t). \qquad (52)
\end{aligned}
$$

This system can be solved in the following way. First, one solves Eq. (51) for $\phi_\perp^2$; then substitution of this solution into (52) gives an equation for ϕ_z that can be solved. Finally, the substitution of the solutions for $\phi_\perp^2$ and ϕ^z into (50) gives an equation for the angle component of the order parameter. Its solution gives the dependence $\varphi(t,\mathbf{x})$.

It is important that there is no dynamics in the system if we put $\phi_\perp=0$. In this case we have only Eq. (52) for ϕ_z that has no time derivatives. Therefore, the vector field should be considered in order to study spinodal decomposition.

First of all, we note that as it follows from Eq. (51), the solution for $f(t,\mathbf{x})=\phi_\perp^2(t,\mathbf{x})$ depends on one variable $x=R^2/t$, where R is the radial component of the vector in the spherical basis. This can be shown in the most simple way in the spherical coordinates. Renormalizing the time and radius variables: $t=t/\gamma_1$, $R=\sqrt{8dT_c/a^2}R$, we get the following equation for f:

$$
\frac{df}{dt} - \left(\frac{d^2}{dR^2} + \frac{d-1}{R}\frac{d}{dR}\right)f = 0. \qquad (53)
$$

In terms of the new variable $x=R^2/t$, we have

$$
xf'' + (x+d)f' = 0. \qquad (54)
$$

This equation has a formal solution,

$$
f(x) = C_1\int_{C_2}^{x} dz\, z^{-d}e^{-z}, \qquad (55)
$$

or

$$
\phi_\perp^2(t,\mathbf{x}) = C_1\int_{C_2}^{R^2/t} dz\, z^{-d}e^{-z}. \qquad (56)
$$

Nevertheless, it is difficult to analyze the solutions of Eq. (52) for ϕ_z and especially of Eq. (50) for the order parameter phase φ. We shall use the following approximation in order to simplify the system of equations (50)–(52). We assume that the vector $\boldsymbol{\phi}$ is precessing around the z axis with a constant frequency:

$$
\varphi(\mathbf{x},t) = \varphi_0 + \omega_1 t. \qquad (57)
$$

This is a natural assumption that follows from the equilibrium theory. It is possible to find the equilibrium value of the frequency ω_1. Really, if we insert $\phi_\perp=\text{const}$, $\phi^z=\text{const}$ and φ given by (57) into (50)–(52), we find the equilibrium value for ϕ^z for a given $\phi_\perp$ from (52), and the expression for the frequency ω_1:

$$\omega_1 = -\frac{1}{\gamma_1}\left(\frac{\tilde{C}^{xxz}}{3}\phi^z + \frac{\tilde{\lambda}^{xxxx}}{3!}\phi_\perp^2 + \frac{\tilde{\lambda}^{xxzz}}{12}\phi^{z2}\right). \quad (58)$$

We assume that initially the angle $\varphi_0(\mathbf{x})$ is the same over the lattice.

Now the system (50)–(52) can be simplified in the following way. Assuming that φ is given by (57), we neglect Eq. (51), since it was obtained by minimizing (49) with respect to φ. We substitute solution (57) into (50), which leads to renormalization of the mass,

$$(\tilde{m}^{xx})^2 \to (\tilde{m}^{xx})^2 + \gamma_1\omega_1. \quad (59)$$

Assuming that the frequency ω_1 is not too large, which is true in the case of small magnetic fields, the mass $(\tilde{m}^{xx})^2$ remains negative, even after the renormalization. Below we shall assume that the plane mass is given by (59), where ω_1 is defined by the equilibrium solution (57). In this case the simplified system of equations is

$$\left(\gamma_1\frac{d^2}{dt^2} + (\tilde{m}^{xx})^2 - \frac{a^2}{8\,dT_c}\boldsymbol{\nabla}^2 + \frac{\tilde{C}^{xxz}}{3}\phi^z(\mathbf{x},t) + \frac{\tilde{\lambda}^{xxxx}}{3!}\phi_\perp^2(\mathbf{x},t) + \frac{\tilde{\lambda}^{xxzz}}{12}\phi^{z2}(\mathbf{x},t)\right)\phi_\perp(\mathbf{x},t) = 0, \quad (60)$$

$$\left((\tilde{m}^{zz})^2 - \frac{a^2}{8dT_c}\boldsymbol{\nabla}^2 + \frac{\tilde{C}^{zzz}}{2}\phi^z(\mathbf{x},t) + \frac{\tilde{\lambda}^{xxzz}}{12}\phi_\perp^2(\mathbf{x},t) + \frac{\tilde{\lambda}^{zzzz}}{3!}\phi^{z2}(\mathbf{x},t)\right)\phi^z(\mathbf{x},t) = \tilde{j}^z - \frac{\tilde{C}^{xxz}}{3!}\phi_\perp^2(\mathbf{x},t). \quad (61)$$

Later, we shall study this system in different physical cases.

A. Free case

The system of equations (60) and (61) has a simple form in the free case:

$$\left(\gamma_1\frac{d^2}{dt^2} + (\tilde{m}^{xx})^2 - \frac{a^2}{8\,dT_c}\boldsymbol{\nabla}^2\right)\phi_\perp(\mathbf{x},t) = 0, \quad (62)$$

$$\left((\tilde{m}^{zz})^2 - \frac{a^2}{8\,dT_c}\boldsymbol{\nabla}^2\right)\phi^z(\mathbf{x},t) = \tilde{j}^z. \quad (63)$$

The equations are therefore decoupled. Only the plane component of the field $\phi_\perp$ has time dependence. It is necessary to note that the masses for the transverse and longitudinal components of the field are different, in general. Moreover, the square of the masses can have different signs for these components. However, in the limit of small H/T they coincide [see (38) and (39)]. This rotational symmetry breaking is caused by the magnetic field.

To find the correlation function for the plane components of the order parameter, one needs to solve the quadratic equation in the momentum representation in analogy with Refs. 23 and 24:

$$\left(\frac{d^2}{dt^2} + (\tilde{m}^{xx})^2 + \mathbf{k}^2\right)U^{\pm}_{\perp\mathbf{k}}(t) = 0, \quad (64)$$

where $U^{\pm}_{\perp\mathbf{k}}(t)$ correspond to positive and negative frequency modes of the field $\phi_\perp(\mathbf{x},t)$ in the momentum representation. $t=t/\sqrt{\gamma_2}$ and $\mathbf{k}=(a/\sqrt{8dT_c})\mathbf{k}$ are dimensionless time and momentum variables. The boundary conditions for the fields are

$$U^{\pm}_{\perp\mathbf{k}}(t) = e^{\mp i\omega_<(\mathbf{k})t}, \quad t<0, \quad (65)$$

where $\omega_<(\mathbf{k})$ is defined in (69). The correlation function in this case is given by

$$\langle\phi_{\perp\mathbf{k}}(t)\phi_{\perp-\mathbf{k}}(t)\rangle = \frac{1}{\omega_<(\mathbf{k})}\coth\left(\frac{\beta_i\omega_<(\mathbf{k})}{2}\right)U^+_{\perp\mathbf{k}}(t)U^-_{\perp\mathbf{k}}(t). \quad (66)$$

