Decay of metastable current states in one-dimensional resonant tunneling devices

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Current switching in a double-barrier resonant tunneling structure is studied in the regime where the current-voltage characteristic exhibits intrinsic bistability, so that in a certain range of bias two different steady states of current are possible. Near the upper boundary V_{th} of the bistable region the upper current state is metastable, and because of the shot noise it eventually decays to the stable lower current state. We find the time of this switching process in strip-shaped devices, with the width small compared to the length. As the bias V is tuned away from the boundary value V_{th} of the bistable region, the mean switching time τ increases exponentially. We show that in long strips $\ln \tau \propto (V_{th} - V)^{5/4}$, whereas in short strips $\ln \tau \propto (V_{th} - V)^{3/2}$. The one-dimensional geometry of the problem enables us to obtain analytically exact expressions for both the exponential and the prefactor of τ. Furthermore, we show that, depending on the parameters of the system, the switching can be initiated either inside the strip, or at its ends.

I. INTRODUCTION

In the last two decades with advances of miniaturization techniques various resonant tunneling structures became the subject of intensive research. It was experimentally observed that the current-voltage characteristics of resonant tunneling devices, such as double-barrier resonant tunneling structures[1–5] (DBRTS) and superlattices,[6–9] exhibit intrinsic bistabilities. Namely, it was shown that for each value of bias in the bistable region of the *I-V* curve the current can take two different values. A bistable *I-V* curve of a double-barrier structure derived theoretically in Ref. 10 is depicted schematically in Fig. 1. It is theoretically established[11,12] that near the boundary V_{th} of the bistable region (e.g., at point *A*) the upper current state is metastable. In recent experiments[8,9] the switching from the metastable to stable current state was studied in superlattices. In particular, the mean switching time τ was measured.

The problems of decay of metastable states were studied theoretically in various fields, such as condensed matter physics,[13–19] quantum field theory,[20–24] and chemical
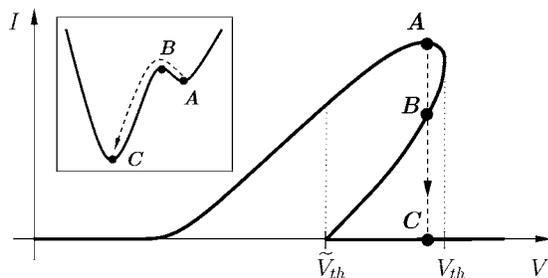

FIG. 1. Current-voltage characteristic of the DBRTS. The bistable region is present in the range of bias between $\widetilde{V}_{th}$ and V_{th}. The inset shows a sketch of generic potential $u(n)$ in the bistable region. The points *A* and *C* on the *I-V* curve correspond to the local and global minima of $u(n)$, respectively. The point *B* on the unstable current branch corresponds to the maximum of u.

kinetics.[25,26] In the context of resonant tunneling structures it was addressed in Refs. 11 and 12. A typical double-barrier resonant tunneling structure consists of three semiconducting layers of GaAs separated by two insulating layers of GaAlAs. In the narrow middle layer of GaAs the electron motion in the direction normal to the layers is quantized, so that a quantum well is formed. In the bistable region (Fig. 1) the two current states correspond to two different values of electron density n in the well. The density n exhibits shot noise fluctuations caused by random events of tunneling of electrons in and out of the well. Thus it becomes possible for the device to switch from one current state to the other. In Refs. 11 and 12 it was shown that there are two regimes of current switching. In the case of relatively small samples the electrons spread uniformly over the well due to diffusion. Then the switching occurs simultaneously in the entire area of the sample, and it was found that the switching time τ is exponentially large, with $\ln \tau \propto (V_{th} - V)^{3/2}$. In larger samples the density in the well is not uniform, and the switching occurs through nucleation mechanism: It initiates in a small region of the quantum well of the characteristic size r_0, which then spreads rapidly to the entire sample. The size of the critical nucleus of the stable current state was found to be bias dependent, $r_0 \propto (V_{th} - V)^{-1/4}$. It has been shown that in the large-sample regime $\ln \tau \propto (V_{th} - V)$.

These results were obtained for samples whose two lateral dimensions are comparable to each other. In this paper we study the interesting case of the devices of strip geometry with $w \ll L$, where w is the width and L is the length of the strip. In these devices there is a special regime where the applied bias is such that $w \ll r_0 \ll L$. Since the scale r_0 gives the characteristic size of the density fluctuations, in this case the density does not vary across the strip, but only along it. Confining the switching process to one dimension alters its properties significantly. Similar to the two-dimensional case, the switching time grows exponentially when bias is tuned inside the bistable region. However, the exponent follows a different dependence which is found in Sec. III.

We show that in long strips the nucleation can occur either inside or at the ends of the device. It turns out that both nucleation regimes can be observed, but the respective decay times are dramatically different. To compare them, one needs to calculate the switching times with the prefactors. Unlike the case of two-dimensional structures, in one dimension the prefactors can be found exactly in the limit of long and narrow strip, Sec. IV.

The decay of the metastable current state is governed by shot noise, and therefore, switching is a stochastic process. We describe this process using the Fokker-Planck equation approach. In Sec. IV A we introduce the Fokker-Planck equation for tunneling and in-plane diffusion in DBRTS. This equation is used to calculate the prefactor of the mean switching time in strip-shaped (Secs. IV B and IV C) and ring-shaped devices (Sec. IV D). We summarize our results and discuss their experimental implications in Sec. V.

II. STATIONARY DISTRIBUTION FUNCTION OF ELECTRON DENSITY

It was shown in Ref. 12 that the stationary distribution function of electron density $n(\mathbf{r})$ in the quantum well takes the form

$$P_0\{n\} = e^{-F\{n\}}, \quad F\{n\} = \int d^2\mathbf{r}[u(n) + \eta(\nabla n)^2]. \quad (1)$$

Here the integration is over the cross section of the well. In the case of uniform density $F\{n\}$ is determined by the effective potential $u(n)$, which describes the tunneling between the well and the leads. The gradient term in the functional $F\{n\}$ accounts for the diffusion in the quantum well. The constant η is positive and proportional to the in-plane conductivity,[12] $\eta \propto \sigma$. By suppressing gradients of n, this term favors the states with uniform density.

The shape of the effective potential $u(n)$ in the bistable region is schematically shown in the inset of Fig. 1. It has the local and global minima at points **A** and **C**, respectively. In the case of uniform density n, these minima result in peaks of the distribution function $P_0\{n\}$, which correspond to the upper and lower branches of the I-V curve. Point **B** on the unstable current branch corresponds to a maximum of u.

In the vicinity of the threshold voltage V_{th} the effective potential $u(n)$ can be approximated by a cubic polynomial

$$u(n) = -\alpha(n - n_{th}) + \frac{\gamma}{3}(n - n_{th})^3, \quad \alpha = a(V_{th} - V). \quad (2)$$

Here γ and a are positive constants. The voltage dependence of α ensures that the local minimum of u, which corresponds to the metastable state, disappears at the bistability threshold. The threshold density n_{th} is defined as the density at point $V = V_{th}$, where the local minimum **A** of $u(n)$ disappears by merging with the maximum **B**.

If the system is in the local minimum of $u(n)$, it will eventually decay to the global minimum. In the limit of large conductivity σ or small sample size the electron density n in the quantum well is uniform, and the gradient term in $F\{n\}$, Eq. (1), can be omitted. In this case Eq. (1) is simplified: The

distribution function P_0 is described by the only variable n and takes the form $P_0(n) = e^{-Su(n)}$, where S is the area of the sample. Initially, the system is in the local minimum of $u(n)$, see point **A** in the inset of Fig. 1. In order to switch to the global minimum the system has to pass through point **B**. As it follows from the expression for $P_0(n)$, the probability of reaching point **B** is exponentially small, with the exponent determined by the barrier height $S(u_B - u_A)$. The latter can be easily found from the expansion (2), and the mean switching time τ_0 takes the form[12]

$$\tau_0 = \tau_0^* \exp\left(\frac{4}{3}\frac{Lw\alpha^{3/2}}{\gamma^{1/2}}\right). \quad (3)$$

Here τ_0^* is a preexponential factor, and we assumed that the cross section of the sample has rectangular shape with the width w and the length L.

