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Competition between composite-fermion-crystal and liquid orders at $\nu=1/5$

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We show that for electrons on a two-dimensional plane exposed to a strong magnetic field, the state at filling factor $\nu=1/5$ is better described as a composite-fermion crystal than as a Laughlin liquid for small systems, but the latter prevails for systems containing more than ten particles. A variational combination of the two wave functions provides an extremely accurate wave function for small systems. This indicates that the $\nu=1/5$ fractional quantum Hall liquid is highly susceptible to formation of composite-fermion crystallites in it. Experimental implications are considered.

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I. INTRODUCTION

Much work has been done investigating the crystalline phase in the lowest Landau level. On the experimental front, a reentrant transition between an insulator and a liquid state was seen more than a decade ago near $\nu=1/5$, and the insulator was interpreted as a pinned crystal.[1] Such an interpretation is consistent with both the close proximity of the phase to a correlated liquid, which is evidence against a single particle localized phase, and radio-frequency spectroscopy experiments that find a sharp resonance in the frequency dependent conductivity in the insulating region,[2] which is naturally interpreted as the pinning mode of the crystal. On the theoretical front, excellent evidence now exists that the lowest Landau level crystal phase at small fillings is a topological quantum crystal of composite fermions.[3–5] One implication is that the existence of several crystal phases, with different numbers of vortices attached to composite fermions, provides a possible explanation for the recent evidence of two different kinds of crystals.[6] In another very interesting experiment in hole doped samples, periodic oscillations in the insulating phase were interpreted as indicating a coexistence of liquid and crystal phases.[7]

The competition between liquid and crystal is especially interesting at $\nu=1/5$. Comparison of trial wave functions with exact results, known for finite systems, has played a crucial and decisive role in clarifying the physics of fractional quantum Hall effect[8] (FQHE) and related states. As seen in Ref. 3, neither Laughlin's liquid wave function[9] nor the composite-fermion (CF) crystal wave function is satisfactory at $\nu=1/5$: the former has an overlap of 0.84 with the exact wave function for $N=6$ electrons in the disk geometry, while the latter has an overlap of 0.94. Neither overlap is as convincing as those for Laughlin's wave function at $\nu=1/3$ or the CF crystal wave functions at $\nu=1/7$ and $\nu=1/9$ (0.98, 0.99, and 0.99, respectively, for $N=6$ electrons on a disk). More disturbing is the fact that the CF crystal has a lower energy and a higher overlap than the liquid wave function (Kashurnikov *et al.*[10] also concluded a crystal state at $\nu=1/5$ from the exact diagonalization for seven electrons), which is inconsistent with the experimental observation of a FQHE plateau at $R_H=h/[(1/5)e^2]$.[1]

In this study, we demonstrate theoretically that the superiority of the CF crystal wave function at $\nu=1/5$ is a finite size effect, and that the liquid prevails for a sufficiently large number of electrons. Further, we seek to construct a better wave function at $\nu=1/5$, which should help clarify the physics of the state here. Experiments[1] have indicated that the liquid state at $\nu=1/5$ is flanked on either side by a crystal, which suggests that the 1/5 liquid may have strong crystal-like correlations. We find that a linear combination of the CF liquid and the CF crystal wave functions provides an excellent account of the actual state for small systems. That indicates a tendency toward formation of CF crystallites within the 1/5 liquid state, the possible experimental manifestations of which will be mentioned.

II. VARIATIONAL WAVE FUNCTIONS

The disk geometry will be employed throughout this work. (The spherical geometry is inappropriate for the issue of our interest, being in general biased against a crystal, because, for a general N, a two-dimensional triangular crystal cannot be wrapped onto a sphere without creating defects.) According to the CF theory, the wave function of the *liquid* state at $\nu=n/(2pn+1)$ is given by[11]

$$\Psi_{\text{liquid}}=\mathcal{P}_{LLL}\prod_{j<k}(z_j-z_k)^{2p}\Phi_n, \tag{1}$$

where $z_k=x_k-iy_k$ represents the position of an electron, Φ_n is the wave function of n filled Landau levels (LLs), and the operator $\mathcal{P}_{LLL}$ projects the wave function into the lowest Landau level, as appropriate for strong magnetic fields. The unit of length is chosen to be the magnetic length $\ell=\sqrt{\hbar c/eB}$, B being the magnetic field. Because the Jastrow factor $\Pi_{j<k}(z_j-z_k)^{2p}$ binds $2p$ quantized vortices to each electron in Φ_n, Ψ_{liquid} is interpreted as the wave function of n filled Λ levels (or quasi-LLs) of composite fermions. At $\nu=1/5$, composite fermions with vorticity $2p=4$ fill the lowest Λ level completely; substituting the explicit form for Φ_1 produces Laughlin's wave function.

