Spin transverse force and intrinsic quantum transverse transport

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The spin-orbit coupling may generate spin transverse force on moving electron spin, which gives a heuristic picture for the quantum transverse transport of electron. A relation between the spin and anomalous Hall conductance and spin force was established, and applied to several systems. It was predicted that the sign change of anomalous Hall conductance can occur in diluted magnetic semiconductors of narrow band and can be applied to identify intrinsic mechanism experimentally.

I. INTRODUCTION

The theory of anomalous Hall effect has a long history since the 1950s.[1] It was realized that several different mechanisms are contributed to the total Hall conductance in ferromagnetic metals and semiconductors. Except for the skew scattering and the side jump from the impurity scattering,[2–5] an intrinsic mechanism tells that the spin-orbit coupling in the electronic band structure of the system may induce a nonzero Berry phase or magnetic monopole in the momentum space and cannot be neglected in the transverse transport of electrons, especially in the diluted magnetic semiconductors.[6,7] On the other hand, the spin aspect of transverse transport was also studied recently, in which an external electric field may drive electrons to form a transverse spin current in the systems with spin-orbit coupling even in paramagnetic electronic systems.[8–10] These two effects reflect the charge and spin aspects of electron transport, respectively, and have some common features as their physical origin stems from the same spin-orbit coupling of conduction electrons. More and more experiments support the intrinsic mechanism.[11–13] Since the spin-orbit coupling reflects the interaction between the electron spin, momentum and external or environmental potential, it was shown that this coupling, as an extension of Ehrenfest's theorem in quantum mechanics, may generate a spin transverse force on the spin current instead of the Lorentz force on electric current in a magnetic field in conventional Hall effect.[14] This spin transverse force provides a new route to get insight into the transverse motion of electrons.

In this paper we focus on the relation between the spin force and intrinsic quantum transverse transport. For the electronic system with spin-orbit coupling and exchange coupling, it was found that a spin transverse force exerts on moving electron spins in the anomalous transverse effect, as a result, a set of formulas based on spin force were derived for intrinsic spin and anomalous Hall conductance. As an intrinsic feature it was predicted that the sign of the anomalous Hall conductance can be changed when the system breaks both the bulk and structural inversion symmetry and time reversal symmetry. It is also shown that spin Hall effect

in the two-dimensional (2D) Luttinger system is equivalent to the intrinsic anomalous and spin Hall problems in a ferromagnetic metal, and its robustness against impurities can be also understood very well from the point of view of spin force.

II. GENERAL FORMULA

To develop a general formula for spin and anomalous Hall conductance, we start with an effective Hamiltonian for electrons with spin $1/2$

$$H = \epsilon(p) + \sum_{\alpha=x,y,z} d_\alpha(p)\sigma_\alpha, \qquad (1)$$

where $\epsilon(p) = p^2/2m^*$ is the kinetic energy with the band electron effective mass m^* and σ_α are the Pauli matrices; $d_\alpha(p)$ are the momentum-dependent coefficients which describe the spin-orbit interactions and exchange interaction of magnetic impurities. The energy eigenvalues are $E_{pn} = \epsilon(p) + \mu d(p)$ with $\mu = \pm$, and $d = \sqrt{d_x^2 + d_y^2 + d_z^2}$, and the corresponding states are denoted by $|p, \mu\rangle$. Using the Heisenberg equation, the kinetic velocity operator ($j = x, y, z$) is defined as

$$v_j = \frac{1}{i\hbar}[r_j, H] = \frac{p_j}{m^*} + \sum_{\alpha=x,y,z} \frac{\partial d_\alpha}{\partial p_j}\sigma_\alpha, \qquad (2)$$

where the first term is the canonical velocity, and second part can be regarded as the spin gauge field or anomalous velocity, $(e/m^*c)\mathcal{A}_j = \sum_{\alpha=x,y,z}\partial d_\alpha/\partial p_j \sigma_\alpha$. As an extension of the Ehrenfest theorem,[14] the spin-dependent force is introduced as the derivative of kinetic momentum operator with the respect to time

$$F_j = \frac{m^*}{i\hbar}[v_j, H] = \frac{2m^*}{\hbar}\epsilon_{\alpha\beta\gamma}\frac{\partial d_\alpha}{\partial p_j}d_\beta\sigma_\gamma. \qquad (3)$$

This spin force is a purely quantum quantity, and has no classical counterpart.