Expression (3) is valid as long as $L, w \ll r_0$, where

$$r_0 = \left(\frac{\eta^2}{\alpha\gamma}\right)^{1/4} \propto (V_{th} - V)^{-1/4} \quad (4)$$

has the meaning of the characteristic spatial scale of stochastic fluctuations of electron density.[12] The scale r_0 can be tuned by changing the bias V. If $L, w \gg r_0$ the switching occurs according to the nucleation scenario. In this case the critical switching density is first achieved in a small part of the sample of size $\sim r_0$. After stochastic creation of the critical nucleus, it grows rapidly in size until it occupies the entire sample.

In this paper we consider the case of a very narrow strip, $w \ll L$. In the regime when the bias is such that $w \ll r_0$, the density may change only along the strip, and the problem becomes one-dimensional.[27]

In the following it will be convenient to express the density $n(x)$ in terms of a dimensionless function $z(\xi)$ that vanishes at the minimum of $u(n)$, namely,

$$n(x) = n_{\min} - 2\sqrt{\frac{\alpha}{\gamma}}z(x/r_0). \quad (5)$$

Here the density at the minimum $n_{\min} = n_{th} + \sqrt{\alpha/\gamma}$. Substituting Eq. (5) into the functional F in Eq. (1), we find

$$F = U_1 \int d\xi\left(\frac{z'^2}{2} + \frac{z^2}{2} - \frac{z^3}{3}\right), \quad (6)$$

where prime denotes differentiation with respect to the dimensionless coordinate $\xi = x/r_0$ along the strip. The characteristic value of the functional F is given by the parameter

$$U_1 = \frac{8w\sqrt{\eta}\alpha^{5/4}}{\gamma^{3/4}}. \quad (7)$$

Its value depends on bias as $U_1 \propto (V_{th} - V)^{5/4}$.

III. THE EXPONENT OF THE MEAN SWITCHING TIME IN LONG STRIPS

In this section we consider the regime $r_0 \lesssim L$. In this case, the density fluctuations along the strip result in a significant
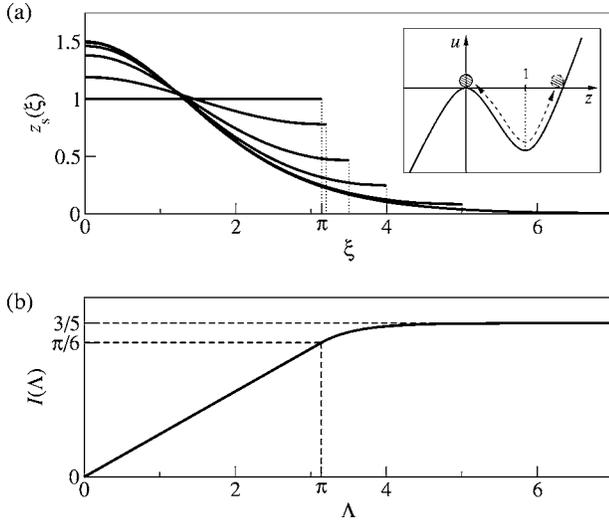
FIG. 2. (a) Saddle point solution $z_s(\xi)$ for strips of length Λ $= \pi$, 3.2, 3.5, 4, 5, and ∞. The allowed motion of a classical particle in the potential $u(z) = -z^2/2 + z^3/3$ is shown in the inset. (b) The dependence $I(\Lambda)$ defined by Eq. (13). At $\Lambda < \pi$ it is linear: $I(\Lambda)$ $= \Lambda/6$; while for $\Lambda > \pi$ it is described by Eq. (14). At $\Lambda \to \infty$ the saddle point $z_s(\xi)$ is given by Eq. (11), and $I = 3/5$.

change of the mean switching time, and the result (3) is no longer applicable. The one-dimensional nature of this problem allows us to obtain an explicit expression for the exponent of the mean switching time τ.

We begin by finding the minimum and the saddle points of the functional F in Eq. (6). They can be found from the condition $\delta F/\delta z = 0$, that is,

$$-\frac{d^2z}{d\xi^2} + z - z^2 = 0. \tag{8}$$

This equation should be solved with the boundary conditions $z'(0) = z'(\Lambda) = 0$ which account for the fact that there is no current flowing through the ends of the strip. Here we have introduced the dimensionless length of the strip $\Lambda = L/r_0$.

Equation (8) can be interpreted as the equation of motion of a classical particle with unit mass in the potential $u(z)$ $= -z^2/2 + z^3/3$, see the inset of Fig. 2(a). In this analogy the coordinate ξ plays the role of time. There are two obvious solutions, $z(\xi) = 0$ and $z(\xi) = 1$, corresponding to the particle staying at the maximum and minimum of $u(z)$, respectively. The minimum of $F\{z\}$ is obviously given by $z(\xi) = 0$, since we defined z in such a way that $z = 0$ at the minimum. The other solution, $z(\xi) = 1$, is a saddle point of $F\{z\}$.

Apart from the two trivial solutions, Eq. (8) may have ξ-dependent solutions corresponding to a moving particle. The boundary conditions $z'(0) = z'(\Lambda) = 0$ require zero velocity at the moments $\xi = 0$ and $\xi = \Lambda$. Thus the particle performs oscillatory motion between turning points $z(0)$ and $z(\Lambda)$. If the particle starts at $z(0) = c_0$ in the range $0 \le c_0 \le 3/2$, one can easily find the other turning point c from the condition $u(c) = u(c_0)$. This equation has two solutions:

$$c_{\pm} = \frac{3 - 2c_0 \pm \sqrt{-12c_0^2 + 12c_0 + 9}}{4}.$$

The turning point $z(\Lambda)$ corresponds to the positive root c_+.

The time Λ required for the particle to travel from one turning point to the other is obviously one-half of the period of oscillations. The period of small amplitude oscillations when the particle starts close to the minimum of the potential $u(z)$, i.e., at $c_0 \to 1$, equals 2π. The period monotonically grows to infinity as $c_0 \to 3/2$ or 0. Therefore, if $\Lambda < \pi$ there are no ξ-dependent solutions, and $z_s(\xi) = 1$. At $\Lambda > \pi$ we have an additional saddle point $z_s(\xi)$ corresponding to a particle moving from $z_s(0) = c_0$ to $z_s(\Lambda) = c_+$.

Equation (8) can be solved analytically in terms of the elliptic integrals. In particular, the inverse function of $z_s(\xi)$ has the form

$$\xi(z_s) = \frac{\sqrt{6}}{\sqrt{c_0 - c_-}} F\left(\arcsin\sqrt{\frac{c_0 - z_s}{c_0 - c_+}}, \sqrt{\frac{c_0 - c_+}{c_0 - c_-}}\right), \tag{9}$$

where $F(\varphi, k)$ is the elliptic integral of the first kind.[28] Using Eq. (9) the length of the strip Λ can be expressed in terms of c_0 as

$$\Lambda = \frac{\sqrt{6}}{\sqrt{c_0 - c_-}} F\left(\frac{\pi}{2}, \sqrt{\frac{c_0 - c_+}{c_0 - c_-}}\right). \tag{10}$$

Solving Eq. (10) with respect to c_0 one can obtain the dependence $c_0(\Lambda)$. Substituting it into Eq. (9) and inverting $\xi(z_s)$ one obtains the saddle point $z_s(\xi)$ for a given Λ. The saddle-point solutions $z_s(\xi)$ for several values of Λ are shown in Fig. 2(a).

