Magnetoelasticity theory of incompressible quantum Hall liquids

I. V. Tokatly[1,2,*]

[1]*Lehrstuhl für Theoretische Festkörperphysik, Universität Erlangen-Nürnberg, Staudtstrasse 7/B2, 91058 Erlangen, Germany*
[2]*Moscow Institute of Electronic Technology, Zelenograd, 124498 Russia*

A simple and physically transparent magnetoelasticity theory is proposed to describe linear dynamics of incompressible fractional quantum Hall states. The theory manifestly satisfies the Kohn theorem and the *f*-sum rule, and predicts a gaped intra-Landau level collective mode with a roton minimum. In the limit of vanishing bare mass m, the correct form of the static structure factor, $s(q) \sim q^4$, is recovered. We establish a connection of the present approach to the fermionic Chern-Simons theory, and discuss further extensions and applications. We also make an interesting analogy of the present theory to the theory of viscoelastic fluids.

I. INTRODUCTION

The fractional quantum Hall (FQH) effect, which occurs in a two-dimensional (2D) electron gas subjected to a strong magnetic field, is one of the most interesting macroscopic manifestations of quantum mechanics.[1,2] It is well established that the perfect quantization of the Hall conductance at particular rational filling factors ν is related to the formation of new strongly correlated states of matter—incompressible quantum liquids. Despite a substantial progress in understanding of the FQH effect (mainly based on a construction of trial wave functions[3–6] or on approximate calculations within Chern-Simons (CS) field theories[6–10]) a simple phenomenological theory of dynamics of FQH states is still lacking.

One of the key phenomenological features of incompressible FQH liquids is a highly collective response to external perturbations. In this respect, FQH liquids look surprisingly similar to the "classical" ideal liquids and solids, despite microscopic mechanisms behind the collective behavior are very different. The collective response of classical condensed matter leads to a great simplification and unification of the long wavelength theory, which takes a universal form of the classical continuum mechanics.[11,12] In the present paper we exploit the above-mentioned similarity, and demonstrate that the formalism of continuum mechanics can be extended to describe collective dynamics in the FQH regime. Using general conservation laws, symmetry, and sum-rules arguments we derive a magnetoelasticity theory of incompressible FQH fluids. For the sake of clarity in this paper, we present a "minimal" construction, which is, in a certain sense, analogous to a single mode approximation (SMA) by Girvin, MacDonald, and Platzman.[13] The minimal theory is parametrized by three elastic constants: the bulk modulus K, the *high-frequency* shear modulus μ^∞ (the static shear modulus of any liquid vanishes), and a "magnetic" modulus Δ that is related to the intra-Landau level (LL) excitation gap.[14] Physically the new elastic modulus is responsible for a "Lorentz shear stress" induced by time-dependent shear deformations in a system with broken time-reversal symmetry. We also show that this three-parameter continuum mechanics can be derived from the fermionic CS theory within the random phase approximation (RPA).[6–9] However, the magneto-elastic theory goes beyond presently available field theoretical approximations. In particular, in its most general form,[15] it predicts two gapped collective modes, which is in agreement with recent experimental observations.[16]

The structure of the paper is the following. In Sec. II we present a simple phenomenological derivation of the basic equations of linear magnetoelasticity theory. We calculate linear response functions and analyze the dispersion relation of collective excitations. In Sec. III we demonstrate that our minimal magnetoelasticity is directly related to the fermionic CS theory when the later is treated at the level of the time-dependent mean field approximation (RPA). The conclusion and the summary of main results are presented in Sec. IV. In this section we also discuss an analogy of the present magnetoelasticity theory to the hydrodynamics of highly viscous fluids.

II. PHENOMENOLOGICAL DERIVATION OF MAGNETOELASTICITY THEORY

Let us consider a 2-D many-particle system in the presence of a strong magnetic field **B** perpendicular to the plane. For definiteness we assume that the particles are confined to the x,y plane, while the magnetic field is pointing in the z direction, $\mathbf{B} = \mathbf{e}_z B$. The dynamics of the system should satisfy the following exact local conservation laws for the number of particles and momentum:

$$(\partial_t + \mathbf{v} \boldsymbol{\nabla})n + n\, \partial_\alpha v_\alpha = 0, \tag{1}$$

