Negative refractions in uniaxially anisotropic chiral media

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Uniaxially anisotropic chiral media are quite easy to be realized artificially, where the chirality appears only in one direction. In this report, we investigate the refractive properties of a plane wave incident from free space to such uniaxially chiral media. We show that different negative phase or group refractions occur in one or two eigenwaves simultaneously or separately. Hence, the uniaxially chiral media may support more kinds of negative refractions than isotropic chiral media and the left-handed materials. In the uniaxially chiral proposal, the condition to realize the negative refraction can be quite loose.

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Recently, more and more interests are focused on the theoretical and experimental studies on the negative refraction index media due to their potential applications in microwave and optics. Negative refraction was first demonstrated when electromagnetic (EM) waves are incident from free space to a left-handed medium[1] (LHM), which has been verified experimentally using a periodic structure of split-ring resonators and conducting wires.[2] Later it was shown that negative refraction is also supported by indefinite media.[3,4] An alternative route to realize negative refractions is to use photonic or phononic crystals with positive indices, which is actually determined by the dispersion characters of wave propagation in periodic structures.[5] Recent studies show that negative refractions can also be achieved for one of the eigenwaves in an isotropic chiral medium[6–8] and a gyrotropic chiral medium,[9,10] where the chirality is a scalar in both cases. While the present report was under review, further investigations on the isotropic chiral media and optically active media were reported,[11,12] showing the possibility to generate subwavelength imaging using the chiral slab.

The chiral medium is well known in the optical frequencies for the *optical activity* phenomenon, where a linearly polarized light is rotated when passing through some crystalline and biological substances.[13] At microwave frequencies, the chiral medium can be realized artificially using miniature wire spirals or conducting springs. In Ref. 6, Pendry proposed a Swiss roll structure to achieve a resonant chiral medium, which can generate negative refractions. Compared with LHM structures, chiral particles have two advantages. First, the artificial LHM uses two sets of resonant structures for electric and magnetic responses, respectively. Such two structures must resonate in the same frequency range to realize negative refractions, which restricts a very narrow LHM band. In the chiral design, however, only one resonance is required.[6] Second, the unit-to-wavelength ratio in the LHM structure is usually worse than 0.1. The smaller the ratio is, the better the structure behaves like a material. In the chiral design proposed in Ref. 6, the ratio can be less than 0.01.

In the previous study of chiral route to negative refraction, an isotropic chiral medium has been assumed,[6–8] which is, however, very difficult to be realized in practice. Usually, the chiral particles are anisotropic. In this report, we will investigate the refractive properties of EM waves in a general case where the host medium is uniaxially anisotropic and the chirality appears only in one direction. Such a uniaxially anisotropic chiral medium is very easily fabricated. Figure 1 describes three typical ways to realize such a medium. In general, the constitutive relations are written as

$$\overline{D} = [\epsilon_t \bar{\bar{I}}_t + \epsilon_z \hat{z}\hat{z}] \times \overline{E} + i\kappa \hat{z}\hat{z} \times \overline{H}, \tag{1}$$

$$\overline{B} = [\mu_t \bar{\bar{I}}_t + \mu_z \hat{z}\hat{z}] \times \overline{H} - i\kappa \hat{z}\hat{z} \times \overline{E}, \tag{2}$$

where $\kappa > 0$, $\mu_t > 0$ and $\mu_z > 0$ have been assumed for easy realization in practice, and ϵ_t and ϵ_z can be either positive or negative depending upon the host medium.