For a strip of infinite length the boundary conditions take the form $c_0 = z_s(0) = 3/2$ and $c_+ = z_s(\infty) = 0$. Then expression (9) can be significantly simplified and yields

$$z_s(\xi) = \frac{3}{2\cosh^2(\xi/2)}. \tag{11}$$

This solution can be easily verified by substitution into Eq. (8).

The mean switching time τ is given by

$$\tau = \tau^* e^{F\{z_s\}}, \tag{12}$$

where τ^* is a preexponential factor. Using the expression for $F\{z\}$, Eq. (6), we find

$$F\{z_s\} = U_1 I(\Lambda), \quad I(\Lambda) = \int_0^\Lambda d\xi\left(\frac{z_s'^2}{2} + \frac{z_s^2}{2} - \frac{z_s^3}{3}\right). \tag{13}$$

At $\Lambda < \pi$, the only saddle point is $z_s = 1$, and therefore, $I(\Lambda) = \Lambda/6$. At $\Lambda > \pi$ the saddle point solution $z_s(\xi)$ given by Eq. (9) corresponds to a smaller value of I. In particular, at $\Lambda \to \infty$ the solution (11) gives $I = 3/5$. In the intermediate region the integral $I(\Lambda)$ can be evaluated analytically,

$$I = \frac{\Lambda}{60}[4c_+^3 - 6(c_+^2 + c_+ + c_0 - c_-) + 9]$$

$$+ \frac{\sqrt{6}}{5}\sqrt{c_0 - c_-}E\left(\sqrt{\frac{c_0 - c_+}{c_0 - c_-}}\right). \tag{14}$$

Here $E(k)$ is the complete elliptic integral of the second kind.[28] The dependence $I(\Lambda)$ is plotted in Fig. 2(b).

Note that apart from the saddle point described by Eq. (9), for $\Lambda > 2\pi$ there are additional ξ-dependent saddle points. For example, there is a ξ-dependent solution corresponding to a particle moving from $z_s(0) = c_0$ to $z_s(\Lambda) = c_+$ and back to c_0. This saddle point is responsible for the processes of switching inside the strip, see Sec. IV C. At $\Lambda > 3\pi$ another saddle point appears which corresponds to a particle moving from c_0 to c_+, returning to c_0, and back to c_+. In general, for Λ between πm and $\pi(m+1)$ there exist m different ξ-dependent solutions. However, the additional saddle points give larger values of $F\{z_s\}$ and thus do not affect the switching time.

IV. PREFACTOR OF THE SWITCHING TIME

The exponential dependence of the mean switching time (12) was obtained from the stationary distribution function $P_0\{n\}$ of electron density. However, to calculate the prefactor τ^*, understanding of the time evolution of the distribution function $P\{n(x),t\}$ is also required. When electrons tunnel in or out of the well, the density $n(x)$ changes in very small increments. Thus the dynamics of $P\{n(x),t\}$ is described by a Fokker-Planck equation.[12] If the strip is very short, the density in the quantum well is uniform, and the system dynamics is described by a single variable n. In this case the Fokker-Planck equation for $P(n,t)$ essentially coincides with the one for a small two-dimensional sample. Then by using conventional techniques[29] the prefactor of the switching time can be found[12] as

$$\tau_0^* = \frac{2\pi}{b\sqrt{\alpha\gamma}}. \tag{15}$$

This result is correct as long as $\Lambda \ll 1$.

In longer strips the density fluctuates, and therefore, the Fokker-Planck equation with one variable n cannot adequately describe the evolution of the distribution function P. In this case P is a functional of $n(x)$, and the Fokker-Planck equation is multidimensional. Then to study the decay of metastable current states one should use a more sophisticated method.

A. Fokker-Planck approach to current switching in DBRTS

To find the prefactor of the mean switching time for the system described by the multidimensional Fokker-Planck equation, we use the first passage time technique.[30,31] This method is applied to the Fokker-Planck equation in the form

$$\frac{\partial P(\mathbf{x},t)}{\partial t} = \mathcal{L}P(\mathbf{x},t), \tag{16a}$$

$$\mathcal{L} = -\sum_i \frac{\partial}{\partial x_i}K_i(\mathbf{x}) + \sum_{i,j}\frac{\partial^2}{\partial x_i \partial x_j}D_{ij}(\mathbf{x}), \tag{16b}$$

where the matrix D_{ij} represents the generic diffusion coefficient, and $\mathbf{K}$ is the drift field. Assuming that the system has a metastable state, one can consider its domain of attraction Ω with the domain boundary $\partial\Omega$ being the separatrix of the field $\mathbf{K}$. For the stochastic process described by Eq. (16), the mean time of the first passage out of the domain Ω has been found in Refs. 30 and 31. The mean switching time is obtained as doubled mean first-passage time and takes the form,[31]

$$\tau = -\frac{2\int_\Omega d^d x P_0(\mathbf{x})}{\sum_i \int_{\partial\Omega} dS_i \sum_j D_{ij}(\mathbf{x})P_0(\mathbf{x})\frac{\partial f(\mathbf{x})}{\partial x_j}}. \tag{17}$$

Here P_0 is the stationary solution of Eq. (16). The form function $f(\mathbf{x})$ is defined as a stationary solution of the adjoint equation,

$$\mathcal{L}^\dagger f(\mathbf{x},t) = \sum_j \left(K_j(\mathbf{x}) + \sum_i D_{ij}(\mathbf{x})\frac{\partial}{\partial x_i}\right)\frac{\partial f(\mathbf{x})}{\partial x_j} = 0. \tag{18}$$

It satisfies the boundary conditions $f(\mathbf{x}) = 1$ inside the domain Ω and $f(\mathbf{x}) = 0$ at the boundary $\partial\Omega$.

The Fokker-Planck equation for the case of nonuniform electron density $n(\mathbf{r})$ in the well was obtained in Ref. 12. It takes the form

$$\frac{\partial P\{n,t\}}{\partial t} = \frac{b}{2}\int d\mathbf{r}\frac{\delta}{\delta n}\left(u'(n) - 2\eta\nabla^2 n + \frac{\delta}{\delta n}\right)P\{n,t\}. \tag{19}$$

It is easy to check that the stationary solution of this equation is given by Eq. (1).

Close to the threshold $u(n)$ can be approximated by the expansion (2), and the Fokker-Planck equation (19) can be represented in terms of dimensionless variable $z(\xi)$, Eq. (5), as

$$\frac{\partial P\{z,\Theta\}}{\partial\Theta} = \int d\xi\frac{\delta}{\delta z}\left(-\frac{d^2 z}{d\xi^2} + z - z^2 + \frac{1}{U_1}\frac{\delta}{\delta z}\right)P\{z,\Theta\}. \tag{20}$$

where $\Theta = 2\pi t/\tau_0^*$ is the dimensionless time. The stationary solution $P_0(z)$ of this equation is e^{-F} with $F\{z\}$ given by Eq. (6). Equation (20) has an infinite number of variables, since the density $z(\xi)$ is different at every point. From now on we can consider a purely mathematical problem of decay of a metastable state for the system governed by dimensionless equation (20). This equation is rather generic and we expect it to describe other one-dimensional problems of decay of metastable states.

