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4*e*-condensation in a fully frustrated Josephson junction diamond chain

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Fully frustrated one-dimensional diamond Josephson chains have been shown [B. Douçot and J. Vidal, Phys. Rev. Lett. **88**, 227005 (2002)] to possess a remarkable property: The superfluid phase occurs through the condensation of pairs of Cooper pairs. By means of Monte Carlo simulations we analyze quantitatively the insulator to 4*e*-superfluid transition. We determine the location of the critical point and discuss the behavior of the phase-phase correlators. For comparison, we also present the case of a diamond chain at zero and 1/3 frustration where the standard 2*e*-condensation is observed.

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Josephson arrays in the quantum regime have been studied extensively,[1] both experimentally and theoretically, as model systems through which to investigate a variety of quantum phase transitions. The application of a magnetic field creates frustration and leads to a number of interesting physical effects.[1,2]

Very recently, renewed interest in frustrated Josephson networks has been stimulated by the work by Vidal *et al.*[3] on the existence of localization in fully frustrated tight binding models with $\mathcal{T}_3$ symmetry. Localization in this case is due the destructive interference for paths circumventing every plaquette. These clusters over which localization takes place were named *Aharonov-Bohm* (AB) *cages*. Experiments in superconducting networks have been performed and the existence of the AB cages has been confirmed through critical current measurements both in wire[4] and junction[5,6] networks. Starting from the original paper by Vidal *et al.*, several aspects of the AB cages both for classical[7–10] and quantum[11] superconducting networks have been highlighted.

The basic mechanism leading to the AB cages is also present in the (simpler) quasi-one-dimensional (1D) lattice shown in Fig. 1. At fully frustration, it has been shown[12] that superconducting coherence is established throughout the system by means of 4*e*-condensation.[13] The global superconducting state is due to the condensation of *pairs* of Cooper pairs. Predictions of the critical current of the diamond chain of Fig. 1 amenable to experimental confirmation have been put forward by Protopopov and Feigel'man.[14,15] Unusual transport properties of these systems have been also predicted in semiconducting samples.[16] In this work, we present the results of our Monte Carlo simulations on the Josephson junction network with the geometry depicted in Fig. 1. Our aim was to perform a detailed quantitative analysis of the phase diagram predicted in Ref. 12. In order to have a fairly complete description of the effect of frustration in this case, we considered the stiffness and phase correlators for three values of the frustration parameter; i.e., $f=0$, $f=1/3$ and $f=1/2$.

The Hamiltonian for a Josephson junction network is

$$\mathcal{H} = E_0 \sum_i n_i^2 - E_J \sum_{\langle i,j\rangle} \cos(\varphi_i - \varphi_j - A_{i,j}). \tag{1}$$

The first term in the Hamiltonian is due to the charging energy. Here for simplicity we consider the case in which the Coulomb interaction is on site; see Ref. 14 for the more realistic case of long-range charging interaction. The second term is the Josephson contribution. The phase of the superconducting order parameter in the ith island is denoted by φ_i, E_0 is the charging energy, and E_J is the Josephson coupling energy. The number n_i and phase φ_i operators are canonically conjugate on each site: $[n_i, e^{\imath\varphi_j}] = \delta_{ij} e^{\imath\varphi_i}$. The gauge-invariant definition of the phase in presence of an external vector potential $\mathbf{A}$ and flux-per-plaquette Φ ($\Phi_0 = hc/2e$ is the flux quantum) contains the term $A_{i,j} = \frac{2\pi}{\Phi_0}\int_i^j \mathbf{A}\cdot d\mathbf{l}$. All the observables are a function of the frustration parameter defined as

$$f = \frac{1}{\Phi_0}\int_P \mathbf{A}\cdot d\mathbf{l} = \frac{1}{2\pi}\sum_P A_{i,j},$$

where the line integral is performed over the elementary plaquette. Due to the periodicity of the model, it is sufficient to consider values of the frustration $0 \leqslant f \leqslant 1/2$.

