

Spin Hall effect of a conserved current: Conditions for a nonzero spin Hall current

Naoyuki Sugimoto,^{1,*} Shigeki Onoda,² Shuichi Murakami,¹ and Naoto Nagaosa^{1,3,4}

¹*Department of Applied Physics, University of Tokyo, Hongo, Bunkyo-ku, Tokyo 113-8656, Japan*

²*Spin Superstructure Project, ERATO, Japan Science and Technology Agency, c/o Department of Applied Physics, University of Tokyo, Hongo, Bunkyo-ku, Tokyo 113-8656, Japan*

³*CERC, AIST, Tsukuba Central 4, Tsukuba 305-8562, Japan*

⁴*CREST, Japan Science and Technology Agency (JST), 4-1-8 Honcho, Kawaguchi-shi, Saitama 332-0012, Japan*

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We study the spin Hall effect taking into account the impurity scattering effect as general as possible with the focus on the definition of the spin current. The conserved bulk spin current (Zhang *et al.*, cond-mat/0503505) satisfying the continuity equation of spin is considered in addition to the conventional one defined by the symmetric product of the spin and velocity operators. Conditions for nonzero spin Hall current are clarified. In particular, it is found that (i) the spin Hall current is nonzero in the Rashba model with a finite-range impurity potential, and (ii) the spin Hall current vanishes in the cubic Rashba model with a δ -function impurity potential.

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Spintronics is one of the most promising new technologies, where the spin degrees of freedom of electrons in semiconductors are manipulated and utilized for functions such as memory, operation, and communication.^{1,2} One of the key routes to spintronics is to invent an efficient method to inject spins into semiconductors. In this respect, the intrinsic spin Hall effect (SHE) has attracted recent intensive attention since its theoretical proposal.^{3,4} This can give a larger effect by orders of magnitude than the extrinsic one based on the impurity scatterings proposed long before.⁵⁻⁷

Recently two experiments have been reported on observations of the SHE in GaAs and related materials.^{8,9} Kato *et al.*⁸ observed the Kerr rotation due to the spin accumulation [$\sim 10 \mu_B(\mu\text{m})^{-3}$] near the edges of the *n*-type GaAs sample. They suggested the extrinsic mechanism of the SHE since it was almost insensitive to the crystal orientation. Wunderlich *et al.*⁹ observed the circularly polarized light emitting diode (LED) signal from spin-polarized interfacial two-dimensional holes in *p*-type GaAs system. From an estimation of transport lifetime, they concluded that the spin accumulation is due to the intrinsic SHE.

The debates on the impurity effect on the intrinsic SHE have continued, which are in parallel to those for the anomalous Hall effect in ferromagnets. In the latter case, the intrinsic mechanism of Karplus-Luttinger¹⁰ was criticized and extrinsic mechanisms due to impurity scatterings were proposed.¹¹ The Hall conductivity is a singular function as the disorder strength approaches zero. In the metallic case, the vertex correction in the diagrammatic language incorporates a deviation of the electronic distribution function from equilibrium, and it represents the dissipative current. This situation is similar also in the spin Hall current.

Actually, the disorder effect on the spin Hall current of the Rashba model in two dimensions has been intensively studied. Sinova *et al.*⁴ obtained the universal value $e/(8\pi)$ ¹² for the spin Hall conductivity (SHC) σ_H^s without disorder. When the self-energy correction due to impurity scattering is taken into account, the SHC σ_H^s is reduced continuously as a function of the disorder strength from the universal value $e/(8\pi)$.¹³ On the other hand, Inoue *et al.*¹⁴ studied the vertex correction and found that σ_H^s vanishes in the clean limit

within the Born approximation. Furthermore, it has been shown that the SHC σ_H^s vanishes for any value of the lifetime τ using the Keldysh formalism,¹⁵⁻¹⁷ the Kubo formula analytically,¹⁸⁻²⁰ and numerically,^{21,22} and the Boltzmann equation.²³ Thus after long debates, people have reached the consensus that the SHC σ_H^s for the Rashba model vanishes for any τ .²⁴