The solution of Eqs. (64) with boundary conditions (65) gives[24]

$$\langle\phi_{\perp\mathbf{k}}(t)\phi_{\perp-\mathbf{k}}(t)\rangle = \frac{1}{\omega_<(\mathbf{k})}\coth\left(\frac{\beta_i\omega_<(\mathbf{k})}{2}\right), \quad t<0, \quad (67)$$

$$\begin{aligned}\langle\phi_{\perp\mathbf{k}}(t)\phi_{\perp-\mathbf{k}}(t)\rangle = \frac{1}{\omega_<(\mathbf{k})}\Bigg[&\left(1+\frac{1}{2}\left(1+\frac{\omega_<^2(\mathbf{k})}{W^2(\mathbf{k})}\right)\times\{\cosh[2W(k)t]-1\}\right)\theta(m_f^2-\mathbf{k}^2)\\ &+\left(1+\frac{1}{2}\left(1-\frac{\omega_<^2(\mathbf{k})}{\omega_>^2(\mathbf{k})}\right)\{\cos[2\omega_>(k)t]-1\}\right)\times\theta(\mathbf{k}^2-m_f^2)\Bigg]\coth(\beta_i\omega_</2), \quad t>0,\end{aligned} \quad (68)$$

where

$$\omega_>(\mathbf{k}) = \sqrt{\mathbf{k}^2-m_f^2}, \quad \omega_<(\mathbf{k}) = \sqrt{\mathbf{k}^2+m_i^2}, \quad (69)$$

$$W(\mathbf{k}) = \sqrt{m_f^2-\mathbf{k}^2}; \quad (70)$$

$\beta_i=1/T_i$ is an initial inverse temperature, which is assumed to be very large. $m_i^2(T)=(\tilde{m}^{xx})^2(T_i)>0$ and $-m_f^2=(\tilde{m}^{xx})^2(T_f)<0$ are the effective masses in the disordered (initial) and in the ordered (final) phases with temperatures $T_i>T_c$ and $T_f<T_c$, correspondingly.

To study how a domain grows in the case of different dimensions, one needs to calculate the scaled correlation function for the transverse field component in the real space representation:

$$\bar{S}(\mathbf{r},t) = \int\frac{d\mathbf{k}}{(2\pi)^d}e^{i\mathbf{kr}}[S(\mathbf{k},t)-S(\mathbf{k},0)]. \quad (71)$$

Approximate analytical expressions for the correlation function (71) can be obtained in the case of different dimensions by using (68). Momentum integration in (71) can be

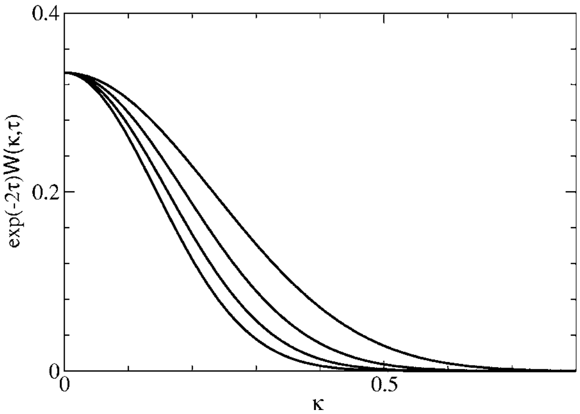

FIG. 2. Momentum dependence of the function $\exp(-2\tau)\tau^{(d-1)/2}W(\kappa,\tau)$, where $W(\kappa,\tau)=\kappa^{d-1}[S(\kappa,\tau)-S(\kappa,0)]$ in the one-dimensional (1D) case at $\tau=10,15,20,25$. Here and in Figs. 3 and 4, the curve moves to the right as time increases.

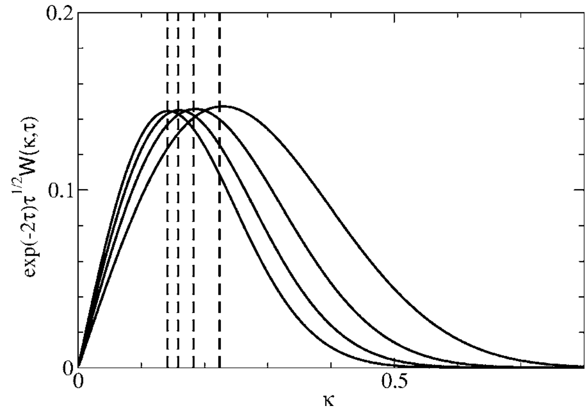

FIG. 3. The same as in Fig. 2 in the 2D case. Vertical lines correspond to the extremum values of the momenta $\kappa_s=1/\sqrt{2\tau}$ at given values of time. Here and in Fig. 4 the curve moves to the left as time increases.

performed for $|\mathbf{k}|<m_f$, since the unstable modes give the main contribution in the integral (71). It is convenient also to introduce dimensionless variables:

$$\kappa = k/m_f, \quad \tau = m_f t, \quad x = r/m_f, \quad L^2 = m_i^2/m_f^2. \tag{72}$$

In this case, the scaled correlation function (71) can be written as

$$\bar{S}(x,\tau) = m_f^{d-2}T_i(L^2+1)\int_0^1 d\kappa\,\kappa^{d-1}\mu_d(\kappa x)\frac{1}{(\kappa^2+L^2)(\kappa^2+1)} \times[\cosh(2\sqrt{1-\kappa^2}\tau)-1], \tag{73}$$

where $\mu_d(\kappa x)$ is the d-dimensional measure of integration,

$$\mu_1(\kappa x) = \frac{1}{\pi}\cos(\kappa x), \quad d=1,$$

$$\mu_2(\kappa x) = \frac{1}{\pi^2}\int_0^{\pi/2} d\varphi\cos(\kappa x\cos(\varphi)), \quad d=2,$$

$$\mu_3(\kappa x) = \frac{1}{2\pi^2}\frac{\sin(\kappa x)}{\kappa x}, \quad d=3. \tag{74}$$

It is easy to show that the function $W(\kappa,\tau)=\kappa^{d-1}[S(\kappa,\tau)-S(\kappa,0)]$ as a function of κ has a sharp maximum at some value of $\kappa=\kappa_s$, which depends on the dimensionality. A simple analysis shows that the value of κ_s is

$$\kappa_s = 0, \quad d=1,$$

$$\kappa_s = \frac{1}{\sqrt{2\tau}}, \quad d=2,$$

$$\kappa_s = \frac{1}{\sqrt{\tau}}, \quad d=3, \tag{75}$$

and

$$W(\kappa_s,\tau) = \kappa_s^{d-1}[S(\kappa_s,\tau)-S(\kappa_s,0)] \sim \exp(2\tau)/\tau^{(d-1)/2}. \tag{76}$$

Numerical calculations confirm these. (See Figs. 2–4.)

