Bilayer self-dual Josephson junction arrays

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We study the zero-temperature physics of coupled bilayer Josephson junction arrays in the self-dual approximation in the presence of external offset charges and magnetic fluxes. Using a Landau-Ginzburg formulation we describe the effect of the electric and magnetic topological excitations on the quantum phase structure of the self-dual model. Through the condensation of boson fields made up of a whole number of topological excitations, this approach captures with ease various phases of Josephson junction arrays including the superconducting phase and the insulating phase, in addition to quantum Hall phases.

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Two-dimensional arrays of Josephson junctions (JJA) exhibit rich quantum dynamics as a result of the competition between two characteristic energy scales: the Josephson energy E_J, associated with the tunneling of Cooper pairs between neighboring islands, and the charging energy E_c, which is the energy needed to add an extra charge to a neutral island.[1] Furthermore, many interesting quantum phases emerge in these systems as one introduces the frustration due to applied magnetic fields and to external offset charges. Among some possible phases discussed in the literature of JJA systems are incompressible quantum fluid states corresponding to quantum Hall states for either charges or vortices.[2–6] These proposals are of paramount interest since they imply the realization of a quantum Hall effect in bosonic systems, which opens different fundamental theoretical understandings.

In the limit $E_c \gg E_J$, Ref. 5 suggests that a dilute Cooper pair fluid in a magnetic field forms Laughlin-type incompressible states. These states give rise to the quantization of the Hall conductance, $\sigma_{xy}=4e^2\nu/h$, where the filling factor ν is the ratio between the charge density q and the magnetic frustration f. In the opposite limit $E_J \gg E_c$, vortices condense[2–4] to form a quantum Hall fluid and the Hall conductivity is also quantized $\sigma_{xy}=2m4e^2/h$. These studies exploit the dual role played by vortices and charges and make use of analogies with the fractional quantum Hall effect in semiconductor heterojunctions. In the analysis of the latter systems, quantum or topological order has been extensively used leading to an elegant explanation of their robust ground state degeneracy and showing the intimate connection between the ground state degeneracy and the anomalous statistics of quasiparticles.[7]

In the context of one-layer JJA networks, the gauge theory formulation of these systems was developed in Ref. 6 and was crucial in showing that these systems support naturally topological ordered states—that Abelian gauge theory gives rise in the self-dual approximation to a periodic mixed Chern-Simons term describing/the charge-vortex coupling. In this frame-work the authors of Ref. 6 also discussed the Hall phases for charges and vortices in the presence of external offset charges and magnetic fluxes showing how the periodicity of the charge-vortex coupling can lead to transitions to anyone superconductivity phase.

In Ref. 8 I re-examined the self-dual approximation of JJA systems in the framework of a new Landau-Ginzburg theory. I showed that quantum disordering[9] of the topological electric and magnetic excitations in the model describes very effectively various phases of this system. One major result of that work is that the quantum Hall phase results from the condensation of composite boson fields made up of a whole number of electric and magnetic topological excitations. Contrary to previous studies,[2–5] the approach in Ref. 8 did not rely on the addition of any Chern-Simons term.

In this paper I extend the same analysis done in Ref. 8 to bilayer Josephson Junction arrays with the aim of describing quantum Hall phases and coherent states. An appealing feature of the coupling two layers of Josephson junction arrays is that the layer index adds a new degree of freedom which leads to a rich phase space and the possibility to tune independently the interlayer capacitance and thereby to control the interaction between vortices and charges. Similar capacitively coupled JJA systems have been investigated in Refs. 10 and 11 with the aim of showing the existence of a duality between charges and vortices. The focus in Ref. 10 was on the situation where one array is in the quasiclassical (vortex) regime while the other is in the quantum (charge) regime. The resulting effective action describes dual charges in one array and vortices in the other, and in contrast to the one-layer problem, these are dynamic degrees of freedom. In Ref. 12, I have investigated these bilayer JJA in the regime when the Josephson coupling is larger than the charging energy. The vortex dynamics was shown to resemble that of massive charged particles interacting with gauge fields, and an analysis of the vortices conductance showed on the one hand the quantization of the Hall conductance in each layer and the Hall drag conductance. On the other hand, the longitudinal vortex conductivity exhibited an enhanced drag effect, resulting into two mirror currents involving pairs of vortices and antivortices bound by the electrostatic energy coupling capacitance C_I.

Contrary to previous studies, here the coupling between the two layers is introduced by imposing from the outset a self-duality between the charges and vortices. The resulting charge-charge correlations comprise intralayer and interlayer interactions that are similar to those introduced in Refs. 10–12, as well as intralayer and interlayer vortex-vortex interactions that emerge from the high degree of symmetry in the model. The resulting dynamics is rich and leads to a large class of possible states.

