Cubic magnets with Dzyaloshinskii-Moriya interaction at low temperature

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Magnetic properties of noncentrosymmetric cubic helimagnets (MnSi, etc.) at low T are studied theoretically using conventional exchange, Dzyaloshinskii-Moriya interaction, anisotropic exchange and magnetic dipolar interaction. Structure in magnetic field and spin-wave spectrum are considered. In low field $g\mu_B H_1=\Delta\sqrt{2}$, where Δ is the spin-wave gap, the helix axis $\mathbf{k}$ turns along the field. A transition to ferromagnetic state occurs at $g\mu_B H_c=Ak^2$, where A is the spin-wave stiffness at momenta $q\gg k$. It is observed that the spin-wave spectrum is strongly anisotropic: excitations with $\mathbf{q}\|\mathbf{k}$ and $\mathbf{q}\perp\mathbf{k}$ have linear and quadratic dispersion at $q\ll k$ respectively if one neglects the gap Δ. This is a result of umklapp interaction connecting the spin-waves with momenta $\mathbf{q}$ and $\mathbf{q}\pm\mathbf{k}$. This interaction leads to infinite set of equations for the Green functions. The simplest $n=1$ approximation is applicable at $q=0$ and $q\gg k$. Further $n=2$ approximation gives correct result at $q\ll k$. Results of both approximations are qualitative at $q\sim k$. The weak field ($H<H_1$) perpendicular to $\mathbf{k}$ deforms the helix and the second harmonics of the spin rotation appears. The spin-wave gap is a result of the spin-wave interaction considered in the Hartree-Fock approximation and cubic anisotropy. Properties of the ESR and neutron scattering are considered. The theoretical results are in agreement with existing experimental data.

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I. INTRODUCTION

Itinerant cubic magnets MnSi, FeGe, FeSiCo, etc. have attracted a lot of attention due to their specific electronic and magnetic properties. The former originate from closeness to quantum phase transition, which is achieved at high pressure (see Refs. 1 and 3 and references therein). The corresponding theory was developed in Ref. 4. The latter stems from the $P2_13$ symmetry which allows Dzyaloshinskii-Moriya interaction (DMI) responsible for magnetic helix structure. In cubic crystals the DMI fixes the sense of the helix (right or left-handed spiral) but cannot determine its direction.[5,6] It is stabilized by very weak anisotropic exchange interaction (AEI).[7,8]

The helix structure is very sensitive to external magnetic field. Two field-induced transitions have been observed (Refs. 9–11 and references therein). In low field it is a transition to the state with the helix axis $\mathbf{k}$ along the field. Then with the field increasing there is the second transition to the "ferromagnetic" state. These transitions have been described by the phenomenological Landau-like theory which contains a large number of parameters.[12,13] It should be noted also that critical properties of MnSi near transition temperature T_c are very unusual as well. Corresponding experimental studies and its theoretical explanation may be found in Refs. 14–16.

In this paper we present theoretical description of low-temperature properties of the cubic magnets with the DMI. We microscopically evaluate the ground-state energy and the spin-wave spectrum. In particular we demonstrate that the critical fields mentioned above are related to the parameters of the spin-wave spectrum in agreement with existing experimental data.

This paper is organized as follows. In Sec. II the theoretical model is formulated. Along with the exchange interaction, the DMI and AEI studied in Refs. 7 and 8 we include also the magnetic dipolar and the Zeeman interactions. Following Ref. 17 we introduce all three spin components in each lattice point of the helix. Section III is devoted to the consideration of the classical ground state energy. The helical wave vector $\mathbf{k}$ and critical field along $\mathbf{k}$ for the transition to the ferromagnetic state H_c are determined. In Sec. IV the spin-wave Hamiltonian is considered. It consists of two parts. The former includes the spin waves with the same $\mathbf{q}$. The second describes the umklapp processes mixing the spin waves with $\mathbf{q}$ and $\mathbf{q}\pm\mathbf{k}$ which have different energies. It is a result of the incommensurate helical structure and low symmetry of the DMI which breaks the total spin conservation law.

The umklapp interaction is analyzed in Sec. V. Two types of the umklapps are considered. The former created by the field perpendicular to $\mathbf{k}$ leads to the first order transition to the state with $\mathbf{k}$ along the field at $g\mu_B H_\perp=\Delta\sqrt{2}$ where Δ is the spin-wave gap. The second gives rise to strong anisotropy of the spin-wave spectrum: excitations with momentum along and perpendicular to $\mathbf{k}$ have different energies.

Spin configuration in weak perpendicular field is discussed in Sec. VI. The origin of the spin-wave gap Δ is studied in Sec. VII. There are two contributions to Δ^2: the spin-wave interaction considered in the Hartree-Fock approximation and the cubic anisotropy. If the latter contribution is negative and sufficiently strong instead of the helical magnetic order the spin-liquid state is realized where the chiral spin fluctuations persist due to the DMI. They can be studied by polarized neutron scattering. This instability may be a reason of the quantum phase transition observed under pressure in MnSi.[2,3,10]

Possibilities of the electron spin resonance (ESR) and neutron scattering studies of the above-mentioned phenomena are considered in Sec. VIII. The theoretical results are summarized and discussed in Sec. IX in connection to existing experimental data. Some details of calculations are presented in Appendixes A–D.

II. MODEL

There are the following principal interactions in cubic magnets without inversion symmetry: (i) conventional exchange interaction, (ii) the DMI, (iii) the AEI, (iv) the magnetic dipolar interaction, (v) the Zeeman energy, and (vi) the cubic anisotropy. The DMI and AEI are responsible for the helical magnetic structure and its orientation relative the cubic axes respectively.[5–8] Weak cubic anisotropy determines the spin-wave gap and some additional peculiarities of the helix axis orientation in the magnetic field. We investigate spin configuration in magnetic field and the spin-wave spectrum using the first five interactions. The cubic anisotropy will be considered in Sec. VII in connection with the origin of the spin-wave gap.

There are four magnetic ions in the unit sell of the magnets under study. We will consider here the total spin of the unit sell as a basic magnetic entity, completely neglecting internal movements of the cell spins and corresponding optical spin-wave branches. Interested in low-energy spin dynamics, we use the magnetic density approximation[18,19] with total spin of the unit cell $\mathbf{S}(\mathbf{R})$ and usual commutation relations

$$[S_\alpha(\mathbf{R}), S_\beta(\mathbf{R})] = i\epsilon_{\alpha\beta\gamma} S_\gamma(\mathbf{R}). \tag{1}$$

The DMI destroys ferromagnetic order and is responsible for the long periodical spin-density wave.[5–8] For each spin we determine right-handed local orthogonal coordinate frame with basic unit vectors $\hat{\xi}_{\mathbf{R}}$, $\hat{\eta}_{\mathbf{R}}$, and $\hat{\xi}_{\mathbf{R}}$ given by[17]

$$\hat{\zeta}_{\mathbf{R}} = \hat{c}\sin\alpha + (\hat{a}\cos\mathbf{k}\cdot\mathbf{R} + \hat{b}\sin\mathbf{k}\cdot\mathbf{R})\cos\alpha, \tag{2}$$

$$\hat{\eta}_{\mathbf{R}} = -\hat{a}\sin\mathbf{kR} + \hat{b}\cos\mathbf{kR},$$

$$\hat{\xi}_{\mathbf{R}} = \hat{c}\cos\alpha - (\hat{a}\cos\mathbf{k}\cdot\mathbf{R} + \hat{b}\sin\mathbf{k}\cdot\mathbf{R})\sin\alpha,$$

where $\hat{a}\times\hat{b}=\hat{c}$, $\hat{b}\times\hat{c}=\hat{a}$, and $\hat{c}\times\hat{a}=\hat{b}$ and we have

$$\mathbf{S}_{\mathbf{R}} = S^{\zeta}_{\mathbf{R}}\hat{\zeta}_{\mathbf{R}} + S^{\eta}_{\mathbf{R}}\hat{\eta}_{\mathbf{R}} + S^{\xi}_{\mathbf{R}}\hat{\xi}_{\mathbf{R}}, \tag{3}$$

where projections of the spin operators have well known form

$$S^{\zeta}_{\mathbf{R}} = S - (a^+a)_{\mathbf{R}},$$

$$S^{\eta}_{\mathbf{R}} = \frac{1}{i}\sqrt{\frac{S}{2}}\left[a_{\mathbf{R}} - a^+_{\mathbf{R}} - \frac{(a^+a^2)_{\mathbf{R}}}{2S}\right],$$

$$S^{\xi}_{\mathbf{R}} = \sqrt{\frac{S}{2}}\left[a_{\mathbf{R}} + a^+_{\mathbf{R}} - \frac{(a^+a^2)_{\mathbf{R}}}{2S}\right]. \tag{4}$$

Here S is a parameter connected to the cell magnetization by $M=g\mu_B S/v_0$ where $\mu_B>0$, $g\simeq 2$, and $v_0=a^3$ is the unit cell volume. For MnSi magnetic moment per spin $0.4\mu_B$ is strongly reduced in comparison with the paramagnetic state value $1.4\mu_B$ and the unit-cell spin $S=0.2\times 4=0.8$.[10]

For the above-mentioned principal interactions in the $\mathbf{R}$ space we have

$$H = H_{EX} + H_{DM} + H_{AE} + H_D + H_Z,$$

$$H_{EX} = -\frac{1}{2}\sum J_{\mathbf{RR}'}\mathbf{S}_{\mathbf{R}}\cdot\mathbf{S}_{\mathbf{R}'},$$

$$H_{DM} = \frac{1}{2}\sum D_{\mathbf{RR}'}(\nabla - \nabla')\mathbf{S}_{\mathbf{R}}\times\mathbf{S}_{\mathbf{R}'},$$

$$H_{AE} = \frac{1}{2}\sum F_{\mathbf{RR}'}[(\nabla_x S^x_{\mathbf{R}})(\nabla_x S^x_{\mathbf{R}'}) + (\nabla_y S^y_{\mathbf{R}})(\nabla_y S^y_{\mathbf{R}'}) + (\nabla_z S^z_{\mathbf{R}}) \times(\nabla_z S^z_{\mathbf{R}'})],$$

$$H_D = \frac{(g\mu_B)^2}{2}\sum\left[\frac{\mathbf{S}_{\mathbf{R}}\cdot\mathbf{S}_{\mathbf{R}'}}{|\mathbf{R}-\mathbf{R}'|^3} - \frac{3\mathbf{S}_{\mathbf{R}}\cdot(\mathbf{R}-\mathbf{R}')\mathbf{S}'_{\mathbf{R}}\cdot(\mathbf{R}-\mathbf{R}')}{|\mathbf{R}-\mathbf{R}'|^5}\right],$$

$$H_Z = \mu_B\mathbf{H}\cdot\sum\mathbf{S}_{\mathbf{R}}. \tag{5}$$

In cubic crystals the Dzyaloshinskii vector has not a definite direction and proportional to vector $\boldsymbol{\nabla}$.[5,6] It changes sign if $\mathbf{R}\leftrightarrow\mathbf{R}'$ and $D_{\mathbf{RR}'}=D_{\mathbf{R}'\mathbf{R}}$. Otherwise we would have $H_{DM}=0$. In the $\mathbf{q}$ space these expressions have the form

$$H_{EX} = \frac{1}{2}\sum J_{\mathbf{q}}\mathbf{S}_{\mathbf{q}}\cdot\mathbf{S}_{-\mathbf{q}},$$

$$H_{DM} = \sum iD_{\mathbf{q}}\mathbf{q}\cdot[\mathbf{S}_{\mathbf{q}}\times\mathbf{S}_{-\mathbf{q}}],$$

$$H_{AE} = \frac{1}{2}\sum_{\nu=x,y,z} F_{\mathbf{q}}q^2_\nu S^\nu_{\mathbf{q}}S^\nu_{-\mathbf{q}},$$

$$H_D = \frac{\omega_0}{2}\sum[(\mathbf{S}_{\mathbf{q}}\cdot\hat{q})(\mathbf{S}_{-\mathbf{q}}\cdot\hat{q}) - 1/3(\mathbf{S}_{\mathbf{q}}\cdot\mathbf{S}_{-\mathbf{q}})],$$

$$H_Z = N^{1/2}\mathbf{H}\cdot\mathbf{S}_0. \tag{6}$$

where N is total number of the cells, $\mathbf{S}_{\mathbf{q}}=N^{-1/2}\Sigma\mathbf{S}_{\mathbf{R}}\exp(-i\mathbf{q}\cdot\mathbf{R})$, $D_{\mathbf{q}}=D_{-\mathbf{q}}$, $\hat{q}=\mathbf{q}/q$, and $\omega_0=4\pi(g\mu_B)^2/v_0$ is the characteristic energy of the dipolar interaction. The anisotropic part of this interaction contains the tensor $\hat{q}_\alpha\hat{q}_\beta$. At $\mathbf{q}=\mathbf{0}$ it must be replaced by demagnetization tensor $N_{\alpha\beta}$.[19,20] The DMI and AEI are of the first and second order of the spin-orbit interaction respectively. So we have $J\gg D/a \gg F/a^2$ where a is the lattice spacing. Value of the dipolar energy $S\omega_0$ for MnSi will be given in the last section. It is less than characteristic helical energy Ak^2 determined in the next section.