As shown in Fig. 1, the whole space is divided into two regions. Region 0 is free space and Region 1 is occupied by

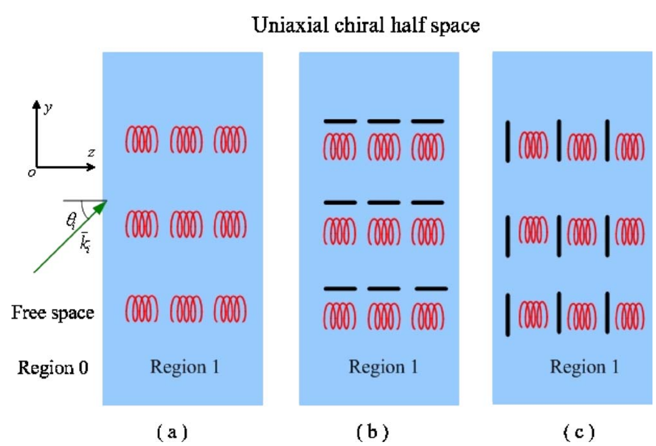

FIG. 1. (Color online) A plane wave incident from free space into a uniaxially anisotropic chiral medium. Such a chiral medium can be easily realized by placing metallic inclusions, e.g., wire spirals or copper springs which are oriented in the z direction in the host medium. (a) The host is conventional right-handed medium, where $\epsilon_t > 0$ and $\epsilon_z > 0$. (b) The host medium contains resonant dipoles directed to the z direction, where $\epsilon_t > 0$ and $\epsilon_z < 0$. Such a chiral medium can also be realized using Swiss-roll array directed to the z direction. (c) The host medium contains resonant dipoles oriented in the transverse direction, where $\epsilon_t < 0$ and $\epsilon_z > 0$.

the uniaxially chiral medium. A time harmonic plane wave is incident from free space to Region 1 at an oblique angle θ_i with respect to the normal direction of the interface. Then, the corresponding wave vector can be expressed as $\overline{k}_i = k_y\hat{y} + k_{0z}\hat{z}$. In the uniaxially chiral medium, the electric and magnetic fields are generally written as

$$\overline{E} = (\overline{E}_{1t} + E_{1z}\hat{z})e^{ik_y y}e^{ik_{1z}z}, \tag{3}$$

$$\overline{H} = (\overline{H}_{1t} + H_{1z}\hat{z})e^{ik_y y}e^{ik_{1z}z}, \tag{4}$$

where the subscripts $1t$ and $1z$ represent the transverse and longitudinal components, respectively. From the Maxwell's equations, we have

$$\nabla \times \overline{E} = i\omega\overline{B}, \tag{5}$$

$$\nabla \times \overline{H} = -i\omega\overline{D}. \tag{6}$$

Letting $\nabla = \nabla_t + ik_{1z}\hat{z}$ and substituting Eqs. (3) and (4) into (5) and (6), we can obtain the transverse fields expressed as functions of E_{1z} and H_{1z}.[14] Furthermore, we get wave equations for the longitudinal electric and magnetic fields as

$$\left\{\nabla_t^2\overline{\overline{I}} + \sigma^2\begin{bmatrix} \epsilon_z/\epsilon_t & i\kappa/\epsilon_t \\ -i\kappa/\mu_t & \mu_z/\mu_t \end{bmatrix}\right\}\begin{bmatrix} E_{1z} \\ H_{1z} \end{bmatrix} = 0, \tag{7}$$

where $\sigma^2 = k_t^2 - k_{1z}^2$, and $k_t^2 = \omega^2\epsilon_t\mu_t$. Since the fields do not have variance in the x direction, we easily obtain $\nabla_t^2 \cdot \overline{E}(\overline{H}) = -k_y^2\overline{E}(\overline{H})$. When connecting this formula with Eq. (7), we obtain the relation that must be obeyed in the uniaxially anisotropic chiral medium

$$k_y^2 = \rho_\pm\sigma^2, \tag{8}$$

which is actually the eigenvalues of the second matrix in Eq. (7), and where

$$\rho_\pm = \frac{1}{2}\left[\frac{\epsilon_z}{\epsilon_t} + \frac{\mu_z}{\mu_t} \pm \sqrt{\left(\frac{\epsilon_z}{\epsilon_t} - \frac{\mu_z}{\mu_t}\right)^2 + 4\frac{\kappa^2}{\epsilon_t\mu_t}}\right]. \tag{9}$$

