Infrared catastrophe and tunneling into strongly correlated electron systems: Beyond the x-ray edge limit

Kelly R. Patton and Michael R. Geller

Department of Physics and Astronomy, University of Georgia, Athens, Georgia 30602-2451, USA

We develop a nonperturbative method to calculate the electron propagator in low-dimensional and strongly correlated electron systems. The method builds on our earlier work using a Hubbard-Stratonovich transformation to map the tunneling problem to the x-ray edge problem, which accounts for the infrared catastrophe caused by the sudden introduction of a new electron into a conductor during a tunneling event. Here we use a cumulant expansion to include fluctuations about this x-ray edge limit. We find that the dominant effect of electron-electron interaction at low energies is to correct the noninteracting Green's function by a factor $e^{-S/\hbar}$, where S can be interpreted as the Euclidean action for a density field describing the time-dependent charge distribution of the newly added electron. Initially localized, this charge distribution spreads in time as the electron is accommodated by the host conductor, and during this relaxation process action is accumulated according to classical electrostatics with a screened interaction. The theory applies to lattice or continuum models of any dimensionality, with or without translational invariance. In one dimension the method correctly predicts a power-law density of states for electrons with short-range interaction and no disorder, and when applied to the solvable Tomonaga-Luttinger model, the exact density of states is obtained.

I. INTRODUCTION

In the conventional many-body theoretic treatment of tunneling, the low-temperature tunneling current between an ordinary metal and a strongly correlated electron system is controlled by the single-particle density of states (DOS) of the correlated system. Tunneling experiments are therefore often used as a probe of the DOS. In a wide variety of low-dimensional and strongly correlated electron systems, including all one-dimensional (1D) metals, the two-dimensional (2D) diffusive metal, the 2D Hall fluid, and the edge of the sharply confined Hall fluid, the DOS exhibits anomalies such as cusps, algebraic suppressions, and pseudogaps at the Fermi energy. We have recently proposed that the origin of these anomalies is the infrared catastrophe caused by the response of the host electron gas to the sudden introduction of a new particle that occurs during a tunneling event.[1]

The infrared catastrophe is a singular screening response of a degenerate Fermi gas to a localized potential applied abruptly in time, which is known to be responsible for the anomalous x-ray optical and photoemission edge spectra of metals,[2,3] the Anderson orthogonality catastrophe,[4,5] and the Kondo effect.[6,7] Imagine an electron in a tunneling process being replaced by a negatively charged, distinguishable particle with mass M. In the $M \to \infty$ limit, the time-dependent potential produced by the particle being added to the origin at time $\tau = 0$ and removed at a later time τ_0 is[8]

$$\phi_{\mathrm{xr}}(\mathbf{r}, \tau) = U(\mathbf{r})\Theta(\tau)\Theta(\tau_0 - \tau), \qquad (1)$$

where U is the two-particle interaction. ϕ_{xr} is identical, up to a sign, to the abruptly turned-on hole potential of the x-ray edge problem, so an infrared catastrophe is expected. Tunneling of a real, finite-mass electron is different because it recoils, softening the potential produced. The real electron is

also an indistinguishable fermion, unable to tunnel into the occupied states below the Fermi energy.

In Ref. 1 we introduced an exact functional integral representation for the interacting propagator, and developed a nonperturbative technique for evaluating it by identifying a "dangerous" scalar field configuration of the form (1), and then treating this special field configuration by using methods developed for the x-ray edge problem. All other field configurations were ignored, thereby reducing the tunneling problem to an x-ray edge problem. Nonetheless, qualitatively correct results were obtained for the 1D electron gas and the 2D Hall fluid using this approach. In this paper we attempt to go beyond this "x-ray edge" limit by including fluctuations about ϕ_{xr}. We find that by including fluctuations through the use of a simple functional cumulant expansion, a qualitatively correct DOS is obtained for electrons with short-range interaction and no disorder in one, two, and three dimensions. We also show that when applied to the solvable Tomonaga-Luttinger model, the low-energy fixed-point Hamiltonian for most 1D metals, the exact DOS is obtained. A preliminary account of this work has been given elsewhere.[9]

II. GENERAL FORMALISM AND CUMULANT EXPANSION METHOD

We assume a grand-canonical Hamiltonian $H = H_0 + V$, where

$$H_0 \equiv \sum_\sigma \int d^D r\, \psi_\sigma^\dagger(\mathbf{r}) \left[\frac{\Pi^2}{2m} + v(\mathbf{r}) - \mu \right] \psi_\sigma(\mathbf{r}) \qquad (2)$$

is the Hartree mean-field Hamiltonian, and

$$V \equiv \frac{1}{2} \int d^D r d^D r' \, \delta n(\mathbf{r}) U(\mathbf{r} - \mathbf{r}') \delta n(\mathbf{r}'). \tag{3}$$

Here $\mathbf{\Pi} \equiv \mathbf{p} + (e/c)\mathbf{A}$, and $v(\mathbf{r})$ is any single-particle potential, which may include a periodic lattice potential or disorder or both, and which also includes the Hartree interaction with the self-consistent density $n_0(\mathbf{r})$. Note in a translationally invariant system the equilibrium density is unaffected by interactions, but in a disordered or inhomogeneous system it will be necessary to distinguish between the approximate Hartree and the exact density distributions. The interaction term is written in terms of the density fluctuation

$$\delta n(\mathbf{r}) \equiv \sum_{\sigma} \psi_{\sigma}^{\dagger}(\mathbf{r}) \psi_{\sigma}(\mathbf{r}) - n_0(\mathbf{r}). \tag{4}$$