B. Nucleation at an end of a strip

To find the prefactor of the mean switching time (12) we use expression (17). We evaluate both integrals in Eq. (17) in

Gaussian approximation. The integral in the numerator of Eq. (17) is dominated by the minimum of $F\{z\}$. To find the expression for F in the vicinity of its minimum, it is convenient to use Fourier expansion

$$z(\xi) = \frac{x_0}{\sqrt{\Lambda}} + \sum_{i=1}^{\infty} x_i \phi_i(\xi), \quad \phi_i(\xi) = \sqrt{\frac{2}{\Lambda}} \cos\left(\frac{\pi i}{\Lambda}\xi\right).$$

(21)

Substituting this expansion into Eq. (6), we find that up to quadratic in x_i terms,

$$F(\mathbf{x}) = \frac{U_1}{2} \sum_{i=0}^{\infty} \lambda_i x_i^2,$$

(22)

where we defined $\lambda_i \equiv (\pi i / \Lambda)^2 + 1$. Using this expression for F, one can easily evaluate the integral in the numerator of Eq. (17),

$$\int_{-\infty}^{\infty} \prod_i dx_i e^{-F(\mathbf{x})} = \prod_i \sqrt{\frac{2\pi}{U_1 \lambda_i}}.$$

(23)

The integral in the denominator of Eq. (17) is dominated by the saddle point $z_s(\xi)$ of the functional $F\{z\}$. To find the expression for F in the vicinity of the saddle point it is convenient to expand $z(\xi)$ near $z_s(\xi)$ as

$$z(\xi) = z_s(\xi) + \sum_{i=0}^{\infty} \tilde{x}_i \tilde{\phi}_i(\xi).$$

(24)

Here $\tilde{\phi}_i(\xi)$ are the normalized solutions of the eigenvalue problem

$$\left(-\frac{d^2}{d\xi^2} - 2z_s(\xi) + 1\right) \tilde{\phi}_i(\xi) = \tilde{\lambda}_i \tilde{\phi}_i(\xi),$$

(25)

with the boundary conditions

$$\tilde{\phi}_i'(0) = 0, \quad \tilde{\phi}_i'(\Lambda) = 0.$$

(26)

Substituting Eq. (24) into $F\{z\}$ given by Eq. (6), and expanding near $\tilde{\mathbf{x}} = 0$ up to the second order in $\tilde{x}_i$, we find

$$F\{\tilde{\mathbf{x}}\} = U_1 I(\Lambda) + \frac{U_1}{2} \sum_{i=0}^{\infty} \tilde{\lambda}_i \tilde{x}_i^2,$$

(27)

where the first term is given by Eq. (13).

Expression (27) implies that it is convenient to calculate the integral in the denominator of Eq. (17) in terms of variables $\tilde{x}_i$. Since the expansion coefficients $\tilde{x}_i$ are related to x_i by orthogonal transformation, the integrals over x_i in the denominator of Eq. (17) can be replaced by those over $\tilde{x}_i$.

To calculate the integral in the denominator of Eq. (17) we also need to find D_{ij} and $\partial f / \partial \tilde{x}_j$. They can be obtained from the $\tilde{\mathbf{x}}$-representation of the Fokker-Planck equation (20). The adjoint to the operator $\mathcal{L}$ of Eq. (20) in the $\tilde{\mathbf{x}}$-representation can be written as

$$\mathcal{L}^{\dagger} = \sum_{i=0}^{\infty} \left(-\tilde{\lambda}_i \tilde{x}_i \frac{\partial}{\partial \tilde{x}_i} + \frac{1}{U_1} \frac{\partial^2}{\partial \tilde{x}_i^2}\right).$$

(28)

Here the terms of higher orders in $\tilde{x}_i$ were neglected.

In the denominator of Eq. (17) the boundary $\partial \Omega$ is orthogonal to the unstable direction $\tilde{x}_0$ on the saddle. Therefore, the sum over i reduces to the only term $i=0$. Using the definition of D_{ij} we find from Eq. (28) that $D_{ij} = U_1^{-1} \delta_{ij}$. The sum over j then reduces to a single term $j=0$, and we need to find only the derivative $\partial f / \partial \tilde{x}_0$, which is given by Eq. (18) in $\tilde{\mathbf{x}}$-representation. The latter equation takes the form

$$\left(-\tilde{\lambda}_0 \tilde{x}_0 + \frac{1}{U_1} \frac{\partial}{\partial \tilde{x}_0}\right) \frac{\partial f}{\partial \tilde{x}_0} = 0.$$

(29)

Since the saddle point is unstable in the $\tilde{x}_0$-direction, the eigenvalue $\tilde{\lambda}_0$ is negative. Thus Eq. (29) can be solved with the boundary conditions $f=1$ at $\tilde{x}_0 \to -\infty$ and $f=0$ at the domain boundary $\tilde{x}_0 = 0$ required by the definition of $f(\tilde{\mathbf{x}})$, cf. Eq. (18). As a result, we obtain $\partial f / \partial \tilde{x}_0 = -(2|\tilde{\lambda}_0|U_1 / \pi)^{1/2}$ at the saddle point.

Substituting Eqs. (23) and (27) into Eq. (17) we express the switching time in a strip-shaped device τ_s as

$$\tau_s = \tau_s^* e^{U_1 I(\Lambda)}, \quad \tau_s^* = \frac{\tau_0^*}{\sqrt{|\tilde{\lambda}_0|}} \sqrt{\frac{\tilde{\lambda}_1}{\lambda_1}} \Upsilon_s,$$

(30)

where

$$\Upsilon_s = \prod_{i=2}^{\infty} \sqrt{\frac{\tilde{\lambda}_i}{\lambda_i}}.$$

(31)

Thus the evaluation of the prefactor of the switching time reduces to solving the eigenvalue problem (25),(26).

1. Very long strip

We first consider the limit of a very long strip, $\Lambda \gg 1$. Then the saddle point $z_s(\xi)$ is given by Eq. (11), so Eq. (25) takes the form

$$\left(-\frac{d^2}{d\xi^2} - \frac{3}{\cosh^2(\xi/2)} + 1\right) \tilde{\phi}_i(\xi) = \tilde{\lambda}_i \tilde{\phi}_i(\xi).$$

(32)

The eigenvalue problem (32) with boundary conditions (26) is solved analytically in the Appendix. In particular, we find that the discrete spectrum consists of two bound states with eigenvalues $\tilde{\lambda}_0 = -5/4$ and $\tilde{\lambda}_1 = 3/4$. At finite Λ the continuous spectrum is discretized according to

$$\tilde{\lambda}_i = 1 + \left(\frac{\pi i}{\Lambda}\right)^2 \left(1 - \frac{\delta(\pi i / \Lambda)}{\pi i}\right)^2,$$

(33)

where the scattering phase shift $\delta(q)$ has the form

$$\delta(q) = \sum_{n=1}^{3} \arctan \frac{n}{2q}.$$

(34)

To evaluate Υ_s it is convenient to take the logarithm of Eq. (31), thereby converting the infinite product to a sum. In the limit $\Lambda \to \infty$, we find

$$\ln \Upsilon_s = -\frac{1}{\pi}\int_0^\infty \frac{q\,\delta(q)}{1+q^2}dq = \frac{1}{2}\ln\frac{2}{15}. \qquad (35)$$

Using this result for Υ_s and the fact that in a long strip $\lambda_1 \to 1$, we find the mean time of switching in a very long strip

$$\tau_e = \frac{\sqrt{2}}{5}\tau_0^* \exp\left(\frac{3U_1}{5}\right). \qquad (36)$$

It is important to note that the above calculation accounts for only one of two equivalent saddle points in a strip-shaped device. Indeed, the saddle point fluctuation (11) is positioned near the left end of the sample, $\xi=0$. Alternatively, the density fluctuation could occur at the right end of the device. This possibility is formally described by considering the saddle point $z_s(\Lambda-\xi)$. Since the two types of processes are equivalent, the total rate of switching at both ends of the device is $2/\tau_e$.