III. CLASSICAL ENERGY

Replacing in Eq. (3) $\mathbf{S}_{\mathbf{R}}$ by $S\hat{\zeta}_{\mathbf{R}}$ we get the classical ground-state energy per unit cell in the following form:

$$E_{cl} = -\frac{S^2}{2}(J_0\sin^2\alpha + J_{\mathbf{k}}\cos^2\alpha + \omega_0/3) - S^2D_0 \times(\mathbf{k}\cdot[\hat{\mathbf{a}}\times\hat{\mathbf{b}}])\cos^2\alpha + \frac{S^2F}{4}[k_x^2(\hat{a}_x^2+\hat{b}_x^2) + k_y^2(\hat{a}_y^2+\hat{b}_y^2) + k_z^2(\hat{a}_z^2+\hat{b}_z^2)]\cos^2\alpha + Sh_\parallel\sin\alpha + \frac{S^2}{2}\omega_0 N_{cc}\sin^2\alpha, \tag{7}$$

here $h_\parallel = g\mu H_\parallel$, $H_\parallel$ is the field component along the c axis and N_{cc} is the corresponding component of the demagnetization tensor. It is important to note that the classical energy depends on the field along c axis only. Meanwhile experiment shows that the system is very sensitive to weak perpendicular field.[9–11] We will explain below the nature of this quantum phenomenon.

We are interested by small k only ($ka \ll 1$) when $J_{\mathbf{k}} \simeq J_0 - \frac{Ak^2}{S}$ and $D_{\mathbf{k}} \simeq D_0$. As we will see below A is the spin-wave stiffness constant at $q \gg k$. For $H_\parallel = 0$ this ground state energy was studied in Ref. 8. From Eq. (7) for components of the vector $\mathbf{k}$ we get

$$Ak_\nu + \frac{SF}{2} k_\nu (\hat{a}_\nu^2 + \hat{b}_\nu^2) = SD_0 [\hat{a} \times \hat{b}]_\nu \tag{8}$$

and

$$Ak^2 + \frac{SF}{2} I(\mathbf{k}) = SD_0 (\mathbf{k} \cdot [\hat{a} \times \hat{b}]), \tag{9}$$

where $I = \Sigma k_\nu^2 (a_\nu^2 + b_\nu^2)$ is a cubic invariant. The classical energy may be represented as

$$E_{cl} = -\frac{S^2}{2}\left(J_0 + \frac{\omega_0}{3}\right) + \left[\frac{SAk^2}{2} + \frac{S^2 FI}{4} - S^2 D_0 (\mathbf{k} \cdot [\hat{a} \times \hat{b}])\right] \times \cos^2 \alpha + Sh_\parallel \sin \alpha + \frac{S^2 N_{cc}}{2} \omega_0 \sin^2 \alpha. \tag{10}$$

Obviously E_{cl} is minimal if $(\mathbf{k} \cdot [\hat{a} \times \hat{b}]) = |\mathbf{k}| D_0 / |D_0|$. In this case the spins rotate in the plane perpendicular to the vector $\mathbf{k}$. For negative F the minimum is for $\mathbf{k}$ along one of the cubic diagonals and $I = 2k^2/3$. For $F > 0$ the vector $\mathbf{k}$ is directed along the cubic edge and $I = 0$.[8] In both cases we have

$$\mathbf{k} = \frac{SD_0 [\hat{a} \times \hat{b}]}{A + SFI/2}. \tag{11}$$

For $D_0 > 0$ and $D_0 < 0$ we have the right and left-handed helix respectively.

The classical energy depends on the field projection onto the vector $\mathbf{k}$ (c axis) only and from Eqs. (9) and (10) we obtain

$$\sin \alpha = \begin{cases} -H_\parallel / H_c, & H_\parallel < H_c \\ -1, & H_\parallel > H_c, \end{cases} \tag{12}$$

where the critical field is given by

$$g\mu_B H_c = h_c = Ak^2 + \frac{SF}{2} I + S\omega_0 N_{cc} \tag{13}$$

Here the last term is a result of the demagnetization. The intrinsic critical field is determined by $H_c^{Int} = H_c - 4\pi N_{cc} M$, where M is the magnetization in the high-field ferromagnetic state determined as $g\mu_B M = S\omega_0$ and $g\mu H_c^{Int} = h_c^{Int} = Ak^2 + SFI/2 \simeq Ak^2$. For $H > H_c$ we have ferromagnetic spin configuration, but according to Eqs. (3) and (4) the perpendicular, spin-wave components of the spin density remain rotating. It has to be taken into account in consideration of the ESR and neutron scattering (see Sec. VIII).

IV. SPIN-WAVES

In the low-q region the hierarchy of the interactions is very important. The ferromagnetic exchange J is the strongest one. The DMI and AEI are results of the weak spin-orbit coupling λ. They are of order λJ and $\lambda^2 J$ respectively. According to Eq. (11) we have $k \sim \lambda$.

We are dealing with incommensurate spin structure where umklapp processes mix excitations with momenta $\mathbf{q}$ and $\mathbf{q} \pm \mathbf{k}$ and different energies. For their consideration it is convenient to represent the unit vectors given by Eq. (2) as

$$\hat{\xi}_{\mathbf{R}} = \hat{c} \sin \alpha + (\mathbf{A} e^{i\mathbf{k}\cdot\mathbf{R}} + \mathbf{A}^* e^{-i\mathbf{k}\cdot\mathbf{R}}) \cos \alpha,$$

$$\hat{\eta}_{\mathbf{R}} = i\mathbf{A} e^{i\mathbf{k}\cdot\mathbf{R}} - i\mathbf{A}^* e^{-i\mathbf{k}\cdot\mathbf{R}},$$

$$\hat{\xi}_{\mathbf{R}} = \hat{c} \cos \alpha - (\mathbf{A} e^{i\mathbf{k}\cdot\mathbf{R}} + \mathbf{A}^* e^{-i\mathbf{k}\cdot\mathbf{R}}) \sin \alpha, \tag{14}$$

where $\mathbf{A} = (\hat{a} - i\hat{b})/2$, $\mathbf{A} \cdot \mathbf{A} = \mathbf{A}^* \cdot \mathbf{A}^* = \mathbf{0}$, $\mathbf{A} \cdot \mathbf{A}^* = 1/2$, $\mathbf{A} \times \hat{c} = -i\mathbf{A}$, $\mathbf{A}^* \times \hat{c} = i\mathbf{A}^*$, and $\mathbf{A} \times \mathbf{A}^* = i\hat{c}/2$ and we have

$$\mathbf{S}_{\mathbf{q}} = S_{\mathbf{q}}^c \hat{c} + S_{\mathbf{q}}^A \mathbf{A} + S_{\mathbf{q}}^{A^*} \mathbf{A}^*,$$

$$S_{\mathbf{q}}^c = S_{\mathbf{q}}^\zeta \sin \alpha + S_{\mathbf{q}}^\xi \cos \alpha,$$

$$S_{\mathbf{q}}^A = S_{\mathbf{q}-\mathbf{k}}^\zeta \cos \alpha - S_{\mathbf{q}-\mathbf{k}}^\xi \sin \alpha + iS_{\mathbf{q}-\mathbf{k}}^\eta,$$

$$S_{\mathbf{q}}^{A^*} = S_{\mathbf{q}+\mathbf{k}}^\zeta \cos \alpha - S_{\mathbf{q}+\mathbf{k}}^\xi \sin \alpha - iS_{\mathbf{q}+\mathbf{k}}^\eta. \tag{15}$$

Using these expressions it is easy to see that H_{DM} given by Eq. (6) contains direct and umklapp terms. The former are proportional to $S_{\mathbf{q}}^c S_{-\mathbf{q}}^c$, $S_{\mathbf{q}-\mathbf{k}}^{A^*} S_{-\mathbf{q}+\mathbf{k}}^A$, and $S_{\mathbf{q}+\mathbf{k}}^A S_{-\mathbf{q}-\mathbf{k}}^{A^*}$. The latter contain $S_{\mathbf{q}}^c S_{-\mathbf{q}-\mathbf{k}}^A$ and $S_{\mathbf{q}}^c S_{-\mathbf{q}+\mathbf{k}}^{A^*}$. They are responsible for umklapps with $\mathbf{q} \to \mathbf{q} \pm \mathbf{k}$. The processes with $\mathbf{q} \to \mathbf{q} \pm 2\mathbf{k}$ appear in the $S^A S^A$ and $S^{A^*} S^{A^*}$ terms. Due to the algebra of the vectors $\hat{c}$, $\mathbf{A}$, and $\mathbf{A}^*$ these double umklapps are in H_{AE} and H_D only and we neglect them below. We will also see that the perpendicular field initiate processes with $\mathbf{q} \pm \mathbf{k}$ also. The first time the umklapps were studied using phenomenological approach in Ref. 13.

We evaluate here the direct contribution to the spin-wave energy. The umklapps will be considered in the next section. The exchange energy does not contain the umklapps and has the form

$$\begin{aligned} H_{EX} = -\frac{1}{2} \sum [& (J_{\mathbf{q}} \sin^2 \alpha + J_{\mathbf{q},\mathbf{k}} \cos^2 \alpha) S_{\mathbf{q}}^\zeta S_{-\mathbf{q}}^\zeta + J_{\mathbf{q},\mathbf{k}} S_{\mathbf{q}}^\eta S_{-\mathbf{q}}^\eta \\ & + (J_{\mathbf{q}} \cos^2 \alpha + J_{\mathbf{q},\mathbf{k}} \sin^2 \alpha) S_{\mathbf{q}}^\xi S_{-\mathbf{q}}^\xi \\ & + i(S_{\mathbf{q}}^\eta S_{-\mathbf{q}}^\zeta - S_{\mathbf{q}}^\zeta S_{-\mathbf{q}}^\eta) N_{\mathbf{q},\mathbf{k}} \cos \alpha \\ & + (J_{\mathbf{q}} - J_{\mathbf{q},\mathbf{k}})(S_{\mathbf{q}}^\zeta S_{-\mathbf{q}}^\xi + S_{\mathbf{q}}^\xi S_{-\mathbf{q}}^\zeta) \\ & \times \sin \alpha \cos \alpha - i(S_{\mathbf{q}}^\eta S_{-\mathbf{q}}^\xi - S_{\mathbf{q}}^\xi S_{-\mathbf{q}}^\eta) N_{\mathbf{q},\mathbf{k}} \sin \alpha], \end{aligned} \tag{16}$$

where $J_{\mathbf{q},\mathbf{k}} = (J_{\mathbf{q}+\mathbf{k}} + J_{\mathbf{q}-\mathbf{k}})/2 \simeq J_0 - A(q^2 + k^2)/S$ and $N_{\mathbf{q},\mathbf{k}} = (J_{\mathbf{q}+\mathbf{k}} - J_{\mathbf{q}-\mathbf{k}})/2 \simeq -2A\mathbf{q}\mathbf{k}/S$.

Direct part of H_{DM} is given by

$$H^d_{DM} = -\sum D_{\mathbf{q}}(\mathbf{k}\cdot\hat{\mathbf{c}})[S^{\zeta}_{\mathbf{q}}S^{\zeta}_{-\mathbf{q}}\cos^2\alpha + S^{\eta}_{\mathbf{q}}S^{\eta}_{-\mathbf{q}} + S^{\xi}_{\mathbf{q}}S^{\xi}_{-\mathbf{q}}\sin^2\alpha - (S^{\zeta}_{\mathbf{q}}S^{\xi}_{-\mathbf{q}} + S^{\xi}_{\mathbf{q}}S^{\zeta}_{-\mathbf{q}})\sin\alpha\cos\alpha] + i\sum D_{\mathbf{q}}(\mathbf{q}\cdot\hat{c}) \times[(S^{\zeta}_{\mathbf{q}}S^{\eta}_{-\mathbf{q}} - S^{\eta}_{\mathbf{q}}S^{\zeta}_{-\mathbf{q}})\cos\alpha + (S^{\eta}_{\mathbf{q}}S^{\xi}_{-\mathbf{q}} - S^{\xi}_{\mathbf{q}}S^{\eta}_{-\mathbf{q}})\sin\alpha], \tag{17}$$

where $\hat{c}=[\hat{a}\times\hat{b}]$.[22]

For the anisotropic exchange we have

$$H^d_{AE} = \frac{F}{2}\sum_{\mathbf{q},\nu=x,y,z}\Bigg\{(c_\nu q_\nu)^2(S^{\zeta}_{\mathbf{q}}\sin\alpha + S^{\xi}_{\mathbf{q}}\cos\alpha)(S^{\zeta}_{-\mathbf{q}}\sin\alpha + S^{\xi}_{-\mathbf{q}}\cos\alpha) + \frac{(q^2_\nu + k^2_\nu)(\hat{a}^2_\nu + \hat{b}^2_\nu)}{2}[(S^{\zeta}_{\mathbf{q}}\cos\alpha - S^{\xi}_{\mathbf{q}}\sin\alpha) \times(S^{\zeta}_{-\mathbf{q}}\cos\alpha - S^{\xi}_{-\mathbf{q}}\sin\alpha) + S^{\eta}_{\mathbf{q}}S^{\eta}_{-\mathbf{q}}] - 2iq_\nu k_\nu(a^2_\nu + b^2_\nu) \times(S^{\zeta}_{\mathbf{q}}\cos\alpha - S^{\xi}_{\mathbf{q}}\sin\alpha)S^{\eta}_{-\mathbf{q}}\Bigg\}, \tag{18}$$

and for the dipolar interaction we obtain

$$H^{(d)}_D = \frac{\omega_0}{2}\sum\Bigg\{(\hat{q}\cdot\hat{c})^2(S^{\zeta}_{\mathbf{q}}\sin\alpha + S^{\xi}_{\mathbf{q}}\cos\alpha) \times(S^{\zeta}_{-\mathbf{q}}\sin\alpha + S^{\xi}_{-\mathbf{q}}\cos\alpha) + \frac{q^2_\perp}{2|\mathbf{q},\mathbf{k}|^2} \times[(S^{\zeta}_{\mathbf{q}}\cos\alpha - S^{\xi}_{\mathbf{q}}\sin\alpha)(S^{\zeta}_{-\mathbf{q}}\cos\alpha - S^{\xi}_{-\mathbf{q}}\sin\alpha) + S^{\eta}_{\mathbf{q}}S^{\eta}_{-\mathbf{q}}]\Bigg\}, \tag{19}$$

where $q^2_\perp = q^2_a + q^2_b$ and $|\mathbf{q},\mathbf{k}|^{-2} = (|\mathbf{q}+\mathbf{k}|^{-2} + |\mathbf{q}-\mathbf{k}|^{-2})/2$. In this expression we have taken into account that the vector $\mathbf{k}$ is perpendicular to the ab plane.