In this case the scaled correlation function (73) can be calculated analytically by making a saddle-point approximation for the expression under the integral

$$\bar{S}(x,\tau) = g\left(\frac{x^2}{\tau}\right)\bar{S}(0,\tau), \tag{77}$$

where the local correlation function is

$$\bar{S}(0,\tau) = \frac{m_f^{d-2}T_i(L^2+1)}{2^d\pi^{d/3}L^2}\frac{e^{2\tau}}{\tau^{d/2}}, \tag{78}$$

and the scaled space dependent part of the correlation function is

$$g\left(\frac{x^2}{\tau}\right) \simeq e^{-x^2/4\tau}, \quad d=1,$$

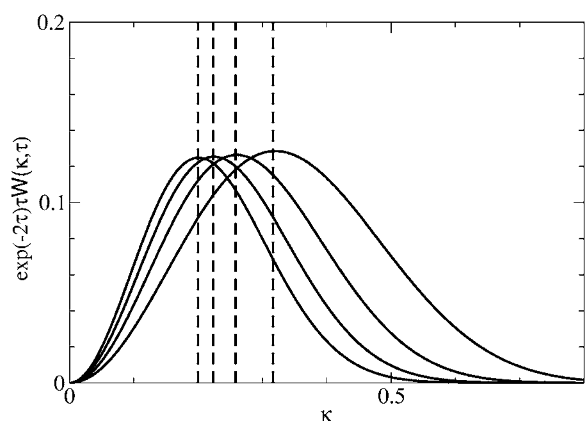

FIG. 4. The same as in Figs. 2 and 3 in the 3D case. Vertical lines correspond to the extremum values of the momenta $\kappa_s=1/\sqrt{\tau}$ at given values of time in this case.

$$g\left(\frac{x^2}{\tau}\right) \simeq e^{-x^2/2\tau}, \quad d=2,$$

$$g\left(\frac{x^2}{\tau}\right) \simeq \frac{\sin(x/\sqrt{\tau})}{x/\sqrt{\tau}} e^{-x^2/4\tau}, \quad d=3. \tag{79}$$

It follows from (78) that the fluctuations inside the domain at long times, $\bar{S}(0,\tau)=\langle\phi^2(0,\tau)\rangle$, are the strongest in the 1D case, and they decrease when the dimensionality of the system is increased, as it should be.

The scaling of the correlation function arguments x^2/τ, shows that the domain size grows as $R\sim\sqrt{t}$ in all the cases with different dimensionalities, similar to the 3D case.[23,24]

B. The interacting case

In the interacting case the system (60) and (61) is much more complicated, since this is a system of coupled nonlinear equations. To solve them, we shall use the Hartree approximation. This approximation was already used to solve the equation in the scalar case.[23,24] Here we use this approximation to solve the system of coupled equations. First of all, we simplify Eqs. (60) and (61) by considering the case $H/T \ll 1$, which corresponds to a rather high temperature below T_c, or to a weak magnetic field. In this case the coefficients $\tilde{j}$ and $\tilde{C}$ are proportional to H/T and are small. The coefficients $\tilde{\lambda}$ are H independent in the lowest order in H/T. It is worth mentioning that the role of the linear and the cubic terms in the free energy, in general, can be important, though it does not change the rate of the domain growth in the classic theory, where only the domain boundary curvature defines the bubble radius velocity (for a review, see, for example, Ref. 30).

Now we apply the Hartree approximation for the system (60) and (61). The coupled system of equations for the local in space correlation functions $\langle\phi_\perp^2(t)\rangle \equiv \langle\phi_\perp(\mathbf{0},t)\phi_\perp(\mathbf{0},t)\rangle$ and $\langle\phi^{z2}(t)\rangle \equiv \langle\phi^z(\mathbf{0},t)\phi^z(\mathbf{0},t)\rangle$ has the following form in this case:

$$\left(\frac{d^2}{dt^2} + \mathbf{k}^2 + m^2 + \frac{\lambda}{2}(\langle\phi_\perp^2(t)\rangle - \langle\phi_\perp^2(0)\rangle + \langle\phi^{z2}(t)\rangle - \langle\phi^{z2}(0)\rangle)\right) U_{\perp\mathbf{k}}^{\pm}(t) = 0, \tag{80}$$

$$\left(m^2 + \mathbf{k}^2 + \frac{\lambda}{2}(\langle\phi_\perp^2(t)\rangle - \langle\phi_\perp^2(0)\rangle + \langle\phi^{z2}(t)\rangle - \langle\phi^{z2}(0)\rangle)\right) \times U_{z\mathbf{k}}^{\pm}(t) = j - C(\langle\phi_\perp^2(t)\rangle - \langle\phi_\perp^2(0)\rangle + \langle\phi^{z2}(t)\rangle - \langle\phi^{z2}(0)\rangle), \tag{81}$$

$$\langle\phi_\perp^2(t)\rangle - \langle\phi_\perp^2(0)\rangle = \int \frac{d^dk}{(2\pi)^d}\frac{1}{2\omega_<(k)}(U_{\perp\mathbf{k}}^{+}(t)U_{\perp\mathbf{k}}^{-}(t) - 1) \times \coth[\beta_i\omega_<(k)/2], \tag{82}$$

$$\langle\phi^{z2}(t)\rangle - \langle\phi^{z2}(0)\rangle = \int \frac{d^dk}{(2\pi)^d}\frac{1}{2\omega_<(k)}(U_{z\mathbf{k}}^{+}(t)U_{z\mathbf{k}}^{-}(t) - 1) \times \coth[\beta_i\omega_<(k)/2], \tag{83}$$

where we have introduced the renormalized current:

$$j = \tilde{j}^z + C(\langle\phi_\perp^2(0)\rangle + \langle\phi^{z2}(0)\rangle), \tag{84}$$

the mass

$$m^2 = 1 - \frac{T_c}{T} + \frac{\lambda}{2}(\langle\phi_\perp^2(0)\rangle + \langle\phi^{z2}(0)\rangle), \tag{85}$$

and the couplings $C=H/(8T^3)$, $\lambda=1/(8T^3)$. We have neglected the term proportional to C in (80); since we assume that $\langle\phi^z(t)\phi_\perp(t)\rangle \ll \langle\phi^{z2}(t)\rangle, \langle\phi_\perp^2(t)\rangle$ [see (60)]. It is important to note that the dimensionality of the system does not enter in the system of equations (82) and (83) after the momentum renormalization. Equation (81) can be formally solved. Substitution of the solution into (83) gives a transcendental equation that connects $\langle\phi^{z2}(t)\rangle - \langle\phi^{z2}(0)\rangle$ and $\langle\phi_\perp^2(t)\rangle - \langle\phi_\perp^2(0)\rangle$:

$$\langle\phi^{z2}(t)\rangle - \langle\phi^{z2}(0)\rangle = [j - C(\langle\phi_\perp^2(t)\rangle - \langle\phi_\perp^2(0)\rangle + \langle\phi^{z2}(t)\rangle - \langle\phi^{z2}(0)\rangle)]^2 \times \int \frac{d^dk}{(2\pi)^d}\frac{1}{2\omega_<(k)}\frac{\coth[\beta_i\omega_<(k)/2]}{m^2 + \mathbf{k}^2 + (\lambda/2)(\langle\phi_\perp^2(t)\rangle - \langle\phi_\perp^2(0)\rangle + \langle\phi^{z2}(t)\rangle - \langle\phi^{z2}(0)\rangle)} - \int \frac{d^dk}{(2\pi)^d}\frac{1}{2\omega_<(k)}\coth[\beta_i\omega_<(k)/2]. \tag{86}$$

Now the problem is reduced to the field plane component correlation function $\langle\phi_\perp^2(t)\rangle - \langle\phi_\perp^2(0)\rangle$ problem (80) and (82), where $\langle\phi^{z2}(t)\rangle - \langle\phi^{z2}(0)\rangle$ is defined by (86). This problem can be solved exactly numerically. Since we consider here the limit of a weak magnetic field, we can find the behavior of the system by an analytic approach.