Following Ref. 1, we use a Hubbard-Stratonovich transformation to write the exact Euclidean propagator

$$G(\mathbf{r}_f \sigma_f, \mathbf{r}_i \sigma_i, \tau_0) \equiv - \langle T \psi_{\sigma_f}(\mathbf{r}_f, \tau_0) \bar{\psi}_{\sigma_i}(\mathbf{r}_i, 0) \rangle_H, \tag{5}$$

in the form

$$G(\mathbf{r}_f \sigma_f, \mathbf{r}_i \sigma_i, \tau_0) = \mathcal{N} \frac{\displaystyle\int D\phi \, e^{-1/2(\int \phi U^{-1} \phi)} g(\mathbf{r}_f \sigma_f, \mathbf{r}_i \sigma_i, \tau_0 | \phi)}{\displaystyle\int D\phi \, e^{-1/2(\int \phi U^{-1} \phi)}}, \tag{6}$$

where

$$g(\phi) \equiv - \langle T \psi_{\sigma_f}(\mathbf{r}_f, \tau_0) \bar{\psi}_{\sigma_i}(\mathbf{r}_i, 0) e^{i \int \phi(\mathbf{r}, \tau) \delta n(\mathbf{r}, \tau)} \rangle_0 \tag{7}$$

is a noninteracting correlation function in the presence of a purely imaginary scalar potential $i\phi(\mathbf{r}, \tau)$, and $\mathcal{N} \equiv \langle T \exp(-\int_0^\beta d\tau V) \rangle_0^{-1}$ is a constant, independent of τ_0. Equation (6) is an exact expression for the interacting Green's function.

The region of function space that contributes to the functional integral in (6) is controlled by the width of the Gaussian, which in the small U limit becomes strongly localized around $\phi = 0$. By expanding (7) in powers of ϕ and evaluating the functional integral term by term, one recovers the naive perturbative expansion for $G(\mathbf{r}_f \sigma_f, \mathbf{r}_i \sigma_i, \tau_0)$ in powers of U. Therefore, it will be necessary to go beyond a perturbative expansion for $g(\mathbf{r}_f \sigma_f, \mathbf{r}_i \sigma_i, \tau_0 | \phi)$. We evaluate $g(\mathbf{r}_f \sigma_f, \mathbf{r}_i \sigma_i, \tau_0 | \phi)$ approximately, using a second-order functional cumulant expansion. Such an expansion amounts to a resummation of the most divergent terms in the perturbation series when $\phi = \phi_{xr}$ and the infrared catastrophe occurs. Indeed, one can view our resulting expression for $g(\mathbf{r}_f \sigma_f, \mathbf{r}_i \sigma_i, \tau_0 | \phi)$ as a functional generalization of Mahan's "perturbative" result for a similar correlation function.[2] Furthermore, for field configurations far from ϕ_{xr}, the cumulant expansion will yield a result that is, by construction, exact through second order in U. After carrying this out we obtain

$$g(\phi) \approx G_0(\mathbf{r}_f \sigma_f, \mathbf{r}_i \sigma_i, \tau_0) e^{\int C_1(\mathbf{r}\tau)\phi(\mathbf{r},\tau) + \int C_2(\mathbf{r}\tau, \mathbf{r}'\tau')\phi(\mathbf{r},\tau)\phi(\mathbf{r}',\tau')}, \tag{8}$$

where

$$C_1(\mathbf{r}\tau) = \frac{g_1(\mathbf{r}\tau)}{G_0(\mathbf{r}_f \sigma_f, \mathbf{r}_i \sigma_i, \tau_0)} \tag{9}$$

and

$$C_2(\mathbf{r}\tau, \mathbf{r}'\tau') = \frac{g_2(\mathbf{r}\tau, \mathbf{r}'\tau')}{G_0(\mathbf{r}_f \sigma_f, \mathbf{r}_i \sigma_i, \tau_0)} - \frac{g_1(\mathbf{r}\tau) g_1(\mathbf{r}'\tau')}{2[G_0(\mathbf{r}_f \sigma_f, \mathbf{r}_i \sigma_i, \tau_0)]^2}. \tag{10}$$

Here

$$g_n(\mathbf{r}_1 \tau_1, \ldots, \mathbf{r}_n \tau_n)$$
$$\equiv - \frac{i^n}{n!} \langle T \psi_{\sigma_f}(\mathbf{r}_f, \tau_0) \bar{\psi}_{\sigma_i}(\mathbf{r}_i, 0) \delta n(\mathbf{r}_1 \tau_1) \cdots \delta n(\mathbf{r}_n \tau_n) \rangle_0 \tag{11}$$

is the coefficient of ϕ^n appearing in the perturbative expansion of (7), as in

$$g(\mathbf{r}_f \sigma_f, \mathbf{r}_i \sigma_i, \tau_0 | \phi) = G_0(\mathbf{r}_f \sigma_f, \mathbf{r}_i \sigma_i, \tau_0)$$
$$+ \sum_{n=1}^{\infty} \int g_n(\mathbf{r}_1 \tau_1, \ldots, \mathbf{r}_n \tau_n) \phi(\mathbf{r}_1, \tau_1) \cdots \phi(\mathbf{r}_n, \tau_n). \tag{12}$$