2. Strip of arbitrary length

Another regime in which analytical expression for τ_s can be obtained is that of $\Lambda<\pi$. In this case the saddle-point density is uniform, $z_s(\xi)=1$. Substituting it into Eq. (25) we find $\tilde{\lambda}_i=\lambda_i-2$. Then expressions (30) and (31) simplify significantly, and the prefactor of τ_s becomes

$$\tau_s^* = \tau_0^*\prod_{i=1}^\infty \sqrt{\frac{(\pi i)^2 - \Lambda^2}{(\pi i)^2 + \Lambda^2}}. \qquad$$

Evaluating the infinite product we obtain

$$\tau_s^* = \tau_0^*\sqrt{\frac{\sin\Lambda}{\sinh\Lambda}}. \qquad (37)$$

This result generalizes the formula (3) for the switching time in short devices, $\Lambda\ll 1$, to the case of any $\Lambda<\pi$.

For a strip of finite length $\Lambda>\pi$, the saddle point density $z_s(\xi)$ is given by Eqs. (9) and (10). In this case the eigenvalues $\tilde{\lambda}_i$ are obtained by solving numerically the eigenvalue problem (25) with boundary conditions (26) for a given Λ. Substituting them into expressions (30) and (31) we find the switching time. The result for the prefactor of τ_s as a function of Λ is shown in Fig. 3.

Let us discuss the behavior of the prefactor at $\Lambda=\pi$. At $\Lambda<\pi$ the functional $F\{z\}$ has only the uniform saddle point $z_s(\xi)=1$, whereas at $\Lambda>\pi$ there are two saddle points, the uniform one and the ξ-dependent one, see Fig. 2(a). These two saddle points merge at $\Lambda=\pi$. Formally, this gives rise to the fact that $\tilde{\lambda}_1$ vanishes at $\Lambda=\pi$, and therefore, the prefactor $\tau_s^*=0$. We believe that the prefactor remains nonzero at $\Lambda=\pi$; however, the evaluation of the prefactor in this case requires a more careful treatment than the Gaussian approximation used in this paper. We leave this problem for future work.

It is worth mentioning that the nonmonotonic behavior of the prefactor, see Fig. 3, does not result in nonmonotonic dependence of the switching time on sample length Λ. The reason is that the prefactor τ_s^* is multiplied by the very large exponential [see Eqs. (12) and (13) and Fig. 2(b)] which
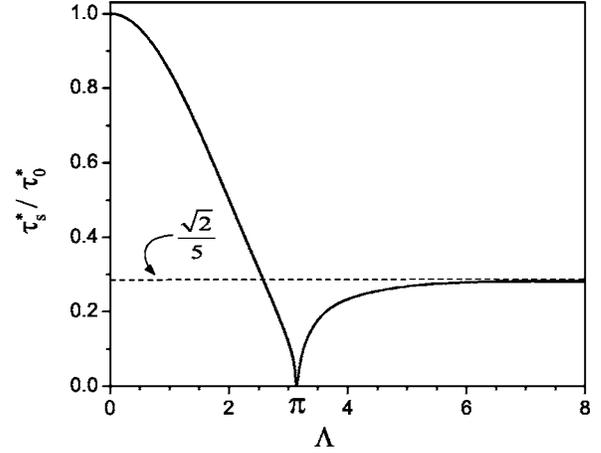

FIG. 3. Prefactor τ_s^* of the mean switching time in units τ_0^* vs the dimensionless length of the strip Λ. At $\Lambda<\pi$ the prefactor is given by Eq. (37), whereas above π it is calculated numerically. At $\Lambda\to\infty$ the prefactor approaches $\sqrt{2}/5$ in agreement with Eq. (36).

monotonically grows with Λ. Therefore, expression (12) with the prefactor τ_s^* and exponent (13) is monotonic everywhere except for the very narrow region $\pi-3/U_1\lesssim\Lambda<\pi$. As discussed in the previous paragraph, a more accurate evaluation of the prefactor is required to evaluate the prefactor near $\Lambda=\pi$, and we expect that it would restore the monotonicity of τ as the function of Λ.

C. Nucleation inside long strips and in rings of large circumference

Apart from the ends of a strip, nucleation can occur inside the sample. Such processes are most important in ring-shaped samples, which have no ends. Thus to study interior switching we model the sample by a strip with periodic boundary conditions. In the following it will be convenient to consider a strip of length 2Λ with boundary conditions $z(-\Lambda)=z(\Lambda)$, $z'(-\Lambda)=z'(\Lambda)$.

To obtain the switching time we first need to find the saddle point of $F\{z\}$. In the mechanical analogy used in Sec. III [see also the inset of Fig. 2(a)] the periodic boundary conditions for the saddle-point equation correspond to a full period of oscillations of the particle, rather than half period as in the case of nucleation at the ends in Sec. IV B. One such solution is given by $z_s(|\xi|)$ on the interval $-\Lambda<\xi<\Lambda$, where $z_s(\xi)$ is defined by Eqs. (9) and (10), see also Fig. 2(a). [Additional solutions are obtained by shifts $z_s(\xi)\to z_s(\xi+\Delta\xi)$.] To find the exponent of the switching time, we need to calculate $F\{z_s\}$ using Eq. (6). Since the saddle-point solution $z_s(|\xi|)$ is a symmetric function on the interval $-\Lambda<\xi<\Lambda$, the integral in $F\{z_s\}$ is doubled compared to that of end switching. Thus the time of the switching inside the strip takes the form

$$\tau_r = \tau_r^* e^{2I(\Lambda)U_1}, \qquad (38)$$

cf. Eq. (30).

The calculation of the prefactor τ_r^* is similar to the one for the switching at an end of the strip (Sec. IV B). As in Sec.

IV B it is convenient to evaluate the denominator of Eq. (17) by expanding $z(\xi)$, Eq. (24), where $\widetilde{\phi}_i(\xi)$ are again the eigenfunctions of Eq. (32), but now with periodic boundary conditions

$$\widetilde{\phi}_i(-\Lambda) = \widetilde{\phi}_i(\Lambda), \quad \widetilde{\phi}_i'(-\Lambda) = \widetilde{\phi}_i'(\Lambda). \tag{39}$$

The eigenvalue problem (32),(39) is solved in the Appendix. The discrete spectrum consists of one negative eigenvalue $\widetilde{\lambda}_0 = -5/4$, one zero eigenvalue $\widetilde{\lambda}_1 = 0$, and $\widetilde{\lambda}_2 = 3/4$; all other eigenvalues are positive and belong to the quasicontinuous spectrum, $\widetilde{\lambda}_i \geq 1$ for $i > 2$. In the denominator of Eq. (17) the integration over the amplitudes $\widetilde{x}_2, \widetilde{x}_3, \widetilde{x}_4, \ldots$ of the modes with positive eigenvalues is easily performed in Gaussian approximation. The integration over the amplitude $\widetilde{x}_1$ of the zero mode is less trivial.