The Monte Carlo simulations have been performed on an effective classical 1+1D XY-model. The effective action is constructed by applying the Trotter-Suzuki time slicing to the partition function associated to the Hamiltonian in Eq. (1).

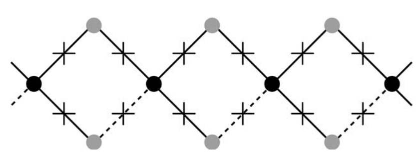

FIG. 1. The diamond chain Josephson network analyzed in the present paper. The crosses represent the Josephson junctions connecting two neighboring superconducting islands. In the chain, there are two types of inequivalent sites with two (gray) and four (black) neighbors. By an appropriate choice of the gauge, the magnetic phase factors $A_{i,j}$ can be chosen to be zero on the three links indicated by continuous lines, and f in the fourth one indicated by a dotted line.

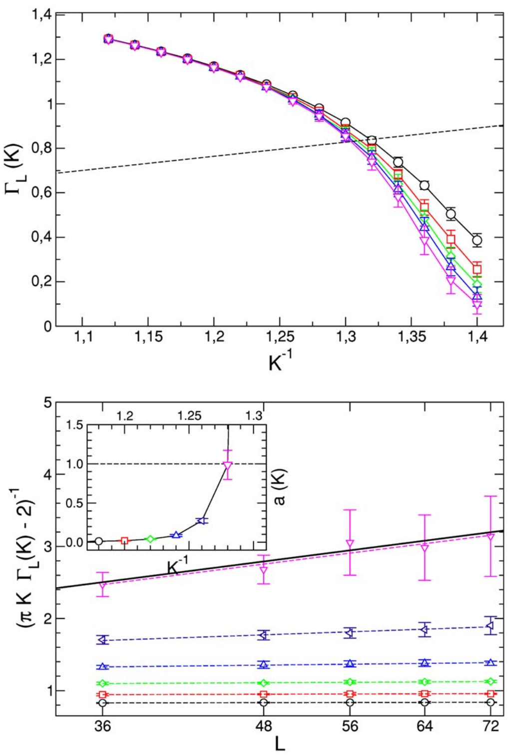

FIG. 2. (Color online) In the upper panel, the stiffness for the case of $f=0$ frustration is plotted against the coupling. Different symbols correspond to different sizes of the chain: circles to $L=36$, squares to $L=48$, diamonds to $L=56$, up triangles to $L=64$, and down triangles to $L=72$. The dashed line with slope $2/\pi$ gives a rough estimate of the transition point. A better estimate is obtained by means of the finite size scaling shown in the lower panel and explained in the text. The thick black line has a slope of exactly 1 and is plotted as a reference guide. The value of l_0 at the transition is 2.9.

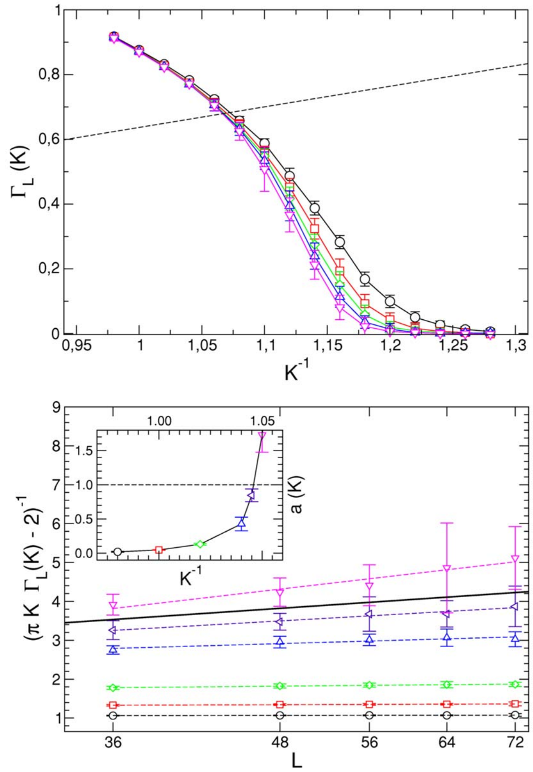

FIG. 3. (Color online) The same plots of Fig. 2 for the case of $f=1/3$. The critical point is $K_c^{-1}=1.045$ (with $l_0 \sim 0.6$).