There are two kinds of terms in Eqs. (16)–(19): (i) Diagonal terms proportional to $S^{\lambda}S^{\lambda}$ and (ii) off diagonal terms containing $S^{\lambda}S^{\mu}$ with $\lambda\neq\mu$. The former are responsible for the direct contribution to the spin-wave spectrum and the four-point spin-wave interaction considered in Appendix D. The latter are not contributed at least in the main order. We begin with the $S^{\zeta}S^{\xi}$ terms. Replacing $S^{\zeta}_{\mathbf{q}}$ by $N^{1/2}S\delta_{\mathbf{q},0}$ we obtain an expression which cancels $H_\parallel S^{\xi}_0$ term in the Zeeman energy. The remaining part gives the odd point spin-wave interaction considered in Appendix D. The $S^{\zeta}S^{\eta}$ and $S^{\xi}S^{\eta}$ terms cancel also by virtue of Eq. (8). So we can restrict here to the diagonal contribution only.

In the linear spin-wave theory from Eqs. (4) and (16)–(19) after rather tedious calculations we obtain

$$H^{(d)}_{SW} = \sum\left[E_{\mathbf{q}}a^+_{\mathbf{q}}a_{\mathbf{q}} + \frac{B_{\mathbf{q}}}{2}(a_{\mathbf{q}}a_{-\mathbf{q}} + a^+_{\mathbf{q}}a^+_{-\mathbf{q}})\right], \tag{20}$$

where at $q\ll 1/a$

$$E_{\mathbf{q}} = Aq^2 + \frac{1}{2}\left(Ak^2 + \frac{SFI}{2}\right)\cos^2\alpha + \frac{S\omega_0}{2} \times\left[\hat{q}^2_c\cos^2\alpha + \frac{q^2_\perp}{|\mathbf{q},\mathbf{k}|^2}(1+\sin^2\alpha)\right] - S(h_c\sin\alpha + h_\parallel)\sin\alpha, \tag{21}$$

where $h_\parallel = g\mu_B H_\parallel$ and h_c is determined by Eq. (13). Using Eq. (12) one can show that expression in the last brackets is zero if $H_\parallel < H_c$. For $B_{\mathbf{q}}$ at small q we have

$$B_{\mathbf{q}} = \frac{1}{2}\left(Ak^2 + \frac{SFI}{2}\right)\cos^2\alpha + \frac{S\omega_0}{2}\left(\hat{q}^2_c - \frac{q^2_\perp}{|\mathbf{q},\mathbf{k}|^2}\right)\cos^2\alpha. \tag{22}$$

At arbitrary $\mathbf{q}$ expressions for $E_{\mathbf{q}}$ and $B_{\mathbf{q}}$ are given in Appendix D.

It is easy to show that the spin-wave energy is given by $\epsilon_{\mathbf{q}} = (E^2_{\mathbf{q}} - B^2_{\mathbf{q}})^{1/2}$. For $H_\parallel > H_c$ we have $B_{\mathbf{q}} = 0$ and

$$\epsilon_{\mathbf{q}} = Aq^2 + h_\parallel - h_c + \frac{S\omega_0 q^2_\perp}{|\mathbf{q},\mathbf{k}|^2}. \tag{23}$$

At $H_\parallel < H_c$ the energy has the form

$$\epsilon_{\mathbf{q}} = \left\{\left(Aq^2 + \frac{S\omega_0 q^2_\perp}{|\mathbf{q},\mathbf{k}|^2}\right)\left[Aq^2 + (Ak^2 + S\omega_0\hat{q}^2_c) \times\cos^2\alpha + \frac{S\omega_0 q^2_\perp}{|\mathbf{q},\mathbf{k}|^2}\sin^2\alpha\right]\right\}^{1/2}, \tag{24}$$

where we neglected small F term in the second brackets. So we have the gapless excitation with linear dispersion at $q \ll k$. In these equations the factor $q^2_\perp/|\mathbf{q},\mathbf{k}|^2$ is singular if $\mathbf{q}\to\pm\mathbf{k}$ and should be replaced by $\hat{q}^2_\perp/2$ or $N_\perp/2 = (N_{aa} + N_{bb})/2$ if $\mathbf{q}_\perp = 0$. This replacement becomes evident if we put $\mathbf{q} = \pm\mathbf{k} + \mathbf{q}_\perp$.

At $q \gg k$ both expressions (23) and (24) may be approximated by $\epsilon_{\mathbf{q}} \simeq Aq^2$. The same holds if one takes into account the umklapp interaction [see Eqs. (35) and (36) below].

V. UMKLAPP INTERACTION

There are the following umklapp contributions to the Hamiltonian: (i) Interaction of the perpendicular magnetic field with S^{ζ} spin component. (ii) Interference of the longitudinal (c) and transverse ($\mathbf{A}$ and $\mathbf{A}^*$) spin components in the DM and dipolar interactions. All these contributions mix excitations with $\mathbf{q}$ and $\mathbf{q}\pm\mathbf{k}$. The dipolar interaction contains also the ($\mathbf{AA}$) and ($\mathbf{A}^*\mathbf{A}^*$) terms which mix waves with $\mathbf{q}$ and $\mathbf{q}\pm 2\mathbf{k}$. We neglect this contribution.[23] We neglect the AEI umklapps also.

The umklapp part of the Hamiltonian consists of terms odd and even in operators a and a^+. It is convenient to write out general expression for both of them and then extract the bilinear part which contribute to the spin-wave spectrum and linear one considered in the next section in connection with the perpendicular field problem. The full umklapp Hamiltonian has the form

$$H^U = H^U_Z + H^U_{DM} + H^U_D,$$

$$H_Z^U = -\mathbf{h}\cdot\mathbf{A}\left(S^{\xi}_{-\mathbf{k}}\sin\alpha - iS^{\eta}_{-\mathbf{k}} + \sum a^{+}_{\mathbf{q}+\mathbf{k}}a_{\mathbf{q}}\cos\alpha\right) - \mathbf{h}\cdot\mathbf{A}^{*}\left(S^{\xi}_{\mathbf{k}}\sin\alpha + iS^{\eta}_{\mathbf{k}} + \sum a^{+}_{\mathbf{q}-\mathbf{k}}a_{\mathbf{q}}\cos\alpha\right),$$

$$H_{DM}^U + H_D^U = \sum\{[-2D_0(\mathbf{q}\cdot\mathbf{A}) + \omega_0(\hat{c}\cdot\hat{q})(\hat{q}\cdot\mathbf{A})]S^c_{\mathbf{q}}S^A_{-\mathbf{q}} + [2D_0(\mathbf{q}\cdot\mathbf{A}^{*}) + \omega_0(\hat{c}\cdot\hat{q})(\hat{q}\cdot\mathbf{A}^{*})]S^c_{\mathbf{q}}S^{A^{*}}_{-\mathbf{q}}\}, \tag{25}$$

where the spin components S^c, S^A, and S^{A^*} are given by Eq. (15). It is important to note that the umklapp Hamiltonian contains the components of the magnetic field and the spin-wave momentum perpendicular to the helix axis c only. So the umklapps cannot change spin-wave spectrum with $\mathbf{q}\|\mathbf{k}$.

The term $S\delta_{\mathbf{q},0}$ in the expression for $S^c_{\mathbf{q}}$ leads to demagnetization of the perpendicular field if $H_{\|}\neq 0$. It appears in nonspherical samples. In this case $-\mathbf{h}\cdot\mathbf{A}$ is replaced by

$$P = -(\mathbf{h}\cdot\mathbf{A} + S\omega_0 N_{cA}\sin\alpha), \tag{26}$$

where $N_{cA}=(N_{ca}-iN_{cb})/2$ is off-diagonal component of the tensor $N_{\alpha\beta}$. This unusual feature must be elucidated. The tensor $N_{\alpha\beta}$ is diagonal if the coordinates are along the principal axes of the ellipsoidal sample. If $\mathbf{k}$ is not along one of these axes the off-diagonal components of $N_{\alpha\beta}$ appear. The $\delta_{\mathbf{q},\pm\mathbf{k}}$ terms in $S^{A,A^*}_{\mathbf{q}}$ do not contribute to H^U due to the condition $\mathbf{q}\cdot\mathbf{A}=0$. As a result for the bilinear spin-wave part of H^U we obtain

$$H^U_{SW} = \sum\{[P\cos\alpha + R_{-\mathbf{q}-\mathbf{k}}(1-\sin\alpha) - R_{\mathbf{q}}(1+\sin\alpha)]\times a^{+}_{\mathbf{q}+\mathbf{k}}a_{\mathbf{q}} + [P^{*}\cos\alpha + R^{*}_{-\mathbf{q}}(1-\sin\alpha) - R^{*}_{\mathbf{q}-\mathbf{k}}(1+\sin\alpha)]a^{+}_{\mathbf{q}-\mathbf{k}}a_{\mathbf{q}} + R_{\mathbf{q}}[a_{\mathbf{q}}a_{-\mathbf{q}-\mathbf{k}}(1-\sin\alpha) - a^{+}_{-\mathbf{q}}a^{+}_{\mathbf{q}+\mathbf{k}}(1+\sin\alpha)] + R^{*}_{-\mathbf{q}}[a^{+}_{-\mathbf{q}}a^{+}_{\mathbf{q}-\mathbf{k}}(1-\sin\alpha) - a_{\mathbf{q}}a_{-\mathbf{q}+\mathbf{k}}(1+\sin\alpha)]\} \tag{27}$$

where

$$R_{\mathbf{q}} = \frac{S\cos\alpha}{2}[-2D_0(\mathbf{q}\cdot\mathbf{A}) + \omega_0(\hat{c}\cdot\hat{q})(\hat{q}\cdot\mathbf{A})]. \tag{28}$$

From these equations we see that the umklapp interaction contributes to the spin-wave energy at $H_{\|}<H_c$ only where $\cos\alpha\neq 0$ and do not affect excitations with $\mathbf{q}$ along the helix axis $\mathbf{k}$. At the same time for $H_{\|}>H_c$ we have $B_{\mathbf{q}}=0$ in Eq. (20) and zero-point vibrations disappear.

For consideration of the umklapp interaction we will use equations of motion

$$\omega G_{A,B}(\omega) + G_{[H,A],B}(\omega) = \langle[A,B]\rangle, \tag{29}$$

where $H=H^{(d)}_{SW}+H^U_{SW}$ and the Green functions are given by

$$G_{A,B}(\omega) = -i\int_0^{\infty} e^{i\omega t}\langle[A(t),B(0)]\rangle = -\langle A,B\rangle_{\omega} \tag{30}$$

where $\langle A,B\rangle_{\omega}$ coincides with conventional determination of the generalized susceptibility.

We will use below the following notations:

$$G_{a_{\mathbf{q}},a^{+}_{\mathbf{q}}} = G_{\mathbf{q}}, \quad G_{a_{\mathbf{q}\pm\mathbf{k}},a^{+}_{\mathbf{q}}} = G_{\pm 1},$$

$$F_{a^{+}_{-\mathbf{q}},a^{+}_{\mathbf{q}}} = F_{\mathbf{q}}, \quad F_{a^{+}_{-\mathbf{q}\mp\mathbf{k}},a^{+}_{\mathbf{q}}} = F_{\pm 1}. \tag{31}$$

Neglecting the umklapp interaction we have

$$G_{\mathbf{q}}(\omega) = \frac{E_{\mathbf{q}}+\omega}{\omega^2-\epsilon^2_{\mathbf{q}}}; \quad F_{\mathbf{q}}(\omega) = -\frac{B_{\mathbf{q}}}{\omega^2-\epsilon^2_{\mathbf{q}}}, \tag{32}$$

where $E_{\mathbf{q}}$, $B_{\mathbf{q}}$, and $\epsilon_{\mathbf{q}}$ are given by Eqs. (21)–(24). For $H_{\|}>H_c$ the function F is zero.

Using Eq. (29) we obtain infinite set of equations which contain along with conventional Green functions G and F the functions $G_{\pm n\mathbf{k}}$ and $F_{\pm n\mathbf{k}}$ where $n=1,2,\ldots$. Below we consider mainly $n=1$ approximation. Its validity will be discussed later. As a result we obtain six linear equations for functions (31) (see Appendix A). In general from their solution is very complicated. So we will consider two limiting cases: $q\to 0$ and $q\gtrsim k$.