I consider a bilayer JJA system in the limit where the nearest-neighbors capacitance C dominates the self-charging capacitance C_0. I include an external magnetic field B_{ext} to induce vortices in the system, and I allow for offset charges on the superconducting grains $Q_x=2e\bar{n}_x$. I model this system, in the self-dual approximation, with an effective Abelian gauge theory describing the fluctuations of charges and vortices. Similar to the one-layer JJA system,[6] the self-dual approximation consists of enlarging the near-duality between the charge and vortex degrees of freedom by adding a bare kinetic term for the vortices. Together with the already existing kinetic term for the charges, these two kinetic terms contribute to form two standard Maxwell terms for the two effective gauge fields embodying the original degrees of freedom. Assuming self-duality between charges and vortices from the start might be seen as a strong statement and warrants some justifications. First one remarks that a kinetic term for the vortices is inevitably induced in the effective action of the system once one integrates out the charge degrees of freedom. Furthermore, one would expect such an approximation to be valid at low energies compared to the two relevant energy scales in the model $(E \ll E_C, E_J)$ given that the duality-breaking terms are suppressed in that energy regime. Alternatively, one might adopt a drastic viewpoint: imposing self-duality allows a suitable gauge theory representation of the model of interacting charges and vortices, and generates especially in the bilayer JJA system the most general interactions between the charges and the vortices. Moreover, in the absence of small parameters to help with the analysis of strong correlations in quantum Josephson arrays, the exploitation of self-dual symmetry and the associated simplification related to the low-energy limit turn out to be very useful theoretical tools for analyzing these systems.

For the bilayer JJA system, the continuum model is formulated in terms of four gauge fields $a_\mu^{(\alpha)}$ and $b_\mu^{(\alpha)}$, $\mu=0,1,2$ is the gauge field component index, and $\alpha=1,2$ is the layer index. These fields describe the conserved currents of charges $(\frac{1}{2}\pi)\varepsilon^{\mu\nu\lambda}\partial_\nu b_\lambda^{(\alpha)}$, and the the conserved currents of vortices $(\frac{1}{2}\pi)\varepsilon^{\mu\nu\lambda}\partial_\nu a_\lambda^{(\alpha)}$.[12] As stated above the coupling between various degrees is imposed by the self-dual symmetry. Here, we generalize the procedure adopted in Ref. 6 for the one-layer JJA to the case of a bilayer JJA. The resulting dynamics is governed by the imaginary-time Lagrangian

$$L=\frac{\kappa_1}{2}[f_i^{(\alpha)}]^2+\frac{\kappa_2}{2}[F_i^{(\alpha)}]^2+i\eta_{\alpha\beta}(b_\mu^{(\alpha)}+\bar{b}_\mu^{(\alpha)})\varepsilon^{\mu\nu\lambda}\partial_\nu(a_\lambda^{(\beta)}+\bar{a}_\lambda^{(\beta)})+ia_\mu^{(\alpha)}Q_\mu^{(\alpha)}+ib_\mu^{(\alpha)}M_\mu^{(\alpha)}, \qquad (1)$$

where $f_i^{(\alpha)}=\varepsilon^{ij}(\partial_j b_o^{(\alpha)}-\partial_o b_j^{(\alpha)})$, $F_i^{(\alpha)}=\varepsilon^{ij}(\partial_j a_o^{(\alpha)}-\partial_o a_j^{(\alpha)})$, and the background gauge potentials $\bar{a}^{(\alpha)}$ and $\bar{b}^{(\alpha)}$ account for the frustration due to an external magnetic field B_{ext} and to offset charges $\bar{n}_q$, and are defined by $\bar{a}_0^{(\alpha)}=0$, $\partial_o\bar{a}_j^{(\alpha)}=0$, $\bar{b}_0^{(\alpha)}=0$, $\partial_o\bar{b}_j^{(\alpha)}=0$, $\vec{\nabla}\wedge\bar{a}^{(1)}=\vec{\nabla}\wedge\bar{a}^{(2)}=-2eB_{ext}$, and $\vec{\nabla}\wedge\bar{b}^{(\alpha)}=2\pi\bar{n}_q^{(\alpha)}$. The coupling constants κ_1 and κ_2 are related to the Josephson energy and to the charging energy $\kappa_1=1/(4\pi^2E_J)$, $\kappa_2=1/(8E_c)$, and $\eta_{\alpha\beta}$ in the mixed Chern-Simons term is a symmetric matrix that we define below.