Equation (8) actually defines the dispersion relations in the uniaxially chiral medium

$$\frac{k_y^2}{\rho_\pm} + k_{1z}^2 = k_t^2. \tag{10}$$

Let $k_y = k_\pm \sin\theta_{p\pm}$ and $k_{1z} = k_\pm \cos\theta_{p\pm}$, where $\theta_{p\pm}$ are phase refraction angles and $k_\pm$ are wave numbers for the eigenwaves in the uniaxially chiral medium. From Eq. (10), we have

$$k_\pm = k_t/\sqrt{\cos\theta_{p\pm}^2 + \sin\theta_{p\pm}^2/\rho_\pm}. \tag{11}$$

On the other hand, the continuity of electric and magnetic fields on the medium interface requires

$$k_0 \sin\theta_i = k_+ \sin\theta_{p+} = k_- \sin\theta_{p-}, \tag{12}$$

where k_0 is the wave number in free space. Hence, it is easy to find

$$\sin\theta_{p\pm} = k_0 \sin\theta_i\sqrt{\cos\theta_{p\pm}^2 + \sin\theta_{p\pm}^2/\rho_\pm}/k_t. \tag{13}$$

Clearly, in the uniaxially chiral medium, the phase refraction angles for both p^+ and p^- waves can be expressed as the transcendental function of the incident angle θ_i.

From Eqs. (5) and (6), all field components in the uniaxially chiral medium can be expressed as functions of E_{1y} once the wave number $k_\pm$ is determined

$$E_{1x} = -i\omega\mu_t\beta_\pm E_{1y}/k_{1z}, \tag{14}$$

$$E_{1z} = -\sigma^2 E_{1y}/(k_y k_{1z}), \tag{15}$$

$$H_{1x} = -\omega\epsilon_t E_{1y}/k_{1z}, \tag{16}$$

$$H_{1y} = -i\beta_\pm E_{1y}, \tag{17}$$

$$H_{1z} = i\beta_\pm\sigma^2 E_{1y}/(k_y k_{1z}), \tag{18}$$

where $\beta_\pm = \epsilon_t(\rho_\pm - \epsilon_z/\epsilon_t)/\kappa$. Considering $\sigma^2 = k_t^2 - k_{1z}^2 = k_y^2/\rho_\pm$, we can obtain the Poynting vectors in the y and z directions as

$$S_{1y} = \frac{\omega k_y}{\rho_\pm k_{1z}^2}|E_{1y}|^2\gamma_\pm, \tag{19}$$

$$S_{1z} = \frac{\omega}{k_{1z}}|E_{1y}|^2\gamma_\pm, \tag{20}$$

in which $\gamma_\pm = \mu_t\beta_\pm^2 + \epsilon_t$. Hence the group refraction angle defined by $\theta_{gp_\pm} = \tan^{-1} S_{1y}/S_{1z}$ can be given as

$$\theta_{gp_\pm} = \tan^{-1}(\tan\theta_{p_\pm}/\rho_\pm). \tag{21}$$

Now let us inspect Eqs. (8)–(10) again. When the plane wave enters the uniaxially anisotropic chiral medium from free space, it will obviously be decomposed into two eigenwaves, which we may call p^+ and p^- waves with the wave numbers defined in Eq. (11), respectively. For different models of uniaxially chiral media illustrated in Fig. 1, the dispersion relations and refraction properties will be quite different.

When wire spirals or copper springs, which are oriented in the z direction, are hosted in a conventional right-handed medium, as shown in Fig. 1(a), we have $\epsilon_t > 0$ and $\epsilon_z > 0$. In most of the traditional analysis, we assume $|\kappa| < \sqrt{\epsilon_z\mu_z}$ so that $\rho_\pm$ are always positive. Thus the dispersion relations in Eq. (10) describe elliptical curves for both p^+ and p^- waves, as demonstrated in Fig. 2(a). In such a condition, S_{1y} is always parallel to the transverse wave number k_y and S_{1z} is always parallel to k_{1z} due to $\gamma_\pm > 0$, which indicates positive phase and group refractions for both p^+ and p^- waves, as illustrated in Fig.3(a).