In an infinite system, a crystal wave function is prepared with a lattice constant a related to the filling factor by $\nu=4\pi\ell^2/(\sqrt{3}a^2)$. In finite systems it is more convenient to work with eigenstates of the total angular momentum, espe-

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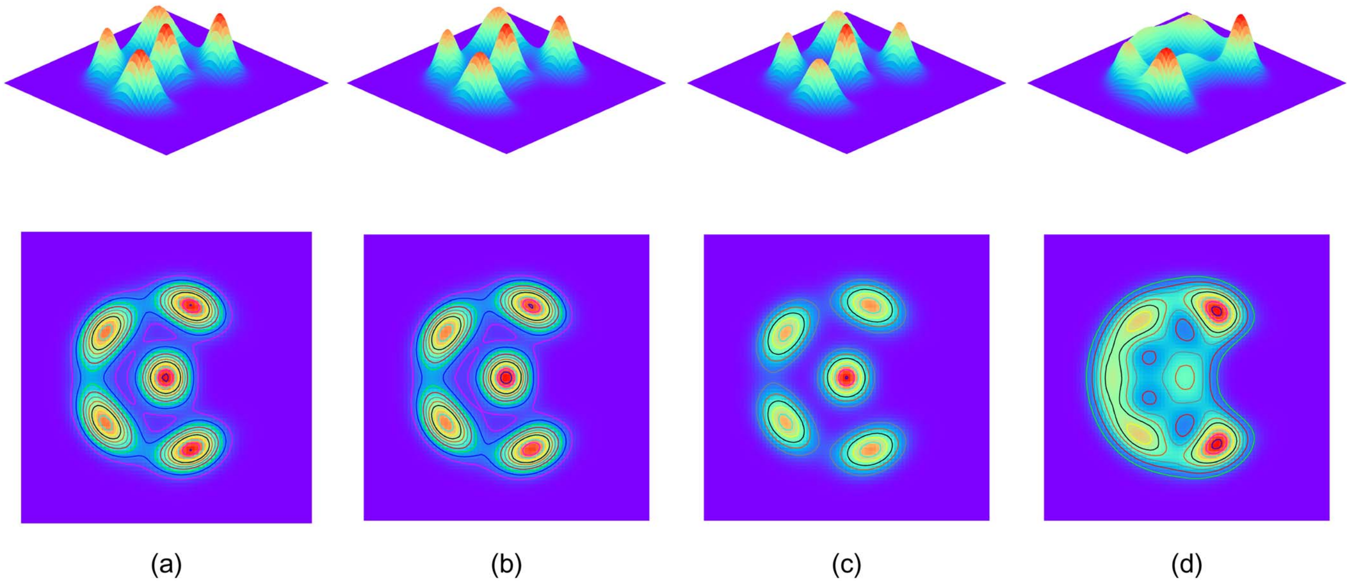

FIG. 1. (Color online) Angular view and contour plot of the pair correlation function for (a) exact, (b) mixed, (c) CFC, and (d) liquid state for $N=6$ particles at $\nu=1/5$. All figures are plotted on the same scale. The missing peak depicts the position $\mathbf{R}$ of the fixed particle in Eq. (8). The liquid plot was shown earlier in Ref. 18, and is reproduced here for comparison. The exact pair correlation function, shown in (a), is indistinguishable from that in (b).

cially for comparing them with the liquid and the exact eigenstates. The CF crystal (CFC) wave function at $\nu=1/5$, following Yi and Fertig[4] and Chang *et al.*,[3] is written as:

$$\Psi_{\mathrm{CFC}}(L) = \prod_{j<k}(z_j - z_k)^2\chi(L^*). \qquad (2)$$

Here, $\chi(L^*)$ is a "projected" Hartree-Fock crystal of *electrons*[12] at angular momentum L^*, and multiplication by the Jastrow factor $\prod_{j<k}(z_j-z_k)^2$ binds two vortices to each electron to produce a crystal of composite fermions carrying two vortices each. The total angular momentum is given by $L=N(N-1)+L^*$, where the first term on the right-hand side is contributed by the Jastrow factor and the second by the projected Hartree-Fock wave function $\chi(L^*)$. We prepare the CFC wave function at the same angular momentum as the liquid 1/5 wave function, namely $L=5N(N-1)/2$. The unprojected Hartree-Fock crystal wave function χ is given by