Let us discuss the physical origin of the zero mode. The functional $F\{z\}$ is translationally invariant on a ring, and therefore, $F\{z_s\}$ does not change if the saddle-point solution is shifted, $z_s(\xi) \rightarrow z_s(\xi+\Delta\xi)$. In other words, the deformation $\delta z_s = \Delta\xi z_s'(\xi)$ does not affect F. On the other hand, according to Eqs. (24) and (27) the deformation $\delta z_s = \widetilde{x}_1\widetilde{\phi}_1(\xi)$ of the saddle-point solution does not change F either, because $\widetilde{\lambda}_1 = 0$. Since the zero mode is unique, we conclude

$$\Delta\xi z_s'(\xi) = \widetilde{x}_1\widetilde{\phi}_1(\xi). \tag{40}$$

Then it follows that[32] $z_s'(\xi) = c_1\widetilde{\phi}_1(\xi)$. The constant c_1 can be found from the normalization condition for the eigenfunctions $\widetilde{\phi}_i(\xi)$,

$$c_1 = \sqrt{\int [z_s'(\xi)]^2 d\xi}. \tag{41}$$

Using Eq. (40) and the relation $z_s'(\xi) = c_1\widetilde{\phi}_1(\xi)$, the integral over the amplitude $\widetilde{x}_1$ of the zero mode takes the form

$$\int d\widetilde{x}_1 = c_1\int d(\Delta\xi) = 2c_1\Lambda. \tag{42}$$

Expressions for $\partial f/\partial\widetilde{x}_0$ and D_{ij} were obtained in Sec. IV independently of the exact form of the saddle-point density and are, therefore, still applicable. Substituting expression (23) for the numerator of Eq. (17) and using Eq. (42) in the denominator, we find

$$\tau_r^* = \frac{\sqrt{\pi}\tau_0^*}{\sqrt{2|\widetilde{\lambda}_0|\lambda_1}c_1\Lambda\sqrt{U_1}}\sqrt{\frac{\widetilde{\lambda}_2}{\lambda_2}}Y_r. \tag{43}$$

Here the infinite product Y_r is similar to Y_s evaluated in Sec. IV B, Eq. (31), but with the eigenvalues λ_i and $\widetilde{\lambda}_i$ calculated with periodic boundary conditions (39). It is shown in the Appendix that at $\Lambda \gg 1$ the quasicontinuous spectrum of the eigenvalue problem (32),(39) is still given by Eq. (33), but becomes doubly degenerate. Therefore, in the limit of an infinite strip the product Y_r can be found as

$$Y_r = Y_s^2 = \frac{2}{15}, \tag{44}$$

cf. Eq. (35). Also, substituting expression (11) for $z_s(\xi)$ into Eq. (41) we find $c_1 = \sqrt{6/5}$ at $\Lambda \gg 1$. Upon substitution of these results into Eq. (43) the mean time of switching inside a long strip takes the form,

$$\tau_i = \frac{\sqrt{\pi}}{15\Lambda}\frac{\tau_0^*}{\sqrt{U_1}}\exp\left(\frac{6U_1}{5}\right), \tag{45}$$

where U_1 and τ_0^* are given, respectively, by Eqs. (7) and (15), and $\Lambda = L/2r_0$. To obtain the exponent we used the fact that the integral $I = 3/5$ at $\Lambda \gg 1$.

In the case of a device of strip geometry, switching can be initiated both at the ends and inside the sample. The exponent of τ_i is a factor of 2 larger than the exponent of τ_e [see Eq. (36)], which makes the switching at the ends generally more favorable. On the other hand, the rate $1/\tau_i$ is proportional to the strip length L. Therefore, in very long strips interior switching denominates.

D. Nucleation in rings of arbitrary circumference

Expression (45) was obtained in the limit $L \gg r_0$. In this section we discuss the case of a ring of finite circumference $L \lesssim r_0$. We will consider separately the cases $L < 2\pi r_0$ and $L > 2\pi r_0$. We will use the expression (38) for the exponential of the switching time, which was obtained for an arbitrary circumference of the ring.

Let us start with the case $L < 2\pi r_0$. In dimensionless units it corresponds to the problem of a ring with the circumference $2\Lambda < 2\pi$. Similarly to the case of a strip of length $\Lambda < \pi$, see Sec. IV B 2, the saddle-point density is uniform, $z_s(\xi) = 1$, and from Eq. (25) we obtain $\widetilde{\lambda}_i = \lambda_i - 2$. Analogously to the result (30),(31) for the strip geometry the prefactor takes the form

$$\tau_r^* = \frac{\tau_0^*}{\sqrt{|\widetilde{\lambda}_0|}}\prod_{i=1}^{\infty}\sqrt{\frac{\widetilde{\lambda}_i}{\lambda_i}}. \tag{46}$$

Since the boundary conditions for the ring are periodic, Eq. (25) has two types of solutions, $\cos(\pi i\xi/\Lambda)$ and $\sin(\pi i\xi/\Lambda)$. The two solutions have the same eigenvalues for any $i > 0$, and thus all eigenvalues except for $\widetilde{\lambda}_0$ are doubly degenerate. Repeating calculations similar to those for Eq. (37), we find

$$\tau_i^* = \tau_0^*\frac{\sin\Lambda}{\sinh\Lambda}. \tag{47}$$

This expression is similar to the result (37) for the strip of length $\Lambda < \pi$. The absence of the square root in the right-hand side of Eq. (47) is due to the double degeneracy of the eigenvalues.

At $\Lambda > \pi$ one needs to solve the eigenvalue problem (25) with nonuniform saddle point $z_s(\xi)$, Eqs. (9) and (10), which can be done numerically. Then one substitutes the eigenvalues $\widetilde{\lambda}_i$ into Eq. (43) to find the prefactor τ_r^*. As we found in the Appendix, in the limit $\Lambda \rightarrow \infty$ the three lowest eigenvalues $\widetilde{\lambda}_i$ are nondegenerate whereas the rest of them are doubly
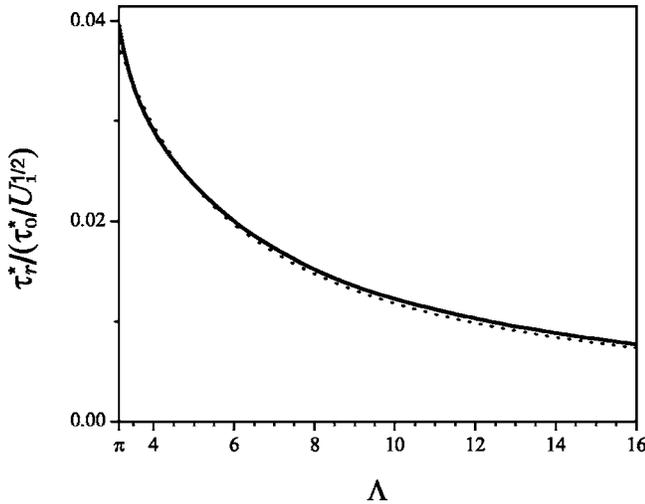
FIG. 4. Prefactor τ_r^* of the mean switching time in units $\tau_0^*/\sqrt{U_1}$ vs the dimensionless circumference of the ring 2Λ above $\Lambda=\pi$. The dotted line shows the asymptote $\sqrt{\pi}/15\Lambda$ of τ_r^* at $\Lambda \to \infty$, see Eq. (45).

degenerate, because the potential $-2z_s(\xi)$ is then reflectionless. Unexpectedly, our numerical calculation shows that the same property holds for any $\Lambda > \pi$. Then similar to Eq. (44) we find $\Upsilon_r = \Upsilon_s^2$, where Υ_s defined by Eq. (31) was computed in Sec. IV B 2. The prefactor τ_i^* above $\Lambda=\pi$ is plotted as a function of Λ in Fig. 4. One can see in Fig. 4 that at $\Lambda \to \infty$ the prefactor approaches 0 as $1/\Lambda$ in agreement with Eq. (45).