The cumulants C_1 and C_2 in terms of G_0 are

and (suppressing spin for clarity)

$$C_2(\mathbf{r}\tau, \mathbf{r}'\tau') = \frac{1}{2 G_0(\mathbf{r}_f, \mathbf{r}_i, \tau_0)^2} \{ [G_0(\mathbf{r}, \mathbf{r}', \tau - \tau') G_0(\mathbf{r}_f, \mathbf{r}_i, \tau_0) $$
$$- G_0(\mathbf{r}, \mathbf{r}_i, \tau) G_0(\mathbf{r}_f, \mathbf{r}', \tau_0 - \tau')][\mathbf{r} \leftrightarrow \mathbf{r}', \tau \leftrightarrow \tau'] \}. \tag{14}$$

The functional integral in (6) can now be done exactly, leading to

$$G(\mathbf{r}_f \sigma_f, \mathbf{r}_i \sigma_i, \tau_0) = \mathcal{A}(\tau_0) G_0(\mathbf{r}_f \sigma_f, \mathbf{r}_i \sigma_i, \tau_0) e^{-S(\tau_0)}, \tag{15}$$

where

$$\mathcal{A} \equiv \mathcal{N} \frac{\displaystyle\int D\phi \, e^{-1/2[\int \phi(U^{-1} - 2C_2)\phi]}}{\displaystyle\int D\phi \, e^{-1/2(\int \phi U^{-1} \phi)}} = \mathcal{N} [\det(1 - 2C_2 U)]^{-1/2}, \tag{16}$$

and

$$S \equiv \frac{1}{2} \int_0^\beta d\tau \, d\tau' \int d^D r \, d^D r' \, \rho(\mathbf{r}, \tau) U_{\text{eff}}(\mathbf{r}\tau, \mathbf{r}'\tau') \rho(\mathbf{r}', \tau'). \tag{17}$$

Here

$$\rho(\mathbf{r}, \tau) \equiv - i C_1(\mathbf{r}\tau) \tag{18}$$

$$= -\sum_\sigma \frac{G_0(\mathbf{r}_f\sigma_f, \mathbf{r}\sigma, \tau_0 - \tau)G_0(\mathbf{r}\sigma, \mathbf{r}_i\sigma_i, \tau)}{G_0(\mathbf{r}_f\sigma_f, \mathbf{r}_i\sigma_i, \tau_0)} \quad (19)$$

and

$$U_{\mathrm{eff}}(\mathbf{r}\tau, \mathbf{r}'\tau') \equiv [U^{-1}(\mathbf{r} - \mathbf{r}')\delta(\tau - \tau') - 2C_2(\mathbf{r}\tau, \mathbf{r}'\tau')]^{-1} \quad (20)$$

is a screened interaction.

Because spin-orbit coupling has been neglected in H, the noninteracting Green's function is diagonal in spin, and

$$\rho(\mathbf{r}, \tau) = -\frac{G_0(\mathbf{r}_f\sigma_i, \mathbf{r}\sigma_i, \tau_0 - \tau)G_0(\mathbf{r}\sigma_i, \mathbf{r}_i\sigma_i, \tau)}{G_0(\mathbf{r}_f\sigma_i, \mathbf{r}_i\sigma_i, \tau_0)}\delta_{\sigma_i\sigma_f}. \quad (21)$$

Equation (15) is the principal result of this work.

III. CHARGE SPREADING INTERPRETATION

We interpret (15) as follows: S is the Euclidean action[10] for a time-dependent charge distribution $\rho(\mathbf{r}, \tau)$. We shall show that $\rho(\mathbf{r}, \tau)$ acts like the charge density associated with an electron being inserted at $\mathbf{r}_i$ at $\tau = 0$ and removed at $\mathbf{r}_f$ at τ_0. This charge density interacts via an effective interaction $U_{\mathrm{eff}}(\mathbf{r}\tau, \mathbf{r}'\tau')$ that accounts for the modification of the electron-electron interaction by dynamic screening.[11] Our result can therefore be regarded as a variant of the intuitive but phenomenological charge spreading picture of Spivak[12] and of Levitov and Shytov.[13] However, here the dynamics of $\rho(\mathbf{r}, \tau)$ is completely determined by the mean-field Hamiltonian, and has the dynamics of essentially noninteracting electrons.