$$\chi = \frac{1}{\sqrt{N!}}\mathcal{A}\prod \phi_{\mathbf{R}_i}(\mathbf{r}_j), \qquad (3)$$

where $\mathcal{A}$ is the antisymmetrization operator, $\phi_{\mathbf{R}}(\mathbf{r})$ is the lowest LL coherent-state wave packet[13] for an electron localized at $\mathbf{R}=(X,Y)$:

$$\phi_{\mathbf{R}}(\mathbf{r}) = \frac{1}{\sqrt{2\pi}}e^{[-(1/4)(\mathbf{r}-\mathbf{R})^2+(i/2)(xY-yX)]}, \qquad (4)$$

and the positions $\mathbf{R}_i$ represent the lowest energy configuration of two-dimensional (2D) classical point charges,[14] which is a triangular lattice in the thermodynamic limit, but can have more complicated structures for finite systems. Following Yannouleas and Landmann[12] we project the wave function χ onto a definite angular momentum L^* to obtain the wave function $\chi(L^*)=\mathcal{P}(L^*)\chi$. The projection operator is defined[12] as $\mathcal{P}(L^*)=\frac{1}{2\pi}\int_0^{2\pi}d\gamma e^{i\gamma(\hat{L}-L^*)}$, where $\hat{L}=\Sigma_i\hat{L}_i$ is the total angular momentum operator. After the projection, $\chi(L^*)$ can be written as a linear combination of lowest LL basis functions composed of the single particle eigenstates of the circular gauge, with $L^*=\Sigma_i L_i$, L_i being the angular momentum of the ith electron.

We shall study how the liquid and the crystal wave functions at $\nu=1/5$ compare to each other and to the exact wave function as a function of N. In addition, we will also consider a trial wave function which is a mixture of the CF liquid and CF crystal state

$$\Psi_{\mathrm{mixed}} = \alpha_{\mathrm{liquid}}\Psi_{\mathrm{liquid}} + \alpha_{\mathrm{CFC}}\Psi_{\mathrm{CFC}}. \qquad (5)$$

The coefficients of Ψ_{mixed} are determined by energy minimization in the 2×2 basis space defined by $\{\Psi_{\mathrm{liquid}},\Psi_{\mathrm{CFC}}\}$. Because Ψ_{liquid} and Ψ_{CFC} are not orthogonal, a Gram-Schmidt orthogonalization is required. To test the various trial wave functions, we evaluate their Coulomb interaction energy per particle and their overlaps with the exact wave function, defined as

$$E = \frac{1}{2N}\sum_{i\neq j}\frac{\left\langle\Psi\left|\frac{1}{|\mathbf{r}_i-\mathbf{r}_j|}\right|\Psi\right\rangle}{\langle\Psi|\Psi\rangle}\left[\frac{e^2}{\epsilon\ell}\right], \qquad (6)$$

$$\mathcal{O} = \frac{|\langle\Psi_{\mathrm{ex}}|\Psi\rangle|}{\sqrt{\langle\Psi_{\mathrm{ex}}|\Psi_{\mathrm{ex}}\rangle}\sqrt{\langle\Psi|\Psi\rangle}}, \qquad (7)$$

where ϵ is the dielectric constant of the host semiconductor and ℓ is the magnetic length. In the above formulas, Ψ could be the mixed, CFC, or liquid wave function. All multidimensional integrals are evaluated by the Metropolis Monte Carlo method.

TABLE I. Interaction energy per particle for the exact (E_{ex}), mixed (E_{mixed}), CFC (E_{CFC}), and liquid (E_{liquid}) wave functions. The energy calculated by CF diagonalization is in the column labeled by E_{CFD}. E_{EC} denotes the energy of the electron crystal wave function $\chi(L^{*})$ (definition in the text), calculated exactly. The unit of energy is $e^2/\epsilon\ell$, where ϵ is the dielectric constant of the host semiconductor and ℓ is the magnetic length. D is the full Fock space dimension in the lowest LL. D_{CFD} is the number of linearly independent CF basis functions. Numbers shown in parentheses are statistical uncertainties arising from Monte Carlo sampling. E_{ex} and E_{liquid} have been calculated earlier (Refs. 17–20) for several values of N; E_{CFC} for $N=6$ was given in Ref. 3; E_{EC} was reported for $N=6$ and 7 in Refs. 17 and 18; these have been reproduced here for completeness.