The Lagrangian in Eq. (1) displays a high degree of symmetry between the charges and the vortices in both layers. The intralayer and interlayer charge-charge and vortex-vortex interactions can be made more explicit after integrating over the temporal components $a_o^{(\alpha)}$ and $b_o^{(\alpha)}$. In the Coulomb gauge such an integration gives

$$L_{charge\text{-}charge}=\frac{(2e)^2}{2}(n^{(\alpha)}+\bar{n}_q^{(\alpha)})[C^{-1}]_{\alpha\beta}(n^{(\beta)}+\bar{n}_q^{(\beta)}), \qquad (2)$$

$$L_{vortex\text{-}vortex}=2\pi^2E_JC(\ell^{(\alpha)}+\bar{\ell})[C^{-1}]_{\alpha\beta}(\ell^{(\beta)}+\bar{\ell}), \qquad (3)$$

where $2\pi n^{(\alpha)}=\vec{\nabla}\wedge\vec{b}^{\alpha}$ is the charge density, $2\pi\ell^{(\alpha)}=\vec{\nabla}\wedge\vec{a}^{\alpha}$. In momentum space, the inverse matrix $\hat{C}^{-1}$ is related to the matrix $\hat{\eta}$ appearing in Eq. (1) as $\hat{C}^{-1}=4\pi^2/(q^2C)\hat{\eta}^2$ and is given by

$$\hat{C}^{-1}=\frac{1}{[C_I+Cq^2]^2-C_I^2}\begin{pmatrix} C_I+Cq^2 & C_I \\ C_I & C_I+Cq^2 \end{pmatrix}, \qquad (4)$$

here C is the capacitance of the junction assumed to be the same in each array, and C_I is the interlayer capacitance between each island in one array coupled parallel to one island (straight coupling) in the other array. Note that the interlayer capacitance C_I not only couples the layers, but also introduces a self-capacitance of each island to the ground.

The mixed Chern-Simons term encodes the interaction between the charges and the vortices. It describes both the Lorentz force exerted by the vortices on the charges and the magnus force exerted by the charges on the vortices. The variables $Q_\mu^{(\alpha)}$ and $M_\mu^{(\alpha)}$ represent the topological electric and magnetic excitations in the model. In the case of one-layer JJA, these degrees were introduced in the lattice model[6] to guarantee that the currents of charges and the current of vortices are integers. Equation (1) shows that the moving particles associated with the current $M_i^{(\alpha)}$ see a "magnetic field" $\eta_{\alpha\beta}\varepsilon^{ij}\partial_i(\bar{b}_j^{(\beta)}+b_j^{(\beta)})$ equal to the sum of the density of bosons and a fluctuating field. Similarly, the moving particles associated with the current $Q_i^{(\alpha)}$ see a "magnetic field" $\eta_{\alpha\beta}\varepsilon^{ij}\partial_i(\bar{a}_j+a_j^{\beta})$ equal to the sum of the density of fluxes and a fluctuating field. In the rest of the paper we shall investigate the role of these topological excitations and their effect on the phase structure of this model.

Next using standard $U(1)$ particle-vortex duality,[8,13] we introduce four complex scalar fields $\phi_C^{(\alpha)}$ and $\phi_M^{(\alpha)}$ to create and annihilate the topological excitations $Q_\mu^{(\alpha)}$ and $M_\mu^{(\alpha)}$ by elaborating the description in (1) to a Landau Ginzburg theory

$$L_D=\tfrac{1}{2}|[\partial_\mu-i(a_\mu^{(\alpha)}-\bar{a}_\mu)]\phi_C^{(\alpha)}|^2+\tfrac{1}{2}|[\partial_\mu-i(b_\mu^{(\alpha)}-\bar{b}_\mu^{(\alpha)})]\phi_M^{(\alpha)}|^2+V(\phi_C^{(\alpha)},\phi_M^{(\alpha)})+\frac{\kappa_1}{2}[f_i^{(\alpha)}]^2+\frac{\kappa_2}{2}[F_i^{(\alpha)}]^2+i\eta_{\alpha\beta}b_\mu^{(\alpha)}\varepsilon^{\mu\nu\lambda}\partial_\nu a_\lambda^{(\beta)}+i\frac{e}{\pi}b_\mu^{(\alpha)}\varepsilon^{\mu\nu\lambda}\partial_\nu A_\lambda^{(\alpha)}. \qquad (5)$$

The potential $V(\phi_{C,M}^{(\alpha)})$ can be expanded as $V(\phi)=r|\phi|^2+u|\phi|^4+\cdots$, and describes the short distance physics contained in the original theory (1) defined on the lattice. As usual these scalar fields are minimally coupled to the fluctuating gauge fields $a_\mu^{(\alpha)}$ and $b_\mu^{(\alpha)}$, and also to background fields $\bar{a}_\mu$ and $\bar{b}_\mu^{(\alpha)}$ that account for the frustrations in the system. The last term in Eq. (5) couples the current charges with an external probing electromagnetic field $A_\mu^{(\alpha)}$.

This dual Landau-Ginzburg representation is convenient to study bilayer JJA systems since its phase structure can be analyzed by considering the condensation of various fields. For example, we can consider a phase in which the topological charge excitations are absent in the ground state, which corresponds to taking r large and positive in the above Landau-Ginzburg description, or a phase in which these degrees created by ϕ_C proliferate and condense: $\langle \phi_C \rangle \neq 0$. More generally, one can consider a situation in which the lowest energy excitation is a composite consisting of $n^{(\alpha)}$ excitations in $\phi_C^{(\alpha)}$ and $m^{(\alpha)}$ excitations in $\phi_M^{(\alpha)}$, $\phi_{\{n,m\}} \sim \phi_C^{n^{(1)}} \phi_C^{n^{(2)}} \phi_M^{m^{(1)}} \phi_M^{m^{(2)}}$, where $n^{(\alpha)}$ and $m^{(\alpha)}$ are integers. In this case we write an effective action as

$$L_D = \frac{1}{2} |[\partial_\mu - i n^{(\alpha)} (a_\mu^{(\alpha)} - \bar{a}_\mu) - i m^{(\alpha)} (b_\mu^{(\alpha)} - \bar{b}_\mu^{(\alpha)})] \phi_{\{n,m\}}|^2 + V(\phi_{\{n,m\}}) + \frac{\kappa_1}{2} [f_i^{(\alpha)}]^2 + \frac{\kappa_2}{2} [F_i^{(\alpha)}]^2 + i \eta_{\alpha\beta} b_\mu^{(\alpha)} \varepsilon^{\mu\nu\lambda} \partial_\nu a_\lambda^{(\beta)} + i \frac{e}{\pi} b_\mu^{(\alpha)} \varepsilon^{\mu\nu\lambda} \partial_\nu A_\lambda^{(\alpha)}. \tag{6}$$

To analyze this effective action, we find it convenient to diagonalize the gauge part of the Lagrangian by using the linear transformation $a_\mu^{(\alpha)} = (X_\mu^{(\alpha)} + Y_\mu^{(\alpha)}) \sqrt{\kappa_1/\kappa_2}$, $b_\mu^{(\alpha)} = (X_\mu^{(\alpha)} - Y_\mu^{(\alpha)}) \sqrt{\kappa_2/\kappa_1}$ and by working with the symmetric and antisymmetric combinations $X_\mu^{\pm} = (X_\mu^{(1)} \pm X_\mu^{(2)})/\sqrt{2}$, $Y_\mu^{\pm} = (Y_\mu^{(1)} \pm Y_\mu^{(2)})/\sqrt{2}$. In terms of the new gauge fields the quadratic gauge part of the Lagrangian becomes in the long wavelength

$$L_g = \frac{1}{2\pi\omega_J} (\partial_i X_0^+ - \partial_0 X_i^+)^2 + \frac{i}{2\pi} X_\mu^+ \varepsilon^{\mu\nu\lambda} \partial_\nu X_\lambda^+ + \frac{1}{2\pi\omega_J} (\partial_i Y_0^+ - \partial_0 Y_i^+)^2 - \frac{i}{2\pi} Y_\mu^+ \varepsilon^{\mu\nu\lambda} \partial_\nu Y_\lambda^+ + \frac{1}{2\pi\omega_J} (\partial_i X_0^- - \partial_0 X_i^-)^2 + \frac{1}{2\pi\omega_J} (\partial_i Y_0^- - \partial_0 Y_i^-)^2, \tag{7}$$

where $\omega_J = \sqrt{8E_C E_J}$ is the Josephson plasma frequency. Note that the symmetric fields X_μ^+ and Y_μ^+ have Chern-Simons terms with opposite coefficients and describe modes with mass equal to the Josephson plasma frequency.