However, the vanishing result of σ_H^s depends on the definition of the spin current. In the previous calculations on the SHC, the spin current is defined *ad hoc* as a symmetrized product of the spin and the velocity $J_s = \frac{1}{2}\{v_y, s_z\}$, where $v_y = \partial H / \partial p_y$, in response to the electric field \mathbf{E} along the *x* axis. However, this “conventional” definition of spin current loses its physical foundation when the spin-orbit coupling is present and the conservation law of the spin is violated. This is an important issue since the concept of “current” depends crucially on conservation; nonlocal effects of the current comes from the fact that an incoming flow goes out without loss. Therefore we need to search for a proper definition of a conserved spin current in the bulk. From this viewpoint, the conserved spin current \mathcal{J}_s ²⁵ deserves scrutiny. If $\int dV \langle \dot{s}_z \rangle = 0$, it satisfies the Onsager’s reciprocity relation and the continuity equation for the spin in the bulk: $\partial_t \langle s_z \rangle + \nabla \cdot \mathcal{J}_s = 0$. Furthermore it is free from an artifact that the spin current is proportional to time derivative of the spin operator;^{18,19} hence, the spin Hall current can be nonzero even for the Rashba model. Even though the experiments on the SHE up to now^{8,9} detect the spin accumulation at the sample edges, we stick here to the SHE defined by the bulk spin current. This is because we consider that the generation of the conserved spin current in the bulk is a more fundamental phenomenon than the spin accumulation at edges, which is not solely determined by this spin current since the continuity equation for the spin is not satisfied there. In principle, there should be other means to detect the SHE without using the spin accumulation such as voltage measurement with the injected spin current.²⁶

In this paper, we study the SHE as generally as possible, taking into account the disorder with the definitions of the conventional and conserved spin current. Applying this consideration, some new results are obtained for the Rashba and

cubic Rashba models. We employ the Keldysh formalism,^{15,27,28} by which the infinite series of the Feynman diagrams both for the self-energy and vertex correction are taken into account compactly, and the expression for the SHC is obtained for both definitions of the spin current described above.

We consider a generic model with spin-orbit coupling with a random impurity potential, this random potential is assumed to be spin-independent, and the time-reversal symmetric model exclusively. In the Keldysh formalism, a Green's function matrix \underline{G} is introduced,

$$\underline{G} = \begin{pmatrix} G^R & G^< \\ 0 & G^A \end{pmatrix}, \quad (1)$$

where the superscripts R , A , and $<$ denote the retarded, advanced, and lesser Green's functions, respectively. The self-energy matrix $\underline{\Sigma}$ is defined similarly. The Green's functions satisfy

$$(G^R)^{-1} \otimes G^< - G^< \otimes (G^A)^{-1} = \Sigma^< \otimes G^A - G^R \otimes \Sigma^<, \quad (2)$$

$$(\mathcal{G}_0^{-1} - \Sigma^{R,A}) \otimes G^{R,A} = \delta(1-2), \quad (3)$$

where $(A \otimes B)(1,2) \equiv \int d^3A(1,3)B(3,2)$ and \mathcal{G}_0 is the unperturbed Green's function. We then separate the center-of-mass and the relative coordinates and perform the Fourier transform to the relative coordinates.^{27,28} The final result is written in terms of the center-of-mass coordinates (T, \mathbf{R}) and the relative momentum (ω, \mathbf{p}) . We put the constant electric field $\mathbf{E} = (E, 0, 0)$ and look for solutions independent of T and \mathbf{R} . Therefore, the quantum Boltzmann equation (QBE) for particles with charge q is written by

$$\begin{aligned} [G, H] = & -iq\mathbf{E} \cdot \nabla_{\mathbf{p}} G - \frac{i}{2}q\mathbf{E} \cdot \{\nabla_{\mathbf{p}} H, \partial_{\omega} G\} \\ & - \frac{i}{2}q\mathbf{E} \cdot (\{\nabla_{\mathbf{p}} \Sigma, \partial_{\omega} G\} - \{\partial_{\omega} \Sigma, \nabla_{\mathbf{p}} G\}) + [\Sigma, G]. \end{aligned} \quad (4)$$