$$m(\partial_t + \mathbf{v} \boldsymbol{\nabla})v_\alpha + B\varepsilon_{\alpha\beta}v_\beta + \frac{1}{n}\partial_\beta P_{\alpha\beta} + \partial_\alpha U_H = F_\alpha, \tag{2}$$

where n and $\mathbf{v} = \mathbf{j}/n$ are the density and the velocity of the quantum fluid, U_H is the Hartree potential, and $\mathbf{F} = -\boldsymbol{\nabla}U + \partial_t\mathbf{a}$ is the force due to external time-dependent scalar and vector potentials.[17] According to mapping theorems of the time-dependent density functional theory (TDDFT),[18,19] the exact stress tensor $P_{\mu\nu}$ is a unique functional of the velocity **v** [for a detailed discussion of a hydrodynamic formulation of TDDFT based on the local conservation laws of Eqs. (1) and (2), see Refs. 20 and 21 and references therein]. For general nonlinear dynamics, the functional $P_{\mu\nu}[\mathbf{v}]$ is un-

known. However, in the linear regime its structure can be easily established from the symmetry considerations. Let us introduce the displacement vector $\mathbf{u}(\mathbf{r}, t)$, which is related to the velocity $\mathbf{v}(\mathbf{r}, t)$, and to the density $n(\mathbf{r}, t) = n_0 + \delta n(\mathbf{r}, t)$, as follows: $\mathbf{v} = \partial_t \mathbf{u}$, and $\delta n = -n_0 \nabla \mathbf{u}$. Linearizing Eqs. (1) and (2) we arrive at the equation of the exact linear continuum mechanics,

$$- m\omega^2 u_\alpha - i\omega B \varepsilon_{\alpha\beta} u_\beta + \frac{1}{n_0} \partial_\beta \delta P_{\alpha\beta} + \partial_\alpha U_H = F_\alpha, \quad (3)$$

where $\delta P_{\alpha\beta}[\mathbf{u}]$ is a linear functional of $\mathbf{u}$. As usual, we decompose the stress tensor $\delta P_{\alpha\beta}$ into a scalar (pressure) and traceless (shear) parts,

$$\delta P_{\alpha\beta} = \delta_{\alpha\beta} \delta P + \pi_{\alpha\beta}, \quad \mathrm{Tr}\, \hat{\pi} = 0. \quad (4)$$

Rotational symmetry in the x, y plane, and the presence of an axial vector perpendicular to the plane dictate the following most general form for linear functionals $\delta P[\mathbf{u}]$ and $\pi_{\alpha\beta}[\mathbf{u}]$ (in fact, for their Fourier components),

$$\delta P = - K u_{\gamma\gamma}, \quad (5)$$

$$\pi_{\alpha\beta} = - \mu(2u_{\alpha\beta} - \delta_{\alpha\beta} u_{\gamma\gamma}) + i\omega\Lambda(\varepsilon_{\alpha\gamma} u_{\gamma\beta} + \varepsilon_{\beta\gamma} u_{\gamma\alpha}), \quad (6)$$

were $u_{\alpha\beta} = (\partial_\alpha u_\beta + \partial_\beta u_\alpha)/2$ is the strain tensor. In general the coefficients K, μ, and Λ in Eqs. (5) and (6) are allowed to be functions of ω and $|\mathbf{q}| = q$. The scalar δP, Eq. (5), defines the trace of the stress tensor $\delta P_{\alpha\beta}$, which is the change of pressure due to density variations. The first term on the right hand side of Eq. (6) is easily recognized as the usual shear stress induced by the shear strain.[12] Hence the coefficients K and μ correspond to the exact (ω- and q-dependent) bulk and shear moduli, respectively. To reveal the physics of the second term in Eq. (6), we note that the expression in brackets in this term is simply a symmetrized cross-product of $\mathbf{e}_z$ and the strain tensor $u_{\alpha\beta}$. Apparently only the shear strain tensor, $S_{\alpha\beta} = 2u_{\alpha\beta} - \delta_{\alpha\beta} u_{\gamma\gamma}$, contributes to that cross-product. Hence Eq. (6) can be rewritten in the following simple structural form:

$$\hat{\pi} = - \mu \hat{S} - \Lambda \partial_t \hat{S} \times \mathbf{e}_z. \quad (7)$$

The representation of Eq. (7) suggests a natural physical interpretation of the second contribution to the shear stress tensor. This is a local "Lorentz stress," which is proportional to the rate of shear strain. The divergence of the Lorentz stress tensor gives a transverse compression force exerted on two neighboring fluid layers moving in opposite directions.