The quantum transverse transport of electrons caused by a weak electric field **E** can be calculated within the framework

of linear response theory. The Kubo formula for the dc conductivity gives[15]

$$\sigma_{ij} = \frac{e^2\hbar}{\Omega} \sum_{p,\mu \neq \mu'} \frac{(f_{p\mu} - f_{p\mu'})\mathrm{Im}(\langle p\mu|v_i|p\mu'\rangle\langle p\mu'|v_j|p\mu\rangle)}{(E_{p\mu} - E_{p\mu'})(E_{p\mu} - E_{p\mu'} + i\delta^+)} \tag{4}$$

with $\delta^+ \to 0^+$, and the Dirac–Fermi distribution function $f_{p,\mu} = 1/\{\exp[\beta(E_\mu(p) - \mu)] + 1\}$. With the formula of the kinetic velocity in Eq. (2)

$$\langle p\mu|v_i|p\mu'\rangle = \sum_{\alpha = x,y,z} \frac{\partial d_\alpha}{\partial p_j}\langle p\mu|\sigma_\alpha|p\mu'\rangle \tag{5}$$

for $\mu \neq \mu'$ and, furthermore, an identity can be proved for this system

$$\mathrm{Im}(\langle p\mu|\sigma_\alpha|p\mu'\rangle\langle p\mu'|\sigma_\beta|p\mu\rangle) = \mu\epsilon_{\alpha\beta\gamma}\frac{d_\gamma}{d}, \tag{6}$$

for $\mu \neq \mu'$. We limit our discussion to the case that two spectra are nondegenerated in the whole momentum space, i.e., $d \neq 0$ in the whole momentum p space such that δ^+ can be taken to be zero before the integral over p.[16] Thus the intrinsic transverse conductance can be expressed as

$$\sigma_{ij} = -\frac{e^2\hbar}{2\Omega} \sum_p \frac{(f_{p,-} - f_{p,+})}{d^3}\epsilon_{\alpha\beta\gamma}\frac{\partial d_\alpha}{\partial p_i}\frac{\partial d_\beta}{\partial p_j}d_\gamma. \tag{7}$$

This recovers the conductance formula in terms of the Berry curvature.[17–19] On the other hand, the anticommutators of the spin force F_j and spin gauge field $\mathcal{A}_i$ give

$$\mathrm{Tr}[\{F_j, \mathcal{A}_i\}] = \frac{8m^{*2}c}{\hbar e}\epsilon_{\alpha\beta\gamma}\frac{\partial d_\alpha}{\partial p_i}\frac{\partial d_\beta}{\partial p_j}d_\gamma, \tag{8}$$

where the trace runs through the spin variables, $\sum_\mu \langle p\mu|\cdots|p\mu\rangle$. In this way we established a relation between the electric conductance and spin force

$$\sigma_{ij} = -\frac{e^3\hbar^2}{16m^{*2}c\Omega} \sum_p \frac{(f_{p,-} - f_{p,+})}{d^3}\mathrm{Tr}[\{F_j, \mathcal{A}_i\}]. \tag{9}$$

Since $\mathrm{Tr}[\{F_j, v_i\}] = \frac{e}{m^*c}\mathrm{Tr}[\{F_j, \mathcal{A}_i\}]$, the anomalous Hall conductance is determined by both the spin force and the spin gauge field, or anomalous part of the velocity. This fact reflects the physical origin of spin-orbit coupling in this effect. Note that $\mathrm{Tr}[\{F_j, \mathcal{A}_i\}] = -\mathrm{Tr}[\{F_i, \mathcal{A}_j\}]$, so $\sigma_{ij} = -\sigma_{ji}$ for $i \neq j$, that is the so-called Onsager relation. It is noticed that $\mathrm{Tr}[\{F_j, \mathcal{A}_i\}] = 0$ if any one of d_α is zero.