Here one can show from Eq. (4) that the time derivative $\dot{\mathcal{O}}$ of an arbitrary operator \mathcal{O} , which is independent of \mathbf{p} , \mathbf{R} , T , and ω , has a vanishing expectation value in the steady state, $\langle \dot{\mathcal{O}} \rangle = 0$, even with a general form of the impurity potential.^{18,19} We start with

$$\begin{aligned} \langle \dot{\mathcal{O}} \rangle = & \frac{1}{i} \int \frac{d\omega}{2\pi} \int \frac{d^2p}{(2\pi)^2} \text{tr}(\dot{\mathcal{O}} G^<) \\ = & \int \frac{d\omega}{2\pi} \int \frac{d^2p}{(2\pi)^2} \text{tr}(\mathcal{O} [G, H]^<), \end{aligned} \quad (5)$$

where $[]^<$ denotes the lesser (upper right) component of the matrix in the Keldysh space. Equation (4) is plugged into Eq. (5), and evaluate the respective terms in the RHS of Eq. (4). The first term $-iq\mathbf{E} \cdot \nabla_{\mathbf{p}} G^<$ becomes zero after an integration over \mathbf{p} . The second term vanishes after an ω -integration. The third and fourth terms vanish after partial integrations in terms of \mathbf{p} and ω . Lastly, to evaluate the last term $([\Sigma, G]^<)$, we need a relationship between the self-energies and Green's

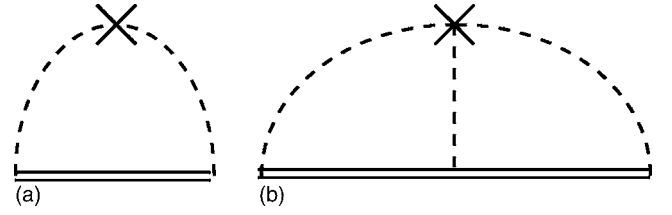


FIG. 1. Diagrammatic representation of the self-energy $\underline{\Sigma}$ in the present self-consistent approximation. Doubled line is the dressed Keldysh Green's function \underline{G} . Dashed line with a cross denotes the average over the impurity positions in the second order for (a) and in the third order (b), respectively.

functions. We employ the self-consistent Born approximation [the diagrams in Figs. 1(a) and 1(b)]²⁹ for the impurity scattering. Up to the second order it is given by

$$\begin{aligned} \underline{\Sigma}(\omega, \mathbf{p}) = & n_i \int \frac{d^2p'}{(2\pi)^2} |V_{\mathbf{p}, \mathbf{p}'}|^2 \underline{G}(\omega, \mathbf{p}') \\ & + n_i \int \frac{d^2p' d^2p''}{(2\pi)^4} V_{\mathbf{p}, \mathbf{p}'} V_{\mathbf{p}', \mathbf{p}''} V_{\mathbf{p}'', \mathbf{p}} \underline{G}(\omega, \mathbf{p}') \underline{G}(\omega, \mathbf{p}''). \end{aligned} \quad (6)$$

where n_i is an impurity density, and $V_{\mathbf{p}, \mathbf{p}'}$ is the Fourier transform of the impurity potential. From the second Born approximation Eq. (6) one can easily show $\langle \dot{\mathcal{O}} \rangle = 0$ for arbitrary forms of impurity potentials.³⁰ This holds true even for higher-order Born approximation. On the other hand, for the charge current $\mathbf{v} = \dot{\mathbf{R}}$, this argument does not apply, and $\langle \mathbf{v} \rangle$ can be nonzero in the steady state as expected.

Now we proceed to a closed form of the SHC. The expectation value of the conventional spin current $J_s \equiv \frac{1}{2} \langle v_y, s_z \rangle$ is obtained as

$$\frac{\langle J_s \rangle}{E} = - \lim_{E \rightarrow 0} \frac{i}{2E} \int \frac{d\omega}{2\pi} \frac{d^2p}{(2\pi)^2} \text{tr} \left[s_z \left\{ \frac{\partial H}{\partial p_y}, \underline{G} \right\} \right]_E^<, \quad (7)$$

where $[]_E$ implies that we retain the terms linear in a uniform electric field $\mathbf{E} = E\hat{x}$.