The Kohn theorem and the f-sum rule bound the possible ω and q dependence of K, μ, and Λ. The Kohn theorem requires the stress force, $\partial_\beta \delta P_{\alpha\beta}$, in Eq. (3) to vanish for a rigid motion. This requirement forbids divergencies of the elastic moduli in the limit $q \to 0$. The f-sum rule is equivalent to the statement that at $\omega \to \infty$ the leading contribution to the left hand side of Eq. (3) is given by the first (acceleration) term. Thus the Kohn theorem as well as the f-sum rule are satisfied if the functions $K(\omega, q)$, $\mu(\omega, q)$, and $\Lambda(\omega, q)$ have finite $q \to 0$ and $\omega \to \infty$ limits. Indeed, in this case the

stress force $\partial_\beta \delta P_{\alpha\beta}$ is of the order of q^2 in the small-q limit, so that at $q \to 0$ the equation of motion, Eq. (3), simplifies as follows:

$$- m\omega^2 \mathbf{u} - i\omega \mathbf{u} \times \mathbf{B} + \mathbf{q}(\mathbf{q}\mathbf{u}) n_0 V(q) + O(q^2) = \mathbf{F}. \quad (8)$$

The third term on the left hand side in Eq. (8) corresponds to the Hartree contribution [$V(q)$ is the interaction potential], while $O(q^2)$ represents the stress force that is proportional to q^2. Apparently a solution to Eq. (8) satisfies both the Kohn theorem and the f-sum rule, provided the coefficients in the $O(q^2)$ term (which are composed of elastic moduli) are finite at $\omega \to \infty$. For a system with Coulomb interaction, $V(q) = 2\pi/q$, the Hartree term in Eq. (8) is proportional to q, which yields a known subleading correction $\sim q$ to the dispersion of the Kohn mode. Finally, one more obvious restriction on K, μ, and Λ is that in the limit of a strong magnetic field ($m \to 0$) all elastic moduli should contain only the interaction energy scale.

Up to this point the discussion was absolutely general. Now we concentrate on a particular case of incompressible FQH liquids. Let us substitute the general form of the stress tensor, Eqs. (5) and (6), into Eq. (3). Considering $m \to 0$ limit (i.e., neglecting the acceleration term) we get the following equation for the displacement vector:

$$- i\omega(n_0 B + \Lambda q^2)\mathbf{u} \times \mathbf{e}_z + \mu q^2 \mathbf{u} + K_q \mathbf{q}(\mathbf{q}\mathbf{u}) = n_0 \mathbf{F}, \quad (9)$$

where $K_q = K + n_0^2 V(q)$. Equation (9) describes the lowest LL dynamics. By setting $\mathbf{F} = 0$, we obtain a linear homogeneous equation that determines intra-LL eigenmodes. The solution of this equation leads to the following, formally exact dispersion equation for allowed frequencies of eigenmodes,

$$\omega^2 = \frac{\mu(\mu + K_q)}{(n_0 + \Lambda q^2 l^2)^2} q^4 l^4. \quad (10)$$

The most important feature of incompressible FQH liquids is the gap in the spectrum of intra-LL excitations. This fact can be used to establish a frequency dependence of the shear modulus μ. The exact dispersion equation of Eq. (10) shows that a finite gap Δ at $q \to 0$ can exist only if the elastic moduli diverge either in the q domain (as $1/q^4$) or in the ω domain (when $\omega \to \Delta$). However, the singularities in q are forbidden by the Kohn theorem. Hence, the only allowed behavior for the exact elastic moduli is to diverge at $\omega \to \Delta$. The most natural assumption would be $\mu \sim (\omega^2 - \Delta^2)^{-1}$. This expectation can be rigorously confirmed by the following arguments. The exact density response function $\chi(\omega, q)$, which straightforwardly follows from the relation $\delta n = -n_0 \nabla \mathbf{u}$, and Eq. (9) with $\mathbf{F} = -\nabla U$, takes the form[22]

$$\chi(\omega, q) = \frac{n_0^2 \mu}{\omega^2(n_0 + \Lambda q^2 l^2)^2 - \mu(\mu + K_q) q^4 l^4} q^4 l^4. \quad (11)$$

In the $q \to 0$ limit, Eq. (11) reduces to $\chi = \mu q^4 l^4/\omega^2$. On the other hand, in that limit one expects[13] to have only one gapped intra-LL mode and the response function of the form $\chi \sim (ql)^4/(\omega^2 - \Delta^2)$. The previous two representations of the

density response function uniquely determine the small-q form of the exact dynamic shear modulus $\mu(\omega)$:

$$\mu(\omega) = \frac{\omega^2}{\omega^2 - \Delta^2} \mu^\infty, \tag{12}$$

where μ^∞ is a constant that has a clear meaning of the high-frequency shear modulus. In the low frequency limit the effective shear modulus μ, Eq. (12), vanishes as it should do in a liquid state.