First consider the integrated charge,

$$Q(\tau) \equiv \int d^D r \rho(\mathbf{r}, \tau) = -\frac{\int d^D r G_0(\mathbf{r}_f, \mathbf{r}, \tau_0 - \tau)G_0(\mathbf{r}, \mathbf{r}_i, \tau)}{G_0(\mathbf{r}_f, \mathbf{r}_i, \tau_0)}. \quad (22)$$

Using exact eigenfunction expansions for the Green's functions we obtain

$$Q(\tau) = -\frac{\sum_\alpha \Phi_\alpha^*(\mathbf{r}_f)\Phi_\alpha(\mathbf{r}_i)e^{-(\epsilon_\alpha - \mu)\tau_0}\{[n_{\mathrm{F}}(\epsilon_\alpha - \mu) - 1]^2\Theta(\tau_0 - \tau)\Theta(\tau) + n_{\mathrm{F}}(\epsilon_\alpha - \mu)[n_{\mathrm{F}}(\epsilon_\alpha - \mu) - 1][\Theta(-\tau) + \Theta(\tau - \tau_0)]\}}{\sum_\alpha \Phi_\alpha^*(\mathbf{r}_f)\Phi_\alpha(\mathbf{r}_i)e^{-(\epsilon_\alpha - \mu)\tau_0}[n_{\mathrm{F}}(\epsilon_\alpha - \mu) - 1]}, \quad (23)$$

where n_{F} is the Fermi distribution function and the Φ_α are the single-particle eigenfunctions of H_0. In the zero-temperature limit

$$Q(\tau) = -\frac{\sum_\alpha \Phi_\alpha^*(\mathbf{r}_f)\Phi_\alpha(\mathbf{r}_i)e^{-(\epsilon_\alpha - \mu)\tau_0}\{(N_\alpha - 1)^2\Theta(\tau_0 - \tau)\Theta(\tau) + N_\alpha(N_\alpha - 1)[\Theta(-\tau) + \Theta(\tau - \tau_0)]\}}{\sum_\alpha \Phi_\alpha^*(\mathbf{r}_f)\Phi_\alpha(\mathbf{r}_i)e^{-(\epsilon_\alpha - \mu)\tau_0}(N_\alpha - 1)}, \quad (24)$$

where N_α is the ground-state occupation number of state α, which in the absence of ground-state degeneracy takes the value of 0 or 1. In this case (24) reduces to

$$Q(\tau) = \Theta(\tau_0 - \tau)\Theta(\tau). \quad (25)$$

When the sum rule (25) holds, the net added charge, as described by $\rho(\mathbf{r}, \tau)$, is unity (in units of the electron charge) for times between 0 and τ_0, and zero otherwise. This behavior correctly mimics the action of the field operators in (5).

At short times, $\tau \ll \tau_0$, the charge density is approximately

$$\rho(\mathbf{r}, \tau) \sim -G_0(\mathbf{r}, \mathbf{r}_i, \tau), \quad (26)$$

which is localized around $\mathbf{r} = \mathbf{r}_i$. As time evolves this distribution relaxes. Then as τ approaches τ_0 the charge density again becomes localized around $\mathbf{r} = \mathbf{r}_f$,

$$\rho(\mathbf{r}, \tau) \sim -G_0(\mathbf{r}_f, \mathbf{r}, \tau_0 - \tau). \quad (27)$$

A plot of $\rho(\mathbf{r}, \tau)$ for the 1D electron gas is given in Fig. 1.

The dynamics of $\rho(\mathbf{r}, \tau)$ can be shown to be governed by the equation of motion

$$[\partial_\tau + H_0(\mathbf{r})]\rho(\mathbf{r}, \tau) = -\delta(\mathbf{r}_f - \mathbf{r})\delta(\tau_0 - \tau) + \delta(\mathbf{r} - \mathbf{r}_i)\delta(\tau). \quad (28)$$

This can be seen by noting that the noninteracting Green's function satisfies

$$[\partial_\tau + H_0(\mathbf{r})]G_0(\mathbf{r}, \mathbf{r}', \tau, \tau') = -\delta(\mathbf{r} - \mathbf{r}')\delta(\tau - \tau'). \quad (29)$$

Then, by using the definition (19) one can obtain (28). Again, we stress that $\rho(\mathbf{r}, \tau)$ describes the dynamics of noninteracting electrons, governed by the mean-field Hamiltonian H_0, which in the absence of disorder or inhomogeneity reduces to the noninteracting Hamiltonian (which is the case for the systems considered here).

Although $\rho(\mathbf{r}, \tau)$ has many properties that make it reasonable to interpret as the charge density associated with the added and subsequently removed electron described in (5), one should not take this interpretation too literally. The tunneling electron is an indistinguishable particle, so one cannot ascribe $\rho(\mathbf{r}, \tau)$ to "the" tunneling particle. (Our formalism, however, does treat the indistinguishability exactly.) Also it
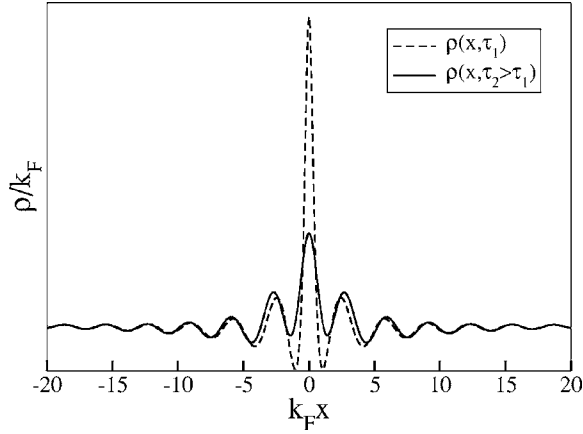
FIG. 1. Charge density $\rho(x,\tau)$ for the 1D electron gas at two times, showing Friedel oscillations and gradual spreading.

should be noted that the sum rule (25) holds only in the zero-temperature limit and in the absence of ground-state degeneracy. And as will be seen in Sec. IV C, $\rho(\mathbf{r},\tau)$ may even be complex valued.