Let us turn to the second definition of the spin current \mathcal{J}_s as proposed by Zhang *et al.*²⁵ \mathcal{J}_s is defined to satisfy $\partial_t \langle s_z \rangle + \nabla \cdot \mathcal{J}_s = 0$, and is divided into $\langle \mathcal{J}_s \rangle + \mathbf{P}_\tau$ where $\mathbf{J}_s = \frac{1}{2} \langle \mathbf{v}, s_z \rangle$. The second term \mathbf{P}_τ is called the torque dipole density, and is required to satisfy $\lim_{\Omega \rightarrow 0} \lim_{\mathbf{Q} \rightarrow 0} (\langle \dot{s}_z(\Omega, \mathbf{Q}) \rangle + i\mathbf{Q} \cdot \mathbf{P}_\tau(\Omega, \mathbf{Q})) = 0$, where (Ω, \mathbf{Q}) are the Fourier components of the center-of-mass coordinates. We put $Q_x = 0$, and take the limit $Q_y = Q \rightarrow 0$, i.e., $\mathbf{P}_\tau \equiv \mathbf{P}_\tau \cdot \hat{\mathbf{y}} = -\lim_{\Omega \rightarrow 0} \lim_{Q \rightarrow 0} (1/iQ) \times \langle \dot{s}_z(\Omega, Q) \rangle$. More explicitly, \mathbf{E} is spatially modulated along the y -axis, $\mathbf{E} = E e^{iQY - i\Omega T} \hat{\mathbf{x}}$.²⁵ Note that $\langle \dot{s}_z(0, 0) \rangle = 0$ from Eq. (5), which means \mathbf{P}_τ is finite and well-defined. In response to the electric field, the Green's function \underline{G} and the self-energy $\underline{\Sigma}$ acquire terms proportional to $e^{iQY - i\Omega T}$. \mathbf{P}_τ is expressed as

$$\mathbf{P}_\tau = -i \lim_{Q \rightarrow 0} \frac{1}{Q} \int \frac{d\omega}{2\pi} \int \frac{d^2p}{(2\pi)^2} \text{tr} [s_z [H, \underline{G}]^<]. \quad (8)$$

We first write down the QBE to the linear order in E and to the linear order in Ω or Q . Next, we replace the term $[H, \underline{G}]^<$

in Eq. (8) with the corresponding term in the QBE. While there arise a number of terms, most of them give no contribution to P_τ after partial integrations over p_i or ω . To calculate the remaining terms, we note that this electric field necessarily accompanies a magnetic field according to the Maxwell equation. To deal with the response to these two fields, it is convenient to consider the corresponding vector potential, $\mathbf{A}=(Ae^{iQY-i\Omega T},0,0)$. Relevant terms in the QBE are classified to those proportional to $i\Omega A=E$ (electric field) or those proportional to $-iQA=B$ (magnetic field). The resulting form is a sum of the contributions from the response to a dc electric field $E\hat{x}$ and that to a dc magnetic field $B\hat{z}$:

$$\frac{\mathcal{J}_s}{E} = \lim_{E \rightarrow 0} \frac{i}{E} \int \frac{d\omega}{2\pi} \frac{d^2p}{(2\pi)^2} \text{tr} \left[\frac{1}{2} s_z \left\{ \frac{\partial \Sigma}{\partial p_y}, \mathcal{G} \right\} \right]_E^< \\ + \lim_{B \rightarrow 0} \frac{i}{B} \int \frac{d\omega}{2\pi} \frac{d^2p}{(2\pi)^2} \text{tr} \left[s_z \mathcal{G} - \frac{1}{2} s_z \left\{ \frac{\partial \Sigma}{\partial \omega}, \mathcal{G} \right\} \right]_B^<. \quad (9)$$

Here $\mathcal{J}_s = \mathcal{J}_s \cdot \hat{\mathbf{y}}$, and $[\]_B$ retains the terms linear in an uniform magnetic field $\mathbf{B} = B\hat{z}$. Because \mathbf{B} does not drive the system off-equilibrium, the relation in equilibrium, $G_0^< = (1 - \tanh(\omega/2k_B T))(G_0^A - G_0^R)/2$, is satisfied even in the presence of $\mathbf{B} = B\hat{z}$. The details of the derivation of Eq. (9) will be presented elsewhere.³¹ Remarkably, in calculating the total conserved spin current $\mathcal{J}_s = \langle J_s \rangle + P_\tau$ there appears a term in P_τ which exactly cancels $\langle J_s \rangle$. We also note that this formula is quite generic and applies to any model. The expression of the charge current is obtained just by replacing s_z by $-e$ in Eq. (9), and the B -term in the final formula is reminiscent of the Streda formula.^{32,33}

We now discuss the explicit models based on the results obtained above. Here we consider the Rashba model

$$H = \frac{\mathbf{p}^2}{2m} + \lambda(\boldsymbol{\sigma} \times \mathbf{p}) \cdot \hat{z} + v, \quad (10)$$

and the cubic Rashba model

$$H = \frac{\mathbf{p}^2}{2m} + \frac{i\lambda}{2}(p_-^3 \sigma_+ - p_+^3 \sigma_-) + v. \quad (11)$$

Here, $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z) = 2\mathbf{s}$ is the Pauli matrix, $p_\pm = p_x \pm ip_y$, $\sigma_\pm = \sigma_x \pm i\sigma_y$, and v is an impurity random potential. We take the unit where $\hbar = c = 1$. The Rashba model represents an n -type semiconductor in two-dimensional heterostructure. The second term in Eq. (10) represents the spin-orbit coupling with an inversion-symmetry-breaking potential along the z direction perpendicular to the plane. As noticed in Refs. 18 and 19, J_s is proportional to \dot{s}_y : $J_s = p_y \sigma_z / 2m = [H, \sigma_y] / 4im\lambda$. Therefore, from the generic argument in Eq. (5), there occurs no spin Hall current when this definition of the spin current is employed for the Rashba model [Table I(a)]. This result is consistent with previous works using various methods, including the calculations in the clean limit by the Kubo formula,^{14,20} and Keldysh formalism.¹⁶ For finite τ ($\epsilon_F \tau \gg 1$), it is also consistent with the analytic^{18,19} and numerical^{21,22} results by the Kubo formula, and with the results by the Keldysh formalism.^{15,17} The calculation in Ref. 19 by the Kubo formula for finite τ ($\epsilon_F \tau \gg 1$) is similar to

TABLE I. Spin Hall effect in the (a) (linear) Rashba model and (b) cubic Rashba model, with various types of the impurity potential for the two different definitions: $\langle J_s \rangle$ is the conventional spin current, $J_s = \frac{1}{2}\{v_y, s_z\}$, and \mathcal{J}_s is the conserved effective spin current, $\mathcal{J}_s = \langle J_s \rangle + P_\tau$. We show the new results in the boldface.

(a) Rashba model			
		Definition of spin current	
Impurity potential	Born approx.	$\langle J_s \rangle$	\mathcal{J}_s
$\delta(\mathbf{r})$	1st/higher	0 ³⁷	0
$V_{\mathbf{p}-\mathbf{p}'}$	1st	0 ³⁸	0
	higher	0	Finite
(b) Cubic Rashba model			
$\delta(\mathbf{r})$	1st/higher	Finite ^{35,36}	0
$V_{\mathbf{p}-\mathbf{p}'}$	1st	Finite	0
	higher	Finite	Finite

ours by the Keldysh formalism. Nevertheless, our approach better reveals the reason why the spin Hall current generally vanishes when $J_s \propto \dot{s}_y$.

The cubic Rashba model, on the other hand, describes the heavy-hole bands of cubic semiconductors in heterostructure. The conventional spin current J_s can no longer be expressed as $\dot{\mathcal{O}}$; hence the previous argument for vanishing spin Hall current does not apply. Indeed, the resulting SHC is nonzero³⁴ [Table I(b)], which is consistent with Refs. 35 and 36. Therefore, the zero or nonzero spin Hall current for the conventional spin current J_s is mostly determined whether it is expressed by the time derivative of some local operator or not.