The Kubo formula for spin Hall conductance is written as

$$\sigma_{ij}^\gamma = \frac{e\hbar}{\Omega} \sum_{p,\mu \neq \mu'} \frac{(f_{p\mu} - f_{p\mu'})\mathrm{Im}\langle p\mu|J_i^\gamma|p\mu'\rangle\langle p\mu'|v_j|p\mu\rangle}{(E_{p\mu} - E_{p\mu'})(E_{p\mu} - E_{p\mu'} + i\delta^+)}, \tag{10}$$

where spin current operator J_i^γ is defined conventionally as $J_i^\gamma = (\hbar/4)\{v_i, \sigma_\gamma\}$. Note that this choice is a natural one but not a unique one in the presence of spin-orbit coupling since this is no continuity equation for spin density as is the case

for charge density. Following the calculation mentioned above, from Eq. (10) one can obtain[18]

$$\sigma_{ij}^\gamma = \frac{e\hbar^2}{4\Omega} \sum \frac{(f_{p,-} - f_{p,+})}{d^3}\frac{\partial\epsilon}{\partial p_i}d_\alpha\frac{\partial d_\beta}{\partial p_j}\epsilon_{\alpha\beta\gamma}. \tag{11}$$

Similarly, the spin conductance can also be expressed in terms of the spin force

$$\sigma_{ij}^\gamma = -\frac{e\hbar^2}{16m^*\Omega} \sum_p \frac{(f_{p,-} - f_{p,+})}{d^3}\mathrm{Tr}[\{F_j, J_i^\gamma\}], \tag{12}$$

which is given by the anticommutator of the spin force and spin current operators J_i^γ with

$$\mathrm{Tr}[\{F_j, J_i^\gamma\}] = -4m^*\frac{\partial\epsilon}{\partial p_i}d_\alpha\frac{\partial d_\beta}{\partial p_j}\epsilon_{\alpha\beta\gamma}. \tag{13}$$

On the other hand, due to the spin-orbit coupling, an electric field can also induce a nonzero spin polarization. Its linear response to the field is

$$\chi_j^\gamma = \frac{\hbar}{2}(\langle\sigma_\gamma\rangle_E - \langle\sigma_\gamma\rangle_{E=0})/E_j$$

$$= -\frac{e\hbar^3}{32m^*\Omega} \sum_p \frac{(f_{p,-} - f_{p,+})}{d^3}\mathrm{Tr}[\{F_j, \sigma_\gamma\}]. \tag{14}$$

In this way we have established an explicit relation between the spin force and the quantum spin and charge transverse transport in a system with spin-orbit coupling.

In general, for any observable O, its linear response to an external electric field $\mathbf{E}$, $\langle O_i\rangle = \chi_{ij}E_j$, where, from the Kubo formula,

$$\chi_{ij} = -\frac{e\hbar^2}{16m^*\Omega} \sum_p \frac{(f_{p,-} - f_{p,+})}{d^3}\mathrm{Tr}[\{F_j, O_i\}]. \tag{15}$$

III. APPLICATIONS OF THE GENERAL FORMULA

A. 2D ferromagnetic system with the wurtzite and zinc-blende structures

Now we apply the general formula to several systems in semiconductors. We first consider an effective Hamiltonian for a two-dimensional ferromagnetic system with the wurtzite and zinc blende structures

$$H = \frac{p^2}{2m^*} + a_0 f(p)(p_y\sigma_x - p_x\sigma_y) + h_0\sigma_z. \tag{16}$$

The exchange field due to the magnetic impurities or Coulomb interaction is taken to be uniform, as a mean field effect, and is normal to the plain $f(p) = 1$ for the wurtzite structure, and $f(p) = p^2$ for the zinc blende structure such as $Hg_{1-x}Mn_xTe$.[20] The components of spin gauge field $\mathcal{A}$ in this system are

$$\mathcal{A}_x = \frac{m^*ca_0}{e}\left[p_y\frac{\partial f}{\partial p_x}\sigma_x - \left(f + p_x\frac{\partial f}{\partial p_x}\right)\sigma_y\right], \tag{17}$$

$$\mathcal{A}_y = \frac{m^* c a_0}{e}\left[\left(f + p_y \frac{\partial f}{\partial p_y}\right)\sigma_x - p_x \frac{\partial f}{\partial p_y}\sigma_y\right] \qquad (18)$$

and the spin force is

$$\mathbf{F}_s = \frac{4m^*}{\hbar^2}a_0^2 f^2 J_s^z \times \hat{z} - \frac{2m^* a_0 h_0}{\hbar}\nabla_p(f\mathbf{p}\cdot\sigma). \qquad (19)$$