When there is no condensate, $\langle \phi_{\{n,m\}} \rangle = 0$, for all $n^{(\alpha)}$ and $m^{(\alpha)}$ we may drop the topological part of the action and integrate out the remaining quadratic terms in $X_\mu^{\pm}$ and $Y_\mu^{\pm}$ to achieve an effective action describing the dynamics of the probing electromagnetic field. Varying with respect to $A_\mu^{\pm}$, we obtain the response functions

$$\sigma_{xy}^{\pm} = 0, \quad \sigma_{xx}^{+} = 0, \quad \sigma_{xx}^{-} = 4e^2 E_J \frac{i}{\omega}. \tag{8}$$

So in this model a phase with zero condensates corresponds to an insulator state with interlayer coherence as evident from the zero frequency singularity.

On the other hand, when the scalar fields condense, one can explore a large number of possible states. Here we focus on the special case when there is a condensation of composite fields formed by n modes in the symmetric charge and m modes in the symmetric vortex part. We further require that the resulting composite fields see a zero background field $n\bar{a}_i^+ + m\bar{b}_i^+ = 0$, which in terms of the total density of offset charges $\bar{n}_q = \bar{n}_q^{(1)} + \bar{n}_q^{(2)}$ and external flux quanta $\bar{n}_\Phi$, translates into a filling factor $\nu = \bar{n}_q/\bar{n}_\Phi = 2n/m$. As a result of the condensation, the Anderson-Higgs effect takes place and the gauge field $na_\mu^+ + mb_\mu^+$ acquires a mass justifying a mean-field Landau-Ginzburg analysis. After integrating out all fluctuating gauge fields, we obtain an effective action of the electromagnetic field $A_\lambda^{\pm}$ from which we derive the response functions of the resulting phase

$$\sigma_{xy}^{+} = \frac{e^2 n}{\pi m}, \quad \sigma_{xx}^{+} = \left(\frac{2\pi n \kappa_1}{m} \right)^2 i\omega \to 0, \quad \sigma_{xx}^{-} = 4e^2 E_J \frac{i}{\omega}. \tag{9}$$

These equations show that the conductivity of bilayer JJA has the properties of the conductivity matrix of a quantum Hall effect (QHE) system, namely, a zero longitudinal conductivity and a quantized Hall conductivity. Furthermore, the zero frequency singularity in σ_{xx}^{-} indicates that the bilayer JJA system is coherent.

In summary, I have used the charge-vortex duality to extend a Landau-Ginzburg theory for bilayer Josephson junction arrays to discuss quantum Hall states and the interlayer coherence. This approach has the advantage of analyzing a large class of possible states. Some results obtained here confirm recent ones derived using a completely different approach.[12] I now comment briefly on the difference between both approaches. First, in Ref. 12 the bilayer Josephson junction array system was considered in the regime $E_J \gg E_C$ and the quantum Hall effect was argued from the dynamics of vortices only. Second, that analysis relied on the introduction of a Chern-Simons term that attaches an even number of fictitious cooper pairs to each vortex. The mean-field state achieved consisted of transformed bosons (composites of vortices and effective flux tubes) in zero magnetic fields due to the cancellation between the effective magnetic field associated with offset charges and the Chern-Simons magnetic field average.

The approach here is more general, it treats both types of excitations (electric and magnetic) on an equal footing by considering composite fields that describe both type of excitations. The filling factor conditions $\nu = 2n/m$ emerge in a natural way by requiring that the background field seen by the composite fields to be zero; this is to be contrasted with the Gauss law constraint resulting in a theory with a Chern-Simons gauge field added. Here no Chern-Simons term is added; I simply exploit the condensation of composite bosons of electric and magnetic topological excitations to capture with ease various phases of the model. In this paper effects of irregularities and randomness in the JJA system were not included. These may suppress the quantum Hall phase found here just as disorder extinguishes the FQHE in two-dimensional electron systems. This will be the subject of a future investigation. Finally, up to now there is no experimental evidence for the quantum Hall effect in Josephson arrays. Still the measurements of Ref. 14 indicate that the Hall effect in these systems is more complicated but exhibits some interesting characteristics such as a periodic Hall resistance with respect to the applied magnetic field and a larger Hall angle consistent with the expectation that the offset charges are responsible for the Hall effect. Notwithstanding experimental challenges, the observation of a Hall voltage in Josephson junction arrays requires very low temperature and the parameters of the array should be such that a balance is achieved between the charging energy and the Josephson energy $E_C \sim E_J$. This is crucial since in a strongly superconducting array ($E_J \gg E_C$) any Hall probes would be shorted leading to a zero Hall voltage. Whereas in an array with a strong Coulomb blockade ($E_C \gg E_J$), the whole array is insulating and therefore the Hall probes are effectively disconnected, and no Hall voltage can be measured.

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