Contrary to the case of a strip, see Fig. 3, the prefactor τ_r^* does not vanish as Λ approaches π from above. This is due to the fact that at $\Lambda \to \pi$ small $\widetilde{\lambda}_2$ in the numerator of Eq. (43) is compensated by small c_1 in the denominator. To see this let us consider the saddle point solution close to $\Lambda=\pi$. At $\Lambda = \pi + \epsilon$ with $\epsilon \ll 1$ this saddle point can be expanded up to its second harmonic as

$$z_s = 1 + \kappa_0 + \kappa_1 \cos(\pi\xi/\Lambda) + \kappa_2 \cos(2\pi\xi/\Lambda), \quad (48)$$

where $\kappa_0 = -\kappa_1^2/2$, $\kappa_2 = \kappa_1^2/6$, and $\epsilon = 5\pi\kappa_1^2/12$. Then from Eq. (41) we obtain $c_1 = \sqrt{\pi}\kappa_1$, and calculating $\widetilde{\lambda}_2$ up to second order in κ_1 we find $\widetilde{\lambda}_2 = 5\kappa_1^2/3$. As a result the ratio $\sqrt{\widetilde{\lambda}_2}/c_1$ in Eq. (43) remains finite, and the prefactor at $\Lambda \to \pi^+$ takes the form

$$\tau_r^* = \frac{\sqrt{5}}{2\sqrt{6}\sinh\pi} \frac{\tau_0^*}{\sqrt{U_1}}. \quad (49)$$

Similar to the case of a finite strip, the singular behavior of τ_r^* near $\Lambda=\pi$ is an artifact of Gaussian approximation. We expect that a more careful treatment will show that the prefactor remains nonzero and continuous at $\Lambda=\pi$.

V. DISCUSSION

We have studied the mean time τ of switching from the metastable to the stable state in one-dimensional double-barrier resonant tunneling structures. We calculated both the

exponentials and prefactors of τ for the strip and ring geometries of the sample. In this section we discuss the behavior of the mean switching time in a strip-shaped sample depending on the distance from the threshold and the structural parameters of DBRTS.

In the very vicinity of the threshold the exponent of the mean switching time is of order unity and the metastable state decays very rapidly. As the voltage V is tuned further inside the bistable region, the decay time becomes exponentially long. Our results are applicable in this regime. Thus the exponents in the expressions for the switching time in the regime of short strip, Eq. (3), and in the regime of long strip, Eqs. (36) and (45), must be much greater than unity. To check whether these conditions are satisfied, it is convenient to introduce a bias-independent characteristic length

$$d \equiv r_0 \left(\frac{3U_1}{5} \right)^{1/5} = \left(\frac{24}{5} \frac{w\eta^3}{\gamma^2} \right)^{1/5} \quad (50)$$

chosen in such a way that the exponent in Eq. (36) takes the simple form $(d/r_0)^5$.

The exponent of the mean switching time in the short-strip regime [Eq. (3)] can be expressed in terms of d and r_0 as

$$\frac{4Lw\alpha^{3/2}}{3\gamma^{1/2}} = \frac{5}{18} \frac{L}{r_0} \left(\frac{d}{r_0} \right)^5. \quad (51)$$

This exponent is much greater than unity at $r_0 \ll (Ld^5)^{1/6}$. Since the regime of short strip is defined by $r_0 \gg L$, it is present only if $L \ll d$. In this case, starting at voltage difference $V_{th} - V$ corresponding to $r_0 \sim (Ld^5)^{1/6}$, there is a region of 3/2-power law dependence of $\ln \tau$ corresponding to the short-strip regime. Then, as $V_{th} - V$ reaches the value corresponding to $r_0 \sim L$, follows the region of 5/4-dependence of $\ln \tau$ for the long-strip regime, see Fig. 5(a).

In the latter case the switching can be initiated either inside or at each of the two ends of the sample. Summing the rates of these processes we find

$$\frac{1}{\tau} = \frac{2}{\tau_e} + \frac{1}{\tau_i}. \quad (52)$$

To determine the type of 5/4-dependence that can be observed in the regime of long strip, $r_0 \ll L$, one should compare the rates of switching inside and at the ends of the strip. Using Eqs. (36) and (45), the ratio of the rates τ_e^{-1} and τ_i^{-1} can be expressed as

$$\frac{\tau_e^{-1}}{\tau_i^{-1}} \sim \frac{d}{L} \exp \left[\left(\frac{d}{r_0} \right)^5 - \frac{7}{2} \ln \frac{d}{r_0} \right]. \quad (53)$$

Using this expression we conclude that at $L \ll d$ the switching always initiates at the ends rather than inside the strip. Thus the region of 5/4-dependence is described by Eq. (36).

At $L \gg d$ there is no region corresponding to the regime of a short strip. One can only observe 5/4-power law dependence of $\ln \tau$ corresponding to the long-strip regime. Analyzing Eq. (53) one can see that at $L \gg d$ there are two distinct regions of 5/4-power law dependences. At $r_0 \lesssim d$ and very large L the switching initiates inside the strip. At very small
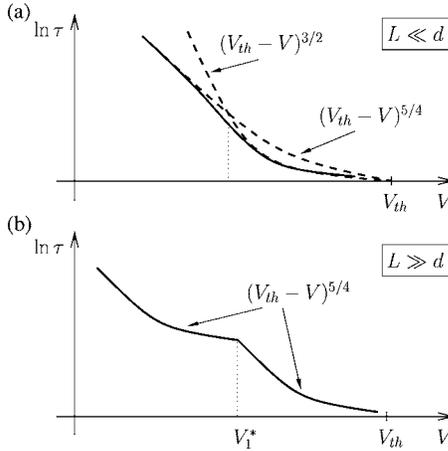
FIG. 5. (a) Logarithm of the mean switching time τ vs voltage at $L \ll d$. Near the threshold there is a region of 3/2-power law dependence of $\ln \tau$ (regime of short strip), then follows the region of 5/4-dependence corresponding to the switching at the ends in the regime of long strip. (b) Logarithm of the switching time τ vs voltage at $L \gg d$. Close to the threshold one first observes 5/4-power law behavior corresponding to the interior switching, then at voltage below V_1^* follows the region of 5/4-dependence corresponding to switching at the ends.

r_0 the exponential in Eq. (53) becomes very large, and therefore, the switching takes place at the ends. The voltage V_1^*, at which the crossover between these two regions occurs, is given by the condition $\tau_i^{-1} = \tau_e^{-1}$ applied to Eq. (53),

$$V_1^* \approx V_{th} - \frac{\eta^2}{\gamma a d^4} \left(\ln \frac{L}{d} \right)^{4/5}. \tag{54}$$

Thus we conclude that in the case of $L \gg d$, starting at V corresponding to $r_0 \sim d$, one first observes 5/4-power law dependence of $\ln \tau$ on voltage corresponding to the switching inside the strip, then at V below V_1^* follows the region of 5/4-dependence corresponding to the switching at the ends, see Fig. 5(b).

ACKNOWLEDGMENTS

APPENDIX: EIGENFUNCTIONS AND EIGENVALUES OF EQ. (32)

In this Appendix we find the eigenfunctions of discrete spectrum and the phase shifts of continuous spectrum of Eq. (32), as well as the respective eigenvalues $\tilde{\lambda}_i$. In order to solve Eq. (32) one needs to diagonalize the Hamiltonian

$$H_m = -\frac{d^2}{d\xi^2} - \frac{m(m+1)}{4 \cosh^2(\xi/2)}, \tag{A1}$$

with $m = 3$.