From the formula we notice that a spin transverse force arises due to the spin-orbit coupling if a spin current exists along the electric field and also an in-plane spin force arises due to the interference between the spin-orbit coupling and the exchange field. To calculate the intrinsic Hall conductance, we notice that the anticommutator

$$\text{Tr}[\{F_y, \mathcal{A}_x\}] = \frac{8m^{*2}ch_0}{\hbar e}a_0^2 f[d(pf)/dp], \qquad (20)$$

which vanishes when $h_0 = 0$. Thus if the system does not violate the time reversal symmetry, no anomalous Hall current circulates. Assuming at zero temperature and taking the bottom subband to be occupied and the top one to be empty, the anomalous Hall conductance is

$$\sigma_{xy} = -\frac{e^2}{2h}(1 - h_0/[a_0^2 x_f^2 + h_0^2]^{1/2}), \qquad (21)$$

where $x_f^2 = p_F^2 f^2$ at the Fermi surface. If $a_0 p_F f \gg h_0$, the anomalous Hall conductance is almost quantized, $\sigma_{xy} \simeq -\frac{e^2}{2h}$. This is consistent with the theory of Berry curvature.[20]

According to Eq. (12), one can obtain the intrinsic spin Hall conductance

$$\sigma_{xy}^z = \frac{e\hbar^2}{4m^*\Omega}\sum_p \frac{(f_{p,-} - f_{p,+})p_x^2 a_0^2 f^2}{[p^2 a_0^2 f^2 + h_0^2]^{3/2}}. \qquad (22)$$

For the Wurtzite structure, i.e., $f(p) = 1$, assuming the Fermi energy μ lies in the gap, the intrinsic spin Hall conductance is

$$\sigma_{xy}^z = \frac{e\hbar^2}{16m^*\pi}\frac{(h_0 - \eta_1)^2}{a_0^2 \eta_1}, \qquad (23)$$

where

$$\eta_1 = (h_0^2 + p_F^2 a_0^2)^{1/2} \qquad (24)$$

with

$$p_F = \sqrt{2}\{[m^{*2}(a_0^4 m^{*2} + h_0^2 + 2a_0^2\mu m^*)]^{1/2} + m^*(a_0^2 m^* + \mu)\}^{1/2}. \qquad (25)$$

For the zinc blende structure, i.e., $f(p) = p^2$,

$$\sigma_{xy}^z = \frac{e\hbar^2}{32m^*\pi}\frac{(h_0 - \eta_2)^2}{a_0^2 \eta_2}, \qquad (26)$$

where

$$\eta_2 = (h_0^2 + p_F^4 a_0^2)^{1/2}, \qquad (27)$$

with

$$p_F = \{2m^*(\mu + [h_0^2 - 4a_0^2 m^{*2}(h_0^2 - \mu^2)]^{1/2})/(1 - 4a_0^2 m^{*2})\}^{1/2}. \qquad (28)$$

B. 2D ferromagnetic electron gas with both Rashba and Dresselhaus coupling

In some materials the spin-orbit coupling has two different contributions when a system has the structural and bulk inversion asymmetry. Here we consider a two-dimensional ferromagnetic electron gas with both Rashba and Dresselhaus coupling

$$H = \frac{p^2}{2m^*} - \lambda(p_x\sigma_y - p_y\sigma_x) - \beta(p_x\sigma_x - p_y\sigma_y) + h_0\sigma_z. \qquad (29)$$

In the language of d_α in Eq. (1)

$$d_x = \lambda p_y - \beta p_x, \qquad (30)$$

$$d_y = -\lambda p_x + \beta p_y, \qquad (31)$$

$$d_z = h_0. \qquad (32)$$

It is straightforward that

$$\text{Tr}[\{F_y, \mathcal{A}_x\}] = \frac{8m^{*2}ch_0}{\hbar e}(\lambda^2 - \beta^2), \qquad (33)$$

which is independent of p. Using the formula for the Hall conductance, we conclude immediately that the anomalous Hall conductance vanishes at $h_0 = 0$. The Hall conductance will change its sign near the point of $\lambda^2 = \beta^2$. Since $h_0 \neq 0$, the band spectra open a finite gap at $p = 0$, and degeneracy of two electron bands is removed. If one assumes that the Fermi surface is below the up band of the spectrum, the anomalous Hall conductance σ_{xy} reads