Really, since the expression in front of the integral in (86) is already proportional to H^2, it means that $|\langle\phi^{z2}(t)\rangle - \langle\phi^{z2}(0)\rangle|$ at initial times is smaller than $|\langle\phi_\perp^2(t)\rangle - \langle\phi_\perp^2(0)\rangle|$ by a factor $(H/T)^2 \ll 1$. Therefore, $\langle\phi^{z2}(t)\rangle - \langle\phi^{z2}(0)\rangle$ on the right hand side of (86) can be neglected. We also neglect $\langle\phi_\perp^2(t)\rangle - \langle\phi_\perp^2(0)\rangle$ in the denominator of the integral, assum-

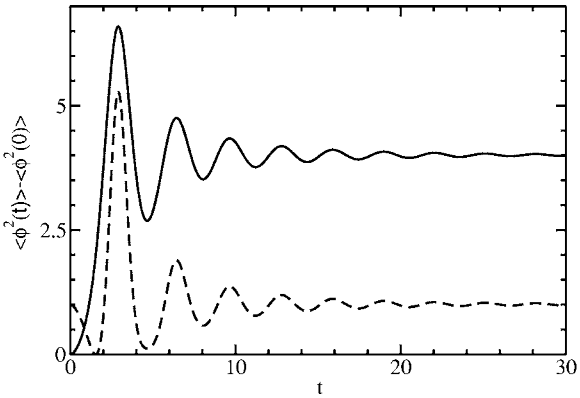

FIG. 5. Time dependencies of the correlation functions for perpendicular field component (solid line) and renormalized parallel component (93) (dashed line) in the 1D case. The model parameters are $m_i^2=1$, $m_f^2=-1$, $T_i=1$, $\lambda=0.5$.

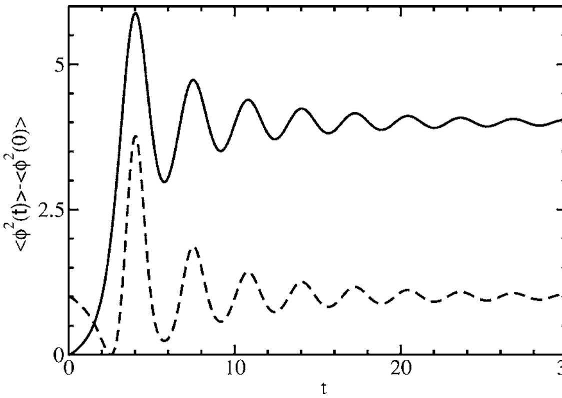

FIG. 6. The same as in Fig. 5 in the 2D case.

ing that $|\langle\phi_\perp^2(t)\rangle-\langle\phi_\perp^2(0)\rangle|/T^2 \ll |m^2|$, which is true when the final temperature is low. In this case we have a simple relation between $\langle\phi_z^2(t)\rangle-\langle\phi_z^2(0)\rangle$ and $\langle\phi_\perp^2(t)\rangle-\langle\phi_\perp^2(0)\rangle$:

$$\langle\phi^{z2}(t)\rangle-\langle\phi^{z2}(0)\rangle=\left(\frac{H}{4T}\right)^2\left(1-\frac{\langle\phi_\perp^2(t)\rangle-\langle\phi_\perp^2(0)\rangle}{2T^2}\right)^2 \times A(m_i^2,m^2,T_i)-B(m_i^2,T_i), \quad (87)$$

where

$$A(m_i^2,m^2,T_i)=\int\frac{d^dk}{(2\pi)^d}\frac{1}{2\omega_<(k)}\frac{1}{m^2+\mathbf{k}^2}\coth[\beta_i\omega_<(k)/2], \quad (88)$$

$$B(m_i^2,T_i)=\int\frac{d^dk}{(2\pi)^d}\frac{1}{2\omega_<(k)}\coth[\beta_i\omega_<(k)/2]. \quad (89)$$

Relation (87) already suggests the answer to how the vector field behaves in the unstable phase. $\langle\phi_\perp^2(t)\rangle$ must be an oscillating function of time with a large amplitude of the oscillations (see later), the amplitude of the oscillations decreases with time and $\langle\phi_\perp^2(t)\rangle$ approaches a positive value. As it follows from (87), $\langle\phi^{z2}(t)\rangle$ is also an oscillating function of time with the same period as $\langle\phi_\perp^2(t)\rangle$. Then $\langle\phi^{z2}(t)\rangle -\langle\phi^{z2}(0)\rangle$ approaches the equilibrium value $\sim[H/(4T)]^2 \simeq[(1/2)\tanh(H/2T)]^2$.

To prove this, one needs to solve (80) and (82) to find $\langle\phi_\perp^2(t)\rangle-\langle\phi_\perp^2(0)\rangle$. It is enough to put $\langle\phi^{z2}(t)\rangle-\langle\phi^{z2}(0)\rangle=0$ in (80), since it is much smaller than $\langle\phi_\perp^2(t)\rangle-\langle\phi_\perp^2(0)\rangle$ at early times. The system of equations for $\langle\phi_\perp^2(t)\rangle-\langle\phi_\perp^2(0)\rangle$ has a simple form in this case

$$\left(\frac{d^2}{dt^2}+m^2+\mathbf{k}^2+\frac{\lambda}{2}(\langle\phi_\perp^2(t)\rangle-\langle\phi_\perp^2(0)\rangle)\right)U_{\perp\mathbf{k}}^{\pm}(t)=0, \quad (90)$$

$$\langle\phi_\perp^2(t)\rangle-\langle\phi_\perp^2(0)\rangle=\int\frac{d^dk}{(2\pi)^d}\frac{1}{2\omega_<(k)}[U_{\perp\mathbf{k}}^{+}(t)U_{\perp\mathbf{k}}^{-}(t)-1] \times\coth[\beta_i\omega_<(k)/2], \quad (91)$$

with the boundary condition

$$\langle\phi_\perp^2(t<0)\rangle-\langle\phi_\perp^2(0)\rangle=0, \quad (92)$$

and (65). This system coincides with the system of equations for the structure factor in the relativistic scalar theory. It was analyzed in Refs. 23 and 24 in the three dimensional case. It was shown that the solution for $\langle\phi_\perp^2(t)\rangle-\langle\phi_\perp^2(0)\rangle$ in the 3D case is an oscillating function with the oscillation amplitude decaying exponentially with time. The domain size increases with time as $\xi_D(t)=(8\sqrt{2})^{1/2}\sqrt{t\xi(0)}$, where in our notation $\xi(0)\sim\sqrt{J/T}$. We present results of calculations for the correlation function for both components of the order parameter $\phi_\perp$ and ϕ^z in the cases of different dimensions in Figs. 5–7. The correlation function for the perpendicular component $\phi_\perp$ was calculated by solving (91) and (92), and the correlation function for the parallel component was calculated by using the approximate expression (87). In Figs. 5–7 we present the renormalized zz-correlation function,

$$\langle\phi^{z2}(t)\rangle-\langle\phi^{z2}(0)\rangle=\left(1-\frac{\langle\phi_\perp^2(t)\rangle-\langle\phi_\perp^2(0)\rangle}{2T^2}\right)^2 \quad (93)$$

[compare to (87)].

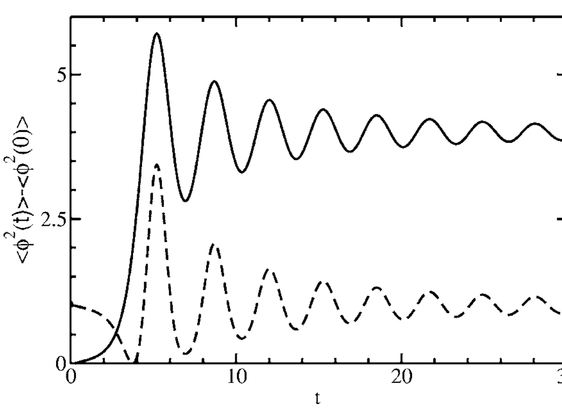

FIG. 7. The same as in Fig. 5 in the 3D case.

As it follows from Figs. 5–7, the fluctuations initially grow until $m^2+\lambda/2(\langle\phi^2(t)\rangle-\langle\phi^2(0)\rangle)$ becomes positive. The condition

$$m^2+\frac{\lambda}{2}(\langle\phi_\perp^2(t)\rangle-\langle\phi_\perp^2(0)\rangle)=0 \tag{94}$$

is used to define the spinodal time or time when the instabilities start to disappear. As it follows From Figs. 5–7, the spinodal time is decreasing when dimensionality of the system becomes lower. Also, the correlations grow faster in low dimensions system, since $\langle\phi^2(t)\rangle\sim t^{-d/2}$ [see (78)].

Shortly after that time, the correlation function starts to oscillate and approaches the equilibrium value,

$$\langle\phi_\perp^2(t)\rangle-\langle\phi_\perp^2(0)\rangle=\frac{2|m^2|}{\lambda}, \tag{95}$$

which is equal to 4 in our case.

As it follows from (93) and Figs. 5–7, the correlation function for the z component has the same time dependence as the perpendicular component, except at a very short time. It means that the correlation functions, and therefore, the bubble radius for both components, must have the same time dependence. Finally, the system relaxes to the equilibrium state at a finite temperature, and the z component of the order parameter approaches its equilibrium value defined by the final temperature and the magnetic field. It is important that the ϕ^z has time dependence only when j and the C parameters are different from zero. Otherwise, Eq. (81) has only the trivial solution $U^{z\pm}(\mathbf{k},t)=0$. The parameters j and C are finite only in the case of a finite external field. Therefore, the magnetization evolves in time to its equilibrium value directed along the z axis only when an external field H is applied. It is enough to have an extremely small magnetic field to get this symmetry broken equilibrium state.

To find the explicit time dependence of the bubble radius in the interacting case, one could solve system (80) and (86), and then substitute the solution for $U_{\perp\mathbf{k}}(t)$ into (71). Then it is necessary to find if the correlation function arguments satisfy some scaling condition, like x^2/t in the free case. This scaled variable will define the domain size time dependence ($x\sim\sqrt{t}$ in the free case). As it follows from our previous analysis, the effective mass remains negative until the spinodal time. Therefore the correlation function in the interacting case has the same time and space dependence as in the free case, if the coupling λ is weak. Our case corresponds to a weakly coupled theory, since we make an expansion of the free energy in powers of the field, and the coupling should be small in this case. Thus, at early times, i.e., at times smaller than the spinodal time, the domains are growing as $\sqrt{t}$ in our effective model. The presence of the cubic terms does not change this result, since the cubic coupling parameter C is also assumed to be small. It would be extremely interesting to generalize these results to intermediate and long times, where the role of the cubic terms can be nontrivial. Also, the ground state value of the field strongly depends on value of the cubic couplings when these couplings are large.

1. Scaling analysis at long times

It is actually possible to show, using a scaling analysis, that the long-time bubble radius time dependence in the Hartree approximation is $R\sim t$. This analysis in the classical scalar case is presented in Ref. 30, for example.

Really, let us consider the system of equations (60) and (61) in the weak magnetic field H limit. In this case the term proportional to $\tilde{C}^{xxz}$ in (60) can be neglected, since it is proportional to H, and it is small compared to the terms proportional to the field-independent terms with $\tilde{\lambda}$'s. Therefore, the system of equations (60) and (61) can be written as

$$\left(\frac{d^2}{dt^2}+m^2-\nabla^2+\frac{\lambda}{2}[\phi_\perp^2(\mathbf{r},t)+\phi^{z2}(\mathbf{r},t)]\right)\phi_\perp(\mathbf{r},t)=0, \tag{96}$$

$$\left(m^2-\nabla^2+\frac{\lambda}{2}[\phi_\perp^2(\mathbf{r},t)+\phi^{z2}(\mathbf{r},t)]\right)\phi^z(\mathbf{r},t)=j-C[\phi_\perp^2(\mathbf{r},t)+\phi^{z2}(\mathbf{r},t)], \tag{97}$$

where m^2 is defined by Eq. (85) and C and λ are defined after this equation. The space and time variables are normalized to have the coefficients equal one in front of d^2/dt^2 and ∇^2.

System (96) and (97) can be solved in the Hartree approximation as follows. First, we define the function

$$a(t)=-m^2-\frac{\lambda}{2}(\langle\phi_\perp^2(\mathbf{r},t)\rangle+\langle\phi^{z2}(\mathbf{r},t)\rangle), \tag{98}$$

which is assumed to be space-independent. Therefore,

$$a(t)=-m^2-\frac{\lambda}{2}\int\frac{d^dk}{(2\pi)^d}(\langle\phi_{\perp\mathbf{k}}(t)\phi_{\perp-\mathbf{k}}(t)\rangle+\langle\phi_{\mathbf{k}}^z(t)\phi_{-\mathbf{k}}^z(t)\rangle). \tag{99}$$

This function approaches to zero when $t\to\infty$, since the equilibrium solution of (96) satisfies

$$m^2+\frac{\lambda}{2}[\phi_\perp^2(\mathbf{r},t)+\phi^{z2}(\mathbf{r},t)]=0. \tag{100}$$

In this case Eq. (96) in the momentum space can be approximated by

$$\left(\frac{d^2}{dt^2}+\mathbf{k}^2-a(t)\right)\phi_{\perp\mathbf{k}}(t)=0. \tag{101}$$

It is enough to solve Eq. (101) for the perpendicular component of the field in order to find the domain size time-dependencies, since the longitudinal component of the field will satisfy the same scaling at long time. Really, $\phi^z(\mathbf{r},t)$ can be found at long times by solving (97). Neglecting space derivatives in this equation, since ∇^2 is small (the fields depend on two variables r/t and t, as it will be shown below, therefore, $\nabla^2\sim t^{-2}\to 0$ at long times) and using (100), one can find:

$$\phi^{z2}(\mathbf{r},t)=j/C-\phi_\perp^2(\mathbf{r},t). \tag{102}$$

Since the system was initially in the disordered state with $T>T_c$, the following initial condition can be used:

$$\langle \phi_\perp(\mathbf{r},0)\phi_\perp(\mathbf{r}',0)\rangle = \langle \phi^z(\mathbf{r},0)\phi^z(\mathbf{r}',0)\rangle = \Delta\delta(\mathbf{r}-\mathbf{r}') \tag{103}$$

or

$$\langle \phi_{\perp\mathbf{k}}(0)\phi_{\perp-\mathbf{k}}(0)\rangle = \langle \phi^z_{\mathbf{k}}(0)\phi^z_{-\mathbf{k}}(0)\rangle = \Delta. \tag{104}$$

Another boundary condition for the correlation functions that can be used to solve the second order differential equation is

$$\langle \phi_\perp(\mathbf{r},t=\infty)\phi_\perp(\mathbf{r}',t=\infty)\rangle = \phi_\perp^{eq2},$$

$$\langle \phi^z(\mathbf{r},t=\infty)\phi^z(\mathbf{r}',t=\infty)\rangle = \phi^{zeq2}, \tag{105}$$

where $\phi_\perp^{eq}$ and ϕ^{zeq} are the equilibrium solutions of system (96) and (97). This condition will be automatically satisfied since the solution is valid when $a(t)\to 0$ at $t\to\infty$.