The $\mathbf{q}\to 0$ *case*. Corresponding equations are analyzed in Appendix B. Neglecting the dipolar interaction we obtain the final result for weak perpendicular field in the form

$$G_{\mathbf{q}}(\omega) = \frac{(Ak^2/2)\cos^2\alpha+\omega}{\omega^2-\epsilon^2_{\mathbf{q}}+(h^2_{\perp}/2)\cos^4\alpha},$$

$$F_{\mathbf{q}}(\omega) = -\frac{(Ak^2/2)\cos^2\alpha}{\omega^2-\epsilon^2_{\mathbf{q}}+(h^2_{\perp}/2)\cos^4\alpha}, \tag{33}$$

where $h^2_{\perp}=(g\mu_B)^2(H^2_a+H^2_b)$. We neglected here the small terms proportional to $h^2_{\perp}$ in numerators as well as small contribution of the mode with the energy $\epsilon\simeq\epsilon_{\mathbf{k}}$ (see Appendix B). The functions $G_{\pm}$ and $F_{\pm}$ are of order of $h_{\perp}$ and are evaluated in Appendix B also. In these expressions the square of the spin-wave energy in the transverse field at $\mathbf{q}=\mathbf{0}$ is given by Eq. (B5)

$$\epsilon^2_0(H_{\perp}) = \Delta^2 - (h^2_{\perp}/2)\cos^4\alpha, \tag{34}$$

where we assume existence of the spin-wave gap Δ. Its origin will be considered in Sec. (7). For the gapless spin waves $\epsilon^2_0(H_{\perp})$ is negative for any $H_{\perp}$ and magnetic subsystem is unstable. It means that the helix axis $\mathbf{k}$ must turn and stand along the field. For finite gap Δ, the spin-wave spectrum remains stable up to $h_{\perp}=g\mu_B H_1=\Delta\sqrt{2}$ and then the first order transition occurs to the parallel state. In the intermediate case, when $H_{\|}\neq 0$ there is rather complex behavior governed by the equation of state derived in the next section. Similar situation takes place in conventional antiferromagnets: If the field is along the sublattice magnetization the first order spin-flop transition occurs at $g\mu_B H=\Delta$ where Δ is the spin-wave gap. For the inclined field there is rather complicated behavior consisting of the sublattice rotation accompanied by the first order transition.[24]

Rotation of the helix axis was observed in[9] FeGe and in MnSi.[10,11] In both cases H_1 is much less than H_c. For MnSi the transition to parallel state happens at $H_{\perp}\sim 0.1$ T and $H_c\simeq 0.6$ T. It is the reason for the $n=1$ approximation (see discussion at the end of Appendix B).

The $q\gtrsim k$ *case*. As above we consider the first-order

umklapps connecting $\mathbf{q}$ and $\mathbf{q}\pm\mathbf{k}$ and restrict ourself to $\mathbf{q}\perp\mathbf{k}$ case where the dipolar umklapps are absent [see Eq. (28)]. We put also $\mathbf{H}=0$ and neglect the spin-wave gap Δ. Then we discuss briefly the role of small $\mathbf{q}_\parallel$. We demonstrate also that $n=2$ approximation does not change qualitatively the $n=1$ results.

If $\mathbf{q}\perp\mathbf{k}$ we have two interacting modes with $\mathbf{q}$ and $\mathbf{q}\pm\mathbf{k}$ which energies according to Eq. (24) are given by

$$\epsilon_{\mathbf{q}} = Aq_\perp(q_\perp^2 + k^2)^{1/2}$$

$$\epsilon_{\mathbf{q}\pm k} = A[(q_\perp^2 + k^2)(q_\perp^2 + 2k^2)]^{1/2}. \tag{35}$$

The excitation $\epsilon_{\mathbf{q}}$ is the gapless and linear in $\mathbf{q}$ at $q \ll k$. The gap of the double degenerated mode is equal to $Ak^2\sqrt{2}$. The umklapp interaction lifts this degeneracy (see Appendix C) and we have three different modes. One of them remains nonrenormalized and coincides with the gapped mode in Eq. (35) and two other are given by

$$\epsilon_\pm = Ak^2\left(1 + \frac{4q_\perp^2}{k^2} + \frac{q_\perp^4}{k^4} \pm \sqrt{1 + \frac{8q_\perp^2}{k^2} + \frac{17q_\perp^4}{k^4} + \frac{8q_\perp^6}{k^6}}\right)^{1/2}, \tag{36}$$

where the plus mode has the same gap as in Eq. (34) but other dependence on $q_\perp$.

The nonrenormalized mode is irrelevant as corresponding expressions for the Green functions given by Eqs. (C5)) do not have poles at this energy.

The gapless mode is strongly renormalized and at small $q_\perp$ we have quadratic dispersion with $\epsilon_- = Aq_\perp^2/2$. At the same time for $\mathbf{q}=q_\parallel$ the energy is given by Eq. (24): $\epsilon_{\mathbf{q}_\parallel} = Aq_\parallel(q_\parallel^2 + k^2)^{1/2}$. We see that umklapps give rise to strong anisotropy of the spin-wave energy in $\mathbf{q}$ space: the excitations with $\mathbf{q}$ along and perpendicular to the helix vector $\mathbf{k}$ have different energies. As was shown in Sec. III the vector $\mathbf{k}$ is directed along the cubic diagonal or edge depending on the sign of the anisotropic exchange. So really we have anisotropy of the spin-wave spectrum with respect to the crystal axes. This anisotropy disappears if $H > H_c$. We do not consider here its evolution at lower fields.

General expression for arbitrary directed $\mathbf{q}$ is very complex. But it becomes simple for small $q_{\perp,\parallel} \ll k$. In this case from Eq. (C3) we get

$$\epsilon_- = A[q_\perp^4/2 + k^2q_\parallel^2]^{1/2}. \tag{37}$$

Cancellation of the $q_\perp^2$ term in the expression for ϵ_-^2 is the most striking feature of $n=1$ approximation. So we have to discuss its validity. Using the same argumentation as in Appendix B one can show that in the case of the gapless mode if $q_\perp \ll k$ any further approximation affects the higher powers of $q_\perp$ [$q_\perp^2$ ($n=1$), $q_\perp^4$ ($n=2$), $q_\perp^6$ ($n=3$), and so on]. In Appendix C this statement is confirmed in $n=2$ approximation and instead of Eq. (37) we have

$$\epsilon_- = A[3q_\perp^4/8 + k^2q_\parallel^2]^{1/2}. \tag{38}$$

At the same time in both approximations the value of the gap equal to $Aq_\perp^2\sqrt{2}$ remains unchanged in the gapped modes given by Eqs. (36) and (37) [see Eq. (C8)].

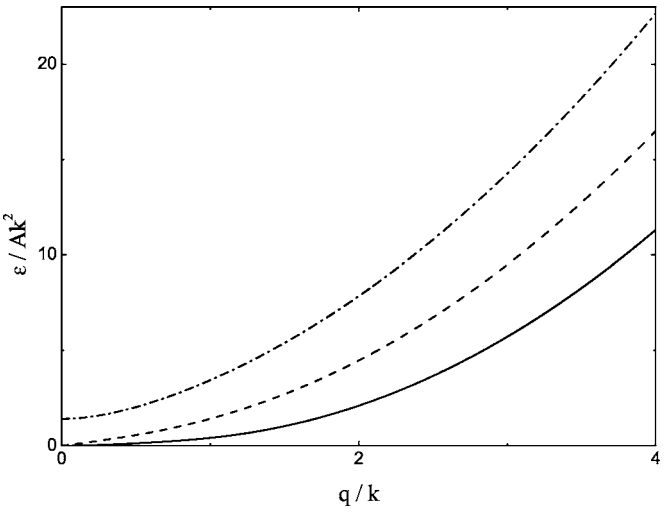

FIG. 1. Spin-wave dispersion for different directions of the wave-vector $\mathbf{q}$. The vector $\mathbf{q}$ is along the helix axis $\mathbf{k}$ (dashed line), gapless and gapped branches for $\mathbf{q}\perp\mathbf{k}$ (solid and dot-dashed lines respectively). Three curves do not merge at $q \gg k$ due to corrections to the asymptotic law $\epsilon_{\mathbf{q}} = Aq^2$ [see Eqs. (34), (35), (C4), and (C10)].

Similar argumentation takes place at $q_\perp \gg k$ and for $\omega \simeq Aq^2$ we have $1/q_\perp$ expansion. All three modes have the same asymptotic behavior Aq^2 and the $n=1$ approximation gives correctly the first $1/q_\perp$ correction to energies of both $\pm$ modes [see Eqs. (C4) and (C10)]. Unfortunately at $q_\perp \sim k$ and $\omega \sim Ak^2$ the results of $n=1$ approximation are qualitatively correct only. The $n=1$ umklapp renormalization is shown in Fig. 1 where the solid and dot-dashed lines correspond to minus and plus modes determined by Eq. (36) and the dashed line is the gapless mode with $\mathbf{q}\parallel\mathbf{k}$.

VI. SPIN CONFIGURATION IN PERPENDICULAR MAGNETIC FIELD

The terms linear in operators a and a^+ cancel in the direct part of the Hamiltonian given by Eqs. (16)–(19) due to equilibrium conditions for the classical energy as is explained in Sec. IV. The interaction with perpendicular field contains operators $a_{\pm\mathbf{k}}$ and $a^+_{\mp\mathbf{k}}$ but we have no conditions for their cancellation. For this part of the interaction from (25) we get

$$H_Z^U = P\left(\frac{S}{2}\right)^{1/2}[-a_{-\mathbf{k}}(1 - \sin\alpha) + a^+_{\mathbf{k}}(1 + \sin\alpha)] + \text{H.c.} \tag{39}$$

This expression may be considered as perturbation which gives additional contribution to the ground-state energy and nonzero average values of the transverse spin components $\langle S^{\xi,\eta}_{\mathbf{K}}\rangle$. But we will use more general method, which leads in the first approximation to the same results. We consider operators $a_{\pm\mathbf{k}}$ and $a^+_{\pm\mathbf{k}}$ as c numbers. Corresponding c number terms appear in Eqs. (20) and (27) and we obtain contribution to the ground-state energy in the following form:

$$E_1 = P\sqrt{S/2}[a_{\mathbf{k}}^*(1+s) - a_{-\mathbf{k}}(1-s)] + [Pc - R_0(1+s)]a_{\mathbf{k}}^* a_0[Pc + R_0(1-s)] \times a_0^* a_{-\mathbf{k}} + R_0[a_0 a_{-\mathbf{k}}(1-s) - a_0^* a_{\mathbf{k}}^*(1+s)] + \text{H.c.} + E_{\mathbf{k}}(|a_{\mathbf{k}}|^2 + |a_{-\mathbf{k}}|^2) + B_{\mathbf{k}}(a_{\mathbf{k}} a_{-\mathbf{k}} + a_{\mathbf{k}}^* a_{-\mathbf{k}}^*) + E_0|a_0|^2 + B_0(a_0^2 + a^{*2})/2, \tag{40}$$

where $s=\sin\alpha$, $c=\cos\alpha$, $R_0=(S/2)\omega_0 N_{cA}\cos\alpha$, and in the classical limit we have $a^+=a^*$.

The energy E_1 is minimal at the obvious equilibrium conditions $\partial E_1/\partial a_{0,\pm\mathbf{k}}^{(*)}=0$ which lead again in the $n=1$ approximation to six linear equations. Corresponding matrix coincides with M given by Eq. (B2) at $\omega=0$ and we get equation

$$M(0)U = V, \tag{41}$$

where two columns are given by

$$U = (a_0, a_0^*, a_{-\mathbf{k}}, a_{\mathbf{k}}^*, a_{\mathbf{k}}, a_{-\mathbf{k}}^*),$$

$$V = (0, 0, -P^*\sqrt{S/2}(1-s), -P^*\sqrt{S/2}(1+s), P\sqrt{S/2}(1+s), P\sqrt{S/2}(1-s)). \tag{42}$$

For $R_0=0$ solution of Eq. (41) is given in Appendix B and using the equilibrium conditions we get

$$E_1 = \frac{P}{2}\sqrt{\frac{S}{2}}[(a_{\mathbf{k}} + a_{\mathbf{k}}^*)(1+s) - (a_{-\mathbf{k}} + a_{-\mathbf{k}}^*)(1-s)] + \text{c.c.}, \tag{43}$$

and for the field depended part of the ground-state energy we obtain

$$E = -\frac{Sh_\parallel^2}{2h_c} - \frac{Sh_\perp^2\Delta^2}{2Ak^2(1+\cos^2\alpha)[\Delta^2 - (h_\perp^2/2)\cos^4\alpha]}, \tag{44}$$

where the first term is the magnetic part of the classical energy (10) for $H<H_c$.

The helix axis has to rotate at $H_\perp \sim H_1$. According to experimental data.[9,11] this rotation was observed at $H_\perp \ll H_c$. So we can put $\cos^2\alpha=1$ and obtain

$$E = \frac{Sh^2}{2Ak^2}\left[-\cos^2\Psi - \frac{\sin^2\Psi}{2[1 - h^2\sin^2\Psi/(2\Delta^2)]}\right], \tag{45}$$

where Ψ is the angle between the helix axis $\mathbf{k}$ and the field. For small $h_\perp$ this energy is minimal at $\Psi=0$, i.e., for the field along $\mathbf{k}$. The real direction of the helix axis is determined by competition between the magnetic energy and anisotropic exchange as well as the cubic anisotropy considered in the next section [see Eq. (51)]. Equation (45) is not valid if $H_\perp$ close to H_1 as it was derived in the linear approximation and the amplitudes a and a^* given by (B7) should be small.

The perpendicular field deforms the helix structure. The higher harmonics and the uniform magnetization along $\mathbf{H}_\perp$ appear. In the $n=1$ approximation the static contribution to the transverse spin components have the form

$$\mathbf{S}_{\mathbf{R}} = i\mathbf{A}e^{i\mathbf{k}\cdot\mathbf{R}}(S_{\mathbf{k}}^\eta e^{i\mathbf{k}\cdot\mathbf{R}} + S_{-\mathbf{k}}^\eta e^{-i\mathbf{k}\cdot\mathbf{R}}] + \text{c.c.}, \tag{46}$$

where $iS_{\pm\mathbf{k}}^\eta = \sqrt{S/2}(a_{\pm\mathbf{k}} - a_{\mp\mathbf{k}}^*)$ and we put $\alpha=0$. From this expression using Eqs. (B7) we obtain the transverse uniform magnetization[25]

$$\mathbf{S}_U^\perp = -\frac{S\mathbf{h}_\perp\Delta^2}{2Ak^2(\Delta^2 - h_\perp^2/2)}. \tag{47}$$

The second harmonic of the helix structure induced by the perpendicular field is given by

$$\mathbf{S}_{\mathbf{R}}^\perp = -\frac{S\Delta^2}{Ak^2(\Delta^2 - h_\perp^2/2)}[\mathbf{A}(\mathbf{A}\cdot\mathbf{h})e^{2i\mathbf{k}\cdot\mathbf{R}} + \mathbf{A}^*(\mathbf{A}^*\cdot\mathbf{h})e^{-2i\mathbf{k}\cdot\mathbf{R}}]. \tag{48}$$

As we have mentioned above corresponding second-order Bragg reflections were observed in Refs. 9 and 11.

VII. CUBIC ANISOTROPY AND THE GAP PROBLEM

We postulated above existence of the spin-wave gap. Now we demonstrate that there are at least two contributions to the square of the gap: cubic anisotropy and interaction between spin-waves considered in the Hartree-Fock (HF) approximation. The former may have arbitrary sign and the latter is positive. So different contributions to the gap may compete. Changing the sign and strength of the cubic anisotropy for example by pressure one can get the quantum phase transition from the ordered to the spin-liquid state. It is a possible explanation of such transition observed in MnSi.[2,10,26]

Expression for the gap at arbitrary $H_\parallel$ is very complicated. So we present here results for $H=0$ and $H>H_c$. We begin with the cubic anisotropy. The single ion cubic anisotropy has the form

$$V = K\sum_{\mathbf{R}}(S_{x,\mathbf{R}}^4 + S_{y,\mathbf{R}}^4 + S_{z,\mathbf{R}}^4). \tag{49}$$

Using Eqs. (4), (14), and (15) for zero magnetic field ($\alpha=0$) we obtain

$$V = 6S^4K\sum_{\nu=x,y,z}|A_\nu|^4 + 6S^3K\sum_{\mathbf{q},\nu=x,y,z}\{2|A_\nu|^2\hat{c}_\nu^2 a_{\mathbf{q}}^+ a_{\mathbf{q}} + [|A_\nu|^2\hat{c}_\nu^2 - 2|A_\nu|^4](a_{\mathbf{q}}a_{-\mathbf{q}} + a_{\mathbf{q}}^+ a_{-\mathbf{q}}^+)\}, \tag{50}$$

and for the part of the classical energy which depends on the helix orientation we have

$$E_{cl} = \frac{S^2F}{4}\sum_{\nu=x,y,z}k_\nu^2(\hat{a}_\nu^2 + \hat{b}_\nu^2) + \frac{3S^4K}{8}\sum_{\nu=x,y,z}(\hat{a}_\nu^2 + \hat{b}_\nu^2)^2, \tag{51}$$

where the first term is the AEI contribution determined by Eq. (10).

The spin-wave terms in Eq. (50) give additional contributions to $E_{\mathbf{q}}$ and $B_{\mathbf{q}}$ in Eq. (20) which are given by

$$\delta E_{\mathbf{q}} = 12S^3K\sum\hat{c}_\nu^2|A_\nu|^2, \quad \delta B_{\mathbf{q}} = 12S^3K\sum(\hat{c}_\nu^2|A_\nu|^2 - 2|A_\nu|^4), \tag{52}$$

and for the anisotropic part of the spin-wave gap we have

$$\Delta_{cub}^2 = \frac{3}{2} S^3 K h_c \sum_{\nu=x,y,z} (\hat{a}_\nu^2 + \hat{b}_\nu^2)^2, \tag{53}$$

where $h_c = E_0 + B_0$ is given by Eq. (13). This equation for the gap holds at $\mathbf{q}=0$ when $\hat{q}_c^2 \to N_{cc}$. For small $\mathbf{q}$ we must replace h_c by $h_c^{Int} + S\omega_0 \hat{q}_c^2$. So at $\mathbf{q}=0$ and $\mathbf{q} \neq 0$ the gap depends on the sample form and angle between $\mathbf{q}$ and the helix axis, respectively. We see that Δ_{cub}^2 is positive for $K>0$ only. In ferromagnets this sign of K corresponds to the easy directions along the cubic diagonals.

Let us consider now the classical energy and the gap for orientations of the helix vector $\mathbf{k}$ along [1,1,1], [1,0,0], and [1,1,0] directions labeled as 1, 2, and 3, respectively. We can choose $\hat{c}_1 = (1,1,1)/3^{1/2}$, $\hat{a}_1 = (1,-1,0)/2^{1/2}$, and $\hat{b}_1 = (1,1,-2)/6^{1/2}$; $\hat{c}_2 = (1,0,0)$, $\hat{a}_2 = (0,1,0)$, and $\hat{b}_2 = (0,0,1)$; $\hat{c}_3 = (1,1,0)/2^{1/2}$, $\hat{a}_3 = (1,-1,0)/2^{1/2}$, and $\hat{b}_3 = (0,0,1)$. For the classical energy we have

$$E_{cl[1,1,1]} = \frac{S^2 F k^2}{6} + \frac{S^4 K}{2}, \quad E_{cl[1,0,0]} = \frac{3S^4 K}{4},$$
$$E_{cl[1,1,0]} = \frac{S^2 F k^2}{8} + \frac{9S^4 K}{16}. \tag{54}$$

From these expressions we see that structures (1,1,1) is realized if $S^2 F k^2 < 3S^4 K/2$. Otherwise we have (1,0,0) structure. The [1,1,0] structure is impossible as two conditions $E_{cl3} < E_{cl1}$ and $E_{cl3} < E_{cl2}$ contradict one another. These expressions are derived at $T=0$. However interplay between two anisotropy energies must determine the helix direction at all T and explain the transition from (1,0,0) to (1,1,1) structure in FeGe with decreasing T.[9]

Corresponding expressions for the gap are given by

$$\Delta_{cub[1,1,1]}^2 = 4S^3 K h_c, \quad \Delta_{cub[1,0,0]}^2 = 6S^3 K h_c,$$
$$\Delta_{cub[1,1,0]}^2 = (9/2) K S^3 h_c. \tag{55}$$

We consider now the contribution to the gap appearing as a result of the interaction between spin waves. It is well known that the total spin conservation law forbids the spin-wave gap in Heisenberg magnets. The DMI violates this law and the gap appears by the same way as in the case of the pseudodipolar interaction in CuO_2 planes in cuprates[27] and dipolar interaction in ferromagnets.[28] The four and three spin-wave interactions can contribute to the gap. Both are examined tn Appendix D. The former is considered in the Hartree-Fock (HF) approximation and the latter as the second-order perturbation. It is zero due to special structure of the triple interaction. From Eqs. (53) and (D9) we obtain the final result

$$\Delta^2 = \frac{3S^3 K h_c}{2} \sum (\hat{a}_\nu^2 + \hat{b}_\nu^2)^2 + \frac{A k^2 h_c}{4SN} \sum \frac{D_q}{D_0}. \tag{56}$$

In strong field $H > H_c$ we have $\cos\alpha = 0$, the spin-wave interaction does not contribute to the gap and

$$E_{cl} = \frac{S\omega_0 N_{cc}}{2} - S h_\parallel + S^4 K \sum \hat{c}_\nu^4,$$
$$\Delta = h_\parallel - h_c + 4S^3 K \sum \left[-\hat{c}_\nu^4 + \tfrac{3}{2} \hat{c}_\nu^2 (\hat{a}_\nu^2 + \hat{b}_\nu^2) \right]. \tag{57}$$

For the cases listed above the contributions of the cubic anisotropy to the classical energy and to the gap are given by

$$\delta E_{cl[1,1,1]} = \frac{KS^4}{3}, \quad \delta E_{cl[1,0,0]} = KS^4, \quad \delta E_{cl[1,1,0]} \frac{KS^4}{2},$$
$$\Delta_{[1,1,1]} = \frac{8KS^3}{3}, \quad \Delta_{[1,0,0]} = -4KS^3, \quad \Delta_{[1,1,0]} = KS^3. \tag{58}$$

VIII. EPR AND NEUTRON SCATTERING

Theory of the polarized neutron elastic scattering from helical magnetic structures was developed many years ago and confirmed experimentally for MnSi[2,11,29] and FeGe.[9] Corresponding expression for the second harmonic scattering in the perpendicular field may be easily evaluated by standard method[30] using Eq. (48).

The EPR and the inelastic magnetic neutron scattering are described by the spin susceptibility. We outline here some of its features. For simplicity we use the $n=1$ approximation.

Using definition (30) we obtain the following general expression for the susceptibility:

$$\begin{aligned}\chi_{\alpha\beta}(\mathbf{q},\omega) = {} & \langle S_{\mathbf{q}}^c, S_{-\mathbf{q}}^c \rangle_\omega \hat{c}_\alpha \hat{c}_\beta + \langle S_{\mathbf{q}}^A, S_{-\mathbf{q}}^{A^*} \rangle_\omega A_\alpha A_\beta^* \\ & + \langle S_{\mathbf{q}}^{A^*}, S_{-\mathbf{q}}^A \rangle_\omega A_\alpha^* A_\beta + \langle S_{\mathbf{q}}^c, S_{-\mathbf{q}}^A \rangle_\omega \hat{c}_\alpha A_\beta \\ & + \langle S_{\mathbf{q}}^A, S_{-\mathbf{q}}^c \rangle_\omega A_\alpha \hat{c}_\beta + \langle S_{\mathbf{q}}^c, S_{-\mathbf{q}}^{A^*} \rangle_\omega \hat{c}_\alpha A_\beta^* \\ & + \langle S_{\mathbf{q}}^{A^*}, S_{\mathbf{q}}^c \rangle_\omega A_\alpha^* \hat{c}_\beta + \langle S_{\mathbf{q}}^A, S_{-\mathbf{q}}^A \rangle_\omega A_\alpha A_\beta \\ & + \langle S_{\mathbf{q}}^{A^*}, S_{-\mathbf{q}}^{A^*} \rangle_\omega A_\alpha^* A_\beta^*, \end{aligned} \tag{59}$$

where $\langle A, B \rangle_\omega$ is the generalized susceptibility $\chi_{AB}(\omega)$ and operators $S_{\mathbf{q}}^\lambda$ are determined by Eq. (14). The first three terms describe the direct processes. All others give the umklapp contribution. We will use below the linear spin-wave theory and consider two limiting cases $H=0$ and $H > H_c$. The main attention will be paid to direct part of the susceptibility. The umklapp part will be discussed briefly in the end of this section.

Zero-field case. There are two contributions to the susceptibility: along $\mathbf{k}$ and in the *ab* plane. For the first using Eqs. (14) and (31) we have[31]

$$\chi_{cc}(\mathbf{q},\omega) = -(S/2)[G_{\mathbf{q}}(\omega) + G_{\mathbf{q}}(-\omega) + 2F_{\mathbf{q}}(\omega)]. \tag{60}$$

In the EPR case when $\mathbf{q}=0$

$$\chi_{cc}(\omega) = -\frac{S\Delta^2}{h_c(\omega^2 - \Delta^2)}. \tag{61}$$

This susceptibility describes responds to the external ac field. Connection between external and intrinsic susceptibilities is given by well known equation (see for example Ref. 32)

$$\chi_{\alpha\beta} = \chi_{\alpha\beta}^{Int} - 4\pi\omega_0 \chi_{\alpha\mu}^{Int} N_{\mu\nu} \chi_{\mu\beta}, \tag{62}$$

and for χ_{cc}^{int} we obtain the same equation as for χ_{cc} with replacement h_c by h_c^{Int} in Eq. (61) and in the expression for

Δ^2. It should be mentioned that there has to be very small contribution of the ϵ_1 mode (more precisely ϵ_+ mode). It appears if in Eq. (B9) we retain terms proportional to $(E_0 - B_0)|R|^2 = \Delta^2|R|^2/(E_0+B_0)$. Moreover in $n=2$ approximation the ϵ_2 mode has to appear also. But we neglect both these contributions.

For $q \gtrsim k$ the susceptibility is strongly anisotropic in q space. For $\mathbf{q} \| \mathbf{k}$ the umklapp interaction is zero and we have

$$\chi_{cc}(q_\parallel, \omega) = -\frac{SAq_\parallel^2}{\omega^2 - \epsilon_{q_\parallel}^2}, \tag{63}$$

where $\epsilon_{q_\parallel}^2 = Aq_\parallel^2[A(q_\parallel^2 + k^2) + S\omega_0 \hat{q}_\parallel^2]$.

With increasing of $\mathbf{q}_\perp$ three different modes appear. However at $q_\parallel = 0$ there are two modes (see Appendix C) given by Eqs. (36) and using Eqs. (5) we get

$$\chi_{cc}(\mathbf{q}_\perp, \omega) = -\frac{SAq_\perp^2}{\epsilon_+^2 - \epsilon_-^2}\left[\frac{\epsilon_+^2 - \epsilon_1^2 + A(2q_\perp^2 + k^2)}{\omega^2 - \epsilon_+^2} + \frac{\epsilon_1^2 - \epsilon_-^2 - A(2q_\perp^2 + k^2)}{\omega^2 - \epsilon_-^2}\right], \tag{64}$$

where ϵ_1 is given by the second line in Eq. (35). In the $n=2$ approximation additional modes appear and so on. However amplitudes of these modes decrease with n at least for $\mathbf{q}_\perp \gg \mathbf{k}$ (see the end of Appendix B).