A resonant frequency dependence of the form Eq. (12) commonly appears in precession dynamics. Therefore it is natural to assume that the shear stress tensor $\pi_{\alpha\beta}$ experiences a torque $\Delta(\hat{\boldsymbol{\pi}} \times \mathbf{e}_z)$, and to guess the following equation of motion for $\pi_{\alpha\beta}$:

$$\partial_t \hat{\boldsymbol{\pi}} + \Delta(\hat{\boldsymbol{\pi}} \times \mathbf{e}_z) = -\mu^\infty \, \partial_t \hat{\mathbf{S}}. \tag{13}$$

The cross product of a symmetric second rank tensor and the unit vector $\mathbf{e}_z$ is defined after Eqs. (6) and (7), namely

$$(\hat{\boldsymbol{\pi}} \times \mathbf{e}_z)_{\alpha\beta} = (\varepsilon_{\alpha\gamma}\pi_{\gamma\beta} + \varepsilon_{\beta\gamma}\pi_{\gamma\alpha})/2.$$

By solving Eq. (13), we indeed obtain tensor $\pi_{\alpha\beta}$ of the required general form, Eq. (6), i.e.,

$$\hat{\boldsymbol{\pi}} = -\mu(\omega)\hat{\mathbf{S}} + i\omega\Lambda(\omega)\hat{\mathbf{S}} \times \mathbf{e}_z,$$

with $\mu(\omega)$ of Eq. (12), and $\Lambda(\omega)$ defined by the following expression

$$\Lambda(\omega) = \frac{\mu^\infty \Delta}{\omega^2 - \Delta^2}. \tag{14}$$

Thus, assuming only the presence of the gap in the spectrum, and making a plausible guess about an equation of motion for the shear stress tensor, we were able to recover the frequency dependence of both the shear modulus μ, and the magnetic modulus Λ. At this point it is worth noting an interesting analogy of the present derivation and the theory of viscoelastic liquids. In fact, the line of arguments, which led us from Eq. (12) to Eq. (13), is very similar to the heuristic derivation of the Maxwellian theory of highly viscous fluids[12] (for a more detailed discussion see Sec. IV).

In the limit $\omega \to 0$ the magnetic modulus $\Lambda(\omega)$ approaches a constant, while the shear modulus $\mu(\omega)$ vanishes as ω^2, which is quite remarkable. Solving Eq. (9) with proper external fields, we find that exactly this low-frequency behavior, $\mu \sim \omega^2$, is required to guarantee the correct low-energy response of FQH liquid, i.e., the proper static Hall conductivity and a creation of a localized fractional charge by the adiabatic insertion of a magnetic flux.

Substituting $\mu(\omega)$, Eq. (12), and $\Lambda(\omega)$, Eq. (14), in the dispersion equation of Eq. (10), and assuming a constant bulk modulus K, we find that in the present theory there exists only one collective mode with the following dispersion:

$$\omega_q^2 = \Delta^2 - 2\,\Delta\bar{\mu}^\infty \, q^2 l^2 + \bar{\mu}^\infty(\bar{\mu}^\infty + \bar{K} + n_0 V(q))q^4 l^4, \tag{15}$$

where $\bar{\mu}^\infty = \mu^\infty/n_0$ and $\bar{K} = K/n_0$ are the high frequency shear modulus and the bulk modulus per particle, respectively. The

density response function $\chi(\omega, q)$ and the static structure factor $s(q)$ take the form

$$\chi(\omega, q) = \frac{\mu^\infty}{\omega^2 - \omega_q^2} q^4 l^4, \quad s(q) = \frac{\mu^\infty}{2\omega_q} q^4 l^4. \tag{16}$$

The collective mode, Eq. (15), is by construction gapped. A much more surprising feature of Eq. (15) is a negative curvature at small q, and a well defined rotonlike minimum at finite q, which is in excellent agreement with the phenomenology of incompressible FQH liquids.