$$\sigma_{xy} = -\frac{e^2}{2h}\text{sgn}(\lambda^2 - \beta^2)(1 - \gamma), \qquad (34)$$

where

$$\gamma = \frac{1}{2\pi}\int_0^{2\pi} d\phi \frac{|\lambda^2 - \beta^2|}{\Pi(\phi)[1 + p_F^2(\phi)\Pi(\phi)/h_0^2]^{1/2}} \qquad (35)$$

with

$$\Pi(\phi) = \lambda^2 + \beta^2 - 2\lambda\beta\sin 2\phi \qquad (36)$$

and p_F is the Fermi momentum

$$p_F = \sqrt{2m^*}\{\mu + m^*\lambda\beta(\lambda\beta - 2\sin 2\phi) + [h_0^2 - \mu^2 + (m^*\lambda^2\beta^2 + \mu - 2m^*\lambda\beta\sin 2\phi)^2]^{1/2}\}^{1/2}. \qquad (37)$$

The dependence σ_{xy} on β of numerical calculations is plotted in Fig. 1. The parameters are taken as:[20] $\lambda = 23$ meV nm/$\hbar$, $h_0 = 1.38$ meV, $\mu = 1.38$ meV, $m^* = 0.9m_0$, and β is comparable with λ. It is shown that the anomalous Hall conductance will change its sign near the point of $\lambda^2 = \beta^2$. If $p_F|\lambda - \beta| \gg h_0$, σ_{xy} is approximately equal to $-\text{sgn}(\lambda^2 - \beta^2)\frac{e^2}{2h}$.

If one assumes that the Fermi surface is below the up band of the spectrum, the intrinsic spin Hall conductance can also be obtained
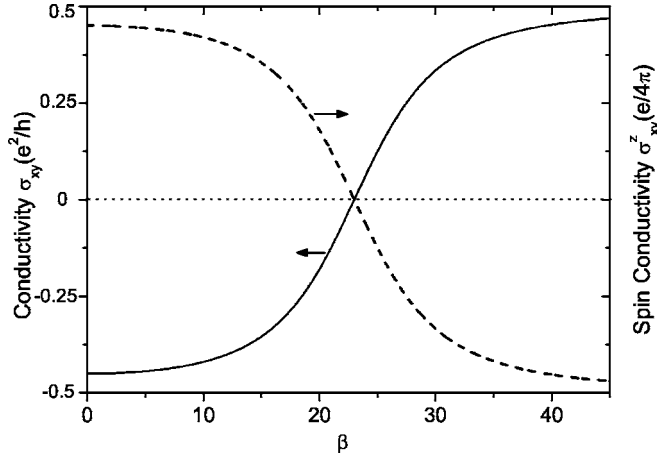
FIG. 1. Variations of the conductivity σ_{xy} and the spin conductivity σ_{xy}^z with the coupling parameter β (meV nm/$\hbar$). The parameters are $\lambda=23$ meV nm/$\hbar$, $h_0=1.38$ meV, $\mu=1.38$ meV, and $m^*=0.9m_0$. The solid line corresponds to σ_{xy}, and the dashed line to σ_{xy}^z.

$$\sigma_{xy}^z = \frac{\hbar^2 e}{4m}\frac{1}{(2\pi)^2}\int_0^{2\pi} d\phi \frac{(\lambda^2-\beta^2)[h_0-\eta(\phi)]^2\cos^2\phi}{\Pi(\phi)^2\eta(\phi)}.$$

(38)

While the Fermi energy is above the gap, the intrinsic spin Hall conductance is

$$\sigma_{xy}^z = \frac{\hbar^2 e}{4m}\frac{1}{(2\pi)^2}\int_0^{2\pi} d\phi \frac{(\lambda^2-\beta^2)\cos^2\phi}{\Pi(\phi)^2\eta_-\eta_+}$$

$$\times (h_0^2 - \eta_-\eta_+)(\eta_+ - \eta_-),$$

(39)

with

$$\eta_\pm = [h_0^2 + p_\pm^2\Pi(\phi)]^{1/2}$$

(40)

and

$$p_\pm = \sqrt{2m^*}\{\mu + m^*\lambda\beta(\lambda\beta - 2\sin 2\phi)$$

$$\mp [h_0^2 - \mu^2 + (m^*\lambda^2\beta^2 + \mu - 2m^*\lambda\beta\sin 2\phi)^2]^{1/2}\}^{1/2}.$$

(41)

The numerical calculation of the intrinsic spin Hall conductance σ_{xy}^z with β is plotted by the dashed line in Fig. 1. When $\mathrm{sgn}(\lambda^2-\beta^2)=+1$, the intrinsic spin Hall conductance σ_{xy}^z is positive; $\mathrm{sgn}(\lambda^2-\beta^2)=-1$, σ_{xy}^z is negative.