The procedure, discussed in detail in Refs. 17 and 18, leads to the action ($\Delta\tau$ is the imaginary time slice)

$$\mathcal{S} = -\frac{1}{E_0\Delta\tau}\sum_{i,\langle kk'\rangle}\cos(\varphi_{i,k}-\varphi_{i,k'}) - E_J\Delta\tau\sum_{\langle ij\rangle,k}\cos(\varphi_{i,k}-\varphi_{j,k}-A_{i,j}). \tag{2}$$

The first term corresponds to charging (and takes into account the quantum fluctuations) while the second is due to the Josephson coupling. Close to the phase transition universality guarantees that the properties are unaffected by rescaling of the underlying space-time lattice. By making the choice $\Delta\tau=1/\sqrt{E_J E_0}$, one obtains an isotropic classical model with coupling $K=\sqrt{E_J/E_0}$.[17,18] Such a choice makes the analysis of the Monte Carlo data considerably simpler; studying the $T=0$ phase diagram implies taking the thermodynamic limit also in the time direction. It is important to emphasize that this rescaling works only for the study of the zero-temperature phase transition. The simulations were performed on $L\times L$ lattice with periodic boundary conditions (the largest lattice was 72×72). The expectation values of the different observables (stiffness and correlation functions) have been obtained, averaging up to 10^7 Monte Carlo configurations, by using a standard Metropolis algorithm. Typically, the first half of configurations in each run were used for thermalization.

The stiffness, related to the critical current, is used to signal the presence of the transition. It is defined through the increase of the free energy $\mathcal{F}$ due to a phase twist δ imposed along the space direction:[19]

$$\Gamma = \frac{\partial^2\mathcal{F}}{\partial\delta^2}.$$

The critical point is expected to be of the Berezinskii-Kosterlitz-Thouless (BKT) universality class.[12,21] Its location can be determined using the following *ansatz* for the size dependence of $\Gamma(K_c)$:[20]

$$\frac{\pi K_c}{2}\Gamma_L(K_c) = 1 + \frac{1}{2\ln(L/l_0)}, \tag{3}$$

where l_0 is the only fit parameter. In the presence of frustration, the universality class of the transition may be different from that of the unfrustrated case. In the case of the two-dimensional fully frustrated XY-model this issue has been investigated in great detail (see Refs. 22 and 23 and refs. therein). Up to date, there is no unanimous consensus on the nature of the transition. However, in this work we suppose that the transition belongs to the BKT universality class, as

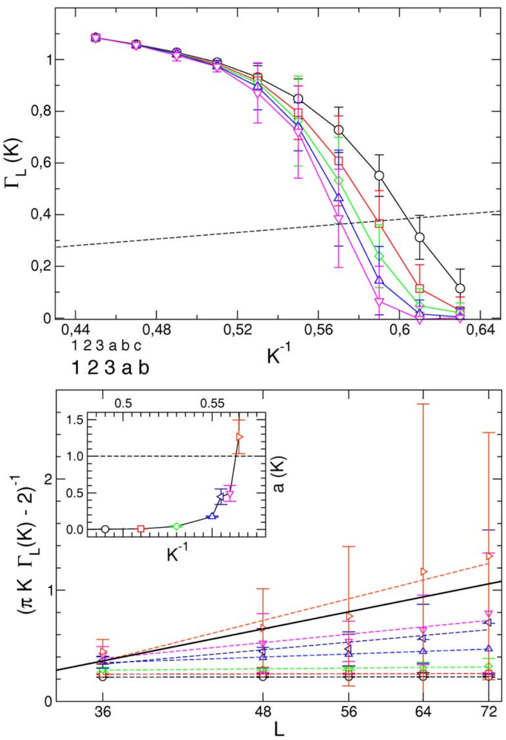

FIG. 4. (Color online) The same plots of Figs. 2 and 3 for $f=1/2$. Compared to the cases of $f=0$ and $f=1/3$, the superfluid region has shrunk considerably. The value of l_0 at the transition is ~ 25.