The final magnetoelasticity theory of incompressible FQH liquids corresponds to the equation of motion, Eq. (3), with $\delta P_{\alpha\beta} = -\delta_{\alpha\beta} K(\nabla \mathbf{u}) + \pi_{\alpha\beta}$, where $\pi_{\alpha\beta}$ is the solution to Eq. (13). Thus the closed set of equations of the theory takes the form

$$m\,\partial_t^2 \mathbf{u} + \partial_t \mathbf{u} \times \mathbf{B} - \frac{K}{n_0}\nabla\,(\nabla\mathbf{u}) + \frac{\nabla\hat{\boldsymbol{\pi}}}{n_0} + \nabla U_H = \mathbf{F}, \tag{17}$$

$$\partial_t \hat{\boldsymbol{\pi}} + \Delta(\hat{\boldsymbol{\pi}} \times \mathbf{e}_z) + \mu^\infty \, \partial_t \hat{\mathbf{S}} = 0. \tag{18}$$

In contrast to the usual continuum mechanics, the shear stress tensor enters the theory as one more dynamic variable. The theory is parametrized by three "elastic" constants, K, μ, and Δ. It satisfies the Kohn theorem and the f-sum rule, reproduces the correct static response (including the creation of Laughlin quasiparticles), and predicts the intra-LL mode with a roton minimum. This theory is the main result of the present paper.

Let us discuss Eqs. (17) and (18) in some more detail. Setting $\Delta = 0$ in Eq. (18) we recover the standard local-in-time shear stress-strain relation, $\hat{\boldsymbol{\pi}} = -\mu^\infty \hat{\mathbf{S}}$. Hence in this limiting case our theory reduces to the usual continuum mechanics of a magnetized elastic medium. It can describe, for example, a long wavelength response of a hexagonal crystal. The collective modes in this regime are gapless magnetophonons. The properties of the system with a nonzero modulus Δ are completely different. In fact, the theory with $\Delta \neq 0$ reproduces most of known phenomenological features of FQH liquids. The source of that nice behavior is a nontrivial "precession" dynamics of the shear stress tensor $\hat{\boldsymbol{\pi}}$, governed by Eq. (18). It is worth mentioning that according to Eq. (18) the system does not respond to a static shear deformation [$\mu(\omega = 0) = 0$], which is an unambiguous signature of a liquid state of matter.

The magnetoelasticity theory should become exact for small wave vectors. The small-q form of Eq. (15),

$$\omega_q \approx \Delta - \bar{\mu}^\infty q^2 l^2,$$

shows that the $q \to 0$ curvature of the dispersion of the intra-LL collective mode is determined solely by the high-frequency shear modulus μ^∞. This fact opens up a possibility to access experimentally the shear modulus of FQH liquids. At small q the static structure factor, Eq. (16), takes the form $s(q) \approx (\mu^\infty/2\Delta)q^4 l^4$ (which coincides with the result of

Ref. 23). Using the static structure factor for the Laughlin state $s(q) \approx n_0[(1-\nu)/8\nu]q^4 l^4$,[13] we can relate μ^∞ to Δ as follows: $\bar{\mu}^\infty = (1-\nu)\Delta/4\nu$. This relation leads to an interesting universal result for the small-q dispersion:

$$\frac{\omega_q}{\Delta} \approx 1 - \frac{1-\nu}{4\nu}q^2 l^2. \qquad (19)$$

The normalized dispersion relation of Eq. (19) is indeed in good agreement with the results of numerical calculations within SMA for $\nu = 1/3, 1/5, 1/7$.[13]

III. CONNECTION TO THE FERMIONIC CHERN-SIMONS THEORY

The main advantage of any continuum mechanics lies in its universality. It should provide a long wavelength limit of any microscopic theory that correctly captures the key physics of the problem. Below we recover the three-parameter magnetoelasticity theory from the fermionic CS theory with the time-dependent mean-field approximation (which is equivalent to RPA).[9]