Consider now the second two terms in Eq. (59). By the same way as above we get

$$\chi_{\alpha\beta}^{AA^*}(\mathbf{q}, \omega) = (\delta_{\alpha\beta} - \hat{c}_\alpha \hat{c}_\beta)\chi_{\perp \mathbf{q}}(\omega) - \frac{i}{2}\epsilon_{\alpha\beta\gamma}\hat{c}_\gamma C_{\mathbf{q}}(\omega), \tag{65}$$

where we put $\hat{a}_\alpha \hat{a}_\beta + \hat{b}_\alpha \hat{b}_\beta = \delta_{\alpha\beta} - \hat{c}_\alpha \hat{c}_\beta$ and $\hat{a}_\alpha \hat{b}_\beta - \hat{a}_\beta \hat{b}_\alpha = \epsilon_{\alpha\beta\gamma}\hat{c}_\gamma$. The susceptibility now acquire antisymmetric or chiral part. It is a result of the DMI.[30] Perpendicular and chiral susceptibilities are determined as

$$\chi_{\perp \mathbf{q}} = \chi_{\mathbf{q}+\mathbf{k}}(\omega) + \chi_{\mathbf{q}-\mathbf{k}}(\omega), \quad C_{\mathbf{q}}(\omega) = \chi_{\mathbf{q}+\mathbf{k}}(\omega) - \chi_{\mathbf{q}-\mathbf{k}}(\omega), \tag{66}$$

where

$$\chi_{\mathbf{Q}}(\omega) = \frac{S}{8}[2F_{\mathbf{Q}}(\omega) - G_{\mathbf{Q}}(\omega) - G_{\mathbf{Q}}(-\omega)]. \tag{67}$$

Chiral part of the susceptibility is now $\mathbf{q}$ odd as it should be in the case of the DMI and $\mathbf{H}=0$.[30] Functions $\mathrm{Im}\,\chi_\perp$ and $\mathrm{Im}\,C$ determine parts of the neutron scattering cross section independent on the neutron polarization $\mathbf{P}_0$ and proportion to it respectively (see for example Ref. 30).

In the EPR case when $\mathbf{q}=\mathbf{0}$ the susceptibility is determined by the Green functions (32) at $\mathbf{Q}=\mathbf{k}$ as according to Eq. (28) the umklapp interaction is zero. The singular terms $\hat{q}_c^2$ and $\mathbf{q}_\perp^2/|\mathbf{q},\mathbf{k}|^2$ in Eqs. (20), (22), and (24) must be replaced by N_{cc} and $N_\perp/2 = N_{aa} + N_{bb}$, respectively [see explanation below Eq. (24)]. As a result we obtain

$$\chi_{\alpha\beta}^{AA^*}(\omega) = -\frac{1}{2}(\delta_{\alpha\beta} - \hat{c}_\alpha \hat{c}_\beta)\frac{S(2Ak^2 + S\omega_0 N_{cc})}{\omega^2 - \epsilon_1^2}, \tag{68}$$

where

$$\epsilon_1 = [2(Ak^2)^2 + S\omega_0 Ak^2 + (S\omega_0)^2 N_{cc}N_\perp/2]^{1/2}, \tag{69}$$

and $C=0$.

For $\mathbf{q} \neq 0$ as above we consider two cases: $\mathbf{q} \| \mathbf{k}$ and $\mathbf{q} \perp \mathbf{k}$. In the first case we have

$$\chi_\perp = -\frac{S}{4}\left(\frac{Z_-}{\omega^2 - \epsilon_-^2} + \frac{Z_+}{\omega^2 - \epsilon_+^2}\right), \quad C = -\frac{S}{4}\left(\frac{Z_-}{\omega^2 - \epsilon_-^2} - \frac{Z_+}{\omega^2 - \epsilon_+^2}\right), \tag{70}$$

where $Z_\pm = A(k \pm q_\parallel)^2 + S\omega_0$ and $\epsilon_\pm^2 = A(k \pm q_\parallel)^2[A(k \pm q_\parallel)^2 + Ak^2 + S\omega_0]$.

For $\mathbf{q} \perp \mathbf{k}$ using (C5) we obtain

$$\chi_\perp = -\frac{S}{4(\epsilon_+^2 - \epsilon_-^2)}\left[\frac{Z_+}{\omega^2 - \epsilon_+^2} + \frac{Z_-}{\omega^2 - \epsilon_-^2}\right], \tag{71}$$

where $\epsilon_\pm$ are given by Eq. (35), $Z_\pm = [2A^2q_\perp^2(2k^2 + q_\perp^2)] \pm A(\epsilon_\pm^2 - \epsilon_1^2)(k^2 + q_\perp^2)$ and $C=0$.

We discuss now briefly off-diagonal terms in Eq. (59). They are a result of the umklapp interaction and has to be less than that considered above. At $\mathbf{q}=\mathbf{0}$ the $\hat{c}\mathbf{A}$ and $\mathbf{AA}$ terms are proportional to off-diagonal components $N_{ca(b)}$ of the demagnetization tensor in the first and the second degree, respectively [see Eq. (B10)]. Possibility of their experimental study is a special problem which is beyond the scope of this paper. At $\mathbf{q} \neq \mathbf{0}$ the umklapp terms are very complicated and we do not analyze them here also.

Ferromagnetic state ($H > H_c$) In the longitudinal cc channel there are the excitations of two spin-waves only which are beyond our consideration. By the same way as above for the transverse susceptibility we obtain

$$\chi_{\perp\alpha\beta} = -\frac{S}{2}(\delta_{\alpha\beta} - \hat{c}_\alpha \hat{c}_\beta)\left(\frac{1}{\omega - \epsilon_{\mathbf{q}+\mathbf{k}} - i\delta} - \frac{1}{\omega + \epsilon_{\mathbf{q}-\mathbf{k}} + i\delta}\right) - \frac{iS}{2}\epsilon_{\alpha\beta\gamma}\hat{c}_\gamma\left(\frac{1}{\omega - \epsilon_{\mathbf{q}+\mathbf{k}} - i\delta} + \frac{1}{\omega + \epsilon_{\mathbf{q}-\mathbf{k}} + i\delta}\right), \tag{72}$$

where the spin-wave energy is given by Eq. (23) and the second term represents the chiral (antisymmetric) part of the susceptibility. According to general theory[30] this part appears in presence of an axial vector interaction. In the considered case we have two such interactions: the Zeeman energy and the DMI. Corresponding axial vectors $\hat{c}$ and $\mathbf{H}$ have the same direction and different t parity: $\hat{c}$ and $\mathbf{H}$ are t even and t odd, respectively. As a result we have more complex symmetry properties than that considered in Ref. 30. The simple H case when $\mathrm{Re}\,C$ and $\mathrm{Im}\,C$ are ω odd and even, respectively, is realized at $\mathbf{q} \perp \mathbf{H}$ and $\mathbf{q}=\mathbf{0}$ only.

At $\mathbf{q}=\mathbf{0}$ the last term in (23) contains the demagnetization and $\epsilon_\mathbf{k} = Ak^2 + S\omega_0(N_{aa} + N_{bb})/2 + g\mu_B(H - H_c)$. Experimentally this contribution could be measured using circularly polarized ac field. For $\mathbf{q} \neq \mathbf{0}$ the neutron scattering has to be maximal at $\mathbf{q} = \pm\mathbf{k}$ in contrast to conventional ferromagnets.

IX. SUMMARY AND DISCUSSION

We begin with a short survey of the main results presented above. We have used the following interactions: the conventional isotropic exchange, the DMI, the anisotropic exchange, the magnetic dipolar interaction, and the Zeeman energy. In the classical approximation these interactions determine the form of the helix and the critical field H_c for the transition to the ferromagnetic spin configuration. This field depends on the sample form due to the demagnetization.

The linear spin-wave theory was developed. It was shown that the spin-wave spectrum depends strongly on the magnetic field. At $H>H_c$ we have quadratic spectrum with the gap linearly increasing with the field [see Eq. (23)]. Below H_c the spectrum is gapless. It is strongly anisotropic due to incommensurate helical structure and low symmetry of the DMI. As a result if the momentum $\mathbf{q}$ has a component perpendicular to the helix vector $\mathbf{k}$ umklapp processes appear which connect the spin waves with momenta $\mathbf{q}$ and $\mathbf{q}\pm\mathbf{k}$ which have different energies. For arbitrary $\mathbf{q}$ the spin-wave energy has a very intricate form. It is determined as a solution of infinite set of linear equations connecting the states with $\mathbf{q}$ and $\mathbf{q}\pm n\mathbf{k}$ where $n=1,2,\ldots$. Restriction to $n=1$ leads to six equations which general solution remains very complicated. For $\mathbf{q}\perp\mathbf{k}$ there are two modes given by Eq. (36). One has the gap equal to $Ak^2\sqrt{2}$ where A is the spin-wave stiffness at $q\gg k$. The second mode is the gapless with quadratic dispersion at small q. Both gapless branches (parallel and perpendicular to $\mathbf{k}$) and the gapped one are shown in Fig. 1. At $q\gg k$ all branches have the same asymptotic Aq^2 but different corrections to it. The $n=2$ approximation do not change these results qualitatively and for $q\ll k$ and $H=0$ the spectrum has a simple form given by Eq. (38). Further approximations ($n>2$) cannot change this expression.

The classical energy depends on the field component along the helix axis $\mathbf{k}$ only. However it was shown experimentally that rather weak perpendicular field $H_\perp\ll H_c$ turns the helix along the field.[9,11] This quantum phenomenon is a consequence of the spiral spin structure when the angle between spin and the field depends on the lattice point. As a result we get the umklapps again and the spin-wave spectrum in the gapless case becomes unstable at infinitesimal perpendicular field. So we must suppose existence of the gap Δ and the spectrum is stable if $g\mu_B H_\perp<\Delta\sqrt{2}$ [see Eq. (33)]. This perpendicular field deforms the helix. The magnetization along $\mathbf{H}_\perp$ and the second harmonic of the spin rotation appear [see Eqs. (47) and (48)]. The latter was observed in Refs. 9 and 11.

We considered two contributions to the gap: cubic anisotropy and interaction between spin waves. The first contribution is proportional to the strength K of the cubic anisotropy. The second appears as a result of breaking of the total spin conservation law by the DMI. It is positive and disappears in the ferromagnetic state at $H>H_c$. But in this region there is the field-induced gap determined by Eq. (23).

In ferromagnets the sign of K determines the direction of the easy axis. In our case the helical structure is stable if Δ^2 given by Eq. (56) is positive. Otherwise we have the spin liquid with chiral spin fluctuations appearing due to the DMI.[30] So there is a question: If change sign of Δ^2 is a reason of the transition to the disordered state in MnSi at high pressure? This problem demands further experimental and theoretical study.

The helix structure leads to peculiar features of the ESR and neutron scattering. In antiferromagnets the ESR frequency is equal to the spin wave gap. In the helical systems due to the umklapps along with this frequency there are more higher resonant excitations corresponding to the spin-waves with $\mathbf{q}=n\mathbf{k}$ where $n=1,2,\ldots$. At $\mathbf{H}\neq 0$ the chiral channel appears. It may be observed using circular polarized ac field.

Below H_c several modes depending on the relative orientations of the momentum transfer $\mathbf{q}$ and the helix axis $\mathbf{k}$ can be studied using the neutron scattering. The chiral channel at $\mathbf{H}=\mathbf{0}$ and $\mathbf{q}\neq\mathbf{0}$ can be studied by polarized neutrons. In strong field the ferromagnetic spin configuration is realized but inelastic neutron scattering has to be maximal at $\mathbf{q}=\pm\mathbf{k}$.

Up to now detailed experimental work was done in the case of MnSi compound only. We now compare some of known experimental results obtained at ambient pressure with the predictions of our theory and discuss possibilities of the further experimental studies. The principal parameters are: lattice spacing $a=4.558$ Å, $T_c\simeq 29$ K, $k\simeq 0.035$ Å^{-1}, saturated magnetization $M=0.4\mu_B/a^3\simeq 0,016$ T ($4\pi M=0.20$ T), critical field[10] $H_c=0.5\div 0.6$ T and spin-wave stiffness $A\simeq 52$ meV Å^2.[33] From these data and Eq. (13) we obtain $H_c\simeq Ak^2/(g\mu_B)\simeq 0.55$ T. This value coincides with experimentally observed critical field. For more precise comparison one must measure all parameters including the demagnetization N_{cc} using single sample.

To the best of my knowledge the ESR in MnSi was studied in Ref. 34 only. Several resonances were observed but only one was studied qualitatively as a function of the magnetic field. Its frequency in zero field is equal to 0.93 T. Using Eq. (69) and taking into account that $S\omega_0=4\pi g\mu_B M$ we obtain 0.85 T. The agreement is within the error bars. According to Eq. (23) at $H=H_c$ the frequency is close to $0.6\text{ T}\simeq H_c$ and then increases linearly with H. This behavior coincides with results of Ref. 34 too. Further experimental studies are necessary. First of all it is essential to observe the gap Δ and its dependence on the perpendicular field, which according to Eq. (34) should be $\Delta=[\Delta^2-(1/2)\times(g\mu_B H_\perp)^2\cos^4\alpha]^{1/2}$. The observation of the umklapp resonances at $\epsilon_{2\mathbf{k}}$ and $\epsilon_{3\mathbf{k}}$ would be important also.

Predicted anisotropy of the spin-wave energy must be verified by inelastic neutron scattering.

The theory developed in this paper explains some experimental findings. It was developed in the linear spin-wave approximation. The spin-wave interaction was used for evaluation contribution to the gap only. We also did not evaluate some experimentally observed quantities such as specific heat, temperature dependence of the site magnetization, electrical resistivity, and their dependence on magnetic field. Corresponding results will be published elsewhere.