IV. APPLICATIONS OF THE CUMULANT METHOD

In the following examples we will assume electrons with a short-range interaction U, no disorder, and no magnetic field. In Sec. IV A we show that our method correctly predicts a constant DOS near the Fermi energy in 2D and in three dimensions (3D), and in Sec. IV B we obtain a power-law DOS in 1D, in qualitative agreement with Luttinger liquid theory.[14–20] Finally, in Sec. IV C we use our method to calculate the DOS for the solvable Tomonaga-Luttinger model, obtaining the exact DOS exponent.

A. 2D and 3D electron gas: Recovery of the Fermi liquid phase

The sum rule (25) allows one to determine the energy dependence of the DOS in D dimensions, asymptotically in the low-energy limit, as follows: In the absence of disorder, a droplet of charge injected into a degenerate Fermi gas with Fermi velocity v_F will relax to a size of order $\ell \sim v_F \tau$ after a time τ. Approximating $\rho(\mathbf{r},\tau)$ to have uniform magnitude in a region of size ℓ and vanishing elsewhere, the sum rule then requires the magnitude of ρ to vary as ℓ^{-D}. The interaction energy of such a charge distribution (assuming a short-range interaction) is

$$E = \frac{U}{2} \int d^D r [\rho(\mathbf{r})]^2 \sim \frac{U}{\ell^D}, \tag{30}$$

which varies with time as τ^{-D}. The action accumulated up to time τ_0 therefore scales as

$$S \sim \frac{U}{\tau_0^{D-1}} \quad \text{if } D \geqslant 2 \tag{31}$$

or

$$S \sim U \ln \tau_0 \quad \text{if } D = 1. \tag{32}$$

The cases (31) and (32) are dramatically different: In 2D and 3D the action vanishes at long times, and the propagator (15) is therefore not appreciably affected by interactions. The resulting DOS is energy independent at low energies, and the expected Fermi liquid behavior is recovered. In 1D, however, the action diverges logarithmically, leading to an algebraic DOS.

B. 1D electron gas: Recovery of the Luttinger liquid phase

The scaling argument of the previous section showed that the DOS in the 1D electron gas with short-range interaction is algebraic, as expected. In this section we calculate the associated exponent.

We proceed in two stages. Initially we keep only the first cumulant C_1, and then afterwards we discuss the effect of C_2. In 1D it is possible to evaluate the action (17) exactly in the long-time asymptotic limit at the first-cumulant level. Setting $x_f = x_i = 0$, we have [see (15)]

$$G(\tau_0) = \text{const} \times G_0(\tau_0)e^{-S(\tau_0)}. \tag{33}$$

Considering a local interaction of the form $U(x-x') = U_0 \lambda \delta(x-x')$ the action is

$$S(\tau_0) = \frac{U_0 \lambda}{2} \int_0^\beta d\tau \int_{-\infty}^{\infty} dx [\rho(x,\tau)]^2. \tag{34}$$

By linearizing the spectrum around the Fermi energy, the zero-temperature propagator at low energy is

$$G_0(x,\tau) = \frac{1}{\pi} \text{Im}\left(\frac{e^{ik_F x}}{x + iv_F \tau} \right) = \frac{x \sin k_F x - v_F \tau \cos k_F x}{\pi(x^2 + v_F^2 \tau^2)}. \tag{35}$$

The charge density (19) in this case is

$$\rho(x,\tau) = \frac{v_F \tau_0}{\pi} \frac{x \sin k_F x - v_F(\tau_0 - \tau)\cos k_F x}{x^2 + v_F^2(\tau - \tau_0)^2}$$
$$\times \frac{x \sin k_F x - v_F \tau \cos k_F x}{x^2 + v_F^2 \tau^2}. \tag{36}$$

Figure 1 shows the charge density $\rho(x,\tau)$ as it spreads in time from its initially localized position.

By a lengthy but straightforward calculation it can be shown that

$$S(\tau_0) = \frac{3}{8} \frac{U_0 \lambda}{v_F \pi} \ln\left(\frac{\tau_0}{a} \right), \tag{37}$$

where a is a microscopic cutoff. This leads to a power-law decay of the interacting propagator as

$$G(\tau) \sim \frac{1}{\tau^\alpha}, \tag{38}$$

where

$$\alpha = \frac{3}{8} \frac{U_0 \lambda}{v_F \pi} + 1 \tag{39}$$

is the propagator exponent. The DOS exponent δ, defined as

$$N(\epsilon) = \text{const} \times \epsilon^{\delta}, \qquad (40)$$

is given in this case by

$$\delta = \frac{3}{8} \frac{U_0 \lambda}{v_F \pi}. \qquad (41)$$

The effects of the second cumulant C_2 are now straightforward to understand: In addition to introducing a slowly varying prefactor $\mathcal{A}$, whose only τ_0 dependence comes from the time dependence of the screening in (20), the second cumulant screens the bare interaction and does not prevent the logarithmic divergence of the action, but it does modify the DOS exponent. It is interesting, however, that in the large U limit, the effective interaction becomes independent of U, a clear indication of nonperturbative behavior.