The eigenstates of the Hamiltonian (A1) can be obtained using an algebraic technique called supersymmetric quantum mechanics, see Refs. 33–35. We begin by introducing the raising and lowering operators

$$a^{\pm} = \mp \frac{d}{d\xi} + \frac{m}{2} \tanh \frac{\xi}{2}. \tag{A2}$$

Hamiltonians H_m and H_{m-1} can be expressed in terms of a^+ and a^- as

$$H_m = a^+ a^- - \frac{m^2}{4}, \tag{A3a}$$

$$H_{m-1} = a^- a^+ - \frac{m^2}{4}. \tag{A3b}$$

Let us consider the function $\Psi(\xi) = 1/\cosh^m(\xi/2)$. It satisfies the condition $a^- \Psi(\xi) = 0$, and therefore, Ψ is an eigenfunction of the Hamiltonian H_m with eigenvalue $\epsilon = -m^2/4$. Since $\Psi(\xi)$ has no zeros, it describes the ground state of H_m. Thus we find the ground-state wave function and energy

$$\Psi_0^{(m)}(\xi) = \sqrt{\frac{\Gamma\left(m + \frac{1}{2}\right)}{2\sqrt{\pi}\Gamma(m)}} \frac{1}{\cosh^m(\xi/2)}, \quad \epsilon_0^{(m)} = -\frac{m^2}{4}. \tag{A4}$$

Applying the raising operator a^+ to $H_{m-1}\Psi_i^{(m-1)} = \epsilon_i^{(m-1)}\Psi_i^{(m-1)}$ we find that $a^+\Psi_i^{(m-1)}$ are eigenfunctions of Hamiltonian H_m with eigenvalues $\epsilon_i^{(m-1)}$. Upon appropriate normalization we obtain

$$\Psi_i^{(m)}(\xi) = \frac{1}{\sqrt{\epsilon_{i-1}^{(m-1)} + m^2/4}} a^+ \Psi_{i-1}^{(m-1)}, \quad \epsilon_i^{(m)} = \epsilon_{i-1}^{(m-1)}. \tag{A5}$$

Conversely, by acting with a^- on $H_m\Psi_i^{(m)} = \epsilon_i^{(m)}\Psi_i^{(m)}$ with $i > 0$, one finds that $a^-\Psi_i^{(m)}$ are eigenfunctions of Hamiltonian H_{m-1} with eigenvalues $\epsilon_i^{(m)}$. Therefore, we conclude that in addition to its ground-state eigenvalue $\epsilon_0^{(m)}$ given by (A4), the spectrum of H_m consists of all the eigenvalues of H_{m-1}.

Using Eqs. (A4) and (A5) one can derive the eigenfunctions and eigenvalues of H_m from those of H_{m-1} and vice versa. To illustrate this, let us consider the Hamiltonian $H_0 = -(d/d\xi)^2$. Its eigenfunctions are $e^{\pm iq\xi}$, and the spectrum is given by $\epsilon_q^{(0)} = q^2$, where q is an arbitrary real wave number. Then using Eq. (A5) we can find the eigenstates of Hamiltonian H_1. The continuous spectrum is obviously $\epsilon_q^{(0)}$, and its normalized eigenfunctions are

$$\Psi_q^{(1)}(\xi) = \frac{1}{\sqrt{2\pi(q^2 + 1/4)}} \left(-\frac{d}{d\xi} + \frac{1}{2} \tanh \frac{\xi}{2} \right) e^{iq\xi}. \tag{A6}$$

The ground-state wave function and eigenvalue of H_1 are given by Eq. (A4) with $m = 1$.

Applying the same technique two more times we find the bound states of H_3,

$$\Psi_0^{(3)}(\xi) = \frac{\sqrt{15}}{4\sqrt{2}}\frac{1}{\cosh^3(\xi/2)}, \quad \epsilon_0^{(3)} = -\frac{9}{4}, \qquad (A7)$$

$$\Psi_1^{(3)}(\xi) = \frac{\sqrt{15}}{2\sqrt{2}}\frac{\sinh(\xi/2)}{\cosh^3(\xi/2)}, \quad \epsilon_1^{(3)} = -1, \qquad (A8)$$

$$\Psi_2^{(3)}(\xi) = \frac{\sqrt{3}}{\sqrt{2}\cosh\dfrac{\xi}{2}} - \frac{5\sqrt{3}}{4\sqrt{2}\cosh^3\dfrac{\xi}{2}}, \quad \epsilon_2^{(3)} = -\frac{1}{4}.$$
$$(A9)$$

The eigenfunctions of continuous spectrum of H_3 are given by

$$\Psi_q^{(3)}(\xi) = \frac{1}{\sqrt{2\pi(q^2+9/4)(q^2+1)(q^2+1/4)}}\left(-\frac{d}{d\xi} + \frac{3}{2}\tanh\frac{\xi}{2}\right)$$
$$\times \left(-\frac{d}{d\xi} + \tanh\frac{\xi}{2}\right)\left(-\frac{d}{d\xi} + \frac{1}{2}\tanh\frac{\xi}{2}\right)e^{iq\xi}. \quad (A10)$$

Noting that H_3 coincides up to a constant with the Hamiltonian of Eq. (32), we conclude that $\widetilde{\phi}_i(\xi) = \Psi_i^{(3)}(\xi)$. Therefore, the discrete spectrum of Eq. (32) has three eigenvalues,

$$\widetilde{\lambda}_0 = -\frac{5}{4}, \quad \widetilde{\lambda}_1 = 0, \quad \widetilde{\lambda}_2 = \frac{3}{4}. \qquad (A11)$$

The continuous spectrum of Eq. (32) is $\widetilde{\lambda}_q = 1+q^2$. The asymptotics of its eigenfunctions (A10) can be expressed in terms of the scattering phase shifts $\delta(q)$ in the following way

$$\widetilde{\phi}_q(\xi) \to \frac{i\,\text{sgn}\,q}{\sqrt{2\pi}}e^{iq\xi}e^{\pm i\delta(q)}, \quad \xi \to \pm\infty. \qquad (A12)$$

Here the absence of the reflected wave illustrates the fact that the Hamiltonian H_m describes scattering in reflectionless potential.[36] Comparing Eqs. (A10) and (A12) one can see that the phase shifts are given by Eq. (34).

For the evaluation of the prefactor of the switching time we need to study a finite system. In Sec. IV C we consider the system of length 2Λ with periodic boundary conditions (39). At $\Lambda \gg 1$ the eigenfunctions of discrete spectrum of Eq. (32) with the boundary conditions (39) are given by Eq. (A11). The quasicontinuous spectrum of a large finite system can be obtained by applying periodic boundary conditions (39) to the eigenfunctions $\widetilde{\phi}_q$ in the asymptotic form (A12). As a result we find the wave number quantization

$$q_l = \frac{\pi l - \delta(q_l)}{\Lambda}, \qquad (A13)$$

where l is an integer. Because the phase shift (34) is an odd function of q, it follows from Eq. (A13) that $q_{-l}=-q_l$. Therefore, the eigenfunctions with wave numbers q_l and q_{-l} have the same eigenvalue $\widetilde{\lambda}=1+q_l^2$. Thus the quasicontinuous spectrum of Eq. (32) with periodic boundary conditions (39) is doubly degenerate. The two real solutions with the same $\widetilde{\lambda}_q$ can be represented as even and odd combinations of $\widetilde{\phi}_q$ and $\widetilde{\phi}_{-q}$.

Transcendental equation (A13) can be solved using iteration procedure. At $\Lambda \gg 1$ it is sufficient to perform only the first iteration. Substituting the result into $\widetilde{\lambda}_i = 1+q_i^2$ we obtain Eq. (33).

In Sec. IV B we consider the system of length Λ with the boundary conditions (26). All solutions of this problem can be obtained from the solutions of the problem with the boundary conditions (39). Indeed, any even solution of the eigenvalue problem (32) with the boundary conditions (39) satisfies the conditions (26). Therefore, from Eqs. (A7)–(A9) and (A11) one concludes that the discrete spectrum consists of two eigenvalues $-5/4$ and $3/4$. Because only even combinations $\widetilde{\phi}_q(\xi)+\widetilde{\phi}_q(-\xi) \equiv \widetilde{\phi}_q(\xi)-\widetilde{\phi}_{-q}(\xi)$ satisfy the conditions (26), the quasicontinuous spectrum is nondegenerate and given by Eq. (33).

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