N	L	D_{ex}	D_{CFD}	E_{ex}	E_{CFD}	E_{mixed}	E_{CFC}	E_{liquid}	E_{EC}
3	15	19	5	0.18416	0.184166(5)	0.18417(4)	0.18417(4)	0.184907(9)	0.18450
4	30	169	10	0.25085	0.250863(6)	0.25094(4)	0.25101(4)	0.252294(8)	0.25202
5	50	1747	17	0.31229	0.312327(10)	0.31245(3)	0.31291(2)	0.313315(9)	0.31527
6	75	19858	28	0.36698	0.367090(16)	0.36709(3)	0.36745(1)	0.368218(9)	0.36993
7	105	239691	43	0.41634	0.416550(26)	0.41636(2)	0.41665(2)	0.418647(8)	0.41878
8	140	3023010	65	0.46425	0.464523(27)	0.46454(2)	0.46484(3)	0.466047(7)	0.46762
9	180	39405777	95		0.509703(30)	0.5097(3)	0.51039(4)	0.510799(9)	
10	225	527142353	137		0.552117(46)	0.5529(6)	0.5606(1)	0.553157(9)	
11	275	7200240116	193		0.592390(38)	0.5925(5)	0.59425(9)	0.593527(7)	
12	330	100048974197	270		0.631012(63)	0.6321(6)	0.6382(7)	0.632143(3)	

We calculate the exact ground state energy and wave function by Lanczos algorithm for $N\leqslant 8$. For larger systems, the dimension of the Fock space becomes too large for an exact calculation. To obtain an independent reference state for $N\geqslant 9$, we use the method of CF diagonalization,[15] wherein the Coulomb interaction is diagonalized within a correlated CF basis. By increasing the basis size gradually, one can improve the approximation systematically. This method has been shown to produce very accurate energies.[16] For example, for $N=7$ at $1/5$, CF diagonalization with a CF basis of dimension 43 produces a ground state energy of $0.416\,550(26)e^2/\epsilon\ell$, which is within 0.05% of the exact energy calculated by diagonalizing the Hamiltonian using 239 691 basis states. Similar level of accuracy is expected for larger systems studied here.

Some of the technical details of our Monte Carlo calculations are as follows. In the energy calculation, a single data point is obtained by averaging over ten independent Monte Carlo runs with typically 10^6-10^7 iterations each. The quoted statistical uncertainty is one standard deviation obtained from the ten independent runs. The liquid state, which contains a single determinant, can be evaluated straightforwardly. The evaluation of the CFC state, however, is more time consuming because it contains many determinants. As an example, the $N=12$ CFC state contains 363 583 determinants.

III. RESULTS AND DISCUSSION

The results of our calculations are summerized in Tables I and II and Figs. 1 and 2.[21] Table I shows the interaction energy per particle for the exact state and all candidate trial wave functions as well as the CF diagonalization energy. The total energy also has contributions from electron-background and background-background interactions; because these contributions are common for all wave functions for a given system size, they are irrelevant in the comparison and are not included in the results shown. In Table II, the overlap of various trial wave functions with the exact ground state are tabulated for $N\leqslant 7$. In the following, we discuss the implications of these results.

For small systems, the CF crystal is superior to the CF liquid at $\nu=1/5$. One of the principal results of our calculation is that for $N\geqslant 10$ particles, the liquid wave function has lower energy. This is a clear indication that the ground state is a liquid in the thermodynamic limit, which is further strengthened by the observation that the crystal is expected to be particularly stable for particle numbers $N=10$ and $N=12$, because here the lowest energy classical crystal has perfect local hexagonal symmetry.[14] We also note that the *electron* crystal is always worse than the CF crystal, and also worse than Laughlin's wave function for $N\geqslant 5$; the electron crystal thus underestimates the correlation length of the solid order at $\nu=1/5$.

Furthermore, for small systems ($N\leqslant 7$), the mixed wave function accurately captures the nature of the exact state, as

TABLE II. The overlap of the mixed ($\mathcal{O}_{\text{mixed}}$), CFC ($\mathcal{O}_{\text{CFC}}$), and liquid ($\mathcal{O}_{\text{liquid}}$) state with the exact state. Numbers shown in the parentheses are statistical uncertainties. The overlap of the CFC state with the exact state for $N=6$ was calculated in Ref. 3. The overlap of the exact state with liquid state was shown for $N=4$ in Ref. 9 and that for $N=6$ was shown in Refs. 3 and 17. Reference 18 calculated the overlap of the exact state with liquid state for $N=7$. (In comparing with previous works, the reader may note that sometimes squares of overlaps are quoted.)