It is illustrative to compare (41) to the prediction of the perturbative x-ray edge limit of Ref. 1, where one neglects all field configurations in (6) except ϕ_{xr}. There we found (for this same short-range interaction model),

$$\delta = 2 \frac{U_0 \lambda}{v_F \pi} \qquad (42)$$

to leading order in U. The DOS exponents (41) and (42) are in qualitative agreement, but the inclusion of fluctuations about ϕ_{xr} in (41) softens the exponent by almost a factor of 4, as one might expect.

The exact DOS exponent is not known for this model. In the next section we apply the cumulant method to the Tomonaga-Luttinger model, for which the exact propagator can be calculated using bosonization.

C. Tomonaga-Luttinger model

We consider the spinless or $U(1)$ Tomonaga-Luttinger model. The noninteracting spectrum is

$$\epsilon_k = \mu + v_F(\pm k - k_F), \qquad (43)$$

where the upper sign refers to the right branch and the lower to the left one. The interaction is

$$V = \frac{1}{2} \int dx \, \delta n_i(x) U_{ij} \delta n_j(x), \qquad (44)$$

$$\delta n_{\pm}(x) \equiv \lim_{a \to 0} :\psi_{\pm}(x+a)\psi_{\pm}(x):, \qquad (45)$$

where the normal ordering is with respect to the noninteracting ground state. The matrix U has the form

$$\mathbf{U} = \begin{pmatrix} U_4 & U_2 \\ U_2 & U_4 \end{pmatrix}. \qquad (46)$$

We want to calculate

$$G_{\pm}(x_f \tau_f, x_i \tau_i) = -\mathcal{N} \langle T \psi_{\pm}(x_f, \tau_f) \bar{\psi}_{\pm}(x_i, \tau_i) e^{-\int d\tau V(\tau)} \rangle. \qquad (47)$$

Make a Hubbard-Stratonovich transformation of the form

$$e^{-1/2(\int \delta n_i U_{ij} \delta n_j)} = \frac{\int D\phi_- D\phi_+ e^{-1/2(\int \phi_i U_{ij}^{-1} \phi_j)} e^{i(\int \phi_i \delta n_i)}}{\int D\phi_- D\phi_+ e^{-1/2(\int \phi_i U_{ij}^{-1} \phi_j)}}, \qquad (48)$$

which leads to

$$G_{\pm}(x_f \tau_f, x_i \tau_i)$$
$$= \mathcal{N} \frac{\int D\phi_- D\phi_+ e^{-1/2(\int \phi_i U_{ij}^{-1} \phi_j)} g_{\pm}(x_f \tau_f, x_i \tau_i | \phi_-, \phi_+)}{\int D\phi_- D\phi_+ e^{-1/2(\int \phi_i U_{ij}^{-1} \phi_j)}}, \qquad (49)$$

where

$$g_{\pm}(x_f \tau_f, x_i \tau_i | \phi_+, \phi_-)$$
$$\equiv -\langle T \psi_{\pm}(x_f \tau_f) \bar{\psi}_{\pm}(x_i \tau_i) e^{i \int_0^\beta d\tau \int dx \phi_i(x,\tau) \delta n_i(x,\tau)} \rangle_0. \qquad (50)$$

The correlation function (50) can also be written as

$$g_{\pm}(x_f \tau_f, x_i \tau_i | \phi_+, \phi_-) = g_{\pm}(x_f \tau_f, x_i \tau_i | \phi_{\pm}) \cdot Z_{\mp}[\phi_{\mp}], \qquad (51)$$

where

$$Z_{\pm}[\phi_{\pm}] \equiv \langle T e^{i \int_0^\beta d\tau \int dx \phi_{\pm}(x,\tau) \delta n_{\pm}(x,\tau)} \rangle_{0,\pm}, \qquad (52)$$

and

$$g_{\pm}(x_f \tau_f, x_i \tau_i | \phi_{\pm}) = -\langle T \psi_{\pm}(x_f \tau_f) \bar{\psi}_{\pm}(x_i \tau_i) e^{i \int_0^\beta d\tau \int dx \phi_{\pm}(x,\tau) \delta n_{\pm}(x,\tau)} \rangle_{0,\pm}. \qquad (53)$$

Next we cumulant expand both (52) and (53) to second order. For (52),

$$Z_{\pm}[\phi_{\pm}] \approx e^{1/2[\int d\tau d\tau' \int dx dx' \Pi_{\pm}(x-x', \tau-\tau') \phi_{\pm} \phi_{\pm}']}. \qquad (54)$$

Here $\Pi_{\pm}$ is the noninteracting density-density correlation function

$$\Pi_{\pm}(x, \tau) \equiv -\langle T \delta n_{\pm}(x, \tau) \delta n_{\pm}(0) \rangle_{0,\pm}, \qquad (55)$$

which can be written as

$$\Pi_{\pm}(x, \tau) = G_{0,\pm}(x, \tau) G_{0,\pm}(-x, -\tau). \qquad (56)$$

The noninteracting chiral propagator is

$$G_{0,\pm}(x, \tau) = \pm \frac{1}{2\pi i} \frac{e^{\pm i k_F x}}{x \pm i v_F \tau}. \qquad (57)$$