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APPENDIX A: EQUATIONS OF MOTION

Using Eqs. (27), (29), and (31) in the $n=1$ approximation for the Green functions (30) we obtain

$$(\omega - E)G - BF - [Pc + R_{-\mathbf{q}}(1-s) - R_{\mathbf{q}-\mathbf{k}}(1+s)]G_- + (R_{-\mathbf{q}} + R_{\mathbf{q}-\mathbf{k}})(1+s)F_- - [P^*c + R^*_{-\mathbf{q}-\mathbf{k}}(1-s) - R^*_{\mathbf{q}}(1+s)]G_+ - (R^*_{\mathbf{q}} + R^*_{-\mathbf{q}-\mathbf{k}})(1-s)F_+ = 1,$$

$$BG + (\omega + E)F + (R_{-\mathbf{q}} + R_{\mathbf{q}-\mathbf{k}})(1-s)G_- + [Pc + R_{\mathbf{q}-\mathbf{k}}(1-s) - R_{-\mathbf{q}}(1+s)]F_- - (R^*_{\mathbf{q}} + R^*_{-\mathbf{q}-\mathbf{k}})(1+s)G_- + [P^*c + R^*_{\mathbf{q}}(1-s) - R^*_{-\mathbf{q}-\mathbf{k}}(1+s)]F_+ = 0,$$

$$-[P^*c + R^*_{-\mathbf{q}}(1-s) - R^*_{\mathbf{q}-\mathbf{k}}(1+s)]G - (R^*_{-\mathbf{q}} + R^*_{\mathbf{q}-\mathbf{k}})(1-s)F + (\omega - E_-)G_- - B_-F_- = 0,$$

$$-(R^*_{-\mathbf{q}} + R^*_{\mathbf{q}-\mathbf{k}})(1+s)G + [P^*c + R^*_{\mathbf{q}-\mathbf{k}}(1-s) - R^*_{-\mathbf{q}}(1+s)]F + B_-G_- + (\omega + E_-)F_- = 0,$$

$$-[Pc + R_{-\mathbf{q}-\mathbf{k}}(1-s) - R_{\mathbf{q}}(1+s)]G + (R_{\mathbf{q}} + R_{-\mathbf{q}-\mathbf{k}})(1+s)F + (\omega - E_+)G_+ - B_+F_+ = 0,$$

$$(R_{\mathbf{q}} + R_{-\mathbf{q}-\mathbf{k}})(1-s)G + [Pc + R_{\mathbf{q}}(1-s) - R_{-\mathbf{q}-\mathbf{k}}(1+s)]F + B_+G_+ + (\omega + E_+)F_+ = 0, \quad \text{(A1)}$$

where $s=\sin\alpha$, $c=\cos\alpha$, $E(B)=E(B)_{\mathbf{q}}$, $E(B)_{\pm}=E(B)_{\mathbf{q}\pm\mathbf{k}}$, and functions E and B are determined by Eqs. (21) and (22).

In matrix form these equations are given by

$$M(\mathbf{q},\omega) = I, \quad \text{(A2)}$$

where column $I=(1,0,0,0,0,0)$. Determinant of the matrix M is even function of ω and has the following general form:

$$\mathrm{Det}[M] = (\omega^2 - \epsilon^2_{\mathbf{q}0})(\omega^2 - \epsilon^2_{\mathbf{q}+})(\omega^2 - \epsilon^2_{\mathbf{q}-}). \quad \text{(A3)}$$

As $\mathrm{Det}[M]$ is the denominator in the expressions for the Green functions they have three poles at the renormalized spin-wave energies. However at $\mathbf{q}\perp\mathbf{k}$ two initial spin-wave energies are equal and only two renormalized branches are physically relevant. Hence the Green functions must have two poles. That it is the case is demonstrated below in Appendixes B and C. There are five renormalized branches in the $n=2$ approximation (see Appendix C) and so on.

General expressions for $\epsilon^2_{\mathbf{q}0}$ and $\epsilon^2_{\mathbf{q}\pm}$ are very complex. We consider below two main cases: (i) Small $\mathbf{q}\to 0$ and (ii) $H=0$, $\mathbf{q}\sim\mathbf{k}$.

APPENDIX B: THE q→0 CASE

From Eqs. (21) and (22) neglecting small contribution of the anisotropic exchange we have

$$E_{\mathbf{q}} = Aq^2 + (Ak^2\cos^2\alpha)/2 + (S\omega_0/2)\hat{q}_c^2\cos^2\alpha,$$
$$B_{\mathbf{q}} = (1/2)(Ak^2 + S\omega_0\hat{q}_c^2)\cos^2\alpha,$$
$$E_1 = Ak^2(2+\cos^2\alpha)/2 + (S\omega_0/2) \times [\hat{q}_c^2\cos^2\alpha + (\hat{q}_\perp^2/2)(1+\sin^2\alpha)],$$
$$B_1 = (1/2)[Ak^2 + (S\omega_0/2)(\hat{q}_c^2 - \hat{q}_\perp^2/2)]\cos^2\alpha, \quad \text{(B1)}$$

where $E_1=E_{\mathbf{k}}$ and $B_1=B_{\mathbf{k}}$. From Eq. (28) we get $R_{\mathbf{q}} = (S\omega_0/2)(\hat{c}\cdot\hat{q})(\hat{q}\cdot\mathbf{A})\cos\alpha$ and $R_{\mathbf{k}}=0$. If $\mathbf{q}\equiv\mathbf{0}$ we must replace $\hat{q}_c^2$, $\hat{q}_\perp^2$ and $(\hat{c}\cdot\hat{q})(\hat{q}\cdot\mathbf{A})$ by N_{cc}, $N_{aa}+N_{bb}$, and $(N_{ca} - iN_{cb})/2$, respectively. The matrix M now has the form

$$\begin{pmatrix} \omega - E_{\mathbf{q}} & -B_{\mathbf{q}} & -Pc - R(1-s) & R(1+s) & -P^*c + R^*(1+s) & -R^*(1-s) \\ B_{\mathbf{q}} & \omega + E_{\mathbf{q}} & R(1-s) & Pc - R(1+s) & -R^*(1+s) & P^*c + R^*(1-s) \\ -P^*c - R^*(1-s) & -R^*(1-s) & \omega - E_1 & -B_1 & 0 & 0 \\ -R^*(1+s) & P^*c - R^*(1+s) & B_1 & \omega + E_1 & 0 & 0 \\ -Pc + R(1+s) & R(1+s) & 0 & 0 & \omega - E_1 & -B_1 \\ R(1-s) & Pc + R(1-s) & 0 & 0 & B_1 & \omega + E_1 \end{pmatrix}. \quad \text{(B2)}$$

For the determinant of this matrix using Program Mathematica 5 we obtain

$$\begin{aligned}\mathrm{Det}[M] = (\omega^2-\epsilon_1^2)\{&(\omega^2-\epsilon_1^2)(\omega^2-\epsilon_{\mathbf{q}}^2)-4|P|^2(E_0E_1+B_0B_1\\&-|P|^2\cos^2\alpha+\omega^2)\cos^2\alpha-8|R|^2(E_0-B_0)(E_1+B_1)\\&+4(PR^*+P^*R)[\omega^2+(E_0-B_0)(E_1-B_1)\\&-2|P|^2\cos^2\alpha]\sin\alpha\cos\alpha-8|R|^2(E_0-B_0)(E_1\\&-B_1)\sin^2\alpha+16|P|^2|R|^2\epsilon_1^2\sin^2\alpha\cos^2\alpha\},\end{aligned} \tag{B3}$$

where at $\mathbf{q}=\mathbf{0}$ we have to replace $\epsilon_0^2=E_0^2-B_0^2$ by Δ^2 (see Secs. V and VII, and Appendix D).

We assume that P, R, and Δ are small in comparison with $Ak^2\simeq g\mu_B H_c$ which is the main parameter of the theory. In this approximation we have

$$\begin{aligned}\mathrm{Det}[M] = (\omega^2-\epsilon_1^2)\Bigg[&(\omega^2-\epsilon_1^2)(\omega^2-\Delta^2)-4|P|^2(E_0E_1+B_0B_1\\&+\omega^2)\cos^2\alpha-8|R|^2\Delta^2\frac{E_1(1+\sin^2\alpha)+B_1\cos^2\alpha}{E_0+B_0}\Bigg].\end{aligned} \tag{B4}$$

As $E_{0,1}\sim B_{0,1}\sim Ak^2>S\omega_0$ we can neglect the last term in square brackets if $h_\perp>\Delta S\omega_0 N_{c\perp}/(Ak^2)$ where $N_\perp=(N_a^2+N_b^2)1/2$. In this case we get

$$\begin{aligned}\epsilon_0^2 &= \Delta^2-\frac{h_\perp^2(Ak^2+S\omega_0N_{cc})\cos^4\alpha}{2Ak^2+S\omega_0N_\perp\cos^2\alpha}\simeq\Delta^2-\frac{h_\perp^2\cos^4\alpha}{2},\\ \epsilon_+^2 &= \epsilon_1^2+h_\perp^2\left[1+\frac{(Ak^2+S\omega_0N_{cc})\cos^2\alpha}{2Ak^2+S\omega_0N_\perp\cos^2\alpha}\right]\cos^2\alpha\\ &\simeq\epsilon_1^2+\frac{h_\perp^2(2+\cos^2\alpha)}{2}\cos^2\alpha,\end{aligned} \tag{B5}$$

where ϵ_1 is given by Eq. (69).

Corresponding expressions for the Green functions are given by

$$\begin{aligned}G(\omega) &= Z^{-1}[(\omega+E_0)(\omega^2-\epsilon_1^2)+(h_\perp^2/2)(E_1-\omega)],\\ F(\omega) &= -Z^{-1}B_0[\omega^2-\epsilon_1^2-(h_\perp^2/2)B_1\cos^2\alpha],\\ G_-(\omega) &= Z^{-1}(h_+/2)[\omega^2-\epsilon_1^2+E_0E_1+B_0B_1\\&\quad-(h_\perp^2/2)\cos^2\alpha]\cos\alpha,\\ F_-(\omega) &= -Z^{-1}(h_+/2)[B_0E_1+B_1E_0+\omega(B_1-B_0)]\cos\alpha,\end{aligned} \tag{B6}$$

where $Z=(\omega^2-\epsilon_+^2)(\omega^2-\epsilon_0^2)$, $h_\pm=(h_a\pm ih_b)/2$, and one gets G_+ and F_+ replacing h_+ by h_- in the expressions for G_- and F_-. Neglecting the dipolar interaction we obtain Eq. (33). We see that Green functions do not contain the pole at $\omega^2=\epsilon_1^2$.

In the same approximation solution of Eq. (41) for the spin deviations frozen in the perpendicular field is given by

$$a_{\mathbf{k}} = -\sqrt{\frac{S}{2}}\frac{(\mathbf{A}\cdot\mathbf{h})(1+\cos^2\alpha+\sin\alpha)\Delta^2}{Ak^2(1+\cos^2\alpha)[\Delta^2-(h_\perp^2/2)\cos^4\alpha]},$$

$$a_{-\mathbf{k}} = \sqrt{\frac{S}{2}}\frac{(\mathbf{A}^*\cdot\mathbf{h})(1+\cos^2\alpha-\sin\alpha)\Delta^2}{Ak^2(1+\cos^2\alpha)[\Delta^2-(h_\perp^2/2)\cos^4\alpha]},$$

$$a_0 = a_0^* = \sqrt{\frac{S}{2}}\frac{h_\perp^2(E_0-B_0)\sin\alpha\cos\alpha}{2Ak^2[\Delta^2-(h_\perp^2/2)\cos^4\alpha]}. \tag{B7}$$

The $\mathbf{H}=\mathbf{0}$ *case*. From Eq. (B4) we have now

$$\mathrm{Det}[M] = (\omega^2-\epsilon_1^2)(\omega^2-\epsilon_0^2)(\omega^2-\epsilon_+^2), \tag{B8}$$

where

$$\epsilon_0^2 = \Delta^2\left[1-\frac{(S\omega_0)^2(N_{ca}^2+N_{cb}^2)}{(Ak^2+S\omega_0N_{cc})(2Ak^2+S\omega_0N_\perp)}\right]\simeq\Delta^2,$$

$$\epsilon_+^2 = \epsilon_1^2+\frac{\Delta^2(S\omega_0)^2(N_{ca}^2+N_{cb}^2)}{(Ak^2+S\omega_0N_{cc})(2Ak^2+S\omega_0N_\perp)}\simeq\epsilon_1^2. \tag{B9}$$

Corresponding expressions for the Green functions are given by

$$\begin{aligned}G(\omega) &= Z^{-1}[(\omega+E_0)(\omega^2-\epsilon_1^2)+4|R|^2(E_1+B_1)],\\ F(\omega) &= -Z^{-1}[B_0(\omega^2-\epsilon_1^2)+4|R|^2(E_1+B_1)],\\ G_-(\omega) &= Z^{-1}R^*(\omega+E_0-B_0)(\omega+E_1+B_1),\\ F_-(\omega) &= -Z^{-1}R^*(\omega+E_0-B_0)(E_1+B_1-\omega),\end{aligned} \tag{B10}$$

where $R=(S\omega_0/4)(N_{ca}-iN_{cb})$, $G_+=-(R/R^*)G_-$, and $F_+=-(R/R^*)F_-$. Note that now it is addition sign minus in expressions for G_+ and F_+ in comparison with the $R=0$ case.

Let us discuss now the validity of the $n=1$ approximation. In the sequence of equations for the Green functions at $\mathbf{q}=\mathbf{0}$ and small $\omega\ll Ak^2$ contributions from $\pm n$th terms are of order $\max(P,R_0)/\epsilon_{nk}\sim\max(P,R_0)/(nAk^2)\ll 1$. As a result in the nth approximation determinant $D_n[M]$ is the nth power polynomial of the ratio $[\max(P,R_0)/Ak^2]^2$. Corresponding expression for ϵ_0^2 has the same structure and we can restrict ourselves to $n=1$ approximation. Similar consideration holds in two other cases considered in main body of the paper: (i) $\mathbf{q}_\perp\ll\mathbf{k}$, $\omega\ll Ak^2$, and (ii) $q\gg k$, $\omega\simeq Aq^2$.