In order to find the exact expressions for the correlation functions,

$$S(\mathbf{k},t) = \langle \phi_{\perp\mathbf{k}}(t)\phi_{\perp-\mathbf{k}}(t)\rangle \tag{106}$$

and

$$S(\mathbf{r},t) = \langle \phi_\perp(\mathbf{r}_1+\mathbf{r},t)\phi_\perp(\mathbf{r}_1,t)\rangle = \int \frac{d^dk}{(2\pi)^d} e^{-i\mathbf{k}\mathbf{r}}\langle \phi_{\perp\mathbf{k}}(t)\phi_{\perp-\mathbf{k}}(t)\rangle, \tag{107}$$

one needs to solve Eq. (101) with initial condition (104). The solution of this equation is

$$\phi_{\perp\mathbf{k}}(t) = \phi_{\perp\mathbf{k}}(0)\frac{\sin(kt)}{kt}\exp g(t), \tag{108}$$

where $g(t)$ is such a function that $\partial^2 g(t)/\partial t^2 = a(t) = -t^{-2}$ (see Appendix B). In this case the momentum correlation function is

$$S(\mathbf{k},t) = \Delta\frac{\sin^2(kt)}{(kt)^2}\exp[2g(t)]. \tag{109}$$

Thus, the correlation function $S(\mathbf{k},t)$ depends on two variables t and kt at long times. In this case the real space correlation function depends on two variables t and r/t: $S(\mathbf{r},t) = S(r/t,t)$, and the domain size must grow as $R\sim t$ in the case of long times.

One can find a simple analytical expression for $S(\mathbf{r},t)$ in the scalar case when $\phi^z(\mathbf{r},t)=0$. Really, in order to get rid of the function $a(t)$ in (109) the following procedure can be used. Since $a(t)$ must be small at long times, it can be put equal to zero on the left hand side of (98). Substitution of (108) into the right hand side of (98) gives in this case,

$$\Delta\exp[2g(t)] = \left(\int \frac{d^dk}{(2\pi)^d}\frac{\sin^2(kt)}{(kt)^2}\right)^{-1}. \tag{110}$$

Therefore, the combination of (109) and (110) yields

$$S(\mathbf{k},t) = \frac{\sin^2(kt)}{(kt)^2}\Bigg/\int \frac{d^dk}{(2\pi)^d}\frac{\sin^2(kt)}{(kt)^2} \equiv \left(\frac{t}{t_0}\right)^d\frac{\sin^2(kt)}{(kt)^2}, \tag{111}$$

where

$$t_0 = \left(\mu_d(0)\int_0^\infty dx\, x^{d-3}\sin^2 x\right)^{1/d}, \tag{112}$$

and the measures of integration $\mu_d(0)$ are defined in (74). It must be noted that the integral (112) is divergent in the three dimensional case. However, this divergence can be removed if one introduces some physical momentum cut off in this case, for example, the inverse lattice spacing.

The substitution of (111) into (107) gives

$$S(\mathbf{r},t) = t_0^{-d}\mu_d(0)\int_0^\infty dx\, x^{d-3}\mu_d\left(\frac{r}{t}x\right)\sin^2 x. \tag{113}$$

Therefore, the solution in the Hartree approximation shows that the magnetic domains in the Heisenberg model should grow with time as $R\sim t$ at long times.

IV. CONCLUSIONS

To conclude, we have studied the process of the quantum spinodal decomposition in an effective vector boson theory of the Heisenberg ferromagnet in a weak external magnetic field. This theory is similar to the relativistic theory with additional linear and cubic terms. It was shown that the magnetic domains grow only in the case when the field contains both parallel and perpendicular field components. The perpendicular component correlations grow faster at early times and at late times it is expected that the equilibrium state is established with the magnetization parallel to the external magnetic field. Both parallel and perpendicular component correlations grow with time as $\sqrt{t}$ at short times in different space dimensionalities. This result is similar to the well-known results for the classical ϕ^4-theory, which corresponds to the Ising model case, to the classical vector model and to the relativistic scalar model at short times. Contrary to the classical cases, the domain grows as t at long times, as we have shown by using the Hartree approximation to solve the equation for the order parameter. There are still some open questions. In particular, it would be very important to solve the problem beyond the Hartree approximation, and for the case of strong magnetic fields.

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APPENDIX A: MAGNETIC SUSCEPTIBILITY AND THE FREE ENERGY IN THE QUADRATIC APPROXIMATION

In this section we derive the frequency dependence of the spin magnetic susceptibility (31) in order to get the local free energy in the quadratic approximation (35). Using results for the Green's function (21), and the expressions for the Green's function components (18), (19), and (22)–(24), it is easy to show that the matrix for the spin magnetic susceptibility has the following structure:

$$\bar{\chi}^{sr}(t_1,t_2) = \begin{pmatrix} \bar{\chi}^{xx}(t_1,t_2) & \bar{\chi}^{xy}(t_1,t_2) & 0 \\ -\bar{\chi}^{xy}(t_1,t_2) & \bar{\chi}^{xx}(t_1,t_2) & 0 \\ 0 & 0 & \bar{\chi}^{zz}(t_1,t_2) \end{pmatrix}, \quad \text{(A1)}$$

where

$$\bar{\chi}^{xx}(t_1,t_2) = \frac{1}{4[\exp(\beta H)+1]}[(e^{\beta H}e^{-iH(t_1-t_2)} + e^{iH(t_1-t_2)}) \times \theta_c(t_1-t_2) + (e^{\beta H}e^{iH(t_1-t_2)} + e^{-iH(t_1-t_2)})\theta_c(t_2-t_1)], \quad \text{(A2)}$$

$$\bar{\chi}^{xy}(t_1,t_2) = \frac{i}{4[\exp(\beta H)+1]}[(e^{\beta H}e^{-iH(t_1-t_2)} - e^{iH(t_1-t_2)}) \times \theta_c(t_1-t_2) - (e^{\beta H}e^{iH(t_1-t_2)} - e^{-iH(t_1-t_2)}) \times \theta_c(t_2-t_1)], \quad \text{(A3)}$$

$$\bar{\chi}^{zz}(t_1,t_2) = \frac{1}{4\cosh^2(\beta H/2)}. \quad \text{(A4)}$$

The zz component of the susceptibility is time independent. We do the following approximation to get the local in time quadratic term of the z component of the field:

$$-i\int_c dt_1 \int_c dt_2\, \phi^z(t_1)\bar{\chi}^{zz}(t_1-t_2)\phi^z(t_2) = -\frac{i}{4\cosh^2(\beta H/2)}\int_c dt_1 \int_c dt_2\, \phi^z(t_1)\phi^z(t_2) \simeq -\frac{i}{4\cosh^2(\beta H/2)}\int_c dt_1 \int_c dt_2\, \phi^{z2}(t_1) = -\frac{\beta}{4\cosh^2(\beta H/2)}\int_c dt_1\, \phi^{z2}(t_1). \quad \text{(A5)}$$

We used the fact that the largest contribution to the two-dimensional integral is obtained at equal times $t_1=t_2$.