For (53),

$$g_{\pm}(x_f \tau_f, x_i \tau_i | \phi_{\pm}) \approx G_{0,\pm}(x_f \tau_f, x_i \tau_i | \phi_{\pm})$$
$$\times e^{\int dx d\tau C_{1,\pm} \phi_{\pm} + \int dx dx' d\tau d\tau' C_{2,\pm} \phi_{\pm} \phi_{\pm}'}, \qquad (58)$$

where

$$C_{1,\pm}(x, \tau) = -i \frac{G_{0,\pm}(x_f, x, \tau_0 - \tau) G_{0,\pm}(x, x_i, \tau)}{G_{0,\pm}(x_f, x_i, \tau_0)}. \qquad (59)$$

$C_{2,\pm}$ for this model reduces to

$$C_{2,\pm}(x-x',\tau-\tau') = \frac{1}{2}\Pi_{\pm}(x-x',\tau-\tau'). \quad (60)$$

Now we solve for U_{eff}, defined by

$$\int dx''d\tau''\mathbf{U}_{\text{eff}}^{-1}(x-x'',\tau-\tau'')\mathbf{U}_{\text{eff}}(x''-x',\tau''-\tau')$$

$$= \delta(x-x')\delta(\tau-\tau')\mathbb{1}, \quad (61)$$

where

$$U_{ij}^{\text{eff}}(x,\tau) = [U_{ij}^{-1}(x,\tau) - \Pi_{\pm}(x,\tau)\delta_{i+}\delta_{j+} - \Pi_{\mp}(x,\tau)\delta_{i-}\delta_{j-}]^{-1}, \quad (62)$$

and $\mathbb{1}$ is a 2×2 identity matrix. To achieve this we Fourier transform (61). This reduces (61) to a matrix equation which gives

$$\mathbf{U}_{\text{eff}}(k,\omega) = \left[\begin{pmatrix} U_4 & U_2 \\ U_2 & U_4 \end{pmatrix}^{-1} - \begin{pmatrix} \Pi_{\pm}(k,\omega) & 0 \\ 0 & \Pi_{\mp}(k,\omega) \end{pmatrix}\right]^{-1}. \quad (63)$$

The $++$ or $--$ component is

$$U_{\text{eff}}(k,\omega) = \frac{U_4 - (U_4^2 - U_2^2)\Pi_{\mp}(k,\omega)}{1 - U_4\Pi_t(k,\omega) + (U_4^2 - U_2^2)\Pi_+(k,\omega)\Pi_-(k,\omega)}, \quad (64)$$

where

$$\Pi_t(k,\omega) = \Pi_+(k,\omega) + \Pi_-(k,\omega) = -\frac{1}{\pi}\frac{k^2}{(\omega+ik)(\omega-ik)}. \quad (65)$$

The effective interaction for right movers is (setting $v_{\text{F}}=1$ temporarily)

$$U_{\text{eff}}(k,\omega) = U_4\frac{(\omega+ik)(\omega-iuk)}{(\omega+ivk)(\omega-ivk)}, \quad (66)$$

where

$$u \equiv 1 + \frac{(U_4/2\pi)^2 - (U_2/2\pi)^2}{(U_4/2\pi)} \quad (67)$$

and

$$v \equiv \sqrt{\left(1 + \frac{U_4}{2\pi}\right)^2 - \left(\frac{U_2}{2\pi}\right)^2}. \quad (68)$$

The action can be written as

$$S = \frac{1}{2}\int\frac{dk\,d\omega}{2\pi\,2\pi}\rho_{\pm}(-k,-\omega)U_{\text{eff}}(k,\omega)\rho_{\pm}(k,\omega). \quad (69)$$

The chiral tunneling charge density is

$$\rho_{\pm}(x,\tau) = -\frac{G_{0,\pm}(x_{\text{f}},x,\tau_0-\tau)G_{0,\pm}(x,x_{\text{i}},\tau)}{G_{0,\pm}(x_{\text{f}},x_{\text{i}},\tau_0)}. \quad (70)$$

We now specialize to the DOS case where $x_{\text{i}}=x_{\text{f}}=0$, and assuming right movers we set $\rho_+ \equiv \rho$. The tunneling charge density is

$$\rho(x,\tau) = \frac{v_{\text{F}}\tau_0}{2\pi}\frac{1}{(x+iv_{\text{F}}\tau)[x+iv_{\text{F}}(\tau-\tau_0)]}, \quad (71)$$

which correctly satisfies the sum rule

$$\int dx\rho(x,\tau) = \Theta(\tau)\Theta(\tau_0-\tau). \quad (72)$$

Fourier transforming, we find that

$$\rho(k,\omega) = \frac{1}{i\omega-v_{\text{F}}k}[(e^{i\omega\tau_0}-1)\Theta(k) + (e^{i\omega\tau_0}-e^{v_{\text{F}}k\tau_0})\Theta(-k)] \quad (73)$$

and

$$\rho(k,\omega)\rho(-k,-\omega)$$

$$= \frac{1}{(\omega+ik)^2}[(1-e^{i\omega\tau_0}e^{-k\tau_0}+e^{-k\tau_0}-e^{-i\omega\tau_0})\Theta(k)$$

$$+ (1-e^{i\omega\tau_0}+e^{k\tau_0}-e^{-i\omega\tau_0}e^{k\tau_0})\Theta(-k)]. \quad (74)$$