However, the above condition just corresponds to a weak coupling between the electric and magnetic fields. When the coupling further increases, say, $|\kappa| > \sqrt{\epsilon_z\mu_z}$, we have $\rho_+ > 0$, $\rho_- < 0$, and $\gamma_\pm > 0$. This implies that the dispersion relation for p^+ wave remains elliptic while the dispersion curves for p^- wave are two-sheet hyperbola, as shown in Fig. 2(b). As a consequence, the p^+ wave will experience positive phase and group refractions with no limitation to the phase refraction angle θ_{p+}, as shown in Fig. 3(b).

For the p^- wave, however, S_{1y} is antiparallel with k_y since $\rho_- < 0$ and $\gamma_- > 0$, but S_{1z} always points to the same direction

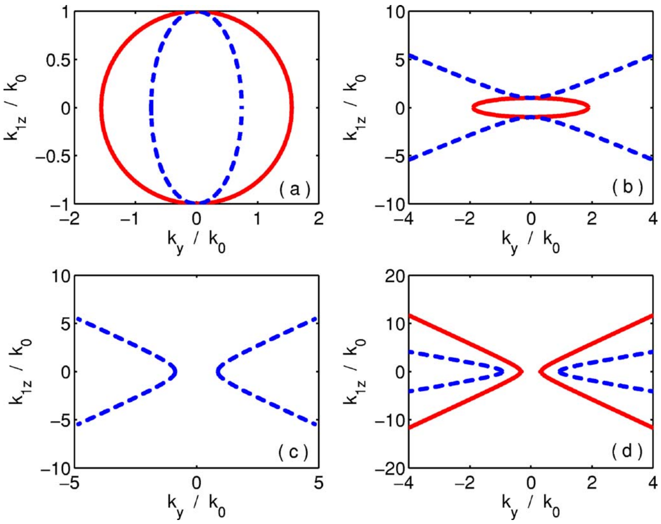

FIG. 2. (Color online) Dispersion curves for the p^+ waves (solid lines) and p^- waves (dashed lines). (a) In the traditional case, both p^+ and p^- waves have elliptical dispersion relations (with different axial ratios), where $\epsilon_t=\epsilon_0$, $\epsilon_z=2\epsilon_0$, $\mu_t=\mu_z=\mu_0$, and $\kappa=0.8\sqrt{\epsilon_0\mu_0}$. (b) In the strong chirality case corresponding to the uniaxial-chiral model shown in Fig. 1(a), the dispersion curve for p^+ wave is an ellipse and the dispersion curves for p^- wave are two-sheet hyperbola, where $\epsilon_t=\epsilon_0$, $\epsilon_z=2\epsilon_0$, $\mu_t=\mu_z=\mu_0$, and $\kappa=2\sqrt{\epsilon_0\mu_0}$. The hyperbola always keep tangent with the ellipse. Such dispersion relations also correspond to the uniaxial-chiral model shown in Fig. 1(b) with an arbitrary chirality. (c) In the weak chirality case corresponding to the uniaxial-chiral model shown in Fig. 1(c), the p^+ wave does not exist, and the dispersion curves for p^- wave are one-sheet hyperbola, where $\epsilon_t=-\epsilon_0$, $\epsilon_z=0.9\epsilon_0$, $\mu_t=\mu_z=\mu_0$, and $\kappa=0.5\sqrt{\epsilon_0\mu_0}$. (d) In the strong chirality case corresponding to the uniaxial-chiral model shown in Fig. 1(c), the dispersion curves for both p^+ and p^- waves are one-sheet hyperbola, where $\epsilon_t=-\epsilon_0$, $\epsilon_z=2\epsilon_0$, $\mu_t=\mu_z=\mu_0$, and $\kappa=1.45\sqrt{\epsilon_0\mu_0}$.

as k_{1z} [see Eq. (20)]. In the mean time, S_{1z} should be $+z$ directed in order to satisfy the radiation condition. Hence, k_{1z} is also along $+z$ direction, and no backward wave is supported in the transmitted direction. From the boundary condition, the transverse wave vector k_y must be continuous at the medium interface. Hence, the wave vector for the p^- wave is positively refracted in the uniaxially chiral medium, which actually stands for a positive phase refraction. Since the Poynting vector S_{1y} has an opposite direction with the transverse wave vector k_y, the power flow for p^- wave will be negatively refracted. This represents a negative group velocity in the uniaxially chiral medium, as shown in Fig. 3(b). Here, we clearly observe that the Poynting vector and the wave vector for p^- wave lie in different sides of the normal direction of the interface, which implies that a negative group refraction is supported at the medium interface when the chirality appears only in one direction. We remark that all phase refraction angles for the p^- wave must be restricted blow the following angle

$$\theta_{p-} < \theta_m = \tan^{-1}\sqrt{|\rho_-|}, \tag{22}$$

because k_- will become imaginary when $\theta_{p-} > \theta_m$.

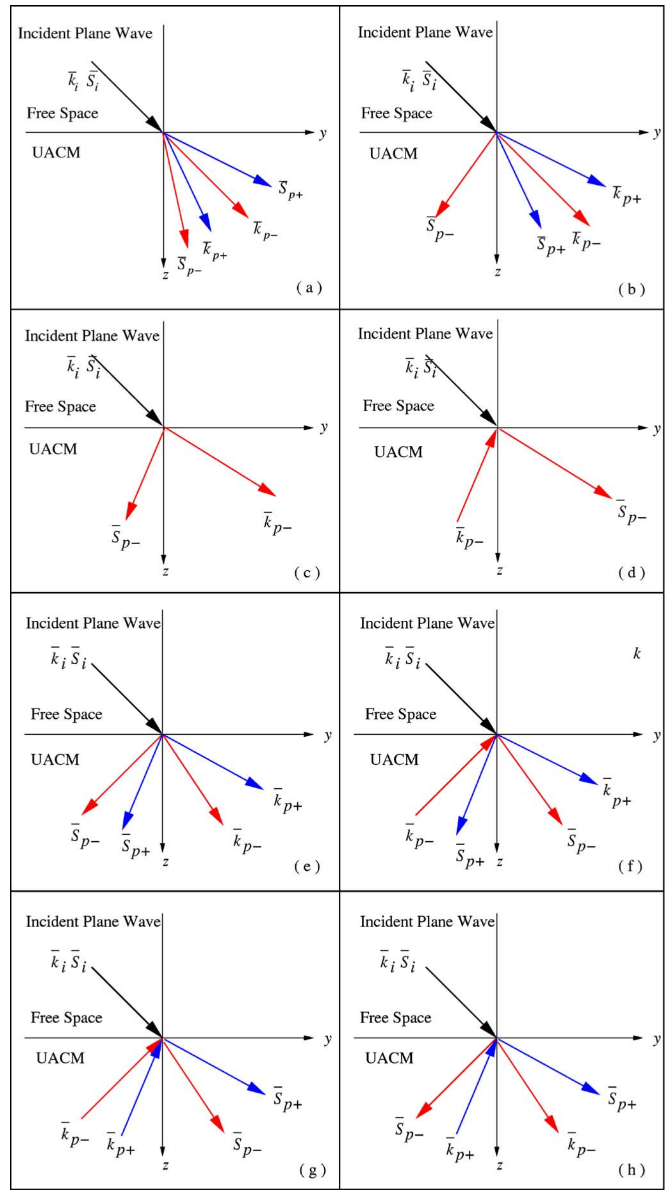

FIG. 3. (Color online) Refraction properties of a plane wave incident from free space to a uniaxially anisotropic chiral medium (UACM). (a) Positive phase and group refractions for both p^+ and p^- waves. (b) Positive phase and group refractions for p^+ wave and positive phase refraction and negative group refraction for p^- wave. (c) Positive phase refraction and negative group refraction for p^- wave. The p^+ wave does not exist. (d) Negative phase refraction and positive group refraction for p^- wave. The p^+ wave does not exist. (e) Positive phase refractions and negative group refractions for both p^+ and p^- waves. (f) Positive phase refraction and negative group refraction for p^+ wave and negative phase refraction and positive group refraction for p^- wave. (g) Negative phase refractions and positive group refractions for both p^+ and p^- waves. (h) Negative phase refraction and positive group refraction for p^+ wave and positive phase refraction and negative group refraction for p^- wave.

When wire spirals or copper springs oriented in the z direction are hosted in a medium containing resonant dipoles directed to the z direction, as shown in Fig. 1(b), we have