Next we turn to the conserved spin current \mathcal{J}_s . Both in the Rashba and the cubic Rashba models, the B -term in Eq. (9) vanishes, because the Hamiltonian lacks a σ_z -term, and the self-energy is independent of the spin. The remaining E -term in Eq. (9) for the Rashba model is calculated as follows. In the first Born approximation [the first term in Eq. (6)], $\mathcal{J}_s = 0$ for general $V_{\mathbf{p}-\mathbf{p}'}$. In the second Born approximation, we obtain

$$\mathcal{J}_s = -\frac{in_i}{4} \int \frac{d\omega}{2\pi} \frac{d^2p}{(2\pi)^2} \frac{d^2p'}{(2\pi)^2} \frac{d^2p''}{(2\pi)^2} \text{tr} \left[\sigma_z [G(\omega, \mathbf{p}) \cdot \mathcal{G}(\omega, \mathbf{p}') \mathcal{G}(\omega, \mathbf{p}'')]]_E^< \frac{\partial (V_{\mathbf{p}-\mathbf{p}'} V_{\mathbf{p}'-\mathbf{p}''} V_{\mathbf{p}''-\mathbf{p}})}{\partial p'_y} \right]. \quad (12)$$

Even for higher-order Born approximation, the formula for \mathcal{J}_s can be written down. We can then see that \mathcal{J}_s always depends on $\partial_{p_y} V$, and the spin Hall current for \mathcal{J}_s is extrinsic for both the Rashba and the cubic Rashba models, depending explicitly on the impurity potential. All these considerations are summarized in Table I. With the δ -impurity potential the conserved spin Hall current vanish both for the Rashba model and the cubic Rashba model. We note that calculated the conserved SHC without disorder is $\sigma_H^s = e/8\pi$ for the Rashba model and $\sigma_H^s = -9e/8\pi$ in the cubic Rashba

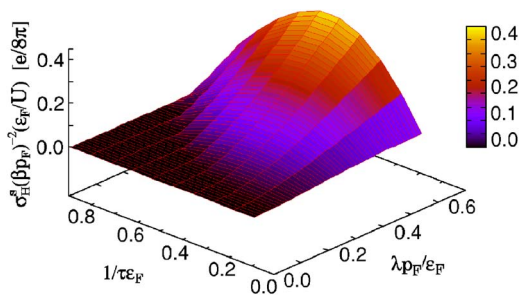


FIG. 2. (Color online) Contour map of the SHC σ_H^s . It has the maximum value nearly along $\alpha=2\lambda p_F\tau=1$. The momentum cutoff is set to be $\Lambda \approx 3p_F$, and $U/\epsilon_F=0.02$.

model.²⁵ These results are reproduced in our calculations by neglecting the self-energy of the lesser Green's function and taking the clean limit.

We now demonstrate the finite spin Hall current for \mathcal{J}_s in the Rashba model with the short-range (not δ -function) impurity potential $V(r)=Ue^{-(r/\beta)^2}$ with $\beta\Lambda \ll 1$, where U is the magnitude of the impurity potential, β is the size of the potential range and Λ is a momentum cutoff. As an approximation, we substitute the Green's functions in Eq. (9) with those for the δ -function impurity potential, and calculate \mathcal{J}_s . Figure 2 shows the results for the SHC σ_H^s in the parameter space of $\lambda p_F/\epsilon_F$ and $\tau\epsilon_F$. Since the SHC σ_H^s is proportional to $(\beta p_F)^2$ and U/ϵ_F within our calculation scheme, the vertical axis is set to be $\sigma_H^s(\beta p_F)^{-2}(\epsilon_F/U)$, and is normalized by

the universal value $e/(8\pi)$. σ_H^s has a maximum value along the line $\alpha=2\lambda p_F\tau \approx 1$ at each λ and the maximum value depends on $\log(\Lambda/p_F)$. This conserved spin current is non-zero even for the Rashba model in two dimensions.

The above results give us a hint to look for systems showing the spin Hall current. One important feature is that the Hamiltonian should involve s_z . The Luttinger model³ satisfies this condition while the cubic Rashba model does not. Therefore, the complete confinement of the electronic motion along one direction is not desirable. We have assumed that the spin-orbit interaction is unchanged by disorder. In reality, the impurity potential $v(\mathbf{r})$ induces a spin-orbit coupling as $(\mathbf{p} \times \nabla v) \cdot \mathbf{s}$. This is also expected to contribute to the extrinsic spin Hall effect.³⁹ This effect is beyond the scope of the present paper.

In conclusion, we derived a general exact formula of the spin Hall conductivity for the two kinds of spin current: (a) The product of spin and velocity operators and (b) the effective conserved spin current. The conditions for the nonzero spin Hall current has been clarified and are applied to the Rashba and cubic Rashba models.

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*Electronic address: sugimoto@appi.t.u-tokyo.ac.jp

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