The sign change of anomalous Hall conductance as that of spin Hall conductance[21] may be observed experimentally in some diluted magnetic GaAs quantum wells where the Rashba and Dresselhaus coupling is usually of the same order of magnitude. The Rashba coupling is adjustable by a gate field perpendicular to the electron gas plane. Thus there is no practical difficulty to achieve the situation near $\lambda=\pm\beta$. Since the Rashba coupling has a relation to the gate voltage, $\lambda=\lambda_0+\delta\lambda V_g$, the gate field can be used to control the direction of intrinsic anomalous Hall current. If the extrinsic contribution to the Hall conductance is comparable with the intrinsic one, at least a jump of the anomalous Hall

conductance should be observed even if the Hall current could not change its direction. This effect can be identified as the intrinsic mechanism for anomalous Hall effect.

C. 2D Luttinger model

We now turn our attention to the Luttinger Hamiltonian for spin $S=3/2$ holes in the valence band of centrosymmetric cubic semiconductors[22]

$$H = \frac{1}{2m}\left(\gamma_1 + \frac{5}{2}\gamma_2\right)p^2 - \frac{\gamma_2}{m}(\mathbf{p}\cdot\mathbf{S})^2,$$

(42)

where γ_1, γ_2 are material-dependent parameters and m is the electron mass. In terms of SO (5) Clifford algebra, the Hamiltonian (42) can be cast into[23]

$$H = \epsilon(\mathbf{p}) + d_a\Gamma^a,$$

(43)

where $\epsilon(\mathbf{p})=\gamma_1 p^2/2m$, Γ^a ($a=1,2,\ldots,5$) the five Dirac Γ matrices and d_a the five d-wave combinations of p are given in Ref. 23. In the two-dimensional case, for the first heavy- and light-hole bands, the confinement in a well of thickness a is approximated by the relation $\langle p_z\rangle=0$, $\langle p_z^2\rangle\simeq(\pi\hbar/a)^2$.[24] In this case, $d_1=d_2=0$ and Γ^a ($a=3,4,5$) forms reducible representation of an SO (3) Clifford subalgebra.[17] In the following, one introduces a new representation under which the expressions of $\mathbf{S}$ matrices and Γ^a ($a=3,4,5$) are given in the Appendix. Under this new representation, the two-dimensional Luttinger model is block diagonal

$$H = \begin{pmatrix} H_+ & 0 \\ 0 & H_- \end{pmatrix},$$

(44)

where

$$H_\mu = \epsilon(p) + \mu\sum_{\alpha=x,y,z} d_a\sigma_\alpha$$

(45)

with $\mu=\pm 1$, and

$$\epsilon(p) = \frac{\gamma_1}{2m}(p_x^2 + p_y^2),$$

(46)

$$d_x = -\frac{\sqrt{3}\gamma_2}{m}p_x p_y,$$

(47)

$$d_y = -\frac{\sqrt{3}\gamma_2}{2m}(p_x^2 - p_y^2),$$

(48)

$$d_z = -\frac{\gamma_2}{m}\left(\langle p_z^2\rangle - \frac{1}{2}p^2\right)$$

(49)