The action therefore is

$$S(\tau_0) = \frac{1}{2}\int\frac{dk\,d\omega}{2\pi\,2\pi}\frac{(\omega+ik)(\omega-iuk)}{(\omega+ivk)(\omega-ivk)}\frac{U_4}{(\omega+ik)^2}$$

$$\times [(1-e^{i\omega\tau_0}e^{-k\tau_0}+e^{-k\tau_0}-e^{-i\omega\tau_0})\Theta(k)$$

$$+ (1-e^{i\omega\tau_0}+e^{k\tau_0}-e^{-i\omega\tau_0}e^{k\tau_0})\Theta(-k)]. \quad (75)$$

The complete action can be written as $S=S_> + S_<$, where

$$S_> = \frac{U_4}{8\pi^2}\int_{0^+}^{\infty}dk\int_{-\infty}^{\infty}d\omega\frac{(\omega-iuk)}{(\omega+ivk)(\omega-ivk)(\omega+ik)}$$

$$\times (1-e^{i\omega\tau_0}e^{-k\tau_0}+e^{-k\tau_0}-e^{-i\omega\tau_0}) \quad (76)$$

and

$$S_< = \frac{U_4}{8\pi^2}\int_{-\infty}^{0^-}dk\int_{-\infty}^{\infty}d\omega\frac{(\omega-iuk)}{(\omega+ivk)(\omega-ivk)(\omega+ik)}$$

$$\times (1-e^{i\omega\tau_0}+e^{k\tau_0}-e^{-i\omega\tau_0}e^{k\tau_0}). \quad (77)$$

$S_> = S_<$ under change of coordinates $k\to-k$ and $\omega\to-\omega$, so $S=2S_>$. To proceed we need the large-τ_0 asymptotic result

$$I(\tau_0) \equiv \int_{0^+}^{\infty}dk\frac{e^{-k\tau_0}}{k} \to -\ln\tau_0, \quad (78)$$

where the additive constant, not shown explicitly, is cutoff dependent. These lead, in the large τ_0 limit, to

$$S = \frac{U_4}{4\pi}\left[\frac{2(1+u)}{(1+v)(1-v)} - \frac{u+v}{v(1-v)}\right]\ln(\tau_0) \quad (79)$$

or

$$S = \frac{U_4}{4\pi}\left[\frac{v-u}{v(1+v)}\right]\ln(\tau_0). \quad (80)$$

Finally, we obtain

$$S = \delta\ln\tau_0 + \text{const} + O(1/\tau_0) \quad (81)$$

and

$$N(\epsilon) = \text{const} \times \epsilon^\delta, \qquad (82)$$

where

$$\delta = \frac{U_4}{4\pi} \frac{v - u}{v(1+v)} = \frac{(u-v)(1-v)}{2v(1+u)} \qquad (83)$$

$$= \frac{1 + \dfrac{U_4}{2\pi v_F} - \sqrt{\left(1 + \dfrac{U_4}{2\pi v_F}\right)^2 - \left(\dfrac{U_2}{2\pi v_F}\right)^2}}{2\sqrt{\left(1 + \dfrac{U_4}{2\pi v_F}\right)^2 - \left(\dfrac{U_2}{2\pi v_F}\right)^2}}. \qquad (84)$$

This is in exact agreement with the bosonization result

$$\delta = \frac{g + g^{-1}}{2} - 1, \qquad (85)$$

with

$$g = \sqrt{\frac{1 + \dfrac{U_4}{2\pi v_F} - \dfrac{U_2}{2\pi v_F}}{1 + \dfrac{U_4}{2\pi v_F} + \dfrac{U_2}{2\pi v_F}}}. \qquad (86)$$

Why does the cumulant method give the exact result for this model? The answer is that a second-order cumulant expansion of the form used here is exact for free bosons, which are the exact eigenstates of the Tomonaga-Luttinger model.[21]

V. DISCUSSION

Our principal result (15) suggests that the dominant effect of interaction on the low-energy DOS in a variety of strongly correlated electron systems is to add a time-dependent charging energy contribution to the total potential barrier seen by a tunneling electron, as in Ref. 13. The energy is computed according to classical electrostatics with a dynamically screened two-particle interaction. In 2D and 3D the added charge is accommodated efficiently and reaches a zero-action state at long times. In 1D the added charge leads to diverging action, and hence suppressed tunneling.

The robustness of the present cumulant method has not been fully explored, although it is known to fail qualitatively in systems with ground-state degeneracy, such as in the quantum Hall fluid. We believe the cause of this failure to be the nonsatisfiability of the sum rule (25) in such situations. In the future we plan to apply this method to other exactly solvable systems, such as the 1D Hubbard and Calogero-Sutherland models.

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[21] This interesting observation was pointed out to us by Giovanni Vignale.