$\epsilon_t > 0$ and $\epsilon_z < 0$. Such a uniaxially chiral medium can also be realized using Swiss-roll array directed to the z direction.[6] In such a case, we always have $\rho_+ > 0$, $\rho_- < 0$, and $\gamma_\pm > 0$ for arbitrary chiral parameter κ. Hence, the dispersion relations for p^+ and p^- waves are an ellipse and a two-sheet hyperbola, respectively, as shown in Fig. 2(b). Again, the p^+ wave experiences positive phase and group refractions, and the p^- wave experiences a positive phase refraction and a negative group refraction at the medium interface, as illustrated in Fig. 3(b). Therefore, a negative group refraction is always supported for the p^- wave in the uniaxially anisotropic chiral medium shown in Fig. 1(b), no matter how small the chirality is. This is a very loose condition and can be easily realized.

Next, we consider another uniaxially anisotropic chiral medium: the wire spirals or copper springs are oriented in the z direction while the host medium contains resonant dipoles directed to the transverse direction, as demonstrated in Fig. 1(c). In such a case, $\epsilon_t < 0$ and $\epsilon_z > 0$, which results in quite different diffraction properties for varied medium parameters.

When the chirality is weak ($|\kappa| < \sqrt{\epsilon_z \mu_z}$), we always obtain $\rho_+ > 0$ and $\rho_- < 0$ for arbitrary $\epsilon_z/\epsilon_t + \mu_z/\mu_t$. Since $k_t^2 = k_0^2 \epsilon_t \mu_t$, the dispersion equation (14) will never be satisfied for the p^+ wave. Hence, the p^+ wave does not exist in such a uniaxially chiral medium. For the p^- wave, the dispersion curves are changed to one-sheet hyperbola, as shown in Fig. 2(c). Because ϵ_t is negative, γ_- can be either positive or negative. If $\gamma_- > 0$, then S_{1y} will be antiparallel with k_y, and S_{1z} will point to the same direction as k_{1z}. As a consequence, a positive phase refraction and a negative group refraction occur at the medium interface for the p^- wave, as illustrated in Fig. 3(c). If $\gamma_- < 0$, however, S_{1y} will have the same direction as k_y and S_{1z} will have opposite direction to k_{1z}. Then a reversed refraction property is achieved for the p^- wave: negative phase refraction and positive group refraction, as shown in Fig. 3(d). Negative phase refraction can be found in isotropic and anisotropic metamaterials.[6,15] We remark that the condition to realize such negative refractions is also very loose since the chirality is required to be small.