(see Refs. 24–26). These two effective Hamiltonians are connected by the time reversal operator $\Theta=-i\sigma_y K$ (K is the complex conjugate operator that forms the complex conjugate of any coefficient that multiplies a ket): $H_+=\Theta H_-\Theta^{-1}$. Thus the energy eigenstates of H are at least doubly degenerated. H_μ has the same form as that for the ferromagnetic metal with exchange coupling since $d_z(p)$ in H_μ contains a nonzero term, $-\frac{\gamma_2}{m}\langle p_z^2\rangle$, but the Pauli operators σ_α here do not
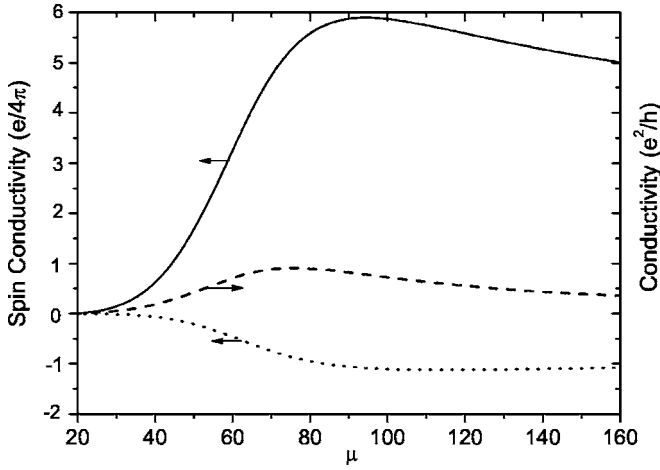
FIG. 2. Variations of $\sigma_{xy}^{(+1)}$, $\sigma_{xy}^{z(+1)}$, and σ_{xy}^z with Fermi level μ (meV). The parameters are $a=8.3$ nm, $\gamma_1=6.92$, and $\gamma_2=2.1$. The solid line corresponds to σ_{xy}^z, the dotted line to $\sigma_{xy}^{z(+1)}$, and the dashed line to $\sigma_{xy}^{(+1)}$.

represent a real spin. To calculate the linear response of the spin current $J_i^z=\frac{1}{2}\{\partial H/\partial p_i,S_z\}$ to an electric current $v_j=\partial H/\partial p_j$, the Kubo formula gives

$$\sigma_{xy}^z=\sum_{\mu=\pm1}\left[-2\sigma_{xy}^{z(\mu)}+\mu\frac{\hbar}{2e}\sigma_{xy}^{(\mu)}\right], \qquad (50)$$

where $\sigma_{xy}^{(\mu)}$ and $\sigma_{xy}^{z(\mu)}$ are the anomalous and spin Hall conductance of H_μ, respectively. The anomalous Hall conductance $\sigma_{ij}=\Sigma_{\mu=\pm1}\sigma_{xy}^{(\mu)}$. These relations are valid even if the nonmagnetic disorder and interaction are taken into account. It was known that the spin Hall conductance is invariant $\sigma_{xy}^{z(+1)}=\sigma_{xy}^{z(-1)}$, but the anomalous Hall conductance changes its sign $\sigma_{xy}^{(+1)}=-\sigma_{xy}^{(-1)}$ under time reversal. *Therefore the calculation of spin Hall conductance in the 2D Luttinger model is reduced to those of spin Hall conductance and anomalous Hall conductance in a ferromagnetic metal or band insulator of H_μ.* For a numerical calculation the material-specific parameters of band structure for GaAs are adopted as $\gamma_1=6.92$ and $\gamma_2=2.1$. The thickness of quantum well $a=8.3$ nm corresponds to the gap between the light- and heavy-hole bands at the Γ-point $\Delta E=40$ meV. Using the formula of Eqs. (9) and (12) $\sigma_{xy}^{(+1)}$, $\sigma_{xy}^{z(+1)}$, and σ_{xy}^z are calculated numerically in terms of chemical potential μ as shown in Fig. 2. It is noted that, for an infinite confinement, $\langle p_z^2\rangle\rightarrow+\infty$, the eigenstates of H_μ become fully saturated.[27] As a result the spin-orbit coupling is suppressed completely, and both spin and anomalous Hall effect vanish, which is in agreement with Bernevig and Zhang.[24]

IV. DISCUSSION

Here we may present a heuristic picture for electronic transverse transports in these systems and the effect of disorder from the point of view of spin force. The effect of disorder on anomalous Hall conductance has been investigated in Refs. 28 and 29. For simplicity we focus on the

system with the Wurtzite structure, i.e., $f=1$ in Eq. (16) or $\beta=0$ in Eq. (29). The spin-orbit coupling induces the spin force, which contains two parts: the transverse force on moving electron spin

$$\mathbf{F}_1=\frac{4m^{*2}\lambda^2}{\hbar^2}J_s^z\times\hat{z} \qquad (51)$$

and the exchange coupling interacting also induces a spin force within the plane, which is relevant to the spin polarization

$$\mathbf{F}_2=-\frac{2m^*\lambda h_0}{\hbar}[\sigma_x\hat{x}+\sigma_y\hat{y}]. \qquad (52)$$