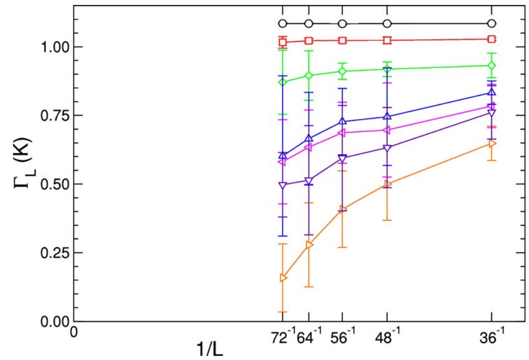

FIG. 5. (Color online) The stiffness is plotted as a function of the system size for different values of K in the critical region. This plot highlights the existence of a transition, although it does not allow us to extract the transition point. For $K^{-1}<0.55$, data seem to scale to a finite value in the thermodynamic limit, whereas over 0.57 it seems clear that they go to zero. Different symbols correspond to values of K^{-1}: circles (0.45), squares (0.49), diamond (0.53), triangles up (0.555), triangles left (0.56), triangles down (0.565), triangles right (0.575).

suggested by Ref. 12, and determine the critical value by means of Eq. (3).

We first analyze the $f=0$ case and extract the value of the critical coupling from the stiffness. This extrapolation has been done by performing a linear fit in logarithmic scale $[\pi K\Gamma_L(K)-2]^{-1}=a(K)\ln L-\ln l_0$ and searching for the coupling value such that $a(K)=1$. This coupling value is then identified with the critical point K_c. The proposed ansatz fits very well the data and the estimated value of the critical coupling is $K_c^{-1}=1.28$ which corresponds to $(E_J/E_0)_c\sim 0.61$. Data are reported in Fig. 2.

The results of the stiffness at $f=1/3$ and $1/2$ are shown in Figs. 3 and 4, respectively. As compared to the unfrustrated case, the critical value of the Josephson coupling required to establish superfluid coherence is slightly larger for $f=1/3$ and further increases for the fully frustrated case $f=1/2$. The ansatz of Eq. (3) seems to provide an accurate estimate of the transition point for $f=0$ and $f=1/3$. In the fully frustrated case, however, the value of $l_0=25$ indicates that we probably need larger chains in order to really enter the critical region. Another indication of this fact emerges in the upper panel of Fig. 4, where the line of slope $2/\pi$ crosses the data when the stiffness decreases to zero. In order to put bounds to the critical point in the fully frustrated case, we plot in Fig. 5 the stiffness as a function of the system size. From the raw data it is possible to bound the transition point in the range $0.55\leqslant K_c^{-1}\leqslant 0.57$.

All these results are summarized in the table below:

f	0	1/3	1/2
K_c^{-1}	1.28 ± 0.01	1.045 ± 0.005	0.56 ± 0.01
$(E_J/E_0)_c$	0.61 ± 0.01	0.91 ± 0.01	3.2 ± 0.1

The ratio of the obtained critical couplings for the unfrustrated and fully frustrated systems is $K_{c,1/2}/K_{c,0}=2.28\pm0.06$ and not 4, as expected from the reduction by a factor of 1/2 of the effective charge of the topological excitation that unbinds at the critical point. This may be due to the fact that the screening of the vortices is different in the unfrustrated and fully frustrated cases, therefore leading to a further correction in the ratio between the two critical points.