Consider a system in an incompressible state with the filling factor $\nu = p/(2sp+1)$. (Within the composite fermion concept, $2s$ is the number of flux quanta attached to every electron, and p is the number of completely filled composite fermion LL[4–6]). Our goal is to describe the long wavelength dynamics in the presence of weak external potentials, $U(\mathbf{r},t)$ and $\mathbf{a}(\mathbf{r},t)$. In the time-dependent mean field approximation, the fermionic CS theory[6,9] reduces to the following equation for the one particle density matrix $\rho(\mathbf{r}_1,\mathbf{r}_2,t)$,

$$i\,\partial_t\rho = [\hat{H}_{CS}, \rho], \qquad (20)$$

where $\hat{H}_{CS}$ is the mean field CS Hamiltonian:

$$\hat{H}_{CS} = \frac{1}{2m^*}[-i\boldsymbol{\nabla} + \boldsymbol{\mathcal{A}}(\mathbf{r},t)]^2 + U_{eff}(\mathbf{r},t). \qquad (21)$$

Here m^* is the effective mass of composite fermions, and $U_{eff} = U_H + U$ and $\boldsymbol{\mathcal{A}} = \mathbf{A} + \mathbf{A}^{CS} + \mathbf{A}^{(1)} + \mathbf{a}$ are the full self-consistent scalar and vector potentials, respectively. The static part, $\mathbf{A}$, of the full vector potential is related to the external constant magnetic field, $B = \varepsilon_{\alpha\beta}\partial_\alpha A_\beta$. The potential $\mathbf{A}^{CS}$ describes the mean CS field that is produced by $2s$ flux quanta, attached to every electron:

$$B^{CS} = \varepsilon_{\alpha\beta}\,\partial_\alpha A_\beta^{CS} = -4\pi s n. \qquad (22)$$

The second self-consistent vector potential,

$$\mathbf{A}^{(1)} = (m^* - m)\mathbf{v}, \qquad (23)$$

plays a role of an F_1 interaction in the Landau Fermi-liquid theory.[6,9] It allows one to introduce an effective mass $m^* \neq m$, but still satisfy the requirements of the Galilean invariance, i.e., the Kohn theorem and the f-sum rule.

To derive the long wavelength limit of Eq. (20), we introduce the following "gauge invariant" Wigner function,

$$f_\mathbf{k}(\mathbf{r},t) = \int d\boldsymbol{\xi}\,\rho(\mathbf{r}+\boldsymbol{\xi}/2, \mathbf{r}-\boldsymbol{\xi}/2, t)e^{-i\boldsymbol{\xi}[\mathbf{k}-\bar{\boldsymbol{\mathcal{A}}}(\mathbf{r},\boldsymbol{\xi},t)]}, \quad (24)$$

where $\overline{\boldsymbol{\mathcal{A}}}(\mathbf{r},\boldsymbol{\xi},t) = \int_{-1}^{1}\boldsymbol{\mathcal{A}}(\mathbf{r}+\lambda\boldsymbol{\xi}/2,t)d\lambda/2$. The density n, the velocity $\mathbf{v}$, and the stress tensor $P_{\alpha\beta}$ are related to zeroth, first, and second moments of the Wigner function:

$$n = \sum_\mathbf{k} f_\mathbf{k}, \quad \mathbf{v} = \frac{1}{n}\sum_\mathbf{k}\frac{\mathbf{k}}{m^*}f_\mathbf{k}, \qquad (25)$$

$$P_{\alpha\beta} = \frac{1}{m^*}\sum_\mathbf{k}(k_\alpha - m^* v_\alpha)(k_\beta - m^* v_\beta)f_\mathbf{k}. \qquad (26)$$

In the homogeneous equilibrium state the Wigner function of Eq. (24) takes the form

$$f_\mathbf{k}^{(0)} = 2e^{-k^2/B^*}\sum_{n=0}^{p-1}(-1)^n L_n^0(2k^2/B^*), \qquad (27)$$

where $L_n^m(x)$ are Laguerre polynomials, and $B^* = B + B_0^{CS}$ is the effective magnetic field that acts on composite fermions ($B_0^{CS} = -4\pi s n_0$). Equation (27) provides the initial condition for our dynamic problem.

Let the characteristic length scale L of the external fields be much larger than the effective magnetic length l^* $= 1/\sqrt{B^*}$. Using Eqs. (20) and (24), we find that the long wavelength dynamics of the Wigner function is governed by the following simple equation of motion,

$$\frac{\partial f_\mathbf{k}}{\partial t} + \frac{k_\alpha}{m^*}\frac{\partial f_\mathbf{k}}{\partial x_\alpha} + \left(\frac{\mathcal{B}}{m^*}\varepsilon_{\alpha\beta}k_\beta - \frac{\partial\mathcal{A}_\alpha}{\partial t} + \frac{\partial U_{eff}}{\partial x_\alpha}\right)\frac{\partial f_\mathbf{k}}{\partial k_\alpha} = 0, \qquad (28)$$

where $\mathcal{B} = \varepsilon_{\alpha\beta}\,\partial_\alpha\mathcal{A}_\beta$. Despite Eq. (28) is exactly of the form of a semiclassical Boltzmann equation, the function $f_\mathbf{k}$ is the full Wigner function, Eq. (24), of the quantum system. Formally, the quantum mechanics enters the problem via the initial condition, Eq. (27), for the equation of motion, Eq. (28). The next step is to linearize Eq. (28) about the equilibrium solution, Eq. (27), and to derive linearized equations of motion for the moments, Eqs. (25) and (26). All calculations closely follow the derivation of generalized hydrodynamics in Refs. 24 and 25. The zeroth moment of the linearized Eq. (28) gives the continuity equation

$$\partial_t\delta n + n_0\,\partial_\alpha v_\alpha = 0. \qquad (29)$$

Similarly, taking the first moment of Eq. (28), we get the force balance equation of the following form:

$$m^*\,\partial_t v_\alpha + B^*\varepsilon_{\alpha\beta}v_\beta - \partial_t A_\alpha^{(1)} - \partial_t A_\alpha^{CS} + \frac{1}{n_0}\partial_\beta\delta P_{\alpha\beta} + \partial_\alpha U_H$$

$$= -\partial_\alpha U + \partial_t a_\alpha. \qquad (30)$$

Substituting the definition of $\mathbf{A}^{(1)}$, Eq. (23) into Eq. (30), we find that the combination of the first and the third terms on the left hand side of Eq. (30) reduces to the correct acceleration term, $m\,\partial_t\mathbf{v}$. In a quite similar fashion the CS electric

field, $-\partial_t \mathbf{A}^{\mathrm{CS}}$, cancels the CS Lorentz force, $\mathbf{v} \times \mathbf{B}^{\mathrm{CS}}$ [one can straightforwardly prove that by calculating the time derivative of Eq. (22), and using the continuity equation]. As a result of these cancellations, the force balance equation takes the standard form, which is required by the Galilean invariance:

$$m\,\partial_t v_\alpha + B\varepsilon_{\alpha\beta}v_\beta + \frac{1}{n_0}\partial_\beta \delta P_{\alpha\beta} + \partial_\alpha U_H = F_\alpha. \quad (31)$$

Finally, calculating the second moment of the linearized equation (28), we obtain the equation of motion for the stress tensor $\delta P_{\alpha\beta} = \delta_{\alpha\beta}\delta P + \pi_{\alpha\beta}$. Decomposition of the scalar and the traceless parts of this equation leads to the following equations of motion for the pressure δP, and for shear stress tensor $\pi_{\alpha\beta}$, respectively,

$$\partial_t \delta P + 2P_0\,\partial_\alpha v_\alpha = 0, \quad (32)$$

$$\partial_t \pi_{\alpha\beta} + (B^*/m^*)(\varepsilon_{\alpha\gamma}\pi_{\gamma\beta} + \varepsilon_{\beta\gamma}\pi_{\gamma\alpha})$$
$$+ P_0(\partial_\alpha v_\beta + \partial_\beta v_\alpha - \delta_{\alpha\beta}\partial_\gamma v_\gamma) = 0, \quad (33)$$

where $P_0 = \Sigma_{\mathbf{k}}k^2 f_{\mathbf{k}}^{(0)}/2m^*$ is the equilibrium (initial) pressure of the composite fermion system. The third moment of the Wigner function, which is, in general, present in the equation for the second moment, introduces higher order gradient corrections that are irrelevant in the long wavelength limit.[25]

The system of Eqs. (29) and (31)–(33) provides a long wavelength "hydrodynamic" representation of the RPA linear response for CS theory.[6–9] To demonstrate an equivalence of the CS hydrodynamics to our phenomenological magnetoelasticity we express the velocity $\mathbf{v}$ in terms of the displacement vector $\mathbf{u}$:

$$\mathbf{v} = \partial_t \mathbf{u}. \quad (34)$$

The representation of Eq. (34) allows us to explicitly integrate the continuity equation, Eq. (29), and the equation for the pressure, Eq. (32),

$$\delta P = -\frac{2P_0}{n_0}\partial_\alpha u_\alpha, \quad \delta n = -n_0\,\partial_\alpha u_\alpha. \quad (35)$$

Substituting Eqs. (34) and (35) into Eqs. (32) and (33), we find that they become identical to the equations of magnetoelasticity theory, Eqs. (17) and (18), with $K=2P_0$, $\mu^\infty=P_0$, and $\Delta=2B^*/m^*$. It is worth mentioning that the ideal gas relation $K=2\mu^\infty$ is a direct consequence of the lack of correlations in RPA. Thus, despite the general structure of the magnetoelastic phenomenology is recovered, the microscopic elastic constants are clearly incorrect.