To get the local free energy for the plane components of the field we use the following approximation:

$$-i\int dt_1 \int dt_2\, \phi^l(t_1)\bar{\chi}^{lm}(t_1-t_2)\phi^m(t_2) \simeq -i\int dt\, \phi^l(t)\left(\bar{\chi}_0^{lm} + i^n\bar{\chi}_n^{lm}\frac{d^n}{dt^n}\right)\phi^m(t), \quad \text{(A6)}$$

where the coefficients $\bar{\chi}_0^{lm}$ and $\bar{\chi}_n^{lm}$ are the coefficients for the low-frequency expansion of the susceptibility in the frequency representation:

$$\bar{\chi}^{lm}(\omega) = \int_{-\infty}^{\infty} dt\, e^{i\omega t}\bar{\chi}_{ii}^{lm}(t) \simeq \bar{\chi}_0^{lm} + \bar{\chi}_n^{lm}\omega^n. \quad \text{(A7)}$$

To prove (A6) we transform the left hand side of this expression by introducing the average and relative time coordinates: $T=(t_1+t_2)/2$, $t=t_1-t_2$. In this case:

$$\begin{aligned}
&-i\int dt_1 \int dt_2\, \phi^l(t_1)\bar{\chi}^{lm}(t_1-t_2)\phi^m(t_2) \\
&= -i\int dT \int dt\, \phi^l\left(T+\frac{t}{2}\right)\bar{\chi}^{lm}(t)\phi^m\left(T-\frac{t}{2}\right) \\
&= -i\int dT \int dt \int \frac{d\omega}{2\pi}e^{-i\omega(T+t/2)}\phi^l(\omega)\int \frac{d\omega_1}{2\pi}e^{-i\omega_1 t}\bar{\chi}^{lm}(\omega_1)\int \frac{d\omega_2}{2\pi}e^{-i\omega_2(T-t/2)}\phi^m(\omega_2) \\
&= -i\int \frac{d\omega}{2\pi}\int d\omega_1 \int \omega_2\delta(\omega+\omega_2)\delta\left(\frac{\omega_2}{2}-\omega_1-\frac{\omega}{2}\right)\phi^l(\omega)\bar{\chi}^{lm}(\omega_1)\phi^m(\omega_2) \\
&= -i\int \frac{d\omega}{2\pi}\phi^l(-\omega)\bar{\chi}^{lm}(\omega)\phi^m(\omega) \\
&\simeq -i\int \frac{d\omega}{2\pi}\phi^l(-\omega)[\bar{\chi}_0^{lm} + \bar{\chi}_n^{lm}\omega^n]\phi^m(\omega) \\
&= -i\int \frac{d\omega}{2\pi}\int dt\, e^{i(-\omega)t}\phi^l(t)\bar{\chi}_0^{lm}\int dt_1\, e^{i(\omega)t_1}\phi^m(t_1) - i\int \frac{d\omega}{2\pi}\int dt\, e^{i(-\omega)t}\phi^l(t)\bar{\chi}_n^{lm}\int dt_1\, e^{i\omega t_1}\int \frac{d\omega_2}{2\pi}e^{-i\omega_2 t_1}\omega_2^n\phi^m(\omega_2) \\
&= -i\int dt\, \phi^l(t)\bar{\chi}_0^{lm}\phi^m(t) - i\int \frac{d\omega}{2\pi}\int dt\, e^{i(-\omega)t}\phi^l(t)\bar{\chi}_n^{lm}\int dt_1\, e^{i\omega t_1}\frac{d^n}{(-i)^n\, dt_1^n}\phi^m(t_1) \\
&= -i\int dt\, \phi^l(t)\bar{\chi}_0^{lm}\phi^m(t) - i\int dt\, \phi^l(t)\bar{\chi}_n^{lm}\frac{d^n}{(-i)^n\, dt^n}\phi^m(t).
\end{aligned} \quad \text{(A8)}$$

This proves the statement (A6). To find the coefficients $\bar{\chi}_n^{lm}$ and $\bar{\chi}_n^{lm}$ in (A6), we need to make the Fourier transform (A7) for $\bar{\chi}^{xx}(t_1-t_2)$ and $\bar{\chi}^{xy}(t_1-t_2)$.

The Fourier transform of the xx component of the susceptibility is

$$\begin{aligned}\bar{\chi}^{xx}(\omega) &= \frac{1}{4[\exp(\beta H)+1]}\int_0^{\infty} dt\, e^{i(\omega+i\delta)t}(e^{\beta H}e^{-iHt}+e^{iHt}) \\ &+ \frac{1}{4[\exp(\beta H)+1]}\int_{-\infty}^{0} dt\, e^{i(\omega-i\delta)t}(e^{\beta H}e^{iHt}+e^{-iHt}) \\ &= \frac{i}{4[\exp(\beta H)+1]}\left(\frac{\exp(\beta H)}{\omega-H+i\delta}+\frac{1}{\omega+H+i\delta} - \frac{\exp(\beta H)}{\omega+H-i\delta}-\frac{1}{\omega-H-i\delta}\right) \\ &= \frac{i}{2}H\tanh(\beta H/2)\frac{1}{\omega^2-H^2}. \end{aligned} \tag{A9}$$

In the same way we show that the mixed component of the susceptibility is

$$\bar{\chi}^{xy}(\omega) = -\frac{1}{2}\omega\tanh(\beta H/2)\frac{1}{\omega^2-H^2}. \tag{A10}$$

In the low frequency limit,

$$\bar{\chi}^{xx}(\omega) \simeq -\frac{i}{2H}\tanh(\beta H/2) - \frac{i}{2H^3}\tanh(\beta H/2)\omega^2, \tag{A11}$$

$$\bar{\chi}^{xy}(\omega) \simeq \frac{1}{2H^2}\tanh(\beta H/2)\omega. \tag{A12}$$

Equations (A6), (A7), (A11), and (A12) give the following approximate expression for the perpendicular part of the free energy in the quadratic approximation:

$$\begin{aligned} &-i\int dt_1\int dt_2\, \phi^l(t_1)\bar{\chi}^{lm}(t_1-t_2)\phi^m(t_2) \\ &\simeq -\frac{\tanh(\beta H/2)}{2H}\int dt\, \boldsymbol{\phi}_\perp(t)\left(1-\frac{1}{H^2}\frac{d^2}{dt^2}\right)\boldsymbol{\phi}_\perp(t) \\ &+ \frac{\tanh(\beta H/2)}{2H^2}\int dt\left(\phi^x(t)\frac{d}{dt}\phi^y(t)-\phi^y(t)\frac{d}{dt}\phi^x(t)\right), \end{aligned} \tag{A13}$$

where $\boldsymbol{\phi}_\perp(t)=[\phi^x(t),\phi^y(t),0]$.

Expressions (A5) and (A13) result in expression (35) for the free energy in the quadratic approximation.