The differences in the fully frustrated case manifest dramatically in the way condensation is achieved. As predicted by Douçot and Vidal,[12] the destructive interference built in the diamond structure prevents the Cooper pair from having (quasi-) long-range order. The superfluid phase is then established via the delocalization of pairs of Cooper pairs. This is at the origin of the 4e-condensation. In order to check this point, the knowledge of the phase-phase correlators is required. Quasi-long-range behavior in a two-point correlation function of the type

$$g_{2n}(|i-j|)=\langle\cos n(\varphi_i-\varphi_j)\rangle \tag{4}$$

signals the existence of condensation of $2n$ charged objects. In Fig. 6 we discuss their properties. In the upper panels, we consider the phase-phase correlator g_2 for two different couplings deep in the superfluid and Mott insulating phases, respectively. What is evident from the figure is that, despite the fact that the system is phase coherent, phase correlations decay very fast almost independently from the value of K. As explained in Ref. 12, this behavior should be ascribed to the existence of the Aharonov-Bohm cages. Even if hopping of single Cooper pairs is forbidden because of quantum inter-

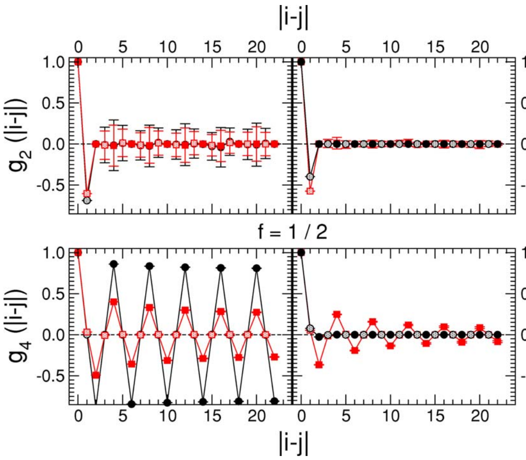

FIG. 6. (Color online) The phase correlators g_2 and g_4 are shown as a function of the distance for the *fully frustrated* case $f=1/2$. Data are plotted for a chain with $L=48$. The distances on the horizontal axis are calculated along the sawtooth path constituted by the lower halves of rhombi in Fig. 1. Black and gray symbols refer to black and gray sites respectively, in Fig. 1. The alternating signs are due to the choice of the gauge. The important feature is the decaying behavior. On the left side, circles correspond to $K^{-1}=0.1$ deep in the ordered phase and squares to $K^{-1}=0.5$ on the border of it. On the right side, squares are $K^{-1}=0.6$ and circles $K^{-1}=1.2$, deep in the Mott insulator phase. Differently from g_2, the correlator g_4 shows quasi-long-range order.

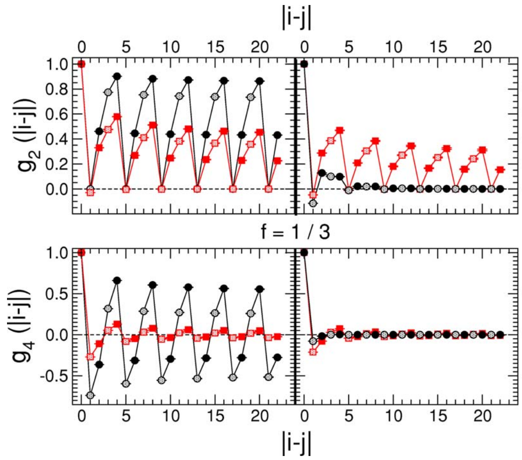

FIG. 7. (Color online) Phase correlators at frustration $f=1/3$. Top: the phase correlator $g_2(|i-j|)$ is shown as a function of the distance between the sites both in the ordered phase (left panel), $K^{-1}=0.3$ (circles) and 1.0 (squares) and in the Mott insulator phase (on the right) $K^{-1}=1.1$ (squares) and 1.4 (circles). Differently from the fully frustrated case, here the phase correlator of Cooper pairs changes its behavior at the critical point. Bottom: the phase correlator $g_4(|i-j|)$ is shown for the same coupling values as at the top.

ference, correlated hopping of two pairs does not suffer the same destructive interference. In the lower panels of the same figure, the space dependence of the correlator g_4 is plotted for the same coupling as upwards. The different behavior between the Mott and the superfluid phase is now evident. The correlator decays exponentially only for $K^{-1}=1.2>K_c$ (right side): in the other panel, differently from g_2, the decay is power-law like. For comparison we report also simulations of the phase correlators for the case $f=1/3$. In this case the "standard" condensation of Cooper pairs is observed as witnessed by the behavior of g_2 shown in Fig. 7.

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