N	L	$\mathcal{O}_{\text{mixed}}$	$\mathcal{O}_{\text{CFC}}$	$\mathcal{O}_{\text{liquid}}$
3	15	0.999982(2)	0.999914(2)	0.98554(9)
4	30	0.9970(2)	0.99428(2)	0.94746(8)
5	50	0.9906(3)	0.9542(5)	0.9098(3)
6	75	0.9916(5)	0.9421(5)	0.8378(8)
7	105	0.9712(1)	0.9501(8)	0.760(3)

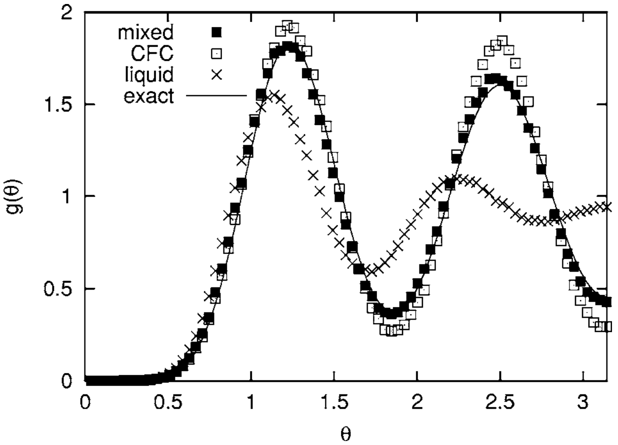

FIG. 2. The pair correlation function $g(\theta)$ of the mixed wave function (solid square), CF crystal (empty square), liquid wave function (cross), and the exact wave function (solid line) at $\nu = 1/5$ for $N=6$ particles. We fix a particle at the position $(5.430\ell,0)$ and plot the pair correlation on a semicircle of radius $R=5.430\ell$ as a function of the radial angle θ.

evident from Tables I and II. Figures 1 and 2 illustrate that it also gives a nearly exact account of the pair correlation function for $N=6$ particles, defined as

$$g(\mathbf{r},\mathbf{R}) \sim \int \prod_{j=3}^{N} d\mathbf{r}_j |\Psi(\mathbf{r},\mathbf{R},\mathbf{r}_3,\ldots,\mathbf{r}_N)|^2, \tag{8}$$

which gives the conditional probability of finding a particle at **r** while one particle is held fixed at **R**. For the system of our interest, we choose $\mathbf{R}=(5.430\ell,0)$ where the density of the system has a local maximum. For a more detailed comparison, we plot the same conditional probability for points on a semicircle of radius $r=5.430\ell$ in Fig. 2.

Reference 16 has shown that the exact state can also be approximated as $\Psi_{\text{exact}} \approx \Psi_{\text{liquid}} + \Psi_{\Lambda}$, where the last term denotes part of the wave function involving Λ level mixing of composite fermions carrying four vortices each. What we have shown here is that the Λ level mixing precisely creates, at $\nu=1/5$, correlations that produce a CF crystallite. It ought to be stressed that, given the liquid nature of the state at large N, our mixed wave function is appropriate only for system sizes that are not large compared to the correlation length of the crystalline order; it can be generalized to larger systems by introducing a correlation length for the crystal part of Eq. (5), but that is beyond the scope of this work.

The incipient crystalline correlations existing in the 1/5 state suggest a high susceptibility to formation of a crystal. We speculate on several possible observable consequences. An exciting possibility raised by our work is that the ever present disorder may create localized CF crystallites in the $\nu=1/5$ state rather than localized quasiparticles or quasiholes. Such crystallites will have their own excitation modes, which may have a lower energy than the magnetoroton excitation of a pure state. It would also be interesting to ask if screening by such crystallites can lead to a reduction in the gap to charged excitations.

Our result also implies that the 1/5 liquid is close to the CF crystal in the phase diagram, consistent with the presence of a crystal at nearby filling factors. Spivak and Kivelson[22] have proposed that the liquid to solid transition in two dimensions generically proceeds through a "microemulsion" phase containing pieces of solid floating in the liquid phase. Such a phase might also be relevant for the CF liquid-CF crystal transition as a function of filling factor in the vicinity of $\nu=1/5$. Should that be the case, that would result in enhanced drag in bilayer systems in the transition region.[23]

In summary, we have established that there are strong short range crystalline correlations present in the 1/5 state, extending over several (~3–4) lattice constants, which are well described in terms of a CF crystal. The competition between the liquid and solid orders results in strong finite size effects, which makes it necessary to go to fairly large systems ($N \geqslant 10$) before one can ascertain the true thermodynamic nature of the state. Possible experimental consequences are listed.

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