If the disorder potential V_{disorder} is taken into account, in a steady state, the spin force may reach at balance

$$\frac{1}{i\hbar}\left\langle\left[\frac{e}{c}\mathcal{A},H+V_{\text{disorder}}\right]\right\rangle=\langle\mathbf{F}_1+\mathbf{F}_2\rangle=0. \qquad (53)$$

This result is independent of the nonmagnetic disorder and interaction because the spin gauge field commutes with nonmagnetic potential V_{disorder}. From the spin force balance we have a relation between spin current and spin polarization

$$\langle J_x^z\rangle=+\frac{\hbar h_0}{2m^*\lambda}\langle\sigma_y\rangle, \qquad (54)$$

$$\langle J_y^z\rangle=-\frac{\hbar h_0}{2m^*\lambda}\langle\sigma_x\rangle. \qquad (55)$$

It becomes obvious that the spin Hall current vanishes in the case of $h_0=0$, which is consistent with previous complicated calculations.[30–33] Of course the purely intrinsic responses of spin current and spin polarization without impurities do not satisfy this relation, and the extrinsic contributions have to be included to reach the balance. Thus, this gives a clear picture for anomalous Hall effect in ferromagnetic metal: When the external electric field is applied along the x axis, it will circulate an electric current $J_{c,x}$, and also a spin current J_x^z, since the charge carriers are partially polarized. The spin-orbit coupling exerts a spin transverse force on the spin current, J_x^z, and generates a drift velocity or the anomalous Hall current $J_{c,y}$. From the Rashba coupling the spin polarization tends to be normal to the momentum or electric current. The electric current $J_{c,x}$ along E induces a nonzero $\langle\sigma_y\rangle$ and the anomalous Hall current $J_{c,y}$ induces nonzero $\langle\sigma_x\rangle$. These nonzero spin polarizations maintain the balance of spin transverse force, and further a nonzero spin current in a steady state. Thus the anomalous electronic transverse transport is robust against the disorder in the ferromagnetic metals and semiconductors. This picture can also be applied to understand the spin Hall effect in the quasi-2D Luttinger system.

ACKNOWLEDGMENTS

APPENDIX

In order to write the two-dimensional Hamiltonian in the form of Eq. (44), one introduces a new representation under which **S** matrices are expressed as

$$
S_z = \begin{pmatrix} -\frac{3}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{3}{2} \end{pmatrix}, \tag{A1}
$$

$$
S_x = \begin{pmatrix} 0 & 0 & -i\frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 & i\frac{\sqrt{3}}{2} \\ i\frac{\sqrt{3}}{2} & 1 & 0 & 0 \\ 0 & -i\frac{\sqrt{3}}{2} & 0 & 0 \end{pmatrix}, \tag{A2}
$$

$$
S_y = \begin{pmatrix} 0 & 0 & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & -i & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & i & 0 & 0 \\ 0 & -\frac{\sqrt{3}}{2} & 0 & 0 \end{pmatrix}. \tag{A3}
$$

Now the Dirac matrices Γ^a ($a=3,4,5$) become

$$
\Gamma_3 = \frac{1}{\sqrt{3}}(S_x S_y + S_y S_x) = \sigma_z \otimes \sigma_x, \tag{A4}
$$

$$
\Gamma_4 = \frac{1}{\sqrt{3}}(S_x^2 - S_y^2) = \sigma_z \otimes \sigma_y, \tag{A5}
$$

$$
\Gamma_5 = S_z^2 - \frac{5}{4} = \sigma_z \otimes \sigma_z \tag{A6}
$$

and furthermore

$$
\Gamma_{12} = \frac{1}{2i}[\Gamma_1, \Gamma_2] = \sigma_z \otimes \mathbf{I}, \tag{A7}
$$

$$
\Gamma_{34} = \frac{1}{2i}[\Gamma_3, \Gamma_4] = \mathbf{I} \otimes \sigma_\mathbf{z}, \tag{A8}
$$

$$
S_z = -\Gamma_{34} - \frac{1}{2}\Gamma_{12}. \tag{A9}
$$

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