When the chirality is strong ($|\kappa| > \sqrt{\epsilon_z \mu_z}$), the signs of ρ_+ and ρ_- will be dependent upon the sign of $\epsilon_z/\epsilon_t + \mu_z/\mu_t$. If $\epsilon_z/\epsilon_t + \mu_z/\mu_t > 0$, we have $\rho_+ > 0$ and $\rho_- > 0$. In such a case, both p^+ and p^- waves do not exist in the chiral medium, and a total reflection occurs at the medium interface. If $\epsilon_z/\epsilon_t + \mu_z/\mu_t < 0$, we have $\rho_+ < 0$ and $\rho_- < 0$. Hence, the dispersion curves for both p^+ and p^- waves are one-sheet hyperbola, as illustrated in Fig. 2(d).

Similar to the earlier case, both γ_+ and γ_- can be either positive or negative. If $\gamma_+ > 0$ and $\gamma_- > 0$, then S_{1y} has the opposite direction to k_y, and S_{1z} has the same direction as k_{1z} for both p^+ and p^- waves, which correspond to positive phase refractions and negative group refractions occur for both waves, as demonstrated in Fig. 3(e). If $\gamma_+ > 0$ and $\gamma_- < 0$, then S_{1y} has the opposite direction to k_y and S_{1z} has the same direction as k_{1z} for the p^+ wave, and opposite phenomenon for the p^- wave. Hence, a positive phase refraction and a negative group refraction occur for p^+ wave and a negative phase refraction and a positive group refraction occur for p^- wave, as shown in Fig. 3(f). If $\gamma_+ < 0$ and $\gamma_- < 0$, a reversed refraction property to that in Fig. 3(e) is achieved: both p^+ and p^- waves experience negative phase refractions and positive group refractions, as illustrated in Fig. 3(g). If $\gamma_+ < 0$ and $\gamma_- > 0$, a reversed refraction phenomenon to that in Fig. 3(f) occurs: a negative phase refraction and positive group refraction for the p^+ wave and a positive phase refraction and negative group refraction for p^- wave, as demonstrated in Fig. 3(h).

In conclusion, we have shown that negative phase or group refractions occur in one or two eigenwaves in the uniaxially anisotropic chiral medium, where the chirality appears only in one direction. Such a uniaxially chiral medium can be easily realized in practice. In the chiral proposals shown in Figs. 1(b) and 1(c), the condition to support the negative refraction is quite loose.

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[1] V. G. Veselago, Sov. Phys. Usp. **10**, 509 (1968).
[2] R. A. Shelby, D. R. Smith, and S. Schultz, Science **292**, 77 (2001).
[3] D. R. Smith and D. Schurig, Phys. Rev. Lett. **90**, 077405 (2003).
[4] D. R. Smith, D. Schurig, J. J. Mock, P. Kolinko, and P. Rye, Appl. Phys. Lett. **84**, 2244 (2004).
[5] M. Notomi, Phys. Rev. B **62**, 10696 (2000).
[6] J. B. Pendry, Science **306**, 1353 (2004).
[7] S. Tretyakov, I. Nefedov, A. Sihvola, S. Maslovski, and C. Simovski, J. Electromagn. Waves Appl. **17**, 695 (2003).
[8] Y. Jin and S. He, Opt. Express **13**, 4974 (2005).
[9] J. Q. Shen, Ph.D. thesis, Zhejiang University, 2005.
[10] J. Q. Shen and S. He, J. Phys. A **39**, 457 (2006).
[11] S. Tretyakov, A. Sihvola, and L. Jylha, Photonics Nanostruct. Fundam. Appl. **3**, 107 (2005).
[12] C. Monzon and D. W. Forester, Phys. Rev. Lett. **95**, 123904 (2005).
[13] D. L. Jaggard, A. R. Mickelson, and C. H. Papas, Appl. Phys. **18**, 211 (1979).
[14] I. V. Lindell, A. H. Sihvola, S. A. Tretyakov, and A. J. Viitanen, *Electromagnetic Waves in Chiral and Bi-Isotropic Media* (Artech House, Boston, 1994).
[15] L. Zhou, C. T. Chan, and P. Sheng, Phys. Rev. B **68**, 115424 (2003).