APPENDIX C: THE $q\gtrsim k$ CASE

It is convenient now to use dimensionless variables $X=\omega/(Ak^2)$, $Y=|q_\perp|/k$, $V=q_\parallel/k$, and $U=-(q_a-iq_b)/k$. In these variables we have $M=(Ak^2)^6 m$ and for m we get

$$\begin{pmatrix} X-W^2-\frac{1}{2} & -\frac{1}{2} & U & 0 & U^* & 0 \\ \frac{1}{2} & X+W^2+\frac{1}{2} & 0 & U & 0 & U^* \\ U^* & 0 & X-W^2+2V-\frac{3}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & U^* & \frac{1}{2} & X+W^2-2V+\frac{3}{2} & 0 & 0 \\ U & 0 & 0 & 0 & X-W^2-2V-\frac{3}{2} & -\frac{1}{2} \\ 0 & U & 0 & 0 & \frac{1}{2} & X+W^2+2V+\frac{3}{2}, \end{pmatrix}, \tag{C1}$$

where $W^2=Y^2+V^2$.

General expression for Det$[m]$ is very complicated. For $V \ll 1$ we have

$$\mathrm{Det}[m]=(X^2-\epsilon_1^2)(X^2-\epsilon_+^2)(X^2-\epsilon_-^2),$$

$$\epsilon_\pm^2=1+4Y^2+Y^4 \pm (1+8Y^2+17Y^4+8Y^6)^{1/2}+4V^2, \tag{C2}$$

where $\epsilon_1^2=(Y^2+1)(Y^2+2)$ is the energy at $\mathbf{q}_\perp \pm \mathbf{k}$ in the dimensionless units. For small Y and V we have

$$X_-^2=\frac{Y^4}{2}+V^2, \quad X_+^2=2+4Y^2. \tag{C3}$$

Asymptotic expressions for $\epsilon_\pm$ at $q_\perp \gg k$ are given by

$$\epsilon_\pm = Aq_\perp^2 \pm \sqrt{2}kq_\perp. \tag{C4}$$

For the Green functions in dimensionless units we have

$$G(X)=Z^{-1}[(X+Y^2+1/2)(X^2-\epsilon_1^2)+2Y^2(Y^2+3/2-X)],$$

$$F(X)=-Z^{-1}(X^2-\epsilon_1^2+2Y^2)/2,$$

$$G_-(X)=-Z^{-1}(Y_a+iY_b)[-(X+Y^2+1/2)(X+Y^2+3/2)+2Y^2+1/4],$$

$$F_-(X)=Z^{-1}(Y_a+iY_b)(2X-1)/2, \tag{C5}$$

where $Z=(X^2-\epsilon_-^2)(X^2-\epsilon_+^2)$ and $G_+(F_+)=[G_-(F_-)](Y_a-iY_b)/(Y_a+iY_b)$. We see that at $q_\parallel=0$ $(V=0)$ the factor $X^2-\epsilon_1^2$ cancels in the expressions for the Green functions.

To illustrate that $n=1$ results are at least qualitatively correct we consider now the $n=2$ approximation. In this case the matrix $M=(Ak^2)^{10}m$ and for m we have

$$\begin{pmatrix} X_- -\frac{1}{2} & -\frac{1}{2} & U & 0 & U^* & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & X_+ +\frac{1}{2} & 0 & U & 0 & U^* & 0 & 0 & 0 & 0 \\ U^* & 0 & X_- -\frac{3}{2} & -\frac{1}{2} & 0 & 0 & U & 0 & 0 & 0 \\ 0 & U^* & \frac{1}{2} & X_+ +\frac{3}{2} & 0 & 0 & 0 & U & 0 & 0 \\ U & 0 & 0 & 0 & X_- -\frac{3}{2} & -\frac{1}{2} & 0 & 0 & U^* & 0 \\ 0 & U & 0 & 0 & \frac{1}{2} & X_+ +\frac{3}{2} & 0 & 0 & 0 & U^* \\ 0 & 0 & U^* & 0 & 0 & 0 & X_- -\frac{9}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & U^* & 0 & 0 & \frac{1}{2} & X_+ +\frac{9}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & U & 0 & 0 & 0 & X_- -\frac{9}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & U & 0 & 0 & \frac{1}{2} & X_+ +\frac{9}{2} \end{pmatrix}, \tag{C6}$$

where $X_\mp = X \mp Y^2$. The determinant of m is given by

$$\begin{aligned}\mathrm{Det}[m]=&1600X^2-1760X^4+564X^6-44X^8+X^{10}+(7760X^2\\&-4868X^4+524X^6-15X^8)Y^2+(-600+9279X^2\\&-2462X^4+167X^6-3X^8)Y^4+(-585+3686X^2\\&-597X^4+22X^6)Y^6+(-519+866X^2-107X^4\\&+3X^6)Y^8+(-528+128X^2-7X^4)Y^{10}\\&+(-144+24X^2-X^4)Y^{12}. \end{aligned}\tag{C7}$$

We see that according to argumentation in Sec. V we have cancellation of the Y^2 term. There is the Y^2X^2 term only.

The equation Det$[m]=0$ determines now five spin-wave energies generated by $\epsilon_{\mathbf{q}}^2=Y^2(Y^2+1)$, $\epsilon_{\mathbf{k}}^2=(Y^2+1)(Y^2+2)$, and

$\epsilon_{2k}=(Y^2+4)(Y^2+5)$. We consider below the three lower branches only.

Small Y case. From Eq. (C6) for the gapless branch we have $X^2=3Y^4/8$ and the factor 1/2 in Eqs. (C3) and (36) has to be replaced by 3/8. Higher approximations with $n>2$ cannot change this result.

Consider now the branches connected to $\epsilon_{\mathbf{k}}$. For $X^2\simeq 2$ the first three terms in Eq. (C7) give $648\chi^2-5904\chi Y^2+9398Y^4=0$, where $\chi=X^2-2$. Solutions of this equation are given by

$$X^2=2+\frac{37Y^2}{18}=2+2.1Y^2, \quad X^2=\frac{127Y^2}{18}=2+7.1Y^2. \tag{C8}$$

We again obtain small changes of the numerical coefficients.

$Y\gg 1$ *case*. For large Y we can write $X=WY^2$ and asymptotically $W\to 1$. Retaining only three main terms in powers of Y we obtain the dispersion equation

$$(W-1)^3-\frac{8(W-1)+15(W-1)^2}{Y^2}+\frac{40}{Y^4W^2}=0. \tag{C9}$$

Its solutions are given by

$$X_{\pm}=Y^2\pm Y\sqrt{2}, \quad X_1=Y^2+\tfrac{5}{2}. \tag{C10}$$

So corrections to the main asymptotic of $\epsilon_{\pm}$ modes remain the same as in $n=1$ approximation and the irrelevant ϵ_1 mode have small correction to the main $Aq_{\perp}^2$ term in complete agreement with the argumentation presented in the end of Appendix B.

APPENDIX D: THE SPIN-WAVE INTERACTION AND THE GAP

The DMI breaks the total spin conservation law. As a result the spin-wave interaction in the Hartree-Fock approximation contribute to the gap. In this approximation momenta $q\sim 1/a$ are important, we can neglect umklapps and instead of Eqs. (21) and (22) use the following expressions:

$$E_{\mathbf{q}}=S[L_0-(L_{\mathbf{q}}+J_{\mathbf{q}}/2\ ], \quad B_{\mathbf{q}}=S(L_{\mathbf{q}}-J_{\mathbf{q}})/2, \tag{D1}$$

where

$$L_{\mathbf{q}}=J_{\mathbf{q},\mathbf{k}}+2D_{\mathbf{q}}(\mathbf{k}\cdot\hat{c}), \tag{D2}$$

and we neglect the AEI and the dipolar interaction.

The interaction Hamiltonian consists of the parts V_4 and V_3 describing four and three spin-waves interactions, respectively. There are also the fifth and sixth order terms. We neglect them as they give further $1/S$ corrections and contain small numerical factors. Using Eqs. (16) and (17) we get

$$V_4=(1/2N)\sum[(L_{\mathbf{3}}-L_{\mathbf{3}-\mathbf{1}})a^+_{\mathbf{1}+\mathbf{2}+\mathbf{3}}a^+_{-\mathbf{3}}a_{\mathbf{1}}a_{\mathbf{2}}+(J_{\mathbf{3}}-L_{\mathbf{3}})a^+_{\mathbf{1}+\mathbf{2}+\mathbf{3}}(a^+_{-\mathbf{3}}+a_{\mathbf{3}})a_{\mathbf{1}}a_{\mathbf{2}}/2], \tag{D3}$$

where $\mathbf{1},\mathbf{2},\mathbf{3}=\mathbf{q}_{1,2,3}$.

This interaction is non-Hermitian and the Hartree-Fock corrections to the E, B coefficients in Eq. (20) are determined as (cf. Ref. 27)

$$\delta E_{\mathbf{q}}=\left\langle\frac{\delta^2V_4}{\delta a^+_{\mathbf{q}}\delta a_{\mathbf{q}}}\right\rangle, \quad \delta B_{\mathbf{q}}=\left\langle\frac{\delta^2V_4}{\delta a^+_{\mathbf{q}}\delta a^+_{-\mathbf{q}}}\right\rangle, \quad \delta B^+_{\mathbf{q}}=\left\langle\frac{\delta^2V_4}{\delta a^+_{-\mathbf{q}}\delta a^+_{\mathbf{q}}}\right\rangle, \tag{D4}$$

and $\delta B^+\neq\delta B^*$. After simple calculations we get

$$\delta E_{\mathbf{q}}=1/N\sum[(L_{\mathbf{1}}+L_{\mathbf{q}}+J_{\mathbf{1}}+J_{\mathbf{q}}-2L_{\mathbf{1}-\mathbf{q}}-2L_0)n_{\mathbf{1}}/2+(J_{\mathbf{q}}-L_{\mathbf{q}}+2J_{\mathbf{1}}-2L_{\mathbf{1}})f_{\mathbf{1}}/4],$$

$$\delta B_{\mathbf{q}}=1/N\sum[(L_{\mathbf{q}}+J_{\mathbf{q}})/2-L_{\mathbf{q}-\mathbf{1}}]f_{\mathbf{1}},$$

$$\delta B^+_{\mathbf{q}}=1/N\sum\{[(L_{\mathbf{1}}+J_{\mathbf{1}})/2-L_{\mathbf{q}-\mathbf{1}}]f_{\mathbf{1}}+[(J_{\mathbf{1}}-L_{\mathbf{1}})/2+J_{\mathbf{q}}-L_{\mathbf{q}}]n_{\mathbf{q}}\}, \tag{D5}$$

where

$$n_{\mathbf{q}}=\langle a^+_{\mathbf{q}}a_{\mathbf{q}}\rangle=\frac{E_{\mathbf{q}}-\epsilon_{\mathbf{q}}}{2\epsilon_{\mathbf{q}}}, \quad f_{\mathbf{q}}=\langle a_{\mathbf{q}}a_{-\mathbf{q}}\rangle=\langle a^+_{-\mathbf{q}}a^+_{\mathbf{q}}\rangle=-\frac{B_{\mathbf{q}}}{2\epsilon_{\mathbf{q}}}. \tag{D6}$$

Using Eqs. (33) one can easily obtain these expressions by standard method.[27,35]

For the spin-wave gap we have now

$$\Delta^2_{SW}=(E_0/N)\sum\{(J_{\mathbf{1}}-L_{\mathbf{1}})n_{\mathbf{1}}/2-[L_0-(J_{\mathbf{1}}+L_{\mathbf{1}})/2]f_{\mathbf{1}}\}. \tag{D7}$$

Neglecting the DMI we get $n_{\mathbf{1}}=f_{\mathbf{1}}=0$ and $J_{\mathbf{1}}-L_{\mathbf{1}}=0$. So the first term in this equation is small and may be omitted. If $1\gg k$ we have $L_0-(L_{\mathbf{1}}+J_{\mathbf{1}})/2\simeq\epsilon_{\mathbf{1}}/S$ and using (D6) we get

$$\Delta^2_{SW}=(E_0/4N)\sum[(J_{\mathbf{1}+\mathbf{k}}+J_{\mathbf{1}-\mathbf{k}})/2-J_{\mathbf{1}}+2D_{\mathbf{1}}(\mathbf{k}\cdot\hat{c})]. \tag{D8}$$

We have $\Sigma J_{\mathbf{1}\pm\mathbf{k}}=0$ and taking into account Eq. (11) we obtain the final result

$$\Delta^2_{SW}=\frac{Ak^2h_c}{4SN}\sum\frac{D_{\mathbf{1}}}{D_0}. \tag{D9}$$

Let us consider now the three spin-wave interactions. Using Eqs. (4), (16), and (17) we obtain

$$V_3=(1/N)\sum C_{\mathbf{q}}a^+_{\mathbf{1}+\mathbf{2}}a_{\mathbf{2}}(a_{\mathbf{1}}-a^+_{-\mathbf{1}}), \tag{D10}$$

where $C_{\mathbf{q}}=-\sqrt{2S}[D_{\mathbf{q}}(\mathbf{q}\cdot\hat{c})+N_{\mathbf{q},\mathbf{k}}/2]$. It is the Hermitian operator and has to be tackled using ordinary perturbation theory.[35] It is easy to show that in the second order corresponding contribution to the gap is zero due to $\mathbf{q}$ oddness of $C_{\mathbf{q}}$.

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