APPENDIX B: LONG-TIME SOLUTION FOR THE ORDER PARAMETER

In order to obtain solution (108) of Eq. (101), we make the following ansatz:

$$\phi_{\perp\mathbf{k}}(t) = \phi_{\perp\mathbf{k}}(0)f(k^m t)\exp[g(t)], \tag{B1}$$

where $f(0)=1$, $g(0)=0$. The substitution of (B1) into (101) gives

$$\frac{\partial^2 g(t)}{\partial t^2}f(x) + 2k^m\frac{\partial g(t)}{\partial t}\frac{\partial f(x)}{\partial x} + k^{2m}\frac{\partial^2 f(x)}{\partial x^2} + k^2 f(x) - a(t)f(x) = 0, \tag{B2}$$

where we have introduced a scale variable $x=k^m t$. In order to obtain an equation for $f(x)$ which depends on the scaled variable x only, one has to choose a function $g(t)$ such that:

$$\frac{\partial^2 g(t)}{\partial t^2} - a(t) = 0. \tag{B3}$$

The lower limit for the integration is chosen to satisfy $g(0)=0$. Actually, the expression for $g(t)$ is not of a special interest, contrary to the expression for $f(x)$ that defines the scaling of the correlation function.

In this case, Eq. (B2) is simplified to

$$2k^m\frac{\partial g(t)}{\partial t}\frac{\partial f(x)}{\partial x} + k^{2m}\frac{\partial^2 f(x)}{\partial x^2} + k^2 f(x) = 0. \tag{B4}$$

As it follows from (B4), the proper choice for the exponent m is $m=1$ in order to get an equation in terms of the dimensionless variable x: In this case (B4) transforms to

$$\frac{\partial^2 f(x)}{\partial x^2} + \frac{2}{k}\frac{\partial g(t)}{\partial t}\frac{\partial f(x)}{\partial x} + f(x) = 0. \tag{B5}$$

To express this equation in terms of x, it is necessary to put

$$\frac{\partial g(t)}{\partial t} = -\frac{a_0}{t}. \tag{B6}$$

It can be shown that $a_0=-1$ [in fact, this choice gives correct time-dependence in the asymptotic solution of Eq. (101) at $t\to\infty$: $\phi_{\mathbf{k}}(t)\sim\sin(kt)$, since $g(t)\simeq\ln(t)$ in Eq. (108) in this case ($a(t)$ can be neglected in Eq. (101) at long times)]. Equation (B5) in this case,

$$\frac{\partial^2 f(x)}{\partial x^2} + \frac{2}{x}\frac{\partial f(x)}{\partial x} + f(x) = 0, \tag{B7}$$

can be simplified by using the ansatz $f(x)=h(x)/x$ to

$$\frac{\partial^2 h(x)}{\partial x^2} + h(x) = 0. \tag{B8}$$

It has a simple solution:

$$h(x) = \alpha \cos(x) + \beta \sin(x), \tag{B9}$$

or

$$f(x) = \frac{1}{x}[\alpha \cos(x) + \beta \sin(x)]. \tag{B10}$$

Since $f(0)=1$, one must have $\alpha=0$, $\beta=1$, and

$$f(x) = \frac{\sin x}{x}. \tag{B11}$$

The substitution of (B11) and (B3) into (B1) gives the solution (108) for the order parameter.

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*Present address: Department of Physics, Georgetown University, Washington, DC 20057, USA.

[1] A. H. Bobeck and E. Della Torre, *Magnetic Bubbles*, Selected Topics in Solid State Physics, edited by E. P. Wahlfarth (North-Holland, Amsterdam, 1975).
[2] A. H. Eschenfelder, *Magnetic Bubble Technology*, Springer Series in Solid-State Sciences, edited by M. Cardona, P. Fulde, and H-J. Queisser (Springer-Verlag, Berlin, 1981).
[3] A. O. Caldeira and K. Furuya, J. Phys. C **21**, 1227 (1988).
[4] F. A. Berezin and M. S. Marinov, Ann. Phys. **104**, 336 (1977).
[5] V. Rocha Vieira, Phys. Rev. B **23**, 6043 (1981).
[6] S. Leibler and H. Orland, Ann. Phys. **132**, 277 (1981).
[7] P. D. S. Sacramento and V. R. Vieira, J. Phys. C **21**, 3099 (1988).
[8] V. R. Vieira, Phys. Rev. B **39**, 7174 (1989).
[9] V. R. Vieira and I. R. Pimentel, Phys. Rev. B **39**, 7196 (1989).
[10] E. V. Podivilov, preprint cond-mat/9402037.
[11] M. Krech, H. Karl, and H. W. Diehl, Physica A **297**, 64 (2001).
[12] V. Turkowski, V. R. Vieira, and P. D. Sacramento, Physica A **327**, 461 (2003).
[13] P. C. Hohenberg and B. I. Halperin, Rev. Mod. Phys. **49**, 435 (1977).
[14] S. Sachdev, *Quantum Phase Transitions* (Cambridge University Press, Cambridge, 1999).
[15] S. M. Allen and J. W. Cahn, Acta Metall. **27**, 1085 (1979).
[16] J. D. Gunton and M. Droz, *Introduction to the Theory of Metastable and Unstable States* (Springer-Verlag, Berlin, 1983).
[17] M. Grant and J. D. Gunton, Phys. Rev. B **28**, 5496 (1983).
[18] E. Oguz, A. Chakrabarti, R. Toral, and J. D. Gunton, Phys. Rev. B **42**, 704 (1990).
[19] K. Kawasaki, M. C. Yalabik, and J. D. Gunton, Phys. Rev. A **17**, 455 (1978).
[20] S. A. Safran, P. S. Sahni, and G. S. Grest, Phys. Rev. B **28**, 2693 (1983).
[21] P. S. Sahni, G. S. Grest, and S. A. Safran, Phys. Rev. Lett. **50**, 60 (1983).
[22] T. Ohta, D. Jasnow, and K. Kawasaki, Phys. Rev. Lett. **49**, 1223 (1982).
[23] D. Boyanovsky, Phys. Rev. E **48**, 767 (1993).
[24] D. Boyanovsky, D.-Sh. Lee, and A. Singh, Phys. Rev. D **48**, 800 (1993).
[25] D. Boyanovsky, H. J. de Vega, and R. Holman, preprint hep-th/9412052.
[26] D. Boyanovsky, H. J. de Vega, and R. Holman, preprint hep-ph/9701304.
[27] Y. Bergner and L. M. A. Bettencourt, Phys. Rev. D **68**, 025014 (2003).
[28] A. J. Bray and S. Puri, Phys. Rev. Lett. **67**, 2670 (1991).
[29] H. Toyoki, Phys. Rev. B **45**, 1965 (1992).
[30] A. J. Bray, Adv. Phys. **43**, 357 (1994).
[31] S. Dattagupta and S. Puri, *Dissipative Phenomena in Condensed Matter: Some Applications* (Springer-Verlag, New York, 2004).
[32] J. Schwinger, J. Math. Phys. **2**, 407 (1961).
[33] K. V. Keldysh, Sov. Phys. JETP **20**, 1018 (1965).
[34] Sh.-K. Ma and G. F. Mazenko, Phys. Rev. B **11**, 4077 (1975).
[35] J. Hertz, Phys. Rev. B **14**, 1165 (1976).
[36] J. C. Le Guillou and J. Zinn-Justin, Phys. Rev. B **21**, 3976 (1980).