IV. CONCLUSION

We proposed a magnetoelasticity theory of incompressible FQH liquids. The theory demonstrates that most of fundamental dynamic properties of FQH liquids (the response to

electric and magnetic fields, the existence of gapped intra-LL collective modes, etc.) can be described in a surprisingly simple and clear fashion. The whole linear response is encoded in the local momentum conservation law, Eq. (17), supplemented by the equation of motion, Eq. (18), for the shear stress tensor $\hat{\boldsymbol{\pi}}$. In contrast to the standard classical elasticity theory, which is a theory of one vector field $\mathbf{u}$, the description of FQH liquids requires one more dynamic variable—the traceless tensor field $\hat{\boldsymbol{\pi}}$. In fact, all unusual dynamic properties of FQH liquids can be traced back to the equation of motion for the tensor field $\hat{\boldsymbol{\pi}}$. In this paper we presented a heuristic derivation of the simplest version of this equation, Eq. (18), which yields a minimal, three-parameter magnetoelasticity theory. Using this theory we were able to reproduce most of the already known results and to make some predictions concerning the dispersion of the intra-LL mode. In particular, we have shown that at small q the dispersion is determined by the high-frequency shear modulus μ^∞, which, in turn, is related the ground state energy E_0 (for a 2-D Coulomb system with a quenched kinetic energy $\mu^\infty=E_0/8$).[20]

Importantly, the time reversal and the rotational symmetry still allow for a generalization of our key equation of motion for the shear stress tensor, Eq. (18). The most general form of the theory, as well as its extensions for crystalline and liquid crystalline quantum Hall states, are presented in a separate paper.[15] The most important result of these generalizations is a prediction of two gapped intra-LL collective modes in FQH liquids. The present formulation (despite a certain lack of generality) has a great advantage to be the simplest, but still nontrivial continuum mechanics of FQH liquids. In fact, one of our aims in this work is to demonstrate a principal possibility of interpreting the behavior of highly unusual FQH liquids using an intuitively clear and physically transparent language of continuum mechanics.

Another advantage of the minimal construction is its clear connection to the well understood mean-field CS theory (Sec. III). In addition to that, there is an interesting analogy of the present formulation of magnetoelasticity theory, Eqs. (17) and (18), and the theory of visco-elastic fluids by Maxwell.[12] Both the Maxwellian hydrodynamics and our magnetoelasticity are formulated in terms of a system of two coupled equations. These are (i) the local momentum conservation law (the equation of motion for the displacement $\mathbf{u}$), and (ii) the equation of motion for the shear stress tensor $\hat{\boldsymbol{\pi}}$. The first equation takes a universal form of Eq. (17). It describes a motion of an infinitesimal fluid element under a combination of the external and internal stress forces. The dynamics of the stress tensor is, however, specific for every system. Physically it is related to a relative motion of particles inside a fluid element, which depends on details of many-body correlations. For highly viscous fluids the equation of motion for $\hat{\boldsymbol{\pi}}$ takes the form[12]

$$\partial_t \hat{\boldsymbol{\pi}} + \frac{1}{\tau}\hat{\boldsymbol{\pi}} + \mu^\infty \partial_t \hat{\mathbf{S}} = 0, \quad (36)$$

while for incompressible FQH liquids the shear stress tensor satisfies Eq. (18). Equation (36) describes the relaxation

dynamics (τ is the relaxation time), which leads to a nonzero shear viscosity, and results in a usual overdamped hydrodynamic mode. In contrast to that, the dynamics of the stress tensor in FQH liquids correspond to a dissipationless precession [see, Eq. (18)]. This immediately opens a gap, making the liquid incompressible, and produces a collective mode with a roton minimum.

*Electronic address: ilya.tokatly@physik.uni-erlangen.de

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