Extension of the Fröhlich method to 4-fermion interactions

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(Received 23 January 2006; published 31 March 2006)

Higher-order terms of the transformed electron-phonon Hamiltonian $H_{e\text{-}ph}$, obtained by performing the Fröhlich transformation, are investigated. The influence of terms discarded by Fröhlich (in particular those proportional to the third power of electron-phonon coupling) on the effective Hamiltonian is examined. To this end a second Fröhlich-type transformation is performed, which yields, among others, an effective four-electron interaction. This interaction is reduced to a form admitting solution of thermodynamics. The form of the coupling of the four-electron interaction is found. By applying standard approximations, it is shown that this interaction is attractive with the interaction coupling given by $-D^6_{\mathbf{k}_F}/\omega^5_{\mathbf{k}_F}$, where $D_{\mathbf{k}}$ is electron-phonon coupling, $\omega_{\mathbf{k}}$ is the phonon energy, and $\mathbf{k}_F$ is the Fermi momentum. The form of higher-order terms of the original Fröhlich-transformed $H_{e\text{-}ph}$ is also found, up to terms proportional to the sixth power of the coupling—that is, up to those which yield the effective four-electron interactions.

DOI: 10.1103/PhysRevB.73.094524 PACS number(s): 74.20.Fg, 74.25.Kc

I. INTRODUCTION

The BCS theory[1] proved extremely effective as a theory of superconductivity. The key idea of this theory—the attractive interaction which binds electrons into Cooper pairs—has its essential origins in earlier conclusions drawn Fröhlich,[2] who performed a unitary transformation U of the electron-phonon Hamiltonian $H_{e\text{-}ph}$. U was adjusted so as to eliminate the electron-phonon interaction as far as possible and replace it by an effective interaction V_{eff} between electrons dressed in the phonon field. V_{eff} proved to be attractive for one-electron energies close to ε_F. A reduced form of V_{eff} was subsequently used by Bardeen, Cooper, and Schrieffer in their theory.[1]

It is worth emphasizing that the Fröhlich transformation is not strictly unitary, because the less significant terms of the resulting expansion of $UH_{e\text{-}ph}U^{-1}$ were discarded. Correctness of the remaining terms, included into the Hamiltonian of a superconductor, was confirmed by the success of the BCS theory.

Unfortunately, BCS theory proved incapable of explaining superconductivity in type-II superconductors, heavy fermions, and high-T_c superconductors (HTSC's). The search for an alternative theory of superconductivity proceeds in various directions, and one of them exploits the idea of extending BCS theory by adding to the BCS Hamiltonian H_{BCS} further interactions. Rickayzen[3] suggested the incorporation of a four-fermion interaction, motivating such a choice by analogies between the theory of superconductivity and nuclear theory, where such interactions had been considered. This idea was also mentioned by Volovik,[4] but remained in the realm of theoretical concepts until the late 1990s, when Maćkowiak and Tarasewicz[5–11] (MT) proposed a Hamiltonian $H_{MT}=H_{BCS}+W+V_{MT}$, where

$$H_{BCS}=T+V_{BCS}=\sum_{\mathbf{k}\sigma}\xi_{\mathbf{k}}n_{\mathbf{k}\sigma}-|\Lambda|^{-1}\sum_{\mathbf{k}\mathbf{k}'}G_{\mathbf{k}\mathbf{k}'}a^*_{\mathbf{k}+}a^*_{-\mathbf{k}-}a_{-\mathbf{k}'-}a_{\mathbf{k}'+}, \tag{1}$$

$$W=\sum_{\mathbf{k}}\gamma_{\mathbf{k}}n_{\mathbf{k}+}n_{\mathbf{k}-}, \tag{2}$$

$$V_{MT}=|\Lambda|^{-1}\sum_{\mathbf{k},\mathbf{k}'}g_{\mathbf{k}\mathbf{k}'}a^*_{\mathbf{k}-}a^*_{\mathbf{k}+}a^*_{-\mathbf{k}-}a^*_{-\mathbf{k}+}a_{-\mathbf{k}'+}a_{-\mathbf{k}'-}a_{\mathbf{k}'+}a_{\mathbf{k}'-}, \tag{3}$$

and $|\Lambda|$ in Eqs. (1)–(3) denotes the system's volume, whereas $g_{\mathbf{k}\mathbf{k}'}$ and $G_{\mathbf{k}\mathbf{k}'}$ are bounded functions and $\gamma_{\mathbf{k}}$, $g_{\mathbf{k}\mathbf{k}'}$, and $G_{\mathbf{k}\mathbf{k}'}$ are invariant under time reversal $\mathbf{k}\to-\mathbf{k}$ or $\mathbf{k}'\to-\mathbf{k}'$.

Considerations which led to the introduction of W were founded on the analysis of the HTSC normal state. There are grounds to believe that this is not a normal Fermi-liquid state. The interaction W, first added to H_{BCS} by Czerwonko,[12,13] guaranteed normal-state behavior characteristic of the so-called statistical spin liquid, considered earlier by Spałek and Wójcik.[14,15]

W is a two-electron interaction. This raises the question whether it can be obtained by a reduction procedure (different than the BCS one) of the interaction derived by Fröhlich. It can be easily verified that this is impossible. More precisely, for the unique possible reduction of momenta, the coupling vanishes, meaning that the nature of W is not phononic.

The introduction of the four-fermion interaction V_{MT} was justified in Ref. 5 by its possible role as an attraction between pairs in HTSC's, mediated by phonons or other quanta.[8] A conjecture put forward in Ref. 5 suggested also that V_{MT} could be expected to arise as one of the higher-order terms of Fröhlich's expansion of $UH_{e\text{-}ph}U^{-1}$. An alternative

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justification was given in Ref. 8, where V_{MT} was viewed as a BCS-type interaction between quasiparticles of a free gas represented by W written in the form $\Sigma_{\mathbf{k}} \gamma_{\mathbf{k}} c^*_{\mathbf{k}} c_{\mathbf{k}}$, with $c_{\mathbf{k}} = a_{\mathbf{k}+} a_{\mathbf{k}-}$. Both of these ideas essentially exploit the concept of phonon-type mediation of interactions. The significance of this mediation in HTSC was stressed by Wysokiński.[16]

The question of the form of higher-order terms of Fröhlich's expansion is interesting itself, not only as providing a possible explanation of the MT extension, but first of all, because these terms could throw some light on further possible extensions of H_{BCS} related to the effects of electron-lattice interactions. Since Fermi-liquid theory will remain the foundation of our formalism, we shall focus our interest on effective electron interactions.

It is worth noting that the possible presence of fermion quadruples in superconductors and superfluids was considered in a number of papers. Schneider and Keller[17] measured the various characteristics of some cuprates and Chevrel-phase superconductors, especially concentrating on the relation between the critical temperature and zero-temperature condensate density. They noticed that the experimental data for, e.g., $YBa_2Cu_3O_{6.602}$ point to similarities with the behavior of a dilute Bose gas. As a result they suggested Bose condensation of weakly interacting fermion pairs as a mechanism of transition from the normal to superconducting state. Bunkov *et al.*[18] pointed to the presence of fermion quadruples in ^{3}He. Their work was devoted to the problem of the influence of spatial disorder on the order parameter in superfluid ^{3}He. By resorting to the work of Volovik,[4] they suggested that the unusual spectra of ^{3}He in aerogel could be explained by a process in which impurities tend to destroy the anisotropic correlations of the order parameter, while correlations of higher symmetry can survive (e.g., four-particle correlations). Recently Schneider *et al.*[19] discovered half-$h/2e$ magnetic flux in superconducting quantum interference devices (SQUID's) fabricated of bicrystalline $YBa_2Cu_3O_{7-\delta}$ films. This situation corresponds to the presence of fermion quadruples in the system. Based on this observation, Aligia *et al.*[20] investigated a model of the interface between two superconductors, based on a one-dimensional boson lattice model and proposed formation of quartets of electrons.

In Sec. II higher-order terms of the expansion of Fröhlich's transformation $UH_{e\text{-}ph}U^{-1}$ are discussed qualitatively in order to exhibit the emerging structure. Owing to the complexity of this procedure, only terms proportional to the third power of the electron-phonon coupling are found in Sec. III. The resulting extended Fröhlich Hamiltonian is transformed in the next section by a second Fröhlich-type transformation, which produces four-electron terms. These are discussed in detail in Sec. V. In particular, a reduction similar to the BCS one is performed, which yields an interaction of the form V_{MT} in Eq. (3). The expression for $g_{\mathbf{kk}'}$ is derived. A detailed analysis of this expression is performed in the next section; in particular, we show, by applying some approximations, that this expression is negative—i.e., V_{MT} is attractive. The original Fröhlich transformation and the resulting terms, up to the sixth power of the coupling, are commented on in Sec. VII. The final section contains a discussion, summary, and open questions.

II. HIGHER-ORDER TERMS OF THE FRÖHLICH TRANSFORMATION

Following Fröhlich[2] (see Appendix A for details), let us consider the electron-phonon Hamiltonian

$$H_{e\text{-}ph} = H_0 + H_{int} = \sum_{\mathbf{k}\sigma} \varepsilon_{\mathbf{k}} a^*_{\mathbf{k}\sigma} a_{\mathbf{k}\sigma} + \sum_{\mathbf{w}} \omega_{\mathbf{w}} b^*_{\mathbf{w}} b_{\mathbf{w}} + i \sum_{\mathbf{w}} D_{\mathbf{w}} (b_{\mathbf{w}} \rho^*_{\mathbf{w}} - b^*_{\mathbf{w}} \rho_{\mathbf{w}}), \tag{4}$$

where

$$\rho_{\mathbf{w}} = \sum_{\mathbf{k}\sigma} a^*_{\mathbf{k}-\mathbf{w}\sigma} a_{\mathbf{k}\sigma} \tag{5}$$

and $a_{\mathbf{k}\sigma}$ ($b_{\mathbf{k}}$) are fermion (boson) operators. The coupling $D_{\mathbf{w}}$ will be assumed small and $\hbar \equiv 1$.

Since the interaction is spin independent, the spin index will be suppressed. Summation over electron momenta will include summation over spins.

Fröhlich performed a unitary transformation of $H_{e\text{-}ph}$ in order to eliminate (as far as possible) the interaction term. The transformed Hamiltonian is

$$H = e^{S^*} H_{e\text{-}ph} e^{S} = H_{e\text{-}ph} - [S, H_{e\text{-}ph}] + \frac{1}{2}[S,[S,H_{e\text{-}ph}]] + \cdots , \tag{6}$$

where

$$S = \sum_{\mathbf{q}} S_{\mathbf{q}} = \sum_{\mathbf{q}} (\gamma^*_{\mathbf{q}} b^*_{\mathbf{q}} - \gamma_{\mathbf{q}} b_{\mathbf{q}}) = -S^*, \tag{7}$$

$$\gamma_{\mathbf{q}} = \sum_{\mathbf{k}} \phi(\mathbf{k},\mathbf{q}) a^*_{\mathbf{k}} a_{\mathbf{k}-\mathbf{q}}, \tag{8}$$

and the unknown function $\phi(\mathbf{k},\mathbf{q}) : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{C}^1$ is adjusted to achieve the cancellation.

Subsequently, a term which is a combination of products, each with f fermion operators and b boson operators, will be written as (f,b). Clearly, f will always be even. For example, H_0 consists of terms (2,0) and (0,2).

The right-hand side (RHS) of Eq. (6) is expressed in terms of commutators $[(f_1,b_1),(f_2,b_2)]$. One easily finds that

$$[(f_1,b_1),(f_2,b_2)] = [f_1,f_2]b_1b_2 + f_2f_1[b_1,b_2] = [f_1,f_2]b_2b_1 + f_1f_2[b_1,b_2]. \tag{9}$$

The necessary commutators $[f_1,f_2]$, $[b_1,b_2]$ are given in Appendix C.

According to Eq. (6), the transformation can be performed, given commutators of the form occurring in Eq. (9) with the first argument equal S. The latter is a (2,1) expression, hence

$$[S,(f,b)] = [(2,1),(f,b)] = (f,b+1) + (f+2,b-1), \tag{10}$$

by virtue of Eqs. (9) and (C1). Clearly $(f,b-1)=0$ for $b=0$.

Based on these grounds, Fröhlich obtained the transformed Hamiltonian (see Appendix A)

$$H_F = \sum_{\mathbf{k}} \varepsilon_{\mathbf{k}} n_{\mathbf{k}} - \frac{1}{2}\sum_{\mathbf{kqw}} \frac{D_{\mathbf{w}}^2[1+\Delta(\mathbf{k},\mathbf{w})][1-\Delta(\mathbf{q},\mathbf{w})]}{\varepsilon_{\mathbf{q}-\mathbf{w}} - \varepsilon_{\mathbf{q}} + \omega_{\mathbf{w}}} \times (a_{\mathbf{k}}^* a_{\mathbf{k}-\mathbf{w}} a_{\mathbf{q}-\mathbf{w}}^* a_{\mathbf{q}} + \text{c.c.}).$$

The second term represents an effective interaction between electrons dressed in the phonon field. If $\varepsilon_{\mathbf{q}-\mathbf{w}} - \varepsilon_{\mathbf{q}} + \omega_{\mathbf{w}} > 0$, this interaction is attractive.

Omission of higher terms in Eq. (6) results in violation of unitarity. The question thus arises whether partial inclusion of these terms (first of all those of third order in the coupling) could improve the agreement between theory and experiment.

Following the general rule for the action of S in consecutive orders, expressed by Eq. (10), one easily finds the form of subsequent terms:

$$\left\{\begin{matrix}(2,0)\\(0,2)\end{matrix}\right\} \xrightarrow{S} (2,1) \xrightarrow{S} \left\{\begin{matrix}(4,0)\\(2,2)\end{matrix}\right\} \xrightarrow{S} (4,1) \xrightarrow{S} \left\{\begin{matrix}(6,0)\\(4,2)\end{matrix}\right\} \xrightarrow{S} (6,1) \xrightarrow{S} \left\{\begin{matrix}(8,0)\\(6,2)\end{matrix}\right\}. \tag{11}$$

In each consecutive step one obtains terms proportional to the next power of the coupling.

From the viewpoint of Fermi-liquid theory, the terms representing effective interelectron interactions are most interesting. According to Eq. (11), new terms (6,0) (proportional to $D_{\mathbf{w}}^4$), describing three-electron interactions, and (8,0) (proportional to $D_{\mathbf{w}}^6$), representing four-electron interactions, appear. Attractive interactions of this type leading to the formation of fermion triples and quadruples could affect the behavior of a superconductor. However, the total spin of an electron triple is nonzero, so such clusters are unstable, as they are not invariant under time inversion. So far, there has been no experimental evidence of such objects.

Most electrons in a superconductor below T_c are paired, so the four-electron interaction between Cooper pairs can be expected to prevail. Furthermore, quadruples with a total spin equal to zero and appropriate one-electron momenta are stable under time inversion. On the other hand, under Fröhlich's conditions for the convergence of series (6), the effect of the terms (6,0) and (8,0) is weaker.

Evaluation of these terms is a complicated procedure. Before doing this, let us first examine the third-order corrections.

III. THIRD ORDER OF THE TRANSFORMATION

Let us consider the effect of the first higher orders discarded by Fröhlich—i.e., terms proportional to $D_{\mathbf{w}}^3$. Then the corrected Hamiltonian takes the form

$$H' = H_0 - ([S,H_0] - H_{int}) + \left(\frac{1}{2}[S,[S,H_0]] - [S,H_{int}]\right) - \left(\frac{1}{6}[S,[S,[S,H_0]]] - \frac{1}{2}[S,[S,H_{int}]]\right) + \cdots . \tag{12}$$

The additional two terms in the last pair of brackets are equal, explicitly,

$$\frac{1}{2}[S,[S,H_{int}]] - \frac{1}{6}[S,[S,[S,H_0]]] = \sum_{\mathbf{qk}} A_{\mathbf{kq}} b_{\mathbf{q}}^* n_{\mathbf{k}} \gamma_{\mathbf{q}}^* + \sum_{\mathbf{qkwk}'} b_{\mathbf{w}}^* \{B_{\mathbf{kqwk}'} a_{\mathbf{k}-\mathbf{w}}^* a_{\mathbf{k}-\mathbf{q}} a_{\mathbf{k}'-\mathbf{q}}^* a_{\mathbf{k}'} + C_{\mathbf{kqwk}'} a_{\mathbf{k}'}^* a_{\mathbf{k}'-\mathbf{q}} a_{\mathbf{k}-\mathbf{q}}^* a_{\mathbf{k}+\mathbf{w}}\} + \text{c.c.} := H_e, \tag{13}$$

where

$$2A_{\mathbf{kq}} = iD_{\mathbf{q}}\phi^*(\mathbf{k},\mathbf{q}) + iD_{\mathbf{q}}\phi(\mathbf{k}+\mathbf{q},\mathbf{q}) + \frac{1}{3}(\varepsilon_{\mathbf{k}-\mathbf{q}} - \varepsilon_{\mathbf{k}} + \omega_{\mathbf{q}})|\phi(\mathbf{k},\mathbf{q})|^2 - \frac{1}{3}(\varepsilon_{\mathbf{k}} - \varepsilon_{\mathbf{k}+\mathbf{q}} + \omega_{\mathbf{q}})|\phi(\mathbf{k}+\mathbf{q},\mathbf{q})|^2 + \text{c.c.}, \tag{14}$$

$$2B_{\mathbf{kqwk}'} = iD_{\mathbf{q}}\{\phi^*(\mathbf{k}',\mathbf{q})\phi^*(\mathbf{k}-\mathbf{q},\mathbf{w}) - \phi^*(\mathbf{k}',\mathbf{q})\phi^*(\mathbf{k},\mathbf{w}) + \phi^*(\mathbf{k},\mathbf{w})\phi(\mathbf{k},\mathbf{q}) - \phi^*(\mathbf{k}-\mathbf{q},\mathbf{w})\phi(\mathbf{k}-\mathbf{w},\mathbf{q})\} + \frac{1}{3}\phi^*(\mathbf{k}',\mathbf{q})\{\phi(\mathbf{k},\mathbf{q})\phi^*(\mathbf{k},\mathbf{w}) \times(\varepsilon_{\mathbf{k}} - \varepsilon_{\mathbf{k}-\mathbf{q}} + \varepsilon_{\mathbf{k}'-\mathbf{q}} - \varepsilon_{\mathbf{k}'}) + \phi(\mathbf{k}-\mathbf{w},\mathbf{q}) \times\phi^*(\mathbf{k}-\mathbf{q},\mathbf{w})(\varepsilon_{\mathbf{k}-\mathbf{w}-\mathbf{q}} - \varepsilon_{\mathbf{k}-\mathbf{w}} - \varepsilon_{\mathbf{k}'-\mathbf{q}} + \varepsilon_{\mathbf{k}'})\}, \tag{15}$$

$$2C_{\mathbf{kqwk}'} = iD_{\mathbf{q}}\{\phi(\mathbf{k}',\mathbf{q})\phi^*(\mathbf{k}+\mathbf{w}-\mathbf{q},\mathbf{w}) - \phi(\mathbf{k}',\mathbf{q}) \times\phi^*(\mathbf{k}+\mathbf{w},\mathbf{w}) - \phi^*(\mathbf{k}+\mathbf{w}-\mathbf{q},\mathbf{w})\phi^*(\mathbf{k}+\mathbf{w},\mathbf{q}) + \phi^*(\mathbf{k}+\mathbf{w},\mathbf{w})\phi^*(\mathbf{k},\mathbf{q})\} + \frac{1}{3}\phi(\mathbf{k}',\mathbf{q}) \times\{\phi^*(\mathbf{k}+\mathbf{w},\mathbf{q})\phi^*(\mathbf{k}+\mathbf{w}-\mathbf{q},\mathbf{w}) \times(\varepsilon_{\mathbf{k}'-\mathbf{q}} - \varepsilon_{\mathbf{k}'} - \varepsilon_{\mathbf{k}+\mathbf{w}-\mathbf{q}} + \varepsilon_{\mathbf{k}+\mathbf{w}}) + \phi^*(\mathbf{k},\mathbf{q}) \times\phi^*(\mathbf{k}+\mathbf{w},\mathbf{w})(\varepsilon_{\mathbf{k}-\mathbf{q}} - \varepsilon_{\mathbf{k}} - \varepsilon_{\mathbf{k}'-\mathbf{q}} + \varepsilon_{\mathbf{k}'})\}. \tag{16}$$

It can be seen that a new phonon index $\mathbf{w}$ has appeared in Eq. (13). It results from the commutator of S with the terms (4,0), so the harmonic approximation has not been violated.

Substitution of the expression (13) into H' yields

$$H' = H_a + H_b + H_c + H_d + H_e,$$

where

$$H_a = \frac{1}{2}\sum_{\mathbf{k}} \varepsilon_{\mathbf{k}} n_{\mathbf{k}} + \frac{1}{2}\sum_{\mathbf{q}} \omega_{\mathbf{q}} b_{\mathbf{q}}^* b_{\mathbf{q}} + \text{c.c.},$$

$$H_b = \sum_{\mathbf{qk}} b_{\mathbf{q}}^* b_{\mathbf{q}} n_{\mathbf{k}} \left\{ iD_{\mathbf{q}}[\phi(\mathbf{k}+\mathbf{q},\mathbf{q}) - \phi(\mathbf{k},\mathbf{q})] - \frac{1}{2}(\varepsilon_{\mathbf{k}} - \varepsilon_{\mathbf{k}+\mathbf{q}} + \omega_{\mathbf{q}})|\phi(\mathbf{k}+\mathbf{q},\mathbf{q})|^2 + \frac{1}{2}(\varepsilon_{\mathbf{k}-\mathbf{q}} - \varepsilon_{\mathbf{k}} + \omega_{\mathbf{q}})|\phi(\mathbf{k},\mathbf{q})|^2 \right\} + \text{c.c.}, \tag{17}$$

$$H_c = \sum_{\mathbf{qk}} b_{\mathbf{q}}[iD_{\mathbf{q}} + (\varepsilon_{\mathbf{k-q}} - \varepsilon_{\mathbf{k}} + \omega_{\mathbf{q}})\phi(\mathbf{k},\mathbf{q})]a^*_{\mathbf{k}}a_{\mathbf{k-q}} + \text{c.c.}, \tag{18}$$

$$H_d = \sum_{\mathbf{qk}'} \left(iD_{\mathbf{q}}\phi(\mathbf{k},\mathbf{q}) + \frac{1}{2}(\varepsilon_{\mathbf{k-q}} - \varepsilon_{\mathbf{k}} + \omega_{\mathbf{q}})\phi(\mathbf{k},\mathbf{q}) \times \phi^*(\mathbf{k}',\mathbf{q})\right) a^*_{\mathbf{k}}a_{\mathbf{k-q}}a^*_{\mathbf{k}'-\mathbf{q}}a_{\mathbf{k}'} + \text{c.c.} \tag{19}$$

H_b is linear in $b^*_{\mathbf{q}}b_{\mathbf{q}}$ but differs from H_a, H_c is part of H_{int} which had been excluded from the transformation to avoid convergence problems, and H_d is the two-electron interaction obtained, whereas H_e contains terms of the form (4, 1) which represent the obtained correction to Fröhlich's Hamiltonian.

The function $\phi(\mathbf{k},\mathbf{q})$ remains in the form (A7), which guarantees minimization of the contribution H_c.

IV. SECOND TRANSFORMATION

The structure of H' bears similarities to that of $H_{e\text{-}ph}$. Both H' and $H_{e\text{-}ph}$ contain terms describing the interaction of electrons with phonons: the counterpart of H_{int} in $H_{e\text{-}ph}$ is H_e in H'. To estimate the effect of H_e, let us repeatedly apply Fröhlich's method and perform a second unitary transformation adjusted to eliminate H_e as far as possible. The form of H_e suggests to take

$$S' = \sum_{\mathbf{u}} (\eta^*_{\mathbf{u}}b^*_{\mathbf{u}} - \eta_{\mathbf{u}}b_{\mathbf{u}} + \xi^*_{\mathbf{u}}b^*_{\mathbf{u}} - \xi_{\mathbf{u}}b_{\mathbf{u}} + \zeta^*_{\mathbf{u}}b^*_{\mathbf{u}} - \zeta_{\mathbf{u}}b_{\mathbf{u}}),$$

where

$$\eta^*_{\mathbf{u}} = \sum_{\mathbf{lm}} \psi^*(\mathbf{l},\mathbf{m},\mathbf{u})a^*_{\mathbf{l}}a_{\mathbf{l}}a^*_{\mathbf{m-u}}a_{\mathbf{m}}, \tag{20}$$

$$\xi^*_{\mathbf{u}} = \sum_{\mathbf{lmt}} \chi^*(\mathbf{l},\mathbf{m},\mathbf{t},\mathbf{u})a^*_{\mathbf{l-u}}a_{\mathbf{l-t}}a^*_{\mathbf{m-t}}a_{\mathbf{m}}, \tag{21}$$

$$\zeta^*_{\mathbf{u}} = \sum_{\mathbf{lmt}} \varphi^*(\mathbf{l},\mathbf{m},\mathbf{t},\mathbf{u})a^*_{\mathbf{l}}a_{\mathbf{l-t}}a^*_{\mathbf{m-t}}a_{\mathbf{m+u}}. \tag{22}$$

The explicit form of the functions ψ, χ, and φ will be found below.

The interaction H_e is third order in $D_{\mathbf{q}}$; therefore, bearing in mind Fröhlich's approach, we shall restrict the expansion of $\exp[S'^*]H'\exp[S']=\hat{H}$ to terms which are sixth order in $D_{\mathbf{q}}$. These terms are

$$\hat{H} = H' - [S',H'] + \frac{1}{2}[S',[S',H_a]] + \cdots . \tag{23}$$

To evaluate the RHS, one needs the commutators

$$[S',H_a] = \sum_{\mathbf{ukl}} b^*_{\mathbf{u}}\psi^*(\mathbf{l},\mathbf{k},\mathbf{u})(\varepsilon_{\mathbf{k}} - \varepsilon_{\mathbf{k-u}} - \omega_{\mathbf{u}})a^*_{\mathbf{l}}a_{\mathbf{l}}a^*_{\mathbf{k-u}}a_{\mathbf{k}} + \sum_{\mathbf{uklt}} b^*_{\mathbf{u}}\chi^*(\mathbf{l},\mathbf{k},\mathbf{t},\mathbf{u})(\varepsilon_{\mathbf{k}} - \varepsilon_{\mathbf{k-t}} + \varepsilon_{\mathbf{l-t}} - \varepsilon_{\mathbf{l-u}} - \omega_{\mathbf{u}}) \times a^*_{\mathbf{l-u}}a_{\mathbf{l-t}}a^*_{\mathbf{k-t}}a_{\mathbf{k}} + \sum_{\mathbf{uklt}} b^*_{\mathbf{u}}\varphi^*(\mathbf{l},\mathbf{k},\mathbf{t},\mathbf{u}) \times(\varepsilon_{\mathbf{k+u}} - \varepsilon_{\mathbf{k-t}} + \varepsilon_{\mathbf{l-t}} - \varepsilon_{\mathbf{l}} - \omega_{\mathbf{u}})a^*_{\mathbf{l}}a_{\mathbf{l-t}}a^*_{\mathbf{k-t}}a_{\mathbf{k+u}} + \text{c.c.},$$

$$[S',H_b] = -\sum_{\mathbf{qk}} b^*_{\mathbf{q}}n_{\mathbf{k}}(\eta^*_{\mathbf{q}} + \xi^*_{\mathbf{q}} + \zeta^*_{\mathbf{q}})(D_{\mathbf{kq}} + D^*_{\mathbf{kq}}) + \text{c.c.},$$

where $D_{\mathbf{kq}}$ is defined by $H_b=\Sigma_{\mathbf{q}}b^*_{\mathbf{q}}b_{\mathbf{q}}n_{\mathbf{k}}D_{\mathbf{kq}}$+c.c. in Eq. (17),

$$[S',H_c] = -\sum_{\mathbf{qk}} E_{\mathbf{k},\mathbf{q}}a^*_{\mathbf{k}}a_{\mathbf{k-q}}(\eta^*_{\mathbf{q}} + \xi^*_{\mathbf{q}} + \zeta^*_{\mathbf{q}}) + \sum_{\mathbf{qkl}} E_{\mathbf{k},\mathbf{q}}b^*_{\mathbf{q}}b_{\mathbf{q}}\{\psi^*(\mathbf{l},\mathbf{k},\mathbf{q})n_{\mathbf{l}}(n_{\mathbf{k-q}} - n_{\mathbf{k}}) + [\psi^*(\mathbf{k},\mathbf{l},\mathbf{q}) - \psi^*(\mathbf{k}-\mathbf{q},\mathbf{l},\mathbf{q})]a^*_{\mathbf{k}}a_{\mathbf{k-q}}a^*_{\mathbf{l-q}}a_{\mathbf{l}}\}$$
$$+ \sum_{\mathbf{qklt}} b^*_{\mathbf{q}}b_{\mathbf{q}}\{[E_{\mathbf{k},\mathbf{q}}\chi^*(\mathbf{l},\mathbf{k},\mathbf{t},\mathbf{q}) - E_{\mathbf{k-t},\mathbf{q}}\chi^*(\mathbf{l},\mathbf{k}-\mathbf{q},\mathbf{t},\mathbf{q})]a^*_{\mathbf{l-q}}a_{\mathbf{l-t}}a^*_{\mathbf{k-t}}a_{\mathbf{k-q}} + [E_{\mathbf{k+q-t},\mathbf{q}}\chi^*(\mathbf{k}+\mathbf{q},\mathbf{l},\mathbf{t},\mathbf{q})$$
$$- E_{\mathbf{k},\mathbf{q}}\chi^*(\mathbf{k},\mathbf{l},\mathbf{t},\mathbf{q})]a^*_{\mathbf{k}}a_{\mathbf{k-t}}a^*_{\mathbf{l-t}}a_{\mathbf{l}} + [E_{\mathbf{k+q+t},\mathbf{q}}\varphi^*(\mathbf{l},\mathbf{k}+\mathbf{t},\mathbf{t},\mathbf{q}) - E_{\mathbf{k},\mathbf{q}}\varphi^*(\mathbf{l},\mathbf{k}+\mathbf{t}-\mathbf{q},\mathbf{t},\mathbf{q})]a^*_{\mathbf{l}}a_{\mathbf{l-t}}a^*_{\mathbf{k}}a_{\mathbf{k+t}} + [E_{\mathbf{k},\mathbf{q}}\varphi^*(\mathbf{k}+\mathbf{t},\mathbf{l},\mathbf{t},\mathbf{q})$$
$$- E_{\mathbf{k+t},\mathbf{q}}\varphi^*(\mathbf{k}+\mathbf{t}-\mathbf{q},\mathbf{l},\mathbf{t},\mathbf{q})]a^*_{\mathbf{k+t}}a_{\mathbf{k-q}}a^*_{\mathbf{l-t}}a_{\mathbf{l+q}}\} + \text{c.c.},$$

where $E_{\mathbf{k},\mathbf{q}}$ is defined by $H_c=\Sigma_{\mathbf{qk}}b_{\mathbf{q}}a^*_{\mathbf{k}}a_{\mathbf{k-q}}E_{\mathbf{k},\mathbf{q}}$+c.c. in Eq. (18),

$$[S',H_d] = \sum_{\mathbf{uqkk}'} b^*_{\mathbf{u}}(F_{\mathbf{k},\mathbf{k}',\mathbf{q}} + F^*_{\mathbf{k}',\mathbf{k},\mathbf{q}})\{a^*_{\mathbf{k}}a_{\mathbf{k-q}}[\eta^*_{\mathbf{u}},a^*_{\mathbf{k}'-\mathbf{q}}a_{\mathbf{k}'}] + [\eta^*_{\mathbf{u}},a^*_{\mathbf{k}}a_{\mathbf{k-q}}]a^*_{\mathbf{k}'-\mathbf{q}}a_{\mathbf{k}'} + a^*_{\mathbf{k}}a_{\mathbf{k-q}}[\xi^*_{\mathbf{u}},a^*_{\mathbf{k}'-\mathbf{q}}a_{\mathbf{k}'}] + [\xi^*_{\mathbf{u}},a^*_{\mathbf{k}}a_{\mathbf{k-q}}]a^*_{\mathbf{k}'-\mathbf{q}}a_{\mathbf{k}'}$$
$$+ a^*_{\mathbf{k}}a_{\mathbf{k-q}}[\zeta^*_{\mathbf{u}},a^*_{\mathbf{k}'-\mathbf{q}}a_{\mathbf{k}'}] + [\zeta^*_{\mathbf{u}},a^*_{\mathbf{k}}a_{\mathbf{k-q}}]a^*_{\mathbf{k}'-\mathbf{q}}a_{\mathbf{k}'}\} + \text{c.c.},$$

where $F_{\mathbf{k},\mathbf{k}',\mathbf{q}}$ is defined by $H_d=\Sigma_{\mathbf{qkk}'}F_{\mathbf{k},\mathbf{k}',\mathbf{q}}a^*_{\mathbf{k}}a_{\mathbf{k-q}}a^*_{\mathbf{k}'-\mathbf{q}}a_{\mathbf{k}'}$+c.c. in Eq. (19),

$$[S',H_e] = -\sum_{\mathbf{qk}} A_{\mathbf{k},\mathbf{q}}b^*_{\mathbf{q}}b_{\mathbf{q}}\{[\eta_{\mathbf{q}},n_{\mathbf{k}}\gamma^*_{\mathbf{q}}] + [\xi_{\mathbf{q}},n_{\mathbf{k}}\gamma^*_{\mathbf{q}}] + [\zeta_{\mathbf{q}},n_{\mathbf{k}}\gamma^*_{\mathbf{q}}]\} - \sum_{\mathbf{qwkk}'} B_{\mathbf{k},\mathbf{q},\mathbf{w},\mathbf{k}'}b^*_{\mathbf{w}}b_{\mathbf{w}}\{[\eta_{\mathbf{w}},a^*_{\mathbf{k-w}}a_{\mathbf{k-q}}a^*_{\mathbf{k}'-\mathbf{q}}a_{\mathbf{k}'}]$$
$$+ [\xi_{\mathbf{w}},a^*_{\mathbf{k-w}}a_{\mathbf{k-q}}a^*_{\mathbf{k}'-\mathbf{q}}a_{\mathbf{k}'}] + [\zeta_{\mathbf{w}},a^*_{\mathbf{k-w}}a_{\mathbf{k-q}}a^*_{\mathbf{k}'-\mathbf{q}}a_{\mathbf{k}'}]\} - \sum_{\mathbf{qwkk}'} C_{\mathbf{k},\mathbf{q},\mathbf{w},\mathbf{k}'}b^*_{\mathbf{w}}b_{\mathbf{w}}\{[\eta_{\mathbf{w}},a^*_{\mathbf{k}'}a_{\mathbf{k}'-\mathbf{q}}a^*_{\mathbf{k-q}}a_{\mathbf{k+w}}]$$
$$+ [\xi_{\mathbf{w}},a^*_{\mathbf{k}'}a_{\mathbf{k}'-\mathbf{q}}a^*_{\mathbf{k-q}}a_{\mathbf{k+w}}] + [\zeta_{\mathbf{w}},a^*_{\mathbf{k}'}a_{\mathbf{k}'-\mathbf{q}}a^*_{\mathbf{k-q}}a_{\mathbf{k+w}}]\} - \sum_{\mathbf{qk}} A_{\mathbf{k},\mathbf{q}}\{\eta_{\mathbf{q}}n_{\mathbf{k}}\gamma^*_{\mathbf{q}} + \xi_{\mathbf{q}}n_{\mathbf{k}}\gamma^*_{\mathbf{q}} + \zeta_{\mathbf{q}}n_{\mathbf{k}}\gamma^*_{\mathbf{q}}\}$$

$$- \sum_{\mathbf{qwkk}'} B_{\mathbf{k},\mathbf{q},\mathbf{w},\mathbf{k}'}\{\eta_{\mathbf{w}} a^*_{\mathbf{k}-\mathbf{w}} a_{\mathbf{k}-\mathbf{q}} a^*_{\mathbf{k}'-\mathbf{q}} a_{\mathbf{k}'} + \xi_{\mathbf{w}} a^*_{\mathbf{k}-\mathbf{w}} a_{\mathbf{k}-\mathbf{q}} a^*_{\mathbf{k}'-\mathbf{q}} a_{\mathbf{k}'} + \zeta_{\mathbf{w}} a^*_{\mathbf{k}-\mathbf{w}} a_{\mathbf{k}-\mathbf{q}} a^*_{\mathbf{k}'-\mathbf{q}} a_{\mathbf{k}'}\}$$
$$- \sum_{\mathbf{qwkk}'} C_{\mathbf{k},\mathbf{q},\mathbf{w},\mathbf{k}'}\{\eta_{\mathbf{w}} a^*_{\mathbf{k}'} a_{\mathbf{k}'-\mathbf{q}} a^*_{\mathbf{k}-\mathbf{q}} a_{\mathbf{k}+\mathbf{w}} + \xi_{\mathbf{w}} a^*_{\mathbf{k}'} a_{\mathbf{k}'-\mathbf{q}} a^*_{\mathbf{k}-\mathbf{q}} a_{\mathbf{k}+\mathbf{w}} + \zeta_{\mathbf{w}} a^*_{\mathbf{k}'} a_{\mathbf{k}'-\mathbf{q}} a^*_{\mathbf{k}-\mathbf{q}} a_{\mathbf{k}+\mathbf{w}}\} + \text{c.c.},$$

where $A_{\mathbf{k},\mathbf{q}}$, $B_{\mathbf{k},\mathbf{q},\mathbf{w},\mathbf{k}'}$, and $C_{\mathbf{k},\mathbf{q},\mathbf{w},\mathbf{k}'}$ are given, respectively, by Eqs. (14)–(16),

$$[S',[S',H_a]] = - \sum_{\mathbf{qk}'} G_{\mathbf{k},\mathbf{k}',\mathbf{q}} b^*_{\mathbf{q}} b_{\mathbf{q}} \{[\eta_{\mathbf{q}}, a^*_{\mathbf{k}'} a_{\mathbf{k}'} a^*_{\mathbf{k}-\mathbf{q}} a_{\mathbf{k}}] + [\xi_{\mathbf{q}}, a^*_{\mathbf{k}'} a_{\mathbf{k}'} a^*_{\mathbf{k}-\mathbf{q}} a_{\mathbf{k}}] + [\zeta_{\mathbf{q}}, a^*_{\mathbf{k}'} a_{\mathbf{k}'} a^*_{\mathbf{k}-\mathbf{q}} a_{\mathbf{k}}]\}$$
$$- \sum_{\mathbf{qk}'\mathbf{t}} H_{\mathbf{k},\mathbf{k}',\mathbf{t},\mathbf{q}} b^*_{\mathbf{q}} b_{\mathbf{q}} \{[\eta_{\mathbf{q}}, a^*_{\mathbf{k}'-\mathbf{q}} a_{\mathbf{k}'-\mathbf{t}} a^*_{\mathbf{k}-\mathbf{t}} a_{\mathbf{k}}] + [\xi_{\mathbf{q}}, a^*_{\mathbf{k}'-\mathbf{q}} a_{\mathbf{k}'-\mathbf{t}} a^*_{\mathbf{k}-\mathbf{t}} a_{\mathbf{k}}] + [\zeta_{\mathbf{q}}, a^*_{\mathbf{k}'-\mathbf{q}} a_{\mathbf{k}'-\mathbf{t}} a^*_{\mathbf{k}-\mathbf{t}} a_{\mathbf{k}}]\}$$
$$- \sum_{\mathbf{qk}'\mathbf{t}} I_{\mathbf{k},\mathbf{k}',\mathbf{t},\mathbf{q}} b^*_{\mathbf{q}} b_{\mathbf{q}} \{[\eta_{\mathbf{q}}, a^*_{\mathbf{k}'} a_{\mathbf{k}'-\mathbf{t}} a^*_{\mathbf{k}-\mathbf{t}} a_{\mathbf{k}+\mathbf{q}}] + [\xi_{\mathbf{q}}, a^*_{\mathbf{k}'} a_{\mathbf{k}'-\mathbf{t}} a^*_{\mathbf{k}-\mathbf{t}} a_{\mathbf{k}+\mathbf{q}}] + [\zeta_{\mathbf{q}}, a^*_{\mathbf{k}'} a_{\mathbf{k}'-\mathbf{t}} a^*_{\mathbf{k}-\mathbf{t}} a_{\mathbf{k}+\mathbf{q}}]\}$$
$$- \sum_{\mathbf{qk}'} G_{\mathbf{k},\mathbf{k}',\mathbf{q}} \{\eta_{\mathbf{q}} n_{\mathbf{k}'} a^*_{\mathbf{k}-\mathbf{q}} a_{\mathbf{k}} + \xi_{\mathbf{q}} n_{\mathbf{k}'} a^*_{\mathbf{k}-\mathbf{q}} a_{\mathbf{k}} + \zeta_{\mathbf{q}} n_{\mathbf{k}'} a^*_{\mathbf{k}-\mathbf{q}} a_{\mathbf{k}}\} - \sum_{\mathbf{qk}'\mathbf{t}} H_{\mathbf{k},\mathbf{k}',\mathbf{t},\mathbf{q}} \{\eta_{\mathbf{q}} a^*_{\mathbf{k}'-\mathbf{q}} a_{\mathbf{k}'-\mathbf{t}} a^*_{\mathbf{k}-\mathbf{t}} a_{\mathbf{k}} + \xi_{\mathbf{q}} a^*_{\mathbf{k}'-\mathbf{q}} a_{\mathbf{k}'-\mathbf{t}} a^*_{\mathbf{k}-\mathbf{t}} a_{\mathbf{k}}$$
$$+ \zeta_{\mathbf{q}} a^*_{\mathbf{k}'-\mathbf{q}} a_{\mathbf{k}'-\mathbf{t}} a^*_{\mathbf{k}-\mathbf{t}} a_{\mathbf{k}}\} - \sum_{\mathbf{qk}'\mathbf{t}} I_{\mathbf{k},\mathbf{k}',\mathbf{t},\mathbf{q}} \{\eta_{\mathbf{q}} a^*_{\mathbf{k}'} a_{\mathbf{k}'-\mathbf{t}} a^*_{\mathbf{k}-\mathbf{t}} a_{\mathbf{k}+\mathbf{q}} + \xi_{\mathbf{q}} a^*_{\mathbf{k}'} a_{\mathbf{k}'-\mathbf{t}} a^*_{\mathbf{k}-\mathbf{t}} a_{\mathbf{k}+\mathbf{q}} + \zeta_{\mathbf{q}} a^*_{\mathbf{k}'} a_{\mathbf{k}'-\mathbf{t}} a^*_{\mathbf{k}-\mathbf{t}} a_{\mathbf{k}+\mathbf{q}}\} + \text{c.c.}, \quad (24)$$

where

$$G_{\mathbf{k},\mathbf{k}',\mathbf{q}} = \psi^*(\mathbf{k}',\mathbf{k},\mathbf{q})(\varepsilon_{\mathbf{k}} - \varepsilon_{\mathbf{k}-\mathbf{q}} - \omega_{\mathbf{q}}), \quad (25)$$

$$H_{\mathbf{k},\mathbf{k}',\mathbf{t},\mathbf{q}} = \chi^*(\mathbf{k}',\mathbf{k},\mathbf{t},\mathbf{q})(\varepsilon_{\mathbf{k}} - \varepsilon_{\mathbf{k}-\mathbf{t}} + \varepsilon_{\mathbf{k}'-\mathbf{t}} - \varepsilon_{\mathbf{k}'-\mathbf{q}} - \omega_{\mathbf{q}}), \quad (26)$$

$$I_{\mathbf{k},\mathbf{k}',\mathbf{t},\mathbf{q}} = \varphi^*(\mathbf{k}',\mathbf{k},\mathbf{t},\mathbf{q})(\varepsilon_{\mathbf{k}+\mathbf{q}} - \varepsilon_{\mathbf{k}-\mathbf{t}} + \varepsilon_{\mathbf{k}'-\mathbf{t}} - \varepsilon_{\mathbf{k}'} - \omega_{\mathbf{q}}). \quad (27)$$

The commutators in the terms containing phonon operators have not been evaluated, as we are interested first of all in expressions containing exclusively electron operators.

Given the transformed Hamiltonian $\hat{H}$, we are now in position to minimize the effect of H_e by imposing, similarly as Fröhlich, the condition $H_e - [S',H_a] = 0$. This leads to equations for ψ, χ, and φ, which determine these functions uniquely—viz.,

$$\psi^*(\mathbf{k}',\mathbf{k},\mathbf{q}) = \frac{\phi^*(\mathbf{k},\mathbf{q}) A_{\mathbf{k}',\mathbf{q}}}{\varepsilon_{\mathbf{k}} - \varepsilon_{\mathbf{k}-\mathbf{q}} - \omega_{\mathbf{q}}}. \quad (28)$$

After substituting ϕ and A given by Eqs. (A7) and (14), one obtains

$$\psi^*(\mathbf{k}',\mathbf{k},\mathbf{q}) = \frac{2iD^3_{\mathbf{q}}[1-\Delta(\mathbf{k},\mathbf{q})]}{3(\varepsilon_{\mathbf{k}-\mathbf{q}} - \varepsilon_{\mathbf{k}} + \omega_{\mathbf{q}})^2}\left(\frac{1-\Delta(\mathbf{k}',\mathbf{q})}{\varepsilon_{\mathbf{k}'-\mathbf{q}} - \varepsilon_{\mathbf{k}'} + \omega_{\mathbf{q}}} - \frac{1-\Delta(\mathbf{k}'+\mathbf{q},\mathbf{q})}{\varepsilon_{\mathbf{k}'} - \varepsilon_{\mathbf{k}'+\mathbf{q}} + \omega_{\mathbf{q}}}\right). \quad (29)$$

The introduction of additional functions in order to preserve convergence is not necessary here, as Δ already guarantees this property.

As for χ and φ, additional functions preserving convergence are indispensable—viz.,

$$\chi^*(\mathbf{k}',\mathbf{k},\mathbf{t},\mathbf{q}) = \frac{B_{\mathbf{k}',\mathbf{t},\mathbf{q},\mathbf{k}}}{\varepsilon_{\mathbf{k}} - \varepsilon_{\mathbf{k}-\mathbf{t}} + \varepsilon_{\mathbf{k}'-\mathbf{t}} - \varepsilon_{\mathbf{k}'-\mathbf{q}} - \omega_{\mathbf{q}}} \times [1 - \tilde{\Delta}(\mathbf{k},\mathbf{k}',\mathbf{t},\mathbf{q})], \quad (30)$$

where

$$\tilde{\Delta}(\mathbf{k},\mathbf{k}',\mathbf{t},\mathbf{q}) = \begin{cases} 1, & \text{if } |\varepsilon_{\mathbf{k}} - \varepsilon_{\mathbf{k}-\mathbf{t}} + \varepsilon_{\mathbf{k}'-\mathbf{t}} - \varepsilon_{\mathbf{k}'-\mathbf{q}} - \omega_{\mathbf{q}}| \leq \tilde{\Gamma}_{\mathbf{q}}, \\ 0, & \text{if } |\varepsilon_{\mathbf{k}} - \varepsilon_{\mathbf{k}-\mathbf{t}} + \varepsilon_{\mathbf{k}'-\mathbf{t}} - \varepsilon_{\mathbf{k}'-\mathbf{q}} - \omega_{\mathbf{q}}| > \tilde{\Gamma}_{\mathbf{q}}. \end{cases} \quad (31)$$

Taking into account Eqs. (A7) and (15), one obtains

$$\chi^*(\mathbf{k}',\mathbf{k},\mathbf{t},\mathbf{q}) = \frac{iD_{\mathbf{t}}^2 D_{\mathbf{q}}}{6(\varepsilon_{\mathbf{k}} - \varepsilon_{\mathbf{k-t}} + \varepsilon_{\mathbf{k}'-\mathbf{t}} - \varepsilon_{\mathbf{k}'-\mathbf{q}} - \omega_{\mathbf{q}})} \times \left(\frac{[1-\Delta(\mathbf{k},\mathbf{t})][1-\Delta(\mathbf{k}',\mathbf{q})]}{(\varepsilon_{\mathbf{k-t}} - \varepsilon_{\mathbf{k}} + \omega_{\mathbf{t}})(\varepsilon_{\mathbf{k}'-\mathbf{q}} - \varepsilon_{\mathbf{k}'} + \omega_{\mathbf{q}})}[2+\Delta(\mathbf{k}',\mathbf{t})] - \frac{[1-\Delta(\mathbf{k},\mathbf{t})][1-\Delta(\mathbf{k}'-\mathbf{t},\mathbf{q})]}{(\varepsilon_{\mathbf{k-t}} - \varepsilon_{\mathbf{k}} + \omega_{\mathbf{t}})(\varepsilon_{\mathbf{k}'-\mathbf{t}-\mathbf{q}} - \varepsilon_{\mathbf{k}'-\mathbf{t}} + \omega_{\mathbf{q}})}[2+\Delta(\mathbf{k}'-\mathbf{q},\mathbf{t})] + \frac{[1-\Delta(\mathbf{k}',\mathbf{t})][1-\Delta(\mathbf{k}',\mathbf{q})]}{(\varepsilon_{\mathbf{k}'-\mathbf{t}} - \varepsilon_{\mathbf{k}'} + \omega_{\mathbf{t}})(\varepsilon_{\mathbf{k}'-\mathbf{q}} - \varepsilon_{\mathbf{k}'} + \omega_{\mathbf{q}})}[4-\Delta(\mathbf{k},\mathbf{t})] - \frac{[1-\Delta(\mathbf{k}'-\mathbf{q},\mathbf{t})][1-\Delta(\mathbf{k}'-\mathbf{t},\mathbf{q})]}{(\varepsilon_{\mathbf{k}'-\mathbf{q}-\mathbf{t}} - \varepsilon_{\mathbf{k}'-\mathbf{q}} + \omega_{\mathbf{t}})(\varepsilon_{\mathbf{k}'-\mathbf{t}-\mathbf{q}} - \varepsilon_{\mathbf{k}'-\mathbf{t}} + \omega_{\mathbf{q}})}[4-\Delta(\mathbf{k},\mathbf{t})] \right)[1-\tilde{\Delta}(\mathbf{k},\mathbf{k}',\mathbf{t},\mathbf{q})], \tag{32}$$

$$\varphi^*(\mathbf{k}',\mathbf{k},\mathbf{t},\mathbf{q}) = \frac{C_{\mathbf{k},\mathbf{t},\mathbf{q},\mathbf{k}'}}{\varepsilon_{\mathbf{k+q}} - \varepsilon_{\mathbf{k-t}} + \varepsilon_{\mathbf{k}'-\mathbf{t}} - \varepsilon_{\mathbf{k}'} - \omega_{\mathbf{q}}}[1-\hat{\Delta}(\mathbf{k},\mathbf{k}',\mathbf{t},\mathbf{q})], \tag{33}$$

where

$$\hat{\Delta}(\mathbf{k},\mathbf{k}',\mathbf{t},\mathbf{q}) = \begin{cases} 1, & \text{if } |\varepsilon_{\mathbf{k+q}} - \varepsilon_{\mathbf{k-t}} + \varepsilon_{\mathbf{k}'-\mathbf{t}} - \varepsilon_{\mathbf{k}'} - \omega_{\mathbf{q}}| \le \hat{\Gamma}_{\mathbf{q}}, \\ 0, & \text{if } |\varepsilon_{\mathbf{k+q}} - \varepsilon_{\mathbf{k-t}} + \varepsilon_{\mathbf{k}'-\mathbf{t}} - \varepsilon_{\mathbf{k}'} - \omega_{\mathbf{q}}| > \hat{\Gamma}_{\mathbf{q}}, \end{cases} \tag{34}$$

and substitution of ϕ and $C_{\mathbf{k},\mathbf{t},\mathbf{q},\mathbf{k}'}$ from Eqs. (A7) and (16) yields

$$\varphi^*(\mathbf{k}',\mathbf{k},\mathbf{t},\mathbf{q}) = \frac{iD_{\mathbf{t}}^2 D_{\mathbf{q}}}{6(\varepsilon_{\mathbf{k+q}} - \varepsilon_{\mathbf{k-t}} + \varepsilon_{\mathbf{k}'-\mathbf{t}} - \varepsilon_{\mathbf{k}'} - \omega_{\mathbf{q}})} \times \left(\frac{[1-\Delta(\mathbf{k}',\mathbf{t})][1-\Delta(\mathbf{k}+\mathbf{q}-\mathbf{t},\mathbf{q})]}{(\varepsilon_{\mathbf{k}'-\mathbf{t}} - \varepsilon_{\mathbf{k}'} + \omega_{\mathbf{t}})(\varepsilon_{\mathbf{k-t}} - \varepsilon_{\mathbf{k+q-t}} + \omega_{\mathbf{q}})}[2+\Delta(\mathbf{k}+\mathbf{q},\mathbf{t})] - \frac{[1-\Delta(\mathbf{k}',\mathbf{t})][1-\Delta(\mathbf{k}+\mathbf{q},\mathbf{q})]}{(\varepsilon_{\mathbf{k}'-\mathbf{t}} - \varepsilon_{\mathbf{k}'} + \omega_{\mathbf{t}})(\varepsilon_{\mathbf{k}} - \varepsilon_{\mathbf{k+q}} + \omega_{\mathbf{q}})}[2+\Delta(\mathbf{k},\mathbf{t})] + \frac{[1-\Delta(\mathbf{k}+\mathbf{q},\mathbf{t})][1-\Delta(\mathbf{k}+\mathbf{q}-\mathbf{t},\mathbf{q})]}{(\varepsilon_{\mathbf{k+q-t}} - \varepsilon_{\mathbf{k+q}} + \omega_{\mathbf{t}})(\varepsilon_{\mathbf{k-t}} - \varepsilon_{\mathbf{k+q-t}} + \omega_{\mathbf{q}})}[4-\Delta(\mathbf{k}',\mathbf{t})] - \frac{[1-\Delta(\mathbf{k},\mathbf{t})][1-\Delta(\mathbf{k}+\mathbf{q},\mathbf{q})]}{(\varepsilon_{\mathbf{k-t}} - \varepsilon_{\mathbf{k}} + \omega_{\mathbf{t}})(\varepsilon_{\mathbf{k}} - \varepsilon_{\mathbf{k+q}} + \omega_{\mathbf{q}})}[4-\Delta(\mathbf{k}',\mathbf{t})] \right)[1-\hat{\Delta}(\mathbf{k},\mathbf{k}',\mathbf{t},\mathbf{q})]. \tag{35}$$

Having established ψ, χ, and φ, let us average $\hat{H}$ over the phonon vacuum:

$$\hat{H}_{\text{av}} = \frac{1}{2}\sum_{\mathbf{k}} \varepsilon_{\mathbf{k}} n_{\mathbf{k}} + \sum_{\mathbf{qkk}'} F_{\mathbf{k},\mathbf{k}',\mathbf{q}} a_{\mathbf{k}}^* a_{\mathbf{k-q}} a_{\mathbf{k}'-\mathbf{q}}^* a_{\mathbf{k}'} + \sum_{\mathbf{qk}} E_{\mathbf{k},\mathbf{q}} a_{\mathbf{k}}^* a_{\mathbf{k-q}} (\eta_{\mathbf{q}}^* + \xi_{\mathbf{q}}^* + \zeta_{\mathbf{q}}^*) + \sum_{\mathbf{qkk}'} \left(A_{\mathbf{k}',\mathbf{q}} \phi^*(\mathbf{k},\mathbf{q}) - \frac{1}{2} G_{\mathbf{k},\mathbf{k}',\mathbf{q}} \right) \{ \eta_{\mathbf{q}} n_{\mathbf{k}'} a_{\mathbf{k-q}}^* a_{\mathbf{k}} + \xi_{\mathbf{q}} n_{\mathbf{k}'} a_{\mathbf{k-q}}^* a_{\mathbf{k}} + \zeta_{\mathbf{q}} n_{\mathbf{k}'} a_{\mathbf{k-q}}^* a_{\mathbf{k}} \} + \sum_{\mathbf{qkk}'\mathbf{t}} \left(B_{\mathbf{k}',\mathbf{t},\mathbf{q},\mathbf{k}} - \frac{1}{2} H_{\mathbf{k},\mathbf{k}',\mathbf{t},\mathbf{q}} \right) \{ \eta_{\mathbf{q}} a_{\mathbf{k}'-\mathbf{q}}^* a_{\mathbf{k}'-\mathbf{t}} a_{\mathbf{k-t}}^* a_{\mathbf{k}} + \xi_{\mathbf{q}} a_{\mathbf{k}'-\mathbf{q}}^* a_{\mathbf{k}'-\mathbf{t}} a_{\mathbf{k-t}}^* a_{\mathbf{k}} + \zeta_{\mathbf{q}} a_{\mathbf{k}'-\mathbf{q}}^* a_{\mathbf{k}'-\mathbf{t}} a_{\mathbf{k-t}}^* a_{\mathbf{k}} \} + \sum_{\mathbf{qkk}'\mathbf{t}} \left(C_{\mathbf{k},\mathbf{t},\mathbf{q},\mathbf{k}'} - \frac{1}{2} I_{\mathbf{k},\mathbf{k}',\mathbf{t},\mathbf{q}} \right) \{ \eta_{\mathbf{q}} a_{\mathbf{k}'}^* a_{\mathbf{k}'-\mathbf{t}} a_{\mathbf{k-t}}^* a_{\mathbf{k+q}} + \xi_{\mathbf{q}} a_{\mathbf{k}'}^* a_{\mathbf{k}'-\mathbf{t}} a_{\mathbf{k-t}}^* a_{\mathbf{k+q}} + \zeta_{\mathbf{q}} a_{\mathbf{k}'}^* a_{\mathbf{k}'-\mathbf{t}} a_{\mathbf{k-t}}^* a_{\mathbf{k+q}} \} + \text{c.c.}, \tag{36}$$

$\hat{H}_{\text{av}}$ contains the free-electron term, the two-electron Fröhlich interaction, and terms representing three-electron and four-electron interactions. The three-electron terms were generated by the second transformation of H_c. The source of these terms is thus the nontransformed part of the original interaction, discarded by BCS theory. If higher-order corrections to BCS theory are of interest, all terms arising from that part—i.e., the three-electron ones—can be therefore neglected. The same conclusion was drawn above on the grounds of unstability.

V. FOUR-FERMION INTERACTIONS

The Hamiltonian $\hat{H}_{\text{av}}$ contains several terms representing four-fermion interactions. Using Eqs. (25), (28), (30), and (33), one finds

$$A_{\mathbf{k}',\mathbf{q}} \phi^*(\mathbf{k},\mathbf{q}) - \frac{1}{2} G_{\mathbf{k},\mathbf{k}',\mathbf{q}} = \frac{1}{2} G_{\mathbf{k},\mathbf{k}',\mathbf{q}} = \frac{1}{2} \phi^*(\mathbf{k},\mathbf{q}) A_{\mathbf{k}',\mathbf{q}}.$$

Additionally, taking into account Eqs. (26) and (27), we get

$$B_{\mathbf{k}',\mathbf{t},\mathbf{q},\mathbf{k}} - \frac{1}{2} H_{\mathbf{k},\mathbf{k}',\mathbf{t},\mathbf{q}} = \frac{1}{2} H_{\mathbf{k},\mathbf{k}',\mathbf{t},\mathbf{q}} = \frac{1}{2} B_{\mathbf{k}',\mathbf{t},\mathbf{q},\mathbf{t}}[1-\tilde{\Delta}(\mathbf{k},\mathbf{k}',\mathbf{t},\mathbf{q})],$$

$$C_{\mathbf{k},\mathbf{t},\mathbf{q},\mathbf{k}'} - \frac{1}{2} I_{\mathbf{k},\mathbf{k}',\mathbf{t},\mathbf{q}} = \frac{1}{2} I_{\mathbf{k},\mathbf{k}',\mathbf{t},\mathbf{q}} = \frac{1}{2} C_{\mathbf{k},\mathbf{t},\mathbf{q},\mathbf{k}'}[1-\hat{\Delta}(\mathbf{k},\mathbf{k}',\mathbf{t},\mathbf{q})].$$

In terms of $G_{\mathbf{k},\mathbf{k}',\mathbf{q}}$, $H_{\mathbf{k},\mathbf{k}',\mathbf{t},\mathbf{q}}$, and $I_{\mathbf{k},\mathbf{k}',\mathbf{t},\mathbf{q}}$ the four-fermion interactions present in $\hat{H}_{\text{av}}$ are expressed as

TABLE I. Reductions of H_4^5.

H_4^5	$\mathbf{l}$	$\mathbf{k}'$	$\mathbf{q}$	$\mathbf{t}$	$\mathbf{w}$	Spins
1	$\mathbf{2m+k}$	$\mathbf{2k+m}$	$\mathbf{2k+2m}$	$\mathbf{k+m}$	$\mathbf{k+m}$	$\sigma''=-\sigma''', \sigma=-\sigma'$
2	$\mathbf{2m+k}$	$\mathbf{m}$	$\mathbf{2m}$	$\mathbf{k+m}$	$\mathbf{k+m}$	$\sigma''=-\sigma'''=-\sigma=\sigma'$
3	$\mathbf{2m-k}$	$\mathbf{m}$	$\mathbf{2m}$	$\mathbf{k+m}$	$\mathbf{m-k}$	$\sigma''=-\sigma'''=\sigma=-\sigma'$
4	$\mathbf{k}$	$\mathbf{2k+m}$	$\mathbf{2k}$	$\mathbf{k+m}$	$\mathbf{k+m}$	$\sigma''=-\sigma=-\sigma'''=\sigma'$
5	$\mathbf{k}$	$\mathbf{m}$	0	$\mathbf{k+m}$	$\mathbf{k+m}$	$\sigma''=-\sigma, \sigma'''=-\sigma'$
6	$-\mathbf{k}$	$\mathbf{m}$	0	$\mathbf{k+m}$	$\mathbf{m-k}$	$\sigma''=-\sigma=\sigma'''=-\sigma'$
7	$\mathbf{k}$	$\mathbf{2k-m}$	$\mathbf{2k}$	$\mathbf{k-m}$	$\mathbf{k+m}$	$\sigma''=-\sigma'=\sigma=-\sigma'''$
8	$\mathbf{k}$	$-\mathbf{m}$	0	$\mathbf{k-m}$	$\mathbf{k+m}$	$\sigma''=-\sigma'=-\sigma=\sigma'''$
9	$-\mathbf{k}$	$-\mathbf{m}$	0	$\mathbf{k-m}$	$\mathbf{m-k}$	$\sigma''=-\sigma', \sigma'''=-\sigma$

$$H_4^1=\sum_{\mathbf{qk'lm}}\frac{1}{2}G_{\mathbf{k,k',q}}\psi(\mathbf{l,m,q})a^*_{\mathbf{m}}a_{\mathbf{m-q}}n_{\mathbf{l}}n_{\mathbf{k'}}a^*_{\mathbf{k-q}}a_{\mathbf{k}}+\text{c.c.},\tag{37}$$

$$H_4^2=\sum_{\mathbf{qk'lmt}}\frac{1}{2}G_{\mathbf{k,k',q}}\chi(\mathbf{l,m,t,q})a^*_{\mathbf{m}}a_{\mathbf{m-t}}a^*_{\mathbf{l-t}}a_{\mathbf{l-q}}n_{\mathbf{k'}}a^*_{\mathbf{k-q}}a_{\mathbf{k}}+\text{c.c.},\tag{38}$$

$$H_4^3=\sum_{\mathbf{qk'lmt}}\frac{1}{2}G_{\mathbf{k,k',q}}\varphi(\mathbf{l,m,t,q})a^*_{\mathbf{m+q}}a_{\mathbf{m-t}}a^*_{\mathbf{l-t}}a_{\mathbf{l}}n_{\mathbf{k'}}a^*_{\mathbf{k-q}}a_{\mathbf{k}}+\text{c.c.},\tag{39}$$

$$H_4^4=\sum_{\mathbf{qk'lmt}}\frac{1}{2}H_{\mathbf{k,k',t,q}}\psi(\mathbf{l,m,q})a^*_{\mathbf{m}}a_{\mathbf{m-q}}n_{\mathbf{l}}a^*_{\mathbf{k'-q}}a_{\mathbf{k'-t}}a^*_{\mathbf{k-t}}a_{\mathbf{k}}+\text{c.c.},\tag{40}$$

$$H_4^5=\sum_{\mathbf{qk'lmtw}}\frac{1}{2}H_{\mathbf{k,k',t,q}}\chi(\mathbf{l,m,w,q})a^*_{\mathbf{m}}a_{\mathbf{m-w}}a^*_{\mathbf{l-w}}a_{\mathbf{l-q}}\times a^*_{\mathbf{k'-q}}a_{\mathbf{k'-t}}a^*_{\mathbf{k-t}}a_{\mathbf{k}}+\text{c.c.},\tag{41}$$

$$H_4^6=\sum_{\mathbf{qk'lmtw}}\frac{1}{2}H_{\mathbf{k,k',t,q}}\varphi(\mathbf{l,m,w,q})a^*_{\mathbf{m+q}}a_{\mathbf{m-w}}a^*_{\mathbf{l-w}}a_{\mathbf{l}}\times a^*_{\mathbf{k'-q}}a_{\mathbf{k'-t}}a^*_{\mathbf{k-t}}a_{\mathbf{k}}+\text{c.c.},\tag{42}$$

$$H_4^7=\sum_{\mathbf{qk'lmt}}\frac{1}{2}I_{\mathbf{k,k',t,q}}\psi(\mathbf{l,m,q})a^*_{\mathbf{m}}a_{\mathbf{m-q}}n_{\mathbf{l}}a^*_{\mathbf{k'}}a_{\mathbf{k'-t}}a^*_{\mathbf{k-t}}a_{\mathbf{k+q}}+\text{c.c.},\tag{43}$$

$$H_4^8=\sum_{\mathbf{qk'lmtw}}\frac{1}{2}I_{\mathbf{k,k',t,q}}\chi(\mathbf{l,m,w,q})a^*_{\mathbf{m}}a_{\mathbf{m-w}}a^*_{\mathbf{l-w}}a_{\mathbf{l-q}}a^*_{\mathbf{k'}}a_{\mathbf{k'-t}}\times a^*_{\mathbf{k-t}}a_{\mathbf{k+q}}+\text{c.c.},\tag{44}$$

$$H_4^9=\sum_{\mathbf{qk'lmtw}}\frac{1}{2}I_{\mathbf{k,k',t,q}}\varphi(\mathbf{l,m,w,q})a^*_{\mathbf{m+q}}a_{\mathbf{m-w}}a^*_{\mathbf{l-w}}a_{\mathbf{l}}a^*_{\mathbf{k'}}a_{\mathbf{k'-t}}\times a^*_{\mathbf{k-t}}a_{\mathbf{k+q}}+\text{c.c.}\tag{45}$$

Five of these interactions—viz., H_4^1, H_4^2, H_4^3, H_4^4, and H_4^7—contain the operator $n_{\mathbf{k}}$; therefore, they are not reducible to the four-fermion MT potential (3). This potential is particularly interesting not only for its possible relevance to the physics of superconductors, but also because the thermodynamics of the Hamiltonian $H_{BCS}+W+V_{MT}$ is exactly solvable (Brankov *et al.*[21]).

Let us consider the eight-fold product of operators in H_4^5 with explicit spin indices:

$$a^*_{\mathbf{m}\sigma''}a_{\mathbf{m-w}\sigma''}a^*_{\mathbf{l-w}\sigma'''}a_{\mathbf{l-q}\sigma'''}a^*_{\mathbf{k'-q}\sigma}a_{\mathbf{k'-t}\sigma}a^*_{\mathbf{k-t}\sigma'}a_{\mathbf{k}\sigma'}.$$

With respect to one-fermion momenta these are nine possible reductions to the form (3), and each of them allows two or four possibilities related to spin reduction. The values assumed by each momentum index are collected in Table I. In four cases $\mathbf{q}=0$ and the coupling vanishes, since $\chi(\mathbf{l},\mathbf{m},\mathbf{w},0)=0$, for all $\mathbf{l}$, $\mathbf{m}$, $\mathbf{w}$. As a consequence, H_4^5 assumes the reduced form

$$\begin{aligned}H^5_{4(red)}=\sum_{\mathbf{k}\neq\mathbf{m}}\{&2H_{\mathbf{k,2k+m,k+m,2k+2m}}\\&\times\chi(\mathbf{2m+k},\mathbf{m},\mathbf{k+m},\mathbf{2k+2m})\\&-H_{\mathbf{k,m,k+m,2m}}[\chi(\mathbf{2m+k},\mathbf{m},\mathbf{k+m},\mathbf{2m})\\&+\chi(\mathbf{2m-k},\mathbf{m},\mathbf{m-k},\mathbf{2m})]-\chi(\mathbf{k},\mathbf{m},\mathbf{k+m},\mathbf{2k})\\&\times(H_{\mathbf{k,2k+m,k+m,2k}}+H_{\mathbf{k,2k-m,k-m,2k}})\}\\&\times a^*_{\mathbf{m}-}a^*_{\mathbf{m}+}a^*_{-\mathbf{m}-}a^*_{-\mathbf{m}+}a_{-\mathbf{k}+}a_{-\mathbf{k}-}a_{\mathbf{k}+}a_{\mathbf{k}-}+\text{c.c.}\end{aligned}\tag{46}$$

Terms with $\mathbf{k}=\mathbf{m}$ have been excluded, similarly as in V_{BCS}. Their contribution is accounted for by a shift of one-fermion energies. The coefficients on the RHS result after performing summation over spins.

This procedure has been also applied to H_4^6, H_4^8, and H_4^9. The corresponding values of one-fermion momenta are given in Tables II–IV. The additional remark $\mathbf{k}\to\mathbf{k}-2\mathbf{m}$ (or simi-

TABLE II. Reductions of H_4^6.

H_4^6	$\mathbf{l}$	$\mathbf{k}'$	$\mathbf{q}$	$\mathbf{t}$	$\mathbf{w}$	Spins	
1	$-\mathbf{k}$	$-\mathbf{m}$	$-2\mathbf{k}-2\mathbf{m}$	$-\mathbf{k}-\mathbf{m}$	$\mathbf{k}+\mathbf{m}$	$\sigma''=-\sigma''', \sigma=-\sigma'$	$\mathbf{m}\to\mathbf{m}-2\mathbf{k}$
2	$\mathbf{k}$	$-\mathbf{m}$	$-2\mathbf{m}$	$\mathbf{k}-\mathbf{m}$	$\mathbf{k}+\mathbf{m}$	$\sigma''=-\sigma'''=-\sigma=\sigma'$	
3	$-\mathbf{k}$	$-\mathbf{m}$	$-2\mathbf{m}$	$\mathbf{k}-\mathbf{m}$	$\mathbf{m}-\mathbf{k}$	$\sigma''=-\sigma'''=\sigma=-\sigma'$	
4	$-\mathbf{k}$	$4\mathbf{k}+\mathbf{m}$	$2\mathbf{k}$	$3\mathbf{k}+\mathbf{m}$	$\mathbf{k}+\mathbf{m}$	$\sigma''=-\sigma=-\sigma'''=\sigma'$	$\mathbf{m}\to\mathbf{m}-2\mathbf{k}$
5	$\mathbf{k}$	$\mathbf{m}$	0	$\mathbf{k}+\mathbf{m}$	$\mathbf{k}+\mathbf{m}$	$\sigma''=-\sigma, \sigma'''=-\sigma'$	
6	$-\mathbf{k}$	$\mathbf{m}$	0	$\mathbf{k}+\mathbf{m}$	$\mathbf{m}-\mathbf{k}$	$\sigma''=-\sigma=\sigma'''=-\sigma'$	
7	$-\mathbf{k}$	$-\mathbf{m}$	$2\mathbf{k}$	$-\mathbf{k}-\mathbf{m}$	$\mathbf{k}+\mathbf{m}$	$\sigma''=-\sigma'=\sigma=-\sigma'''$	$\mathbf{m}\to\mathbf{m}-2\mathbf{k}$
8	$\mathbf{k}$	$-\mathbf{m}$	0	$\mathbf{k}-\mathbf{m}$	$\mathbf{k}+\mathbf{m}$	$\sigma''=-\sigma'=-\sigma=\sigma'''$	
9	$-\mathbf{k}$	$-\mathbf{m}$	0	$\mathbf{k}-\mathbf{m}$	$\mathbf{m}-\mathbf{k}$	$\sigma''=-\sigma', \sigma'''=-\sigma$	

lar) indicates that a translation of one momentum index is necessary after reduction. After reduction, H_4^6, H_4^8, and H_4^9 take the following forms:

$$
\begin{aligned}
H^6_{4(red)} = \sum_{\mathbf{k}\neq\mathbf{m}} \{ & 2H_{\mathbf{k},2\mathbf{k}+\mathbf{m},\mathbf{k}+\mathbf{m},2\mathbf{k}+2\mathbf{m}} \\
& \times \varphi(-\mathbf{k},-2\mathbf{k}-\mathbf{m},-\mathbf{k}-\mathbf{m},2\mathbf{k}+2\mathbf{m}) \\
& - H_{\mathbf{k},\mathbf{m},\mathbf{k}+\mathbf{m},2\mathbf{m}}[\varphi(\mathbf{k},-\mathbf{m},\mathbf{k}-\mathbf{m},2\mathbf{m}) \\
& + \varphi(-\mathbf{k},-\mathbf{m},-\mathbf{m}-\mathbf{k},2\mathbf{m})] \\
& - \varphi(-\mathbf{k},\mathbf{m}-2\mathbf{k},-\mathbf{k}+\mathbf{m},2\mathbf{k}) \\
& \times (H_{\mathbf{k},2\mathbf{k}+\mathbf{m},\mathbf{k}+\mathbf{m},2\mathbf{k}} + H_{\mathbf{k},2\mathbf{k}-\mathbf{m},\mathbf{k}-\mathbf{m},2\mathbf{k}})\} \\
& \times a^*_{\mathbf{m}-}a^*_{\mathbf{m}+}a^*_{-\mathbf{m}-}a^*_{-\mathbf{m}+}a_{-\mathbf{k}+}a_{-\mathbf{k}-}a_{\mathbf{k}+}a_{\mathbf{k}-} + \text{c.c.},
\end{aligned} \tag{47}
$$

$$
\begin{aligned}
H^8_{4(red)} = \sum_{\mathbf{k}\neq\mathbf{m}} \{ & 2I_{-\mathbf{k}-2\mathbf{m},-\mathbf{m},-\mathbf{k}-\mathbf{m},2\mathbf{k}+2\mathbf{m}} \\
& \times \chi(\mathbf{k}+2\mathbf{m},\mathbf{m},\mathbf{k}+\mathbf{m},2\mathbf{k}+2\mathbf{m}) \\
& - I_{\mathbf{k}-2\mathbf{m},-\mathbf{m},\mathbf{k}-\mathbf{m},2\mathbf{m}}[\chi(\mathbf{k}+2\mathbf{m},\mathbf{m},\mathbf{k}+\mathbf{m},2\mathbf{m}) \\
& + \chi(-\mathbf{k}+2\mathbf{m},\mathbf{m},\mathbf{m}-\mathbf{k},2\mathbf{m})] - \chi(\mathbf{k},\mathbf{m},\mathbf{k}+\mathbf{m},2\mathbf{k}) \\
& \times (I_{-\mathbf{k},\mathbf{m},-\mathbf{k}+\mathbf{m},2\mathbf{k}} + I_{-\mathbf{k},-\mathbf{m},-\mathbf{k}-\mathbf{m},2\mathbf{k}})\} \\
& \times a^*_{\mathbf{m}-}a^*_{\mathbf{m}+}a^*_{-\mathbf{m}-}a^*_{-\mathbf{m}+}a_{-\mathbf{k}+}a_{-\mathbf{k}-}a_{\mathbf{k}+}a_{\mathbf{k}-} + \text{c.c.},
\end{aligned} \tag{48}
$$

$$
\begin{aligned}
H^9_{4(red)} = \sum_{\mathbf{k}\neq\mathbf{m}} \{ & 2I_{-\mathbf{k}-2\mathbf{m},-\mathbf{m},-\mathbf{k}-\mathbf{m},2\mathbf{k}+2\mathbf{m}} \\
& \times \varphi(-\mathbf{k},-2\mathbf{k}-\mathbf{m},-\mathbf{k}-\mathbf{m},2\mathbf{k}+2\mathbf{m}) \\
& - I_{\mathbf{k}-2\mathbf{m},-\mathbf{m},\mathbf{k}-\mathbf{m},2\mathbf{m}}[\varphi(\mathbf{k},-\mathbf{m},\mathbf{k}-\mathbf{m},2\mathbf{m}) \\
& + \varphi(-\mathbf{k},-\mathbf{m},-\mathbf{k}-\mathbf{m},2\mathbf{m})] \\
& - \varphi(-\mathbf{k},-2\mathbf{k}+\mathbf{m},-\mathbf{k}+\mathbf{m},2\mathbf{k}) \\
& \times (I_{-\mathbf{k},\mathbf{m},-\mathbf{k}+\mathbf{m},2\mathbf{k}} + I_{-\mathbf{k},-\mathbf{m},-\mathbf{k}-\mathbf{m},2\mathbf{k}})\} \\
& \times a^*_{\mathbf{m}-}a^*_{\mathbf{m}+}a^*_{-\mathbf{m}-}a^*_{-\mathbf{m}+}a_{-\mathbf{k}+}a_{-\mathbf{k}-}a_{\mathbf{k}+}a_{\mathbf{k}-} + \text{c.c.}
\end{aligned} \tag{49}
$$

Collecting all terms in Eqs. (46)–(49), one obtains a V_{MT} interaction as in Eq. (3) with

$$
g_{\mathbf{mk}} = -4\omega_{2\mathbf{k}+2\mathbf{m}}\Lambda_{\mathbf{km}}\Lambda^*_{\mathbf{mk}} + 2\Theta_{\mathbf{km}} + 2\Theta^*_{\mathbf{mk}}, \tag{50}
$$

where

$$
\begin{aligned}
\Lambda_{\mathbf{km}} = {} & \chi^*(2\mathbf{k}+\mathbf{m},\mathbf{k},\mathbf{k}+\mathbf{m},2\mathbf{k}+2\mathbf{m}) \\
& + \varphi^*(-\mathbf{m},-\mathbf{k}-2\mathbf{m},-\mathbf{k}-\mathbf{m},2\mathbf{k}+2\mathbf{m}),
\end{aligned} \tag{51}
$$

$$
\begin{aligned}
\Theta_{\mathbf{km}} = {} & \omega_{\mathbf{m}}[\chi^*(\mathbf{m},\mathbf{k},\mathbf{k}+\mathbf{m},2\mathbf{m}) \\
& + \varphi^*(-\mathbf{m},\mathbf{k}-2\mathbf{m},\mathbf{k}-\mathbf{m},2\mathbf{m})] \\
& \times[\chi(2\mathbf{m}+\mathbf{k},\mathbf{m},\mathbf{k}+\mathbf{m},2\mathbf{m}) + \chi(2\mathbf{m}-\mathbf{k},\mathbf{m},\mathbf{m} \\
& - \mathbf{k},2\mathbf{m})
\end{aligned}
$$

TABLE III. Reductions of H_4^8.

H_4^8	$\mathbf{l}$	$\mathbf{k}'$	$\mathbf{q}$	$\mathbf{t}$	$\mathbf{w}$	Spins	
1	$-\mathbf{k}$	$-\mathbf{m}$	$-2\mathbf{k}-2\mathbf{m}$	$\mathbf{k}+\mathbf{m}$	$-\mathbf{k}-\mathbf{m}$	$\sigma''=-\sigma''', \sigma=-\sigma'$	$\mathbf{k}\to\mathbf{k}-2\mathbf{m}$
2	$\mathbf{k}+4\mathbf{m}$	$-\mathbf{m}$	$2\mathbf{m}$	$\mathbf{k}+\mathbf{m}$	$\mathbf{k}+3\mathbf{m}$	$\sigma''=-\sigma'''=-\sigma=\sigma'$	$\mathbf{k}\to\mathbf{k}-2\mathbf{m}$
3	$-\mathbf{k}$	$-\mathbf{m}$	$2\mathbf{m}$	$\mathbf{k}+\mathbf{m}$	$-\mathbf{m}-\mathbf{k}$	$\sigma''=-\sigma'''=\sigma=-\sigma'$	$\mathbf{k}\to\mathbf{k}-2\mathbf{m}$
4	$-\mathbf{k}$	$\mathbf{m}$	$-2\mathbf{k}$	$\mathbf{k}+\mathbf{m}$	$-\mathbf{k}+\mathbf{m}$	$\sigma''=-\sigma=\sigma'=-\sigma'''$	
5	$\mathbf{k}$	$\mathbf{m}$	0	$\mathbf{k}+\mathbf{m}$	$\mathbf{k}+\mathbf{m}$	$\sigma''=-\sigma, \sigma'''=-\sigma'$	
6	$-\mathbf{k}$	$\mathbf{m}$	0	$\mathbf{k}+\mathbf{m}$	$\mathbf{m}-\mathbf{k}$	$\sigma''=-\sigma=-\sigma'=\sigma'''$	
7	$-\mathbf{k}$	$-\mathbf{m}$	$-2\mathbf{k}$	$\mathbf{k}-\mathbf{m}$	$-\mathbf{k}+\mathbf{m}$	$\sigma''=-\sigma'=\sigma=-\sigma'''$	
8	$\mathbf{k}$	$-\mathbf{m}$	0	$\mathbf{k}-\mathbf{m}$	$\mathbf{k}+\mathbf{m}$	$\sigma''=-\sigma'=-\sigma=\sigma'''$	
9	$-\mathbf{k}$	$-\mathbf{m}$	0	$\mathbf{k}-\mathbf{m}$	$\mathbf{m}-\mathbf{k}$	$\sigma''=-\sigma', \sigma'''=-\sigma$	

TABLE IV. Reductions of H_4^9.

H_4^9	$\mathbf{l}$	$\mathbf{k}'$	$\mathbf{q}$	$\mathbf{t}$	$\mathbf{w}$	Spins	
1	$-\frac{1}{3}\mathbf{k}+\frac{2}{3}\mathbf{m}$	$\frac{2}{3}\mathbf{k}-\frac{1}{3}\mathbf{m}$	$-\frac{2}{3}\mathbf{k}-\frac{2}{3}\mathbf{m}$	$\frac{1}{3}\mathbf{k}+\frac{1}{3}\mathbf{m}$	$\frac{1}{3}\mathbf{k}+\frac{1}{3}\mathbf{m}$	$\sigma''=-\sigma''', \sigma=-\sigma'$	$\mathbf{k}\to 3\mathbf{k}+2\mathbf{m}, \mathbf{m}\to\mathbf{m}-2\mathbf{k}$
2	$\mathbf{k}-2\mathbf{m}$	$\mathbf{m}$	$-2\mathbf{m}$	$\mathbf{k}-\mathbf{m}$	$\mathbf{k}-\mathbf{m}$	$\sigma''=-\sigma'''=-\sigma=\sigma'$	$\mathbf{k}\to\mathbf{k}+2\mathbf{m}$
3	$2\mathbf{m}-\mathbf{k}$	$\mathbf{m}$	$-2\mathbf{m}$	$\mathbf{k}-\mathbf{m}$	$3\mathbf{m}-\mathbf{k}$	$\sigma''=-\sigma'''=\sigma=-\sigma'$	$\mathbf{k}\to\mathbf{k}+2\mathbf{m}$
4	$\mathbf{k}$	$-2\mathbf{k}+\mathbf{m}$	$-2\mathbf{k}$	$-\mathbf{k}+\mathbf{m}$	$-\mathbf{k}+\mathbf{m}$	$\sigma''=-\sigma=\sigma'=-\sigma'''$	$\mathbf{m}\to\mathbf{m}+2\mathbf{k}$
5	$\mathbf{k}$	$\mathbf{m}$	0	$\mathbf{k}+\mathbf{m}$	$\mathbf{k}+\mathbf{m}$	$\sigma''=-\sigma, \sigma'''=-\sigma'$	
6	$-\mathbf{k}$	$\mathbf{m}$	0	$\mathbf{k}+\mathbf{m}$	$\mathbf{m}-\mathbf{k}$	$\sigma''=-\sigma=-\sigma'=\sigma'''$	
7	$\mathbf{k}$	$2\mathbf{k}-\mathbf{m}$	$-2\mathbf{k}$	$3\mathbf{k}-\mathbf{m}$	$-\mathbf{k}+\mathbf{m}$	$\sigma''=-\sigma'=\sigma=-\sigma'''$	$\mathbf{m}\to\mathbf{m}+2\mathbf{k}$
8	$\mathbf{k}$	$-\mathbf{m}$	0	$\mathbf{k}-\mathbf{m}$	$\mathbf{k}+\mathbf{m}$	$\sigma''=-\sigma'=-\sigma=\sigma'''$	
9	$-\mathbf{k}$	$-\mathbf{m}$	0	$\mathbf{k}-\mathbf{m}$	$\mathbf{m}-\mathbf{k}$	$\sigma''=-\sigma', \sigma'''=-\sigma$	

$$+\varphi(\mathbf{k},-\mathbf{m},\mathbf{k}-\mathbf{m},2\mathbf{m})+\varphi(-\mathbf{k},-\mathbf{m},-\mathbf{k}-\mathbf{m},2\mathbf{m})]. \tag{52}$$

VI. FOUR-FERMION INTERACTION COUPLING

The most important question about the four-fermion interaction is the positive or negative valuedness of $g_{\mathbf{mk}}$. In particular, it would be of interest to determine the domain of $g_{\mathbf{mk}}$ in momentum space where it is attractive. Unfortunately, the form of $g_{\mathbf{mk}}$ given by Eq. (50) is extremely complicated, so this problem cannot be resolved in general.

First, let us specify the quantities occurring in Eq. (50). We assume that $\varepsilon_{\mathbf{k}}=ak^2$, $\omega_{\mathbf{k}}=bk$, $\tilde{\Gamma}_{\mathbf{k}}=ck$, $\Gamma_{\mathbf{k}}=dk$, and $\hat{\Gamma}_{\mathbf{k}}=ek$, where a, b, c, d, and e are real, positive constants.

Following Davydov,[22] we apply some approximations in order to estimate $g_{\mathbf{mk}}$. We assume that the most significant contribution to the four-fermion interaction (analogously as in the BCS theory) comes from one-electron momenta **m** and **k** which satisfy the condition $m\approx k$.

The case of exact equality is, of course, excluded [in accordance with the restriction on summation in Eqs. (46)–(49)], so we use $g_{\mathbf{kk}}$ only as an abbreviation for $g_{\mathbf{mk}}$ under our approximation $m\approx k$.

Thus, the interaction coupling $g_{\mathbf{kk}}$ is given by

$$g_{\mathbf{kk}}=-4\omega_{4\mathbf{k}}\Lambda_{\mathbf{kk}}\Lambda^*_{\mathbf{kk}}+2\Theta_{\mathbf{kk}}+2\Theta^*_{\mathbf{kk}}. \tag{53}$$

We take into account only vectors with almost compatible directions; otherwise, $g_{\mathbf{kk}}$ vanishes.

We can thus rewrite Eq. (53) in the form

$$g_{\mathbf{kk}}=-16\omega_{\mathbf{k}}|\Lambda_{\mathbf{kk}}|^2+4\,\mathrm{Re}\,\Theta_{\mathbf{kk}}, \tag{54}$$

where the first term is always nonpositive and "Re" means the real part. The second term requires detailed analysis. Detailed calculations are performed in Appendix B. We have found $g_{\mathbf{kk}}$ in all cases for different values of the constants a, b, c, d, and e. Most interesting is the cases $b>c$, $b>e$, and $b>d$, because the essential part of initial Hamiltonian is transformed under these conditions. Under a further approximation, we find the form of $g_{\mathbf{kk}}$ for $k\approx k_F$, where k_F is Fermi momentum. Additionally we put $c=d=e$ and $\varepsilon_{\mathbf{k}_F}\gg\omega_{\mathbf{k}_F}$ (this is at least true for metals). Under these assumptions,

$$\Theta_{\mathbf{k}_F\mathbf{k}_F}\approx\frac{-D^6_{\mathbf{k}_F}}{4\omega^5_{\mathbf{k}_F}},\quad \Lambda_{\mathbf{k}_F\mathbf{k}_F}\approx\frac{iD^3_{\mathbf{k}_F}}{12\omega^2_{\mathbf{k}_F}\varepsilon_{\mathbf{k}_F}},$$

which implies

$$g_{\mathbf{k}_F\mathbf{k}_F}\approx\frac{-D^6_{\mathbf{k}_F}}{\omega^5_{\mathbf{k}_F}}<0.$$

Now we can compare the magnitudes of the coupling constants of the four-fermion interaction and two-fermion interaction (under the same assumptions). It is shown in Appendix A that $G_{\mathbf{k}_F\mathbf{k}_F}=-D^2_{\mathbf{k}_F}/\omega_{\mathbf{k}_F}$. Thus the interaction coupling for four- and two-fermion (BCS) interactions fulfills the relation $g_{\mathbf{k}_F\mathbf{k}_F}=G^3_{\mathbf{k}_F\mathbf{k}_F}/\omega^2_{\mathbf{k}_F}$. As a consequence, for a strong pairing, there is a significant contribution from four-fermion interactions.

Moreover, the four-fermion interaction coupling $g_{\mathbf{kk}}$ is also negative in most cases without imposing the approximations $c=d=e$ and $k\approx k_F$. This is shown in Appendix B.

VII. FOUR-FERMION INTERACTIONS IN FRÖHLICH'S EXPANSION

As demonstrated in Sec. III, four-fermion interaction terms appear in sixth order of the expansion of Fröhlich's original transformation. These terms are derived in Appendix C. The various resulting four-fermion terms considerably outnumber those obtained by applying a second transformation in Sec. V. Again a reduction procedure to V_{MT} is possible for all four-fermion expressions (C5), which do not contain the particle number operator and possess three phonon indices. However, the resulting couplings are also complicated functions and, therefore, will not be examined in detail.

The disadvantage of this method is appearance of three-fermion interactions, which in the double transformation method gave a small contribution. Here additional arguments must be used to discard these terms.

VIII. CONCLUDING REMARKS

We have extended Fröhlich's transformation of $H_{e\text{-}ph}$ to higher-order terms. This has been done by performing a second transform of the first terms discarded by Fröhlich. The resulting interactions are of three- and four-fermion type. The three-fermion terms can be expected to be inessential in superconductivity because of their instability under time inversion. The four-fermion terms are, in general, reducible to V_{MT}, a BCS-type four-fermion interaction. The resulting four-fermion coupling is extremely complicated, but under reasonable approximations it is negative valued. Moreover, for one-fermion momenta in the neighborhood of Fermi momentum, it has the simple form $-D^6_{\mathbf{k}_F}/\omega^5_{\mathbf{k}_F}=G^3_{\mathbf{k}_F\mathbf{k}_F}/\omega^2_{\mathbf{k}_F}$, where $G_{\mathbf{k}_F\mathbf{k}_F}$ is the two-fermion coupling. This fact implies that four-fermion interactions are significant for systems with strong pairing and allows us to estimate (relative to two-fermion coupling and phonon energy at Fermi momentum) their magnitude.

As in BCS theory, where the relation between the gap parameter Δ and coupling constant G of the BCS interaction allows one to estimate the magnitude of G, it can be expected that the detailed thermodynamics of H_{MT} will provide a relation between $|g_{\mathbf{mk}}|$ and other parameters of the theory, thereby allowing one to estimate the magnitude of $g_{\mathbf{mk}}$. Another emerging question is Cooper's problem[23] for a bound quadruple in the presence of V_{MT} or $V_{BCS}+V_{MT}$. Kamei and Miyake deal with this question in a recent work.[24]

The double transformation has unveiled the structure of three-, four-, and five-fermion interactions. Since they are proportional to the fourth, sixth, and eighth power of $D_{\mathbf{w}}$, they are relatively weak, so inclusion of these terms, apart from four-fermion ones, appears unjustified at present, although Schneider *et al.*[19] suggest the existence of both quadruples and sextets in some HTSC's.

The higher-order expansion terms of the transformed $H_{e\text{-}ph}$, including quadruple, sextet, etc., interactions can be expected to reveal themselves in materials with extremely strong pairing correlations between spin-1/2 fermions, since the presence of strongly correlated pairs implies at least some kind of weak interaction between them.

On the other hand, our extension of Fröhlich's transformation shows that if phonon mediation exists in a superconductor, the four-fermion and six-fermion interactions are always present as supplementary to the BCS one. Unfortunately, the Hamiltonian $H_{BCS}+V_{MT}$, although exactly solvable, leads to intricate mean-field equations.

Our results obtained can be generalized in many respects. First of all, the process of averaging over phonon vacuum could be replaced by averaging over the phonon equilibrium state, which could be justified at higher temperatures. This would already lead to an additional one-fermion term in Fröhlich's Hamiltonian and modification of two-fermion and three-fermion terms in our method. Another extension would result by going beyond the harmonic approximation and including all products of phonon operators. Method of Bogoliubov[25] could be applied to "dangerous terms" (divergent), omitted in Fröhlich's method. These questions will be dealt with in further investigations.

ACKNOWLEDGMENT

The authors would like to thank J. Maćkowiak for valuable comments and suggestions.

APPENDIX A: FRÖHLICH'S TRANSFORMATION

Let us recall Fröhlich's method,[2] using in some details the more elegant approach due to Davydov.[22]

Consider the electron-phonon Hamiltonian

$$H_{e\text{-}ph} = H_0 + H_{int} = \sum_{\mathbf{k}\sigma} \varepsilon_{\mathbf{k}} a^*_{\mathbf{k}\sigma} a_{\mathbf{k}\sigma} + \sum_{\mathbf{w}} \omega_{\mathbf{w}} b^*_{\mathbf{w}} b_{\mathbf{w}} + i \sum_{\mathbf{w}} D_{\mathbf{w}} (b_{\mathbf{w}} \rho^*_{\mathbf{w}} - b^*_{\mathbf{w}} \rho_{\mathbf{w}}),$$

where

$$\rho_{\mathbf{w}} = \sum_{\mathbf{k}\sigma} a^*_{\mathbf{k}-\mathbf{w}\sigma} a_{\mathbf{k}\sigma}$$

and $a_{\mathbf{k}\sigma}$ ($b_{\mathbf{k}}$) are fermion (boson) operators. The coupling $D_{\mathbf{w}}$ will be assumed small and $\hbar \equiv 1$.

The form of the interaction term of $H_{e\text{-}ph}$ arises under a number of assumptions: the ions of the lattice move collectively, the coupling depends only on $\mathbf{w}$, and electrons interact only with longitudinal phonons for which $\omega_{\mathbf{w}}=ws$, s denoting the velocity of sound. Our interest is focused on the behavior of electrons; therefore, variations of the phonon spectrum will be accounted for only through s.

Fröhlich performed a unitary transformation of $H_{e\text{-}ph}$ in order to eliminate (as far as possible) the interaction term. The transformed Hamiltonian is

$$H = \mathrm{e}^{S^*} H_{e\text{-}ph} \mathrm{e}^{S} = H_{e\text{-}ph} - [S, H_{e\text{-}ph}] + \frac{1}{2}[S,[S,H_{e\text{-}ph}]] + \cdots , \tag{A1}$$

where

$$S = \sum_{\mathbf{q}} S_{\mathbf{q}} = \sum_{\mathbf{q}} (\gamma^*_{\mathbf{q}} b^*_{\mathbf{q}} - \gamma_{\mathbf{q}} b_{\mathbf{q}}) = -S^*,$$

$$\gamma_{\mathbf{q}} = \sum_{\mathbf{k}} \phi(\mathbf{k},\mathbf{q}) a^*_{\mathbf{k}} a_{\mathbf{k}-\mathbf{q}},$$

and the unknown function $\phi(\mathbf{k},\mathbf{q}): \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{C}^1$ is adjusted to achieve the cancellation.

Collecting terms of the same order in the coupling $D_{\mathbf{w}}$, one obtains Fröhlich's expansion:

$$H = H_0 - ([S,H_0] - H_{int}) + \left(\frac{1}{2}[S,[S,H_0]] - [S,H_{int}]\right) + \cdots . \tag{A2}$$

Subsequently, a term which is a combination of products, each with f fermion operators and b boson operators will be written as (f,b). Clearly, f will always be even. For example, H_0 consists of terms (2,0) and (0,2).

The RHS of Eq. (A2) is expressed in terms of commutators $[(f_1,b_1),(f_2,b_2)]$. One easily finds that

$$[(f_1,b_1),(f_2,b_2)] = [f_1,f_2]b_1b_2 + f_2f_1[b_1,b_2] = [f_1,f_2]b_2b_1 + f_1f_2[b_1,b_2]. \tag{A3}$$

The necessary commutators $[f_1,f_2]$, $[b_1,b_2]$ are given in Appendix C.

According to Eq. (A1), the transformation can be performed, given commutators of the form occurring in Eq. (A3) with the first argument equal S. The latter is a (2,1) expression, hence

$$[S,(f,b)] = [(2,1),(f,b)] = (f,b+1) + (f+2,b-1),$$

by virtue of Eqs. (A3) and (C1). Clearly $(f,b-1)=0$ for $b=0$.

One finds

$$[S,H_0] = [(2,1),(2,0)] + [(2,1),(0,2)] = (2,1),$$

so the term arising from H_0 in first order has the same form as the one arising from H_{int} in zeroth order. These terms are used to eliminate the interaction in $H_{e\text{-}ph}$.

Similarly, for H_{int},

$$[S,H_{int}] = [(2,1),(2,1)] = (2,2) + (4,0).$$

Fröhlich additionally assumed that introduction of experimentally measured velocity of sound s allows to discard all terms containing $b_{\mathbf{w}}^2$, $(b_{\mathbf{w}}^*)^2$, $b_{\mathbf{w}}b_{\mathbf{v}}$, $b_{\mathbf{w}}^*b_{\mathbf{v}}$ if $\mathbf{w}\neq\mathbf{v}$, etc. In other words, all (f,b) terms for $b\geqslant 3$ and part of $(f,2)$ terms are neglected. As a consequence, the expansion (A1) up to second order reduces to[2]

$$[S,H_0] = \sum_{\mathbf{kq}} b_{\mathbf{q}}(\varepsilon_{\mathbf{k}} - \varepsilon_{\mathbf{k-q}} - \omega_{\mathbf{q}})\phi(\mathbf{k},\mathbf{q})a_{\mathbf{k}}^*a_{\mathbf{k-q}} + \text{c.c.}, \tag{A4}$$

$$[S,H_{int}] = \sum_{\mathbf{q}} iD_{\mathbf{q}}\gamma_{\mathbf{q}}\rho_{\mathbf{q}} + \sum_{\mathbf{kq}} iD_{\mathbf{q}}b_{\mathbf{q}}^*b_{\mathbf{q}}\phi(\mathbf{k},\mathbf{q})(n_{\mathbf{k}} - n_{\mathbf{k-q}}) + \text{c.c.}, \tag{A5}$$

$$\begin{aligned}[S,[S,H_0]] = &\sum_{\mathbf{kq}} (\varepsilon_{\mathbf{k-q}} - \varepsilon_{\mathbf{k}} + \omega_{\mathbf{q}})b_{\mathbf{q}}^*b_{\mathbf{q}}|\phi(\mathbf{k},\mathbf{q})|^2(n_{\mathbf{k}} - n_{\mathbf{k-q}}) \\ &+ \sum_{\mathbf{kq}} (\varepsilon_{\mathbf{k-q}} - \varepsilon_{\mathbf{k}} + \omega_{\mathbf{q}})\phi(\mathbf{k},\mathbf{q})a_{\mathbf{k}}^*a_{\mathbf{k-q}}\gamma_{\mathbf{q}}^* + \text{c.c.}\end{aligned} \tag{A6}$$

ϕ is now adjusted so as to minimize the contribution of terms (2,1). This is achieved by the choice[2]

$$\phi(\mathbf{k},\mathbf{q}) = \frac{-iD_{\mathbf{q}}}{\varepsilon_{\mathbf{k-q}} - \varepsilon_{\mathbf{k}} + \omega_{\mathbf{q}}}[1 - \Delta(\mathbf{k},\mathbf{q})], \tag{A7}$$

where

$$\Delta(\mathbf{k},\mathbf{q}) = \begin{cases} 1, & \text{if } |\varepsilon_{\mathbf{k-q}} - \varepsilon_{\mathbf{k}} + \omega_{\mathbf{q}}| \leqslant \Gamma_{\mathbf{q}}, \\ 0, & \text{if } |\varepsilon_{\mathbf{k-q}} - \varepsilon_{\mathbf{k}} + \omega_{\mathbf{q}}| > \Gamma_{\mathbf{q}}. \end{cases} \tag{A8}$$

$\Delta(\mathbf{k},\mathbf{q})$ is introduced to avoid divergence of the series (A1) and $\Gamma_{\mathbf{q}}$ is positive energy choosing for convergence.

Equations (A4) to (A7) yield

$$\begin{aligned}H = &\sum_{\mathbf{k}} \varepsilon_{\mathbf{k}}a_{\mathbf{k}}^*a_{\mathbf{k}} + \sum_{\mathbf{w}} \omega_{\mathbf{w}}b_{\mathbf{w}}^*b_{\mathbf{w}} + \sum_{\mathbf{wk}} b_{\mathbf{w}}^*b_{\mathbf{w}}(n_{\mathbf{k}} - n_{\mathbf{k-w}}) \\ &\times\left(\frac{1}{2}(\varepsilon_{\mathbf{k-w}} - \varepsilon_{\mathbf{k}} + \omega_{\mathbf{w}})|\phi(\mathbf{k},\mathbf{w})|^2 - iD_{\mathbf{w}}\phi(\mathbf{k},\mathbf{w}) + \text{c.c.}\right) \\ &+ i\sum_{\mathbf{kw}} D_{\mathbf{w}}(b_{\mathbf{w}}a_{\mathbf{k}}^*a_{\mathbf{k-w}} - b_{\mathbf{w}}^*a_{\mathbf{k-w}}^*a_{\mathbf{k}})\Delta(\mathbf{k},\mathbf{w}) \\ &- \frac{1}{2}\sum_{\mathbf{kqw}} \frac{D_{\mathbf{w}}^2[1 + \Delta(\mathbf{k},\mathbf{w})][1 - \Delta(\mathbf{q},\mathbf{w})]}{\varepsilon_{\mathbf{q-w}} - \varepsilon_{\mathbf{q}} + \omega_{\mathbf{w}}} \\ &\times(a_{\mathbf{k}}^*a_{\mathbf{k-w}}a_{\mathbf{q-w}}^*a_{\mathbf{q}} + \text{c.c.}).\end{aligned} \tag{A9}$$

Discarding the nontransformed part of the initial interaction (fourth sum on the RHS) and taking the average in phonon vacuum, one obtains the Fröhlich Hamiltonian

$$\begin{aligned}H_{\mathrm{F}} = &\sum_{\mathbf{k}} \varepsilon_{\mathbf{k}}n_{\mathbf{k}} \\ &- \frac{1}{2}\sum_{\mathbf{kqw}} \frac{D_{\mathbf{w}}^2[1 + \Delta(\mathbf{k},\mathbf{w})][1 - \Delta(\mathbf{q},\mathbf{w})]}{\varepsilon_{\mathbf{q-w}} - \varepsilon_{\mathbf{q}} + \omega_{\mathbf{w}}}(a_{\mathbf{k}}^*a_{\mathbf{k-w}}a_{\mathbf{q-w}}^*a_{\mathbf{q}} \\ &+ \text{c.c.}).\end{aligned} \tag{A10}$$

The second term represents an effective interaction between electrons dressed in the phonon field. If $\varepsilon_{\mathbf{q-w}} - \varepsilon_{\mathbf{q}} + \omega_{\mathbf{w}} > 0$, this interaction is attractive.

APPENDIX B. CALCULATION OF $g_{\mathbf{kk}}$

Our objective is to find the explicit form of the following expression:

$$g_{\mathbf{kk}} = -16\omega_{\mathbf{k}}|\Lambda_{\mathbf{kk}}|^2 + 4\,\mathrm{Re}\,\Theta_{\mathbf{kk}}. \tag{B1}$$

Taking into account Eq. (52), we have

$$\begin{aligned}\Theta_{\mathbf{kk}} = &\omega_{\mathbf{k}}[\chi^*(\mathbf{k},\mathbf{k},2\mathbf{k},2\mathbf{k}) + \varphi^*(-\mathbf{k},-\mathbf{k},0,2\mathbf{k})] \\ &\times[\chi(3\mathbf{k},\mathbf{k},2\mathbf{k},2\mathbf{k}) + \chi(\mathbf{k},\mathbf{k},0,2\mathbf{k}) + \varphi(\mathbf{k},-\mathbf{k},0,2\mathbf{k}) \\ &+ \varphi(-\mathbf{k},-\mathbf{k},-2\mathbf{k},2\mathbf{k})] = \omega_{\mathbf{k}}\chi^*(\mathbf{k},\mathbf{k},2\mathbf{k},2\mathbf{k}) \\ &\times[\chi(3\mathbf{k},\mathbf{k},2\mathbf{k},2\mathbf{k}) + \varphi(-\mathbf{k},-\mathbf{k},-2\mathbf{k},2\mathbf{k})],\end{aligned} \tag{B2}$$

because $\varphi^*(-\mathbf{k},-\mathbf{k},0,2\mathbf{k}) = \chi(\mathbf{k},\mathbf{k},0,2\mathbf{k}) = \varphi(\mathbf{k},-\mathbf{k},0,2\mathbf{k}) = 0$, as can be seen from the explicit form, e.g., of the first of these functions [see Eq. (33)]:

$$\begin{aligned}\varphi^*(-\mathbf{k},-\mathbf{k},0,2\mathbf{k}) = &\frac{-iD_0^2D_{2\mathbf{k}}}{6\omega_{\mathbf{k}}^2\omega_0} \\ &\times[1 - \hat{\Delta}(-\mathbf{k},-\mathbf{k},0,2\mathbf{k})][1 - \Delta(\mathbf{k},2\mathbf{k})] \\ &\times\{[1 - \Delta(-\mathbf{k},0)][-4 + \Delta(\mathbf{k},0)] \\ &+ [1 - \Delta(\mathbf{k},0)][4 - \Delta(-\mathbf{k},0)]\} = 0,\end{aligned}$$

because $\Delta(-\mathbf{k},0) = \Delta(\mathbf{k},0) = 1$, for all $\mathbf{k}$ [see Eq. (A8)], similarly for $\chi(\mathbf{k},\mathbf{k},0,2\mathbf{k})$ and $\varphi(\mathbf{k},-\mathbf{k},0,2\mathbf{k})$.

Similarly, from Eq. (51), we get

$$\Lambda_{\mathbf{kk}} = \chi^*(3\mathbf{k},\mathbf{k},2\mathbf{k},4\mathbf{k}) + \varphi^*(-\mathbf{k},-3\mathbf{k},-2\mathbf{k},4\mathbf{k}). \tag{B3}$$

On the grounds of Eq. (30), we obtain

$$\chi^*(\mathbf{k},\mathbf{k},2\mathbf{k},2\mathbf{k}) = \frac{-i\sqrt{2}D_{\mathbf{k}}^3}{12\omega_k}[1-\tilde{\Delta}(\mathbf{k},\mathbf{k},2\mathbf{k},2\mathbf{k})] \times \Bigg\{ \frac{3[1-\Delta(\mathbf{k},2\mathbf{k})]}{\omega_{\mathbf{k}}^2} - \frac{[1-\Delta(\mathbf{k},2\mathbf{k})][1-\Delta(-\mathbf{k},2\mathbf{k})]}{2\omega_{\mathbf{k}}(4\varepsilon_{\mathbf{k}}+\omega_{\mathbf{k}})} \times[2+\Delta(-\mathbf{k},2\mathbf{k})] - \frac{1-\Delta(-\mathbf{k},2\mathbf{k})}{2(4\varepsilon_{\mathbf{k}}+\omega_{\mathbf{k}})^2} \times[4-\Delta(\mathbf{k},2\mathbf{k})]\Bigg\}. \tag{B4}$$

The functions $\tilde{\Delta}$ and Δ are equal 0 or 1, depending on their argument [see Eqs. (31) and (A8)], so we obtain several conditions. The most important one is

$$\tilde{\Delta}(\mathbf{k},\mathbf{k},2\mathbf{k},2\mathbf{k}) = 0 \Leftrightarrow \omega_{\mathbf{k}} > \tilde{\Gamma}_{\mathbf{k}} \Leftrightarrow b > c \tag{B5}$$

and $\chi^*(\mathbf{k},\mathbf{k},2\mathbf{k},2\mathbf{k})=0$ otherwise. Other conditions are not so strong, because they do not destroy all of parts of Eq. (B4). We have

$$\Delta(\mathbf{k},2\mathbf{k}) = 0 \Leftrightarrow \omega_{\mathbf{k}} > \Gamma_{\mathbf{k}} \Leftrightarrow b > d, \tag{B6}$$

$$\Delta(-\mathbf{k},2\mathbf{k}) = 0 \Leftrightarrow |4\varepsilon_{\mathbf{k}}+\omega_{\mathbf{k}}| > \Gamma_{\mathbf{k}} \Leftrightarrow k > \frac{d-b}{4a}. \tag{B7}$$

Taking into account Eqs. (B5)–(B7) we get the following.

If $b \leqslant c$, then $\chi^*(\mathbf{k},\mathbf{k},2\mathbf{k},2\mathbf{k})=0$.

If $b>c$ and $b>d$, then

$$\chi^*(\mathbf{k},\mathbf{k},2\mathbf{k},2\mathbf{k}) = \frac{-i\sqrt{2}D_{\mathbf{k}}^3\varepsilon_{\mathbf{k}}(5\omega_{\mathbf{k}}+12\varepsilon_{\mathbf{k}})}{3\omega_{\mathbf{k}}^3(\omega_{\mathbf{k}}+4\varepsilon_{\mathbf{k}})^2}.$$

If $b>c$, $b\leqslant d$, and $k\in\left(0,\frac{d-b}{4a}\right]$, then $\chi^*(\mathbf{k},\mathbf{k},2\mathbf{k},2\mathbf{k})=0$.

If $b>c$, $b\leqslant d$, and $k\in\left(\frac{d-b}{4a},\infty\right)$, then

$$\chi^*(\mathbf{k},\mathbf{k},2\mathbf{k},2\mathbf{k}) = \frac{i\sqrt{2}D_{\mathbf{k}}^3}{8\omega_{\mathbf{k}}(\omega_{\mathbf{k}}+4\varepsilon_{\mathbf{k}})^2}.$$

Proceeding similarly, we get the form of other functions occurring in Eqs. (B2) and (B3). For $\chi(3\mathbf{k},\mathbf{k},2\mathbf{k},2\mathbf{k})$, we have the following.

If $b\leqslant c$, then $\chi(3\mathbf{k},\mathbf{k},2\mathbf{k},2\mathbf{k})=0$.

If $b>c$, $b>d$, and $k\in\left(0,\frac{b-d}{4a}\right)\cup\left(\frac{b+d}{4a},\infty\right)$, then

$$\chi(3\mathbf{k},\mathbf{k},2\mathbf{k},2\mathbf{k}) = \frac{i\sqrt{2}D_{\mathbf{k}}^3\varepsilon_{\mathbf{k}}(5\omega_{\mathbf{k}}-12\varepsilon_{\mathbf{k}})}{3\omega_{\mathbf{k}}^3(\omega_{\mathbf{k}}-4\varepsilon_{\mathbf{k}})^2}.$$

If $b>c$, $b>d$, and $k\in\left[\frac{b-d}{4a},\frac{b+d}{4a}\right]$, then

$$\chi(3\mathbf{k},\mathbf{k},2\mathbf{k},2\mathbf{k}) = -\frac{i\sqrt{2}D_{\mathbf{k}}^3}{4\omega_{\mathbf{k}}^3}.$$

If $b>c$, $b\leqslant d$, and $k\in\left[0,\frac{b+d}{4a}\right]$, then $\chi(3\mathbf{k},\mathbf{k},2\mathbf{k},2\mathbf{k})=0$.

If $b>c$, $b\leqslant d$, and $k\in\left(\frac{b+d}{4a},\infty\right)$, then

$$\chi(3\mathbf{k},\mathbf{k},2\mathbf{k},2\mathbf{k}) = \frac{i\sqrt{2}D_{\mathbf{k}}^3}{8\omega_{\mathbf{k}}(\omega_{\mathbf{k}}-4\varepsilon_{\mathbf{k}})^2}.$$

For $\varphi(-\mathbf{k},-\mathbf{k},-2\mathbf{k},2\mathbf{k})$, we have the following.

If $b\leqslant e$, then $\varphi(-\mathbf{k},-\mathbf{k},-2\mathbf{k},2\mathbf{k})=0$.

If $b>e$, $b>d$, and $k\in\left(0,\frac{b-d}{4a}\right)\cup\left(\frac{b+d}{4a},\infty\right)$, then

$$\varphi(-\mathbf{k},-\mathbf{k},-2\mathbf{k},2\mathbf{k}) = \frac{i\sqrt{2}D_{\mathbf{k}}^3\varepsilon_{\mathbf{k}}(\omega_{\mathbf{k}}+12\varepsilon_{\mathbf{k}})}{3\omega_{\mathbf{k}}^3(\omega_{\mathbf{k}}^2-16\varepsilon_{\mathbf{k}}^2)}.$$

If $b>e$, $b>d$, and $k\in\left[\frac{b-d}{4a},\frac{b+d}{4a}\right]$, then

$$\varphi(-\mathbf{k},-\mathbf{k},-2\mathbf{k},2\mathbf{k}) = -\frac{i\sqrt{2}D_{\mathbf{k}}^3}{4\omega_{\mathbf{k}}^3}.$$

If $b>e$, $b\leqslant d$, and $k\in\left[0,\frac{b+d}{4a}\right]$, then $\varphi(-\mathbf{k},-\mathbf{k},-2\mathbf{k},2\mathbf{k})=0$.

If $b>e$, $b\leqslant d$, and $k\in\left[\frac{b+d}{4a},\infty\right]$, then

$$\varphi(-\mathbf{k},-\mathbf{k},-2\mathbf{k},2\mathbf{k}) = \frac{i\sqrt{2}D_{\mathbf{k}}^3}{8\omega_{\mathbf{k}}(\omega_{\mathbf{k}}^2-16\varepsilon_{\mathbf{k}}^2)}.$$

For $\chi^*(3\mathbf{k},\mathbf{k},2\mathbf{k},4\mathbf{k})$, we have the following.

If $b\leqslant c$, then $\chi^*(3\mathbf{k},\mathbf{k},2\mathbf{k},4\mathbf{k})=0$.

If $b>c$, $b<d/3$, and $k\in\left(0,\frac{d-b}{2a}\right]$, then $\chi^*(3\mathbf{k},\mathbf{k},2\mathbf{k},4\mathbf{k})=0$.

If $b>c$, $b<d/3$, and $k\in\left(\frac{d-b}{2a},\frac{d+b}{2a}\right)$, then

$$\chi^*(3\mathbf{k},\mathbf{k},2\mathbf{k},4\mathbf{k}) = \frac{iD_{\mathbf{k}}^3}{16\omega_{\mathbf{k}}(\omega_{\mathbf{k}}+2\varepsilon_{\mathbf{k}})(\omega_{\mathbf{k}}+4\varepsilon_{\mathbf{k}})} \coloneqq \chi^{(1)}.$$

If $b>c$, $b<d/3$, and $k\in\left(\frac{d+b}{2a},\infty\right)$, then

$$\chi^*(3\mathbf{k},\mathbf{k},2\mathbf{k},4\mathbf{k}) = \frac{-3iD_{\mathbf{k}}^3\varepsilon_{\mathbf{k}}}{4(\omega_{\mathbf{k}}^2-16\varepsilon_{\mathbf{k}}^2)(\omega_{\mathbf{k}}^2-4\varepsilon_{\mathbf{k}}^2)} \coloneqq \chi^{(2)}.$$

If $b>c$, $d/3\leqslant b\leqslant d$, and $k\in\left(0,\frac{d-b}{2a}\right]$, then $\chi^*(3\mathbf{k},\mathbf{k},2\mathbf{k},4\mathbf{k})=0$.

If $b>c$, $d/3\leqslant b\leqslant d$, and $k\in\left(\frac{d-b}{2a},\frac{d+b}{2a}\right]$, then $\chi^*(3\mathbf{k},\mathbf{k},2\mathbf{k},4\mathbf{k})=\chi^{(1)}$.

If $b>c$, $d/3\leqslant b\leqslant d$, and $k\in\left(\frac{d+b}{2a},\infty\right)$, then $\chi^*(3\mathbf{k},\mathbf{k},2\mathbf{k},4\mathbf{k})=\chi^{(2)}$.

If $b>c$, $d<b\leqslant 3d$, and $k\in\left(0,\frac{b-d}{4a}\right)\cup\left(\frac{b+d}{2a},\infty\right)$, then

$$\chi^*(3\mathbf{k},\mathbf{k},2\mathbf{k},4\mathbf{k}) = \frac{-iD_{\mathbf{k}}^3\varepsilon_{\mathbf{k}}(7\omega_{\mathbf{k}}^2-16\varepsilon_{\mathbf{k}}^2)}{6\omega_{\mathbf{k}}^2(\omega_{\mathbf{k}}^2-16\varepsilon_{\mathbf{k}}^2)(\omega_{\mathbf{k}}^2-4\varepsilon_{\mathbf{k}}^2)} \coloneqq \chi^{(3)}.$$

If $b>c$, $d<b\leqslant 3d$, and $k\in\left[\frac{b-d}{4a},\frac{b-d}{2a}\right)$, then

$$\chi^*(3\mathbf{k},\mathbf{k},2\mathbf{k},4\mathbf{k}) = \frac{-iD_{\mathbf{k}}^3(-3\omega_{\mathbf{k}}^2+40\varepsilon_{\mathbf{k}}^2+22\omega_{\mathbf{k}}\varepsilon_{\mathbf{k}})}{48\omega_{\mathbf{k}}^2(\omega_{\mathbf{k}}^2-4\varepsilon_{\mathbf{k}}^2)(\omega_{\mathbf{k}}+4\varepsilon_{\mathbf{k}})} \coloneqq \chi^{(4)}.$$

If $b>c$, $d<b\leqslant 3d$, and $k\in\left[\frac{b-d}{2a},\frac{b+d}{2a}\right]$, then

$$\chi^*(3\mathbf{k},\mathbf{k},2\mathbf{k},4\mathbf{k}) = \frac{iD_{\mathbf{k}}^3(3\omega_{\mathbf{k}}+4\varepsilon_{\mathbf{k}})}{24\omega_{\mathbf{k}}^2(\omega_{\mathbf{k}}+2\varepsilon_{\mathbf{k}})(\omega_{\mathbf{k}}+4\varepsilon_{\mathbf{k}})} \coloneqq \chi^{(5)}.$$

If $b>c$, $b>3d$, and $k\in\left(0,\frac{b-d}{4a}\right)\cup\left(\frac{b+d}{4a},\frac{b-d}{2a}\right)\cup\left(\frac{b+d}{2a},\infty\right)$, then $\chi^*(3\mathbf{k},\mathbf{k},2\mathbf{k},4\mathbf{k})=\chi^{(3)}$.

If $b>c$, $b>3d$, and $k\in\left[\frac{b-d}{4a},\frac{b+d}{4a}\right]$, then $\chi^*(3\mathbf{k},\mathbf{k},2\mathbf{k},4\mathbf{k})=\chi^{(4)}$.

If $b>c$, $b>3d$, and $k\in\left[\frac{b-d}{2a},\frac{b+d}{2a}\right]$, then $\chi^*(3\mathbf{k},\mathbf{k},2\mathbf{k},4\mathbf{k})=\chi^{(5)}$.

For $\varphi^*(-\mathbf{k},-3\mathbf{k},-2\mathbf{k},4\mathbf{k})$, we have the following.

If $b \le e$, then $\varphi^*(-\mathbf{k},-3\mathbf{k},-2\mathbf{k},4\mathbf{k})=0$.

If $b>e$, $b<d/3$, and $k\in\left(0,\frac{d-b}{2a}\right]$, then $\varphi^*(-\mathbf{k},-3\mathbf{k},-2\mathbf{k},4\mathbf{k})=0$.

If $b>e$, $b<d/3$, and $k\in\left(\frac{d-b}{2a},\frac{d+b}{2a}\right]$, then

$$\varphi^*(-\mathbf{k},-3\mathbf{k},-2\mathbf{k},4\mathbf{k})=\frac{iD_{\mathbf{k}}^3}{16\omega_{\mathbf{k}}(\omega_{\mathbf{k}}+2\varepsilon_{\mathbf{k}})(\omega_{\mathbf{k}}-4\varepsilon_{\mathbf{k}})}:=\varphi^{(1)}.$$

If $b>e$, $b<d/3$, and $k\in\left(\frac{d+b}{2a},\infty\right)$, then

$$\varphi^*(-\mathbf{k},-3\mathbf{k},-2\mathbf{k},4\mathbf{k})=\frac{iD_{\mathbf{k}}^3\varepsilon_{\mathbf{k}}}{4(\omega_{\mathbf{k}}^2-16\varepsilon_{\mathbf{k}}^2)(\omega_{\mathbf{k}}^2-4\varepsilon_{\mathbf{k}}^2)}:=\varphi^{(2)}.$$

If $b>e$, $d/3\le b\le d$, and $k\in\left(0,\frac{d+b}{4a}\right]$, then $\varphi^*(-\mathbf{k},-3\mathbf{k},-2\mathbf{k},4\mathbf{k})=0$.

If $b>e$, $d/3\le b\le d$, and $k\in\left(\frac{d+b}{4a},\frac{d+b}{2a}\right]$, then $\varphi^*(-\mathbf{k},-3\mathbf{k},-2\mathbf{k},4\mathbf{k})=\varphi^{(1)}$.

If $b>e$, $d/3\le b\le d$, and $k\in\left(\frac{d+b}{2a},\infty\right)$, then $\varphi^*(-\mathbf{k},-3\mathbf{k},-2\mathbf{k},4\mathbf{k})=\varphi^{(2)}$.

If $b>e$, $d<b\le 3d$, and $k\in\left(0,\frac{b-d}{4a}\right)\cup\left[\frac{b+d}{2a},\infty\right)$, then

$$\varphi^*(-\mathbf{k},-3\mathbf{k},-2\mathbf{k},4\mathbf{k})=\frac{iD_{\mathbf{k}}^3\varepsilon_{\mathbf{k}}(\omega_{\mathbf{k}}^2+16\varepsilon_{\mathbf{k}}^2)}{6\omega_{\mathbf{k}}^2(\omega_{\mathbf{k}}^2-4\varepsilon_{\mathbf{k}}^2)(\omega_{\mathbf{k}}^2-16\varepsilon_{\mathbf{k}}^2)}:=\varphi^{(3)}.$$

If $b>e$, $d<b\le 3d$, and $k\in\left[\frac{b-d}{4a},\frac{b-d}{2a}\right)$, then

$$\varphi^*(-\mathbf{k},-3\mathbf{k},-2\mathbf{k},4\mathbf{k})=\frac{-iD_{\mathbf{k}}^3(3\omega_{\mathbf{k}}^2+14\varepsilon_{\mathbf{k}}\omega_{\mathbf{k}}+40\varepsilon_{\mathbf{k}}^2)}{48\omega_{\mathbf{k}}^2(\omega_{\mathbf{k}}^2-4\varepsilon_{\mathbf{k}}^2)(\omega_{\mathbf{k}}+4\varepsilon_{\mathbf{k}})}:=\varphi^{(4)}.$$

If $b>e$, $d<b\le 3d$, and $k\in\left[\frac{b-d}{2a},\frac{b+d}{4a}\right]$, then

$$\varphi^*(-\mathbf{k},-3\mathbf{k},-2\mathbf{k},4\mathbf{k})=\frac{iD_{\mathbf{k}}^3}{16\omega_{\mathbf{k}}^2(\omega_{\mathbf{k}}+2\varepsilon_{\mathbf{k}})}.$$

If $b>e$, $d<b\le 3d$, and $k\in\left(\frac{b+d}{4a},\frac{b+d}{2a}\right)$, then

$$\varphi^*(-\mathbf{k},-3\mathbf{k},-2\mathbf{k},4\mathbf{k})=\frac{iD_{\mathbf{k}}^3(3\omega_{\mathbf{k}}-4\varepsilon_{\mathbf{k}})}{24\omega_{\mathbf{k}}^2(\omega_{\mathbf{k}}+2\varepsilon_{\mathbf{k}})(\omega_{\mathbf{k}}-4\varepsilon_{\mathbf{k}})}:=\varphi^{(5)}.$$

If $b>e$, $3d<b$, and $k\in\left(0,\frac{b-d}{4a}\right)\cup\left(\frac{b+d}{4a},\frac{b-d}{2a}\right)\cup\left(\frac{b+d}{2a},\infty\right)$, then $\varphi^*(-\mathbf{k},-3\mathbf{k},-2\mathbf{k},4\mathbf{k})=\varphi^{(3)}$.

If $b>e$, $3d<b$, and $k\in\left[\frac{b-d}{4a},\frac{b+d}{4a}\right]$, then $\varphi^*(-\mathbf{k},-3\mathbf{k},-2\mathbf{k},4\mathbf{k})=\varphi^{(4)}$.

If $b>e$, $3d<b$, and $k\in\left[\frac{b-d}{2a},\frac{b+d}{2a}\right]$, then $\varphi^*(-\mathbf{k},-3\mathbf{k},-2\mathbf{k},4\mathbf{k})=\varphi^{(5)}$.

On the basis of this results and Eq. (B1), we get the four-fermion interaction coupling $g_{\mathbf{kk}}$:

If $b\le c$ and $b\le e$, then $g_{\mathbf{kk}}=0$.

If $b\le c$, $b>e$, $b<d/3$, and $k\in\left(0,\frac{d-b}{2a}\right]$, then $g_{\mathbf{kk}}=0$.

If $b\le c$, $b>e$, $b<d/3$, and $k\in\left(\frac{d-b}{2a},\frac{d+b}{2a}\right]$, then

$$g_{\mathbf{kk}}=\frac{-D_{\mathbf{k}}^6}{16\omega_{\mathbf{k}}(\omega_{\mathbf{k}}+2\varepsilon_{\mathbf{k}})^2(\omega_{\mathbf{k}}-4\varepsilon_{\mathbf{k}})^2}:=g_{\mathbf{kk}}^{(1)}<0.$$

If $b\le c$, $b>e$, $b<d/3$, and $k\in\left(\frac{d+b}{2a},\infty\right)$, then

$$g_{\mathbf{kk}}=\frac{-D_{\mathbf{k}}^6\omega_{\mathbf{k}}\varepsilon_{\mathbf{k}}^2}{(\omega_{\mathbf{k}}^2-4\varepsilon_{\mathbf{k}}^2)^2(\omega_{\mathbf{k}}^2-16\varepsilon_{\mathbf{k}}^2)^2}:=g_{\mathbf{kk}}^{(2)}<0.$$

If $b\le c$, $b>e$, $d/3\le b\le d$, and $k\in\left(0,\frac{d+b}{4a}\right]$, then $g_{\mathbf{kk}}=0$.

If $b\le c$, $b>e$, $d/3\le b\le d$, and $k\in\left(\frac{d+b}{4a},\frac{d+b}{2a}\right]$, then $g_{\mathbf{kk}}=g_{\mathbf{kk}}^{(1)}$.

If $b\le c$, $b>e$, $d/3\le b\le d$, and $k\in\left(\frac{d+b}{2a},\infty\right)$, then $g_{\mathbf{kk}}=g_{\mathbf{kk}}^{(2)}$.

If $b\le c$, $b>e$, $d<b\le 3d$, and $k\in\left(0,\frac{b-d}{4a}\right)\cup\left[\frac{b+d}{2a},\infty\right)$, then

$$g_{\mathbf{kk}}=-\frac{4D_{\mathbf{k}}^6\varepsilon_{\mathbf{k}}^2(\omega_{\mathbf{k}}^2+16\varepsilon_{\mathbf{k}}^2)^2}{9\omega_{\mathbf{k}}^3(\omega_{\mathbf{k}}^2-4\varepsilon_{\mathbf{k}}^2)^2(\omega_{\mathbf{k}}^2-16\varepsilon_{\mathbf{k}}^2)^2}:=g_{\mathbf{kk}}^{(3)}<0.$$

If $b\le c$, $b>e$, $d<b\le 3d$, and $k\in\left[\frac{b-d}{4a},\frac{b-d}{2a}\right)$, then

$$g_{\mathbf{kk}}=\frac{-D_{\mathbf{k}}^6(3\omega_{\mathbf{k}}^2+14\varepsilon_{\mathbf{k}}\omega_{\mathbf{k}}+40\varepsilon_{\mathbf{k}}^2)^2}{144\omega_{\mathbf{k}}^3(\omega_{\mathbf{k}}^2-4\varepsilon_{\mathbf{k}}^2)^2(\omega_{\mathbf{k}}+4\varepsilon_{\mathbf{k}})^2}:=g_{\mathbf{kk}}^{(4)}<0.$$

If $b\le c$, $b>e$, $d<b\le 3d$, and $k\in\left[\frac{b-d}{2a},\frac{b+d}{4a}\right]$, then

$$g_{\mathbf{kk}}=-\frac{D_{\mathbf{k}}^6}{16\omega_{\mathbf{k}}^3(\omega_{\mathbf{k}}+2\varepsilon_{\mathbf{k}})^2}<0.$$

If $b\le c$, $b>e$, $d<b\le 3d$, and $k\in\left(\frac{b+d}{4a},\frac{b+d}{2a}\right)$, then

$$g_{\mathbf{kk}}=\frac{-D_{\mathbf{k}}^6(3\omega_{\mathbf{k}}-4\varepsilon_{\mathbf{k}})^2}{36\omega_{\mathbf{k}}^3(\omega_{\mathbf{k}}+2\varepsilon_{\mathbf{k}})^2(\omega_{\mathbf{k}}-4\varepsilon_{\mathbf{k}})^2}:=g_{\mathbf{kk}}^{(5)}<0.$$

If $b\le c$, $b>e$, $3d<b$, and $k\in\left(0,\frac{b-d}{4a}\right)\cup\left(\frac{b+d}{4a},\frac{b-d}{2a}\right)\cup\left(\frac{b+d}{2a},\infty\right)$, then $g_{\mathbf{kk}}=g_{\mathbf{kk}}^{(3)}$.

If $b\le c$, $b>e$, $3d<b$, and $k\in\left[\frac{b-d}{4a},\frac{b+d}{4a}\right]$, then $g_{\mathbf{kk}}=g_{\mathbf{kk}}^{(4)}$.

If $b\le c$, $b>e$, $3d<b$, and $k\in\left[\frac{b-d}{2a},\frac{b+d}{2a}\right]$, then $g_{\mathbf{kk}}=g_{\mathbf{kk}}^{(5)}$.

If $b>c$, $b\le e$, $b<d/3$, and $k\in\left(0,\frac{b+d}{4a}\right]$, then $g_{\mathbf{kk}}=0$.

If $b>c$, $b\le e$, $b<d/3$, and $k\in\left(\frac{b+d}{4a},\frac{d-b}{2a}\right]$, then

$$g_{\mathbf{kk}}=\frac{-D_{\mathbf{k}}^6}{8\omega_{\mathbf{k}}(\omega_{\mathbf{k}}^2-16\varepsilon_{\mathbf{k}}^2)^2}<0.$$

If $b>c$, $b\le e$, $b<d/3$, and $k\in\left(\frac{d-b}{2a},\frac{b+d}{2a}\right]$, then

$$g_{\mathbf{kk}}=\frac{-3D_{\mathbf{k}}^6(\omega_{\mathbf{k}}^2+8\varepsilon_{\mathbf{k}}^2)}{16\omega_{\mathbf{k}}(\omega_{\mathbf{k}}+2\varepsilon_{\mathbf{k}})^2(\omega_{\mathbf{k}}^2-16\varepsilon_{\mathbf{k}}^2)^2}:=g_{\mathbf{kk}}^{(6)}<0.$$

If $b>c$, $b\le e$, $b<d/3$, and $k\in\left(\frac{d+b}{2a},\infty\right)$, then

$$g_{\mathbf{kk}}=\frac{-D_{\mathbf{k}}^6(\omega_{\mathbf{k}}^4+64\omega_{\mathbf{k}}^2\varepsilon_{\mathbf{k}}^2+16\varepsilon_{\mathbf{k}}^4)}{8\omega_{\mathbf{k}}(\omega_{\mathbf{k}}^2-16\varepsilon_{\mathbf{k}}^2)^2(\omega_{\mathbf{k}}^2-4\varepsilon_{\mathbf{k}}^2)^2}:=g_{\mathbf{kk}}^{(7)}<0.$$

If $b>c$, $b\le e$, $d/3\le b\le d$, and $k\in\left(0,\frac{d-b}{2a}\right]$, then $g_{\mathbf{kk}}=0$.

If $b>c$, $b\le e$, $d/3\le b\le d$, and $k\in\left(\frac{d-b}{2a},\frac{d+b}{4a}\right]$, then

$$g_{\mathbf{kk}}=\frac{-D_{\mathbf{k}}^6}{16\omega_{\mathbf{k}}(\omega_{\mathbf{k}}+2\varepsilon_{\mathbf{k}})^2(\omega_{\mathbf{k}}+4\varepsilon_{\mathbf{k}})^2}:=g_{\mathbf{kk}}^{(8)}<0.$$

If $b>c$, $b\le e$, $d/3\le b\le d$, and $k\in\left(\frac{d+b}{4a},\frac{d+b}{2a}\right]$, then $g_{\mathbf{kk}}=g_{\mathbf{kk}}^{(6)}$.

If $b>c$, $b\le e$, $d/3\le b\le d$, and $k\in\left(\frac{d+b}{2a},\infty\right)$, then $g_{\mathbf{kk}}=g_{\mathbf{kk}}^{(7)}$.

If $b>c$, $b\le e$, $d<b\le 3d$, and $k\in\left(0,\frac{b-d}{4a}\right)\cup\left(\frac{b+d}{2a},\infty\right)$, then

$$g_{\mathbf{kk}}=-\frac{4D_{\mathbf{k}}^6\varepsilon_{\mathbf{k}}^2(-\omega_{\mathbf{k}}^6+464\omega_{\mathbf{k}}^4\varepsilon_{\mathbf{k}}^2-2848\omega_{\mathbf{k}}^2\varepsilon_{\mathbf{k}}^4+4608\varepsilon_{\mathbf{k}}^6)}{9\omega_{\mathbf{k}}^5(\omega_{\mathbf{k}}^2-4\varepsilon_{\mathbf{k}}^2)^2(\omega_{\mathbf{k}}^2-16\varepsilon_{\mathbf{k}}^2)^2}:=g_{\mathbf{kk}}^{(9)}.$$

We see that $g_{\mathbf{kk}}^{(9)}<0$ iff

$$\omega_{\mathbf{k}}^6 + 464\omega_{\mathbf{k}}^4\varepsilon_{\mathbf{k}}^2 - 2848\omega_{\mathbf{k}}^2\varepsilon_{\mathbf{k}}^4 + 4608\varepsilon_{\mathbf{k}}^6 > 0.$$

Taking into account $\omega_{\mathbf{k}}=bk$ and $\varepsilon_{\mathbf{k}}=ak^2$, we obtain a third-order algebraic inequality

$$4608l^3 - 2848xl^2 + 464x^2l - x^3 > 0,$$

for $l=a^2k^2$, $x=b^2$. This can be easily solved[26]—there exists only one real root $l_0^{(9)} \approx 0,002x$, so

$$g_{\mathbf{kk}}^{(9)} < 0 \quad \text{iff} \quad k > k_0^{(9)} \approx 0.047b/a < b/4a.$$

If $b>c$, $b\leqslant e$, $d<b\leqslant 3d$, and $k\in\left[\frac{b-d}{4a},\frac{b-d}{2a}\right)$, then

$$g_{\mathbf{kk}} = -\frac{D_{\mathbf{k}}^6(9\omega_{\mathbf{k}}^4 + 144\omega_{\mathbf{k}}^3\varepsilon_{\mathbf{k}} + 784\omega_{\mathbf{k}}^2\varepsilon_{\mathbf{k}}^2 + 1632\omega_{\mathbf{k}}\varepsilon_{\mathbf{k}}^3 + 1152\varepsilon_{\mathbf{k}}^4)}{36\omega_{\mathbf{k}}^5(\omega_{\mathbf{k}}+2\varepsilon_{\mathbf{k}})^2(\omega_{\mathbf{k}}+4\varepsilon_{\mathbf{k}})^2} < 0.$$

$$g_{\mathbf{kk}} = -\frac{D_{\mathbf{k}}^6}{144\omega_{\mathbf{k}}^5(\omega_{\mathbf{k}}^2 - 4\varepsilon_{\mathbf{k}}^2)^2(\omega_{\mathbf{k}}+4\varepsilon_{\mathbf{k}})^2} \times (9\omega_{\mathbf{k}}^6 + 348\omega_{\mathbf{k}}^5\varepsilon_{\mathbf{k}} + 1396\omega_{\mathbf{k}}^4\varepsilon_{\mathbf{k}}^2 - 2080\omega_{\mathbf{k}}^3\varepsilon_{\mathbf{k}}^3 - 7616\omega_{\mathbf{k}}^2\varepsilon_{\mathbf{k}}^4 + 7680\omega_{\mathbf{k}}\varepsilon_{\mathbf{k}}^5 + 18432\varepsilon_{\mathbf{k}}^6) \coloneqq g_{\mathbf{kk}}^{(10)}.$$

It turns out that $g_{\mathbf{kk}}^{(10)}<0$ for all $k>0$, because ($l=ak$) the equation

$$18432l^6 + 7680bl^5 - 7616b^2l^4 - 2080b^3l^3 + 1396b^4l^2 + 348b^5l + 9b^6 = 0$$

has only two real roots and both are negative: $l_1^{(10)} \approx -0.208b$ and $l_1^{(10)}=-0.029b$.

If $b>c$, $b\leqslant e$, $d<b\leqslant 3d$, and $k\in\left[\frac{b-d}{2a},\frac{b+d}{4a}\right]$, then

If $b>c$, $b\leqslant e$, $d<b\leqslant 3d$, and $k\in\left(\frac{b+d}{4a},\frac{b+d}{2a}\right]$, then

$$g_{\mathbf{kk}} = -\frac{D_{\mathbf{k}}^6}{36\omega_{\mathbf{k}}^5(\omega_{\mathbf{k}}+2\varepsilon_{\mathbf{k}})^2(\omega_{\mathbf{k}}^2 - 16\varepsilon_{\mathbf{k}}^2)^2} \times (9\omega_{\mathbf{k}}^6 - 48\omega_{\mathbf{k}}^5\varepsilon_{\mathbf{k}} - 832\omega_{\mathbf{k}}^4\varepsilon_{\mathbf{k}}^2 - 2944\omega_{\mathbf{k}}^3\varepsilon_{\mathbf{k}}^3 + 1664\omega_{\mathbf{k}}^2\varepsilon_{\mathbf{k}}^4 + 18432\omega_{\mathbf{k}}\varepsilon_{\mathbf{k}}^5 + 18432\varepsilon_{\mathbf{k}}^6) \coloneqq g_{\mathbf{kk}}^{(11)}.$$

$g_{\mathbf{kk}}^{(11)}<0$ iff $k\in(0,0.073b/a)\cup(0.412b/a,\infty)$.

If $b>c$, $b\leqslant e$, $b>3d$, and $k\in\left(0,\frac{b-d}{4a}\right)\cup\left(\frac{b+d}{4a},\frac{b-d}{2a}\right)\cup\left(\frac{b+d}{2a},\infty\right)$, then $g_{\mathbf{kk}}=g_{\mathbf{kk}}^{(9)}$.

If $b>c$, $b\leqslant e$, $b>3d$, and $k\in\left[\frac{b-d}{4a},\frac{b+d}{4a}\right]$, then $g_{\mathbf{kk}}=g_{\mathbf{kk}}^{(10)}$.

If $b>c$, $b\leqslant e$, $b>3d$, and $k\in\left[\frac{b-d}{2a},\frac{b+d}{2a}\right]$, then $g_{\mathbf{kk}}=g_{\mathbf{kk}}^{(11)}$.

If $b>c$, $b>e$, $b<d/3$, and $k\in\left(0,\frac{b+d}{4a}\right]$, then $g_{\mathbf{kk}}=0$.

If $b>c$, $b>e$, $b<d/3$, and $k\in\left(\frac{b+d}{4a},\frac{d-b}{2a}\right)$, then

$$g_{\mathbf{kk}} = -\frac{D_{\mathbf{k}}^6}{4(\omega_{\mathbf{k}}+4\varepsilon_{\mathbf{k}})^3(\omega_{\mathbf{k}}-4\varepsilon_{\mathbf{k}})^2} < 0.$$

If $b>c$, $b>e$, $b<d/3$, and $k\in\left[\frac{d-b}{2a},\frac{b+d}{2a}\right]$, then

$$g_{\mathbf{kk}} = -\frac{D_{\mathbf{k}}^6(\omega_{\mathbf{k}}^2 + 4\omega_{\mathbf{k}}\varepsilon_{\mathbf{k}} + 2\varepsilon_{\mathbf{k}}^2)}{2(\omega_{\mathbf{k}}^2 - 16\varepsilon_{\mathbf{k}}^2)^2(\omega_{\mathbf{k}}+2\varepsilon_{\mathbf{k}})^2(\omega_{\mathbf{k}}+4\varepsilon_{\mathbf{k}})} \coloneqq g_{\mathbf{kk}}^{(12)} < 0.$$

If $b>c$, $b>e$, $b<d/3$, and $k\in\left(\frac{d+b}{2a},\infty\right)$, then

$$g_{\mathbf{kk}} = -\frac{D_{\mathbf{k}}^6(\omega_{\mathbf{k}}^4 - 4\omega_{\mathbf{k}}^2\varepsilon_{\mathbf{k}}^2 + 16\omega_{\mathbf{k}}\varepsilon_{\mathbf{k}}^3 + 16\varepsilon_{\mathbf{k}}^4)}{4(\omega_{\mathbf{k}}^2 - 16\varepsilon_{\mathbf{k}}^2)^2(\omega_{\mathbf{k}}^2 - 4\varepsilon_{\mathbf{k}}^2)^2(\omega_{\mathbf{k}}+4\varepsilon_{\mathbf{k}})} \coloneqq g_{\mathbf{kk}}^{(13)}.$$

$g_{\mathbf{kk}}^{(13)}<0$ for all $k>0$.

If $b>c$, $b>e$, $d/3\leqslant b\leqslant d$, and $k\in\left(0,\frac{d-b}{2a}\right]$, then $g_{\mathbf{kk}}=0$.

If $b>c$, $b>e$, $d/3\leqslant b\leqslant d$, and $k\in\left(\frac{d-b}{2a},\frac{d+b}{4a}\right]$, then $g_{\mathbf{kk}}=g_{\mathbf{kk}}^{(8)}$.

If $b>c$, $b>e$, $d/3\leqslant b\leqslant d$, and $k\in\left(\frac{d+b}{4a},\frac{d+b}{2a}\right]$, then $g_{\mathbf{kk}}=g_{\mathbf{kk}}^{(12)}$.

If $b>c$, $b>e$, $d/3\leqslant b\leqslant d$, and $k\in\left(\frac{d+b}{2a},\infty\right)$, then $g_{\mathbf{kk}}=g_{\mathbf{kk}}^{(13)}$.

If $b>c$, $b>e$, $d<b\leqslant 3d$, and $k\in\left(0,\frac{b-d}{4a}\right)$, then

$$g_{\mathbf{kk}} = -\frac{32D_{\mathbf{k}}^6\varepsilon_{\mathbf{k}}^2}{9\omega_{\mathbf{k}}^5(\omega_{\mathbf{k}}^2 - 16\varepsilon_{\mathbf{k}}^2)^2(\omega_{\mathbf{k}}^2 - 4\varepsilon_{\mathbf{k}}^2)^2(\omega_{\mathbf{k}}+4\varepsilon_{\mathbf{k}})} \times (-3\omega_{\mathbf{k}}^7 - 20\omega_{\mathbf{k}}^6\varepsilon_{\mathbf{k}} + 84\omega_{\mathbf{k}}^5\varepsilon_{\mathbf{k}}^2 + 400\omega_{\mathbf{k}}^4\varepsilon_{\mathbf{k}}^3 - 568\omega_{\mathbf{k}}^3\varepsilon_{\mathbf{k}}^4 - 2400\omega_{\mathbf{k}}^2\varepsilon_{\mathbf{k}}^5 + 1152\omega_{\mathbf{k}}\varepsilon_{\mathbf{k}}^6 + 4608\varepsilon_{\mathbf{k}}^7) \coloneqq g_{\mathbf{kk}}^{(14)}.$$

$g_{\mathbf{kk}}^{(14)}<0$ iff $k\in(0.268b/a,\infty)$.

If $b>c$, $b>e$, $d<b\leqslant 3d$, and $k\in\left[\frac{b-d}{4a},\frac{b-d}{2a}\right)$, then

$$g_{\mathbf{kk}} = -\frac{16D_{\mathbf{k}}^6\varepsilon_{\mathbf{k}}(60\omega_{\mathbf{k}}^5 + 225\omega_{\mathbf{k}}^4\varepsilon_{\mathbf{k}} - 120\omega_{\mathbf{k}}^3\varepsilon_{\mathbf{k}}^2 - 752\omega_{\mathbf{k}}^2\varepsilon_{\mathbf{k}}^3 + 960\omega_{\mathbf{k}}\varepsilon_{\mathbf{k}}^4 + 2304\varepsilon_{\mathbf{k}}^5)}{9\omega_{\mathbf{k}}^5(\omega_{\mathbf{k}}^2 - 4\varepsilon_{\mathbf{k}}^2)^2(\omega_{\mathbf{k}}+4\varepsilon_{\mathbf{k}})^2} \coloneqq g_{\mathbf{kk}}^{(15)}.$$

$g_{\mathbf{kk}}^{(15)}<0$ for all $k>0$.

If $b>c$, $b>e$, $d<b\leqslant 3d$, and $k\in\left[\frac{b-d}{2a},\frac{b+d}{4a}\right]$, then

$$g_{\mathbf{kk}} = -\frac{D_{\mathbf{k}}^6(81\omega_{\mathbf{k}}^4 + 1320\omega_{\mathbf{k}}^3\varepsilon_{\mathbf{k}} + 6544\omega_{\mathbf{k}}^2\varepsilon_{\mathbf{k}}^2 + 13056\omega_{\mathbf{k}}\varepsilon_{\mathbf{k}}^3 + 9216\varepsilon_{\mathbf{k}}^4)}{9\omega_{\mathbf{k}}^5(\omega_{\mathbf{k}}+4\varepsilon_{\mathbf{k}})^2(\omega_{\mathbf{k}}+2\varepsilon_{\mathbf{k}})^2} < 0.$$

If $b>c$, $b>e$, $d<b\leqslant 3d$, and $k\in\left(\frac{b+d}{4a},\frac{b+d}{2a}\right]$, then

$$g_{\mathbf{kk}} = -\frac{D_{\mathbf{k}}^6}{9\omega_{\mathbf{k}}^5(\omega_{\mathbf{k}}^2-16\varepsilon_{\mathbf{k}}^2)^2(\omega_{\mathbf{k}}+4\varepsilon_{\mathbf{k}})(\omega_{\mathbf{k}}+2\varepsilon_{\mathbf{k}})^2} \times(9\omega_{\mathbf{k}}^7 + 36\omega_{\mathbf{k}}^6\varepsilon_{\mathbf{k}} - 336\omega_{\mathbf{k}}^5\varepsilon_{\mathbf{k}}^2 - 2560\omega_{\mathbf{k}}^4\varepsilon_{\mathbf{k}}^3 - 3264\omega_{\mathbf{k}}^3\varepsilon_{\mathbf{k}}^4 + 14592\omega_{\mathbf{k}}^2\varepsilon_{\mathbf{k}}^5 + 46080\omega_{\mathbf{k}}\varepsilon_{\mathbf{k}}^6 + 36864\varepsilon_{\mathbf{k}}^7) \coloneqq g_{\mathbf{kk}}^{(16)}.$$

$g_{\mathbf{kk}}^{(16)}<0$ iff $k\in(0,0.142b/a)\cup(0.345b/a,\infty)$.

If $b>c$, $b>e$, $d<b\leqslant 3d$, and $k\in\left(\frac{b+d}{2a},\infty\right)$, then $g_{\mathbf{kk}}=g_{\mathbf{kk}}^{(14)}$.

If $b>c$, $b>e$, $b>3d$, and $k\in\left(0,\frac{b-d}{4a}\right)\cup\left(\frac{b+d}{4a},\frac{b-d}{2a}\right)\cup\left(\frac{b+d}{2a},\infty\right)$, then $g_{\mathbf{kk}}=g_{\mathbf{kk}}^{(14)}$.

If $b>c$, $b>e$, $b>3d$, and $k\in\left[\frac{b-d}{4a},\frac{b+d}{4a}\right]$, then $g_{\mathbf{kk}}=g_{\mathbf{kk}}^{(15)}$.

If $b>c$, $b>e$, $b>3d$, and $k\in\left[\frac{b-d}{2a},\frac{b+d}{2a}\right]$, then $g_{\mathbf{kk}}=g_{\mathbf{kk}}^{(16)}$.

APPENDIX C: HIGHER-ORDER TERMS OF FRÖHLICH'S TRANSFORMATION

Evaluation of successive commutators of the expansion (A1) is a time-consuming task. Below the final results are given for the relevant orders.

Since $S=B-B^*$ in Eq. (7), all expansion terms have the form $g+g^*$. We focus on the first summand.

The following formulas for boson operators,

$$[b_1,b_2b_3]=[b_1,b_2]b_3+b_2[b_1,b_3],$$

$$[b_1b_2,b_3b_4]=b_1b_3[b_2,b_4]+b_1[b_2,b_3]b_4+b_3[b_1,b_4]b_2+[b_1,b_3]b_4b_2,$$

and fermion operators,

$$[f_1,f_2f_3]=\{f_1,f_2\}f_3-f_2\{f_1,f_3\},$$

$$[f_1f_2,f_3f_4]=\{f_1,f_3\}f_4f_2-f_3\{f_1,f_4\}f_2+f_1\{f_2,f_3\}f_4-f_1f_3\{f_2,f_4\},$$

will be used. In particular, for fermion creation and annihilation operators,

$$[a_k^*a_l,a_q^*a_r]=\delta_{lq}a_k^*a_r-\delta_{kr}a_q^*a_l. \tag{C1}$$

1. Fourth order

To calculate fourth-order terms, let us rewrite the third-order expression (13) as a sum of $H_{\text{int}}^{(3)}=\frac{1}{2}[S,[S,H_{\text{int}}]]$ and $H_0^{(3)}=-\frac{1}{6}[S,[S,[S,H_0]]]$:

$$H_{int,0}^{(3)}=\sum_{\mathbf{qk}}A_{\mathbf{kq}}^{int,0}b_{\mathbf{q}}^*n_{\mathbf{k}}\gamma_{\mathbf{q}}^*+\sum_{\mathbf{qkwk'}}b_{\mathbf{w}}^*\{B_{\mathbf{kqwk'}}^{int,0}a_{\mathbf{k-w}}^*a_{\mathbf{k-q}}a_{\mathbf{k'-q}}^*a_{\mathbf{k'}}+C_{\mathbf{kqwk'}}^{int,0}a_{\mathbf{k'}}^*a_{\mathbf{k'-q}}a_{\mathbf{k-q}}^*a_{\mathbf{k+w}}\}+\text{c.c.},$$

with

$$A_{\mathbf{kq}}^{int}=\frac{1}{2}iD_{\mathbf{q}}[\phi^*(\mathbf{k},\mathbf{q})+\phi(\mathbf{k+q},\mathbf{q})]+\text{c.c.},$$

$$A_{\mathbf{kq}}^{0}=\frac{1}{6}(\varepsilon_{\mathbf{k-q}}-\varepsilon_{\mathbf{k}}+\omega_{\mathbf{q}})|\phi(\mathbf{k},\mathbf{q})|^2-\frac{1}{6}(\varepsilon_{\mathbf{k}}-\varepsilon_{\mathbf{k+q}}+\omega_{\mathbf{q}})|\phi(\mathbf{k+q},\mathbf{q})|^2+\text{c.c.},$$

$$B_{\mathbf{kqwk'}}^{int}=\frac{1}{2}iD_{\mathbf{q}}[\phi^*(\mathbf{k'},\mathbf{q})\phi^*(\mathbf{k-q},\mathbf{w})-\phi^*(\mathbf{k'},\mathbf{q})\phi^*(\mathbf{k},\mathbf{w})+\phi^*(\mathbf{k},\mathbf{w})\phi(\mathbf{k},\mathbf{q})-\phi^*(\mathbf{k-q},\mathbf{w})\phi(\mathbf{k-w},\mathbf{q})],$$

$$B_{\mathbf{kqwk'}}^{0}=\frac{1}{6}\phi^*(\mathbf{k'},\mathbf{q})\{\phi(\mathbf{k},\mathbf{q})\phi^*(\mathbf{k},\mathbf{w})(\varepsilon_{\mathbf{k}}-\varepsilon_{\mathbf{k-q}}+\varepsilon_{\mathbf{k'-q}}-\varepsilon_{\mathbf{k'}})+\phi(\mathbf{k-w},\mathbf{q})\phi^*(\mathbf{k-q},\mathbf{w})(\varepsilon_{\mathbf{k-w-q}}-\varepsilon_{\mathbf{k-w}}-\varepsilon_{\mathbf{k'-q}}+\varepsilon_{\mathbf{k'}})\},$$

$$C_{\mathbf{kqwk'}}^{int}=\frac{1}{2}iD_{\mathbf{q}}[\phi(\mathbf{k'},\mathbf{q})\phi^*(\mathbf{k+w-q},\mathbf{w})-\phi(\mathbf{k'},\mathbf{q})\phi^*(\mathbf{k+w},\mathbf{w})-\phi^*(\mathbf{k+w-q},\mathbf{w})\times\phi^*(\mathbf{k+w},\mathbf{q})+\phi^*(\mathbf{k+w},\mathbf{w})\phi^*(\mathbf{k},\mathbf{q})],$$

$$C_{\mathbf{kqwk'}}^{0}=\frac{1}{6}\phi(\mathbf{k'},\mathbf{q})\{\phi^*(\mathbf{k+w},\mathbf{q})\phi^*(\mathbf{k+w-q},\mathbf{w})\times(\varepsilon_{\mathbf{k'-q}}-\varepsilon_{\mathbf{k'}}-\varepsilon_{\mathbf{k+w-q}}+\varepsilon_{\mathbf{k+w}})+\phi^*(\mathbf{k},\mathbf{q})\phi^*(\mathbf{k+w},\mathbf{w})(\varepsilon_{\mathbf{k-q}}-\varepsilon_{\mathbf{k}}-\varepsilon_{\mathbf{k'-q}}+\varepsilon_{\mathbf{k'}})\}.$$

The fourth-order term in Eq. (6) equals

$$H^{(4)}=-\frac{1}{6}[S,[S,[S,H_{int}]]]+\frac{1}{24}[S,[S,[S,[S,H_0]]]]=-\frac{1}{3}[S,H_{int}^{(3)}]-\frac{1}{4}[S,H_0^{(3)}]\coloneqq H_{int}^{(4)}+H_0^{(4)}.$$

Explicitly,

$$H_{int,0}^{(4)} = \sum_{\mathbf{kq}} A_{int,0}^{(4)} \gamma_{\mathbf{q}} n_{\mathbf{k}} \gamma_{\mathbf{q}}^{*} + \sum_{\mathbf{kqk'w}} B_{int,0}^{(4)} \gamma_{\mathbf{w}} a_{\mathbf{k-w}}^{*} a_{\mathbf{k-q}} a_{\mathbf{k'-q}}^{*} a_{\mathbf{k'}} + \sum_{\mathbf{kqk'w}} C_{int,0}^{(4)} \gamma_{\mathbf{w}} a_{\mathbf{k'}}^{*} a_{\mathbf{k'-q}} a_{\mathbf{k-q}}^{*} a_{\mathbf{k+w}} + \sum_{\mathbf{kqk'}} D_{int,0}^{(4)} b_{\mathbf{q}}^{*} b_{\mathbf{q}} n_{\mathbf{k}} n_{\mathbf{k'}} + \sum_{\mathbf{kq}} E_{int,0}^{(4)} b_{\mathbf{q}}^{*} b_{\mathbf{q}} a_{\mathbf{k}}^{*} a_{\mathbf{k-q}} \gamma_{\mathbf{q}}^{*} + \sum_{\mathbf{kqk'w}} F_{int,0}^{(4)} b_{\mathbf{w}}^{*} b_{\mathbf{w}} a_{\mathbf{k-w}}^{*} a_{\mathbf{k-q}} a_{\mathbf{k'-q}}^{*} a_{\mathbf{k'-w}} + \sum_{\mathbf{kqwk'}} G_{int,0}^{(4)} b_{\mathbf{w}}^{*} b_{\mathbf{w}} a_{\mathbf{k'+w}}^{*} a_{\mathbf{k'-q}} a_{\mathbf{k-q}}^{*} a_{\mathbf{k+w}} + \sum_{\mathbf{kqwk'}} I_{int,0}^{(4)} b_{\mathbf{w}}^{*} b_{\mathbf{w}} a_{\mathbf{k}}^{*} a_{\mathbf{k-q}} a_{\mathbf{k'-q}}^{*} a_{\mathbf{k'}} + \text{c.c.}, \tag{C2}$$

where

$$A_{int}^{(4)} = \frac{1}{3} A_{\mathbf{kq}}^{int}, \quad A_{0}^{(4)} = \frac{1}{4} A_{\mathbf{kq}}^{0}, \quad B_{int}^{(4)} = \frac{1}{3} B_{\mathbf{kqwk'}}^{int}, \quad B_{0}^{(4)} = \frac{1}{4} B_{\mathbf{kqwk'}}^{0}.$$

Other coefficients in Eq. (C2) arise according to the same scheme—viz.,

$$C_{int}^{(4)} = \frac{1}{3} C_{\mathbf{kqwk'}}^{int}, \quad D_{int}^{(4)} = \frac{1}{3} A_{\mathbf{kq}}^{int} [|\phi(\mathbf{k'},\mathbf{q})|^2 - |\phi(\mathbf{k'}+\mathbf{q},\mathbf{q})|^2],$$

$$E_{int}^{(4)} = \frac{1}{3} (A_{\mathbf{k-q},\mathbf{q}}^{int} - A_{\mathbf{kq}}^{int}) \phi(\mathbf{k},\mathbf{q}),$$

$$F_{int}^{(4)} = \frac{1}{3} [B_{\mathbf{k},\mathbf{q},\mathbf{w},\mathbf{k'-w}}^{int} \phi(\mathbf{k'}-\mathbf{q},\mathbf{w}) - B_{\mathbf{k},\mathbf{q},\mathbf{w},\mathbf{k'}}^{int} \phi(\mathbf{k'},\mathbf{w})],$$

$$G_{int}^{(4)} = \frac{1}{3} [C_{\mathbf{k},\mathbf{q},\mathbf{w},\mathbf{k'}}^{int} \phi(\mathbf{k'}+\mathbf{w},\mathbf{w}) - C_{\mathbf{k},\mathbf{q},\mathbf{w},\mathbf{k'+w}}^{int} \phi(\mathbf{k'}+\mathbf{w}-\mathbf{q},\mathbf{w})],$$

$$I_{int}^{(4)} = \frac{1}{3} [B_{\mathbf{k},\mathbf{q},\mathbf{w},\mathbf{k'}}^{int} \phi(\mathbf{k},\mathbf{w}) - B_{\mathbf{k+w},\mathbf{q},\mathbf{w},\mathbf{k'}}^{int} \phi(\mathbf{k}+\mathbf{w}-\mathbf{q},\mathbf{w}) + C_{\mathbf{k'-w},\mathbf{q},\mathbf{w},\mathbf{k}}^{int} \phi(\mathbf{k'}-\mathbf{q},\mathbf{w}) - C_{\mathbf{k'},\mathbf{q},\mathbf{w},\mathbf{k}}^{int} \phi(\mathbf{k'}+\mathbf{w},\mathbf{w})]$$

and similarly for $C_0^{(4)}$, $D_0^{(4)}$, etc.

2. Fifth order

Proceeding analogously as with the fourth order, one obtains

$$H^{(5)} = \frac{1}{24}[S,[S,[S,[S,H_{int}]]]] - \frac{1}{120}[S,[S,[S,[S,[S,H_0]]]]] = -\frac{1}{4}[S,H_{int}^{(4)}] - \frac{1}{5}[S,H_0^{(4)}] := H_{int}^{(5)} + H_0^{(5)},$$

$$\begin{aligned}
H_{int,0}^{(5)} = & \sum_{\mathbf{qkwk'}} A_{int,0}^{(5)} b_{\mathbf{w}}^{*} \gamma_{\mathbf{q}} n_{\mathbf{k}} a_{\mathbf{k'-w}}^{*} a_{\mathbf{k'+q}} + \sum_{\mathbf{qkw}} B_{int,0}^{(5)} b_{\mathbf{w}}^{*} \gamma_{\mathbf{q}} a_{\mathbf{k}}^{*} a_{\mathbf{k-w}} \gamma_{\mathbf{q}}^{*} + \sum_{\mathbf{qkwk'}} C_{int,0}^{(5)} b_{\mathbf{w}}^{*} a_{\mathbf{k'-w}} a_{\mathbf{k'-q}} n_{\mathbf{k}} \gamma_{\mathbf{q}}^{*} \\
& + \sum_{\mathbf{qkwk'u}} D_{int,0}^{(5)} b_{\mathbf{u}}^{*} \gamma_{\mathbf{w}} a_{\mathbf{k-w}}^{*} a_{\mathbf{k-q}} a_{\mathbf{k'-q}}^{*} a_{\mathbf{k'+u}} + \sum_{\mathbf{qkwk'u}} E_{int,0}^{(5)} b_{\mathbf{u}}^{*} \gamma_{\mathbf{w}} a_{\mathbf{k-w-u}}^{*} a_{\mathbf{k-q}} a_{\mathbf{k'-q}}^{*} a_{\mathbf{k'}} + \sum_{\mathbf{qkwk'ul}} F_{int,0}^{(5)} b_{\mathbf{u}}^{*} a_{\mathbf{l-u}}^{*} a_{\mathbf{l-w}} a_{\mathbf{k-w}}^{*} a_{\mathbf{k-q}} a_{\mathbf{k'-q}}^{*} a_{\mathbf{k'}} \\
& + \sum_{\mathbf{qkwk'u}} G_{int,0}^{(5)} b_{\mathbf{u}} \gamma_{\mathbf{w}} a_{\mathbf{k-w}}^{*} a_{\mathbf{k-q}} a_{\mathbf{k'-q}}^{*} a_{\mathbf{k'-u}} + \sum_{\mathbf{qkwk'u}} I_{int,0}^{(5)} b_{\mathbf{u}} \gamma_{\mathbf{w}} a_{\mathbf{k-w}}^{*} a_{\mathbf{k-q-u}} a_{\mathbf{k'-q}}^{*} a_{\mathbf{k'}} + \sum_{\mathbf{qkwk'ul}} J_{int,0}^{(5)} b_{\mathbf{u}} a_{\mathbf{l+w}}^{*} a_{\mathbf{l-u}} a_{\mathbf{k-w}}^{*} a_{\mathbf{k-q}} a_{\mathbf{k'-q}}^{*} a_{\mathbf{k'}} \\
& + \sum_{\mathbf{qkwk'u}} K_{int,0}^{(5)} b_{\mathbf{u}}^{*} \gamma_{\mathbf{w}} a_{\mathbf{k'}}^{*} a_{\mathbf{k'-q}} a_{\mathbf{k-q-u}}^{*} a_{\mathbf{k+w}} + \sum_{\mathbf{qkwk'u}} L_{int,0}^{(5)} b_{\mathbf{u}}^{*} \gamma_{\mathbf{w}} a_{\mathbf{k'-u}}^{*} a_{\mathbf{k'-q}} a_{\mathbf{k-q}}^{*} a_{\mathbf{k+w}} + \sum_{\mathbf{qkwk'ul}} M_{int,0}^{(5)} b_{\mathbf{u}}^{*} a_{\mathbf{l-u}}^{*} a_{\mathbf{l-q}} a_{\mathbf{k'}}^{*} a_{\mathbf{k'-q}} a_{\mathbf{k-q}}^{*} a_{\mathbf{k'+w}} \\
& + \sum_{\mathbf{qkwk'u}} N_{int,0}^{(5)} b_{\mathbf{u}} \gamma_{\mathbf{w}} a_{\mathbf{k'}}^{*} a_{\mathbf{k'-q}} a_{\mathbf{k-q}}^{*} a_{\mathbf{k+w-u}} + \sum_{\mathbf{qkwk'u}} O_{int,0}^{(5)} b_{\mathbf{u}} \gamma_{\mathbf{w}} a_{\mathbf{k'+u}}^{*} a_{\mathbf{k'-q}} a_{\mathbf{k-q}}^{*} a_{\mathbf{k+w}} + \sum_{\mathbf{qkwk'ul}} P_{int,0}^{(5)} b_{\mathbf{u}} a_{\mathbf{l+w}}^{*} a_{\mathbf{l-u}} a_{\mathbf{k'}}^{*} a_{\mathbf{k'-q}} a_{\mathbf{k-q}}^{*} a_{\mathbf{k+w}} \\
& + \sum_{\mathbf{qkk'}} R_{int,0}^{(5)} b_{\mathbf{q}}^{*} n_{\mathbf{k}} n_{\mathbf{k'}} \gamma_{\mathbf{q}}^{*} + \sum_{\mathbf{qk}} S_{int,0}^{(5)} b_{\mathbf{q}}^{*} a_{\mathbf{k}}^{*} a_{\mathbf{k-q}} \gamma_{\mathbf{q}}^{*} \gamma_{\mathbf{q}}^{*} + \sum_{\mathbf{qk}} S_{int,0}^{(5)} b_{\mathbf{q}} \gamma_{\mathbf{q}} a_{\mathbf{k}}^{*} a_{\mathbf{k-q}} \gamma_{\mathbf{q}}^{*} + \sum_{\mathbf{qkwk'}} T_{int,0}^{(5)} b_{\mathbf{w}}^{*} a_{\mathbf{k-w}}^{*} a_{\mathbf{k-q}} a_{\mathbf{k'-q}}^{*} a_{\mathbf{k'-w}} \gamma_{\mathbf{w}}^{*} \\
& + \sum_{\mathbf{qkwk'}} U_{int,0}^{(5)} b_{\mathbf{w}}^{*} a_{\mathbf{k'+w}}^{*} a_{\mathbf{k'-q}} a_{\mathbf{k-q}}^{*} a_{\mathbf{k+w}} \gamma_{\mathbf{w}}^{*} + \sum_{\mathbf{qkwk'}} W_{int,0}^{(5)} b_{\mathbf{w}}^{*} a_{\mathbf{k}}^{*} a_{\mathbf{k-q}} a_{\mathbf{k'-q}}^{*} a_{\mathbf{k'}} \gamma_{\mathbf{w}}^{*} + \text{c.c.},
\end{aligned} \tag{C3}$$

where

$$A^{(5)}_{int,0} = -\left\{\frac{1}{4},\frac{1}{5}\right\}(A^{(4)}_{int,0} + A^{(4)*}_{int,0}) \times [\phi^*(\mathbf{k}',\mathbf{w})\phi^*(\mathbf{k}'+\mathbf{q},\mathbf{q}) - \phi^*(\mathbf{k}'+\mathbf{q},\mathbf{w})\phi^*(\mathbf{k}'+\mathbf{q}-\mathbf{w},\mathbf{q})],$$

$$B^{(5)}_{int,0} = -\left\{\frac{1}{4},\frac{1}{5}\right\}(A^{(4)}_{int,0} + A^{(4)*}_{int,0} - A^{(4)\mathbf{k}\to\mathbf{k}-\mathbf{w}}_{int,0} - A^{(4)*\mathbf{k}\to\mathbf{k}-\mathbf{w}}_{int,0})\phi^*(\mathbf{k},\mathbf{w}),$$

$$C^{(5)}_{int,0} = -\left\{\frac{1}{4},\frac{1}{5}\right\}(A^{(4)}_{int,0} + A^{(4)*}_{int,0})[\phi^*(\mathbf{k}',\mathbf{w})\phi^*(\mathbf{k}',\mathbf{q}) - \phi^*(\mathbf{k}'-\mathbf{q},\mathbf{w})\phi^*(\mathbf{k}'-\mathbf{w},\mathbf{q})],$$

$$D^{(5)}_{int,0} = -\left\{\frac{1}{4},\frac{1}{5}\right\}[B^{(4)\mathbf{k}'\to\mathbf{k}'+\mathbf{u}}_{int,0}\phi^*(\mathbf{k}'+\mathbf{u}-\mathbf{q},\mathbf{u}) - B^{(4)}_{int,0}\phi^*(\mathbf{k}'+\mathbf{u},\mathbf{u})],$$

$$E^{(5)}_{int,0} = -\left\{\frac{1}{4},\frac{1}{5}\right\}[B^{(4)}_{int,0}\phi^*(\mathbf{k}-\mathbf{w},\mathbf{u}) - B^{(4)\mathbf{k}\to\mathbf{k}-\mathbf{u}}_{int,0}\phi^*(\mathbf{k}-\mathbf{q},\mathbf{u})],$$

$$F^{(5)}_{int,0} = -\left\{\frac{1}{4},\frac{1}{5}\right\}B^{(4)}_{int,0}[\phi^*(\mathbf{l},\mathbf{u})\phi(\mathbf{l},\mathbf{w}) - \phi^*(\mathbf{l}-\mathbf{w},\mathbf{u})\phi(\mathbf{l}-\mathbf{u},\mathbf{w})],$$

$$G^{(5)}_{int,0} = \left\{\frac{1}{4},\frac{1}{5}\right\}[B^{(4)\mathbf{k}'\to\mathbf{k}'-\mathbf{u}}_{int,0}\phi(\mathbf{k}'-\mathbf{q},\mathbf{u}) - B^{(4)}_{int,0}\phi(\mathbf{k}',\mathbf{u})],$$

$$I^{(5)}_{int,0} = \left\{\frac{1}{4},\frac{1}{5}\right\}[B^{(4)\mathbf{k}\to\mathbf{k}-\mathbf{u}}_{int,0}\phi(\mathbf{k}-\mathbf{w},\mathbf{u}) - B^{(4)}_{int,0}\phi(\mathbf{k}-\mathbf{q},\mathbf{u})],$$

$$J^{(5)}_{int,0} = \left\{\frac{1}{4},\frac{1}{5}\right\}B^{(4)}_{int,0}[\phi(\mathbf{l}+\mathbf{w},\mathbf{u})\phi(\mathbf{l}-\mathbf{u}+\mathbf{w},\mathbf{w}) - \phi(\mathbf{l},\mathbf{u})\phi(\mathbf{l}+\mathbf{w},\mathbf{w})],$$

$$K^{(5)}_{int,0} = -\left\{\frac{1}{4},\frac{1}{5}\right\}[C^{(4)}_{int,0}\phi^*(\mathbf{k}-\mathbf{q},\mathbf{u}) - C^{(4)\mathbf{k}\to\mathbf{k}-\mathbf{u}}_{int,0}\phi^*(\mathbf{k}+\mathbf{w},\mathbf{u})],$$

$$L^{(5)}_{int,0} = -\left\{\frac{1}{4},\frac{1}{5}\right\}[C^{(4)}_{int,0}\phi^*(\mathbf{k}',\mathbf{u}) - C^{(4)\mathbf{k}'\to\mathbf{k}'-\mathbf{u}}_{int,0}\phi^*(\mathbf{k}'-\mathbf{q},\mathbf{u})],$$

$$M^{(5)}_{int,0} = -\left\{\frac{1}{4},\frac{1}{5}\right\}C^{(4)}_{int,0}[\phi^*(\mathbf{l},\mathbf{u})\phi(\mathbf{l},\mathbf{w}) - \phi^*(\mathbf{l}-\mathbf{w},\mathbf{u})\phi(\mathbf{l}-\mathbf{u},\mathbf{w})],$$

$$N^{(5)}_{int,0} = \left\{\frac{1}{4},\frac{1}{5}\right\}[C^{(4)\mathbf{k}\to\mathbf{k}-\mathbf{u}}_{int,0}\phi(\mathbf{k}-\mathbf{q},\mathbf{u}) - C^{(4)}_{int,0}\phi(\mathbf{k}+\mathbf{w},\mathbf{u})],$$

$$O^{(5)}_{int,0} = \left\{\frac{1}{4},\frac{1}{5}\right\}[C^{(4)}_{int,0}\phi(\mathbf{k}'+\mathbf{u},\mathbf{u}) - C^{(4)\mathbf{k}'\to\mathbf{k}'+\mathbf{u}}_{int,0}\phi(\mathbf{k}'+\mathbf{u}-\mathbf{q},\mathbf{u})],$$

$$P^{(5)}_{int,0} = \left\{\frac{1}{4},\frac{1}{5}\right\}C^{(4)}_{int,0}[\phi(\mathbf{l}+\mathbf{w},\mathbf{u})\phi(\mathbf{l}-\mathbf{u}+\mathbf{w},\mathbf{w}) - \phi(\mathbf{l},\mathbf{u})\phi(\mathbf{l}+\mathbf{w},\mathbf{w})],$$

$$R^{(5)}_{int,0} = \left\{\frac{1}{4},\frac{1}{5}\right\}(D^{(4)}_{int,0} + D^{(4)*}_{int,0}), \quad S^{(5)}_{int,0} = \left\{\frac{1}{4},\frac{1}{5}\right\}E^{(4)}_{int,0},$$

$$T^{(5)}_{int,0} = \left\{\frac{1}{4},\frac{1}{5}\right\}(F^{(4)}_{int,0} + F^{(4)*\mathbf{k}\leftrightarrows\mathbf{k}'}_{int,0}), \quad U^{(5)}_{int,0} = \left\{\frac{1}{4},\frac{1}{5}\right\}(G^{(4)}_{int,0} + G^{(4)*\mathbf{k}\leftrightarrows\mathbf{k}'}_{int,0}),$$

$$W_{int,0}^{(5)} = \left\{ \frac{1}{4}, \frac{1}{5} \right\} (I_{int,0}^{(4)} + I_{int,0}^{(4)*\mathbf{k} \leftrightarrows \mathbf{k}'}).$$

Here 1/4 in the curly brackets refers to int terms and 1/5 to 0 terms.

3. Sixth order

Proceeding analogously as with the fourth, and the fifth orders, one obtains

$$H^{(6)} = -\frac{1}{120}[S,[S,[S,[S,[S,H_{int}]]]]] + \frac{1}{720}[S,[S,[S,[S,[S,[S,H_0]]]]]] = -\frac{1}{5}[S,H_{int}^{(5)}] - \frac{1}{6}[S,H_0^{(5)}] := H_{int}^{(6)} + H_0^{(6)}, \tag{C4}$$

$$\begin{aligned} H_{int,0}^{(6)} = \left\{ \frac{1}{5}, \frac{1}{6} \right\} \Bigg\{ & \sum_{\mathbf{qkwk}'} A_{int,0}^{(5)} \gamma_{\mathbf{w}} \gamma_{\mathbf{q}} n_{\mathbf{k}} a_{\mathbf{k}'-\mathbf{w}}^{*} a_{\mathbf{k}'+\mathbf{q}} + \sum_{\mathbf{qkw}} B_{int,0}^{(5)} \gamma_{\mathbf{w}} \gamma_{\mathbf{q}} a_{\mathbf{k}}^{*} a_{\mathbf{k}-\mathbf{w}} \gamma_{\mathbf{q}}^{*} + \sum_{\mathbf{qkwk}'} C_{int,0}^{(5)} \gamma_{\mathbf{w}} a_{\mathbf{k}'-\mathbf{w}} a_{\mathbf{k}'-\mathbf{q}} n_{\mathbf{k}} \gamma_{\mathbf{q}}^{*} \\ & + \sum_{\mathbf{qkwk}'\mathbf{u}} D_{int,0}^{(5)} \gamma_{\mathbf{u}} \gamma_{\mathbf{w}} a_{\mathbf{k}-\mathbf{w}}^{*} a_{\mathbf{k}-\mathbf{q}} a_{\mathbf{k}'-\mathbf{q}}^{*} a_{\mathbf{k}'+\mathbf{u}} + \sum_{\mathbf{qkwk}'\mathbf{u}} E_{int,0}^{(5)} \gamma_{\mathbf{u}} \gamma_{\mathbf{w}} a_{\mathbf{k}-\mathbf{w}-\mathbf{u}}^{*} a_{\mathbf{k}-\mathbf{q}} a_{\mathbf{k}'-\mathbf{q}}^{*} a_{\mathbf{k}'} + \sum_{\mathbf{qkwk}'\mathbf{ul}} F_{int,0}^{(5)} \gamma_{\mathbf{u}} a_{\mathbf{l}-\mathbf{u}}^{*} a_{\mathbf{l}-\mathbf{w}} a_{\mathbf{k}-\mathbf{w}}^{*} a_{\mathbf{k}-\mathbf{q}} a_{\mathbf{k}'-\mathbf{q}}^{*} a_{\mathbf{k}'} \\ & + \sum_{\mathbf{qkwk}'\mathbf{u}} G_{int,0}^{(5)} \gamma_{\mathbf{w}} a_{\mathbf{k}-\mathbf{w}}^{*} a_{\mathbf{k}-\mathbf{q}} a_{\mathbf{k}'-\mathbf{q}}^{*} a_{\mathbf{k}'-\mathbf{u}} \gamma_{\mathbf{u}}^{*} + \sum_{\mathbf{qkwk}'\mathbf{u}} I_{int,0}^{(5)} \gamma_{\mathbf{w}} a_{\mathbf{k}-\mathbf{w}}^{*} a_{\mathbf{k}-\mathbf{q}-\mathbf{u}} a_{\mathbf{k}'-\mathbf{q}}^{*} a_{\mathbf{k}'} \gamma_{\mathbf{u}}^{*} + \sum_{\mathbf{qkwk}'\mathbf{ul}} J_{int,0}^{(5)} a_{\mathbf{l}+\mathbf{w}}^{*} a_{\mathbf{l}-\mathbf{u}} a_{\mathbf{k}-\mathbf{w}}^{*} a_{\mathbf{k}-\mathbf{q}} a_{\mathbf{k}'-\mathbf{q}}^{*} a_{\mathbf{k}'} \gamma_{\mathbf{u}}^{*} \\ & + \sum_{\mathbf{qkwk}'\mathbf{u}} K_{int,0}^{(5)} \gamma_{\mathbf{u}} \gamma_{\mathbf{w}} a_{\mathbf{k}'}^{*} a_{\mathbf{k}'-\mathbf{q}} a_{\mathbf{k}-\mathbf{q}-\mathbf{u}}^{*} a_{\mathbf{k}+\mathbf{w}} + \sum_{\mathbf{qkwk}'\mathbf{u}} L_{int,0}^{(5)} \gamma_{\mathbf{u}} \gamma_{\mathbf{w}} a_{\mathbf{k}'-\mathbf{u}}^{*} a_{\mathbf{k}'-\mathbf{q}} a_{\mathbf{k}-\mathbf{q}}^{*} a_{\mathbf{k}+\mathbf{w}} \\ & + \sum_{\mathbf{qkwk}'\mathbf{ul}} M_{int,0}^{(5)} \gamma_{\mathbf{u}} a_{\mathbf{l}-\mathbf{u}}^{*} a_{\mathbf{l}-\mathbf{q}} a_{\mathbf{k}'}^{*} a_{\mathbf{k}'-\mathbf{q}} a_{\mathbf{k}-\mathbf{q}}^{*} a_{\mathbf{k}'+\mathbf{w}} + \sum_{\mathbf{qkwk}'\mathbf{u}} N_{int,0}^{(5)} \gamma_{\mathbf{w}} a_{\mathbf{k}'}^{*} a_{\mathbf{k}'-\mathbf{q}} a_{\mathbf{k}-\mathbf{q}}^{*} a_{\mathbf{k}+\mathbf{w}-\mathbf{u}} \gamma_{\mathbf{u}}^{*} \\ & + \sum_{\mathbf{qkwk}'\mathbf{u}} O_{int,0}^{(5)} \gamma_{\mathbf{w}} a_{\mathbf{k}'+\mathbf{u}}^{*} a_{\mathbf{k}'-\mathbf{q}} a_{\mathbf{k}-\mathbf{q}}^{*} a_{\mathbf{k}+\mathbf{w}} \gamma_{\mathbf{u}}^{*} + \sum_{\mathbf{qkwk}'\mathbf{ul}} P_{int,0}^{(5)} a_{\mathbf{l}+\mathbf{w}}^{*} a_{\mathbf{l}-\mathbf{u}} a_{\mathbf{k}'}^{*} a_{\mathbf{k}'-\mathbf{q}} a_{\mathbf{k}-\mathbf{q}}^{*} a_{\mathbf{k}+\mathbf{w}} \gamma_{\mathbf{u}}^{*} + \sum_{\mathbf{qkk}'} R_{int,0}^{(5)} \gamma_{\mathbf{q}} n_{\mathbf{k}} n_{\mathbf{k}'} \gamma_{\mathbf{q}}^{*} \\ & + \sum_{\mathbf{qk}} S_{int,0}^{(5)} \gamma_{\mathbf{q}} a_{\mathbf{k}}^{*} a_{\mathbf{k}-\mathbf{q}} \gamma_{\mathbf{q}}^{*} \gamma_{\mathbf{q}}^{*} + \sum_{\mathbf{qk}} S_{int,0}^{(5)} \gamma_{\mathbf{q}} a_{\mathbf{k}}^{*} a_{\mathbf{k}-\mathbf{q}} \gamma_{\mathbf{q}}^{*} \gamma_{\mathbf{q}}^{*} + \sum_{\mathbf{qkwk}'} T_{int,0}^{(5)} \gamma_{\mathbf{w}} a_{\mathbf{k}-\mathbf{w}}^{*} a_{\mathbf{k}-\mathbf{q}} a_{\mathbf{k}'-\mathbf{q}}^{*} a_{\mathbf{k}'-\mathbf{w}} \gamma_{\mathbf{w}}^{*} \\ & + \sum_{\mathbf{qkwk}'} U_{int,0}^{(5)} \gamma_{\mathbf{w}} a_{\mathbf{k}'+\mathbf{w}}^{*} a_{\mathbf{k}'-\mathbf{q}} a_{\mathbf{k}-\mathbf{q}}^{*} a_{\mathbf{k}+\mathbf{w}} \gamma_{\mathbf{w}}^{*} + \sum_{\mathbf{qkwk}'} W_{int,0}^{(5)} \gamma_{\mathbf{w}} a_{\mathbf{k}}^{*} a_{\mathbf{k}-\mathbf{q}} a_{\mathbf{k}'-\mathbf{q}}^{*} a_{\mathbf{k}'} \gamma_{\mathbf{w}}^{*} + b^{*} b\{\,\} + \text{c.c.} \Bigg\}. \end{aligned} \tag{C5}$$

The last term denotes all (6, 2) expressions which are irrelevant after averaging over the phonon vacuum. The coefficients are the same as in the fifth-order terms, except for the factors 1/5 and 1/6.

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[1] J. Bardeen, L. N. Cooper, and J. R. Schrieffer, Phys. Rev. **108**, 1175 (1957).

[2] H. Fröhlich, Proc. R. Soc. London, Ser. A **215**, 291 (1952).

[3] G. Rickayzen, *Theory of Superconductivity* (Interscience, New York, 1964), Chap. 4, p. 168.

[4] G. E. Volovik, *Exotic Properties of Superfluid 3He* (World Scientific, Singapore, 1992).

[5] J. Maćkowiak and P. Tarasewicz, Mol. Phys. Rep. **15/16**, 61 (1996).

[6] J. Maćkowiak and P. Tarasewicz, Mol. Phys. Rep. **20**, 145 (1997).

[7] J. Maćkowiak and P. Tarasewicz, Acta Phys. Pol. A **93**, 659 (1998).

[8] J. Maćkowiak and P. Tarasewicz, Physica C **331**, 25 (2000).

[9] P. Tarasewicz, Eur. Phys. J. B **41**, 185 (2004).

[10] P. Tarasewicz, J. Supercond. **17**, 431 (2004).

[11] P. Tarasewicz and J. Maćkowiak, Physica C **329**, 130 (2000).

[12] J. Czerwonko, Physica C **235-240**, 2337 (1994).

[13] J. Czerwonko, Mol. Phys. Rep. **12**, 79 (1995).

[14] J. Spałek and W. Wójcik, Solid State Commun. **60**, 119 (1986).

[15] J. Spałek and W. Wójcik, Phys. Rev. B **37**, 1532 (1988).

[16] K. I. Wysokiński, in *New Developments in High Temperature Superconductivity*, edited by J. Klamut, B. W. Veal, B. M. Dabrowski, P. W. Klamut, and M. Kazimierski, Vol. 475 of *Lecture Notes in Physics* (Springer, Berlin, Heidelberg, 1996), p. 243.

[17] T. Schneider and H. Keller, Int. J. Mod. Phys. B **8**, 487 (1993).

[18] Y. M. Bunkov, A. S. Chen, D. J. Cousins, and H. Godfrin, Phys. Rev. Lett. **85**, 3456 (2000).

[19] C. Schneider, G. Hammerl, G. Logvenov, T. Kopp, J. Kirtley, P. Hirschfeld, and J. Mannhart, Europhys. Lett. **68**, 86 (2004).

[20] A. A. Aligia, A. P. Kampf, and J. Mannhart, Phys. Rev. Lett. **94**, 247004 (2005).
[21] J. Brankov, J. Maćkowiak, and V. Zagrebnov (unpublished).
[22] A. S. Davydov, *Quantum Mechanics* (Pergamon Press, Oxford, 1976).
[23] L. N. Cooper, Phys. Rev. **104**, 1189 (1956).
[24] H. Kamei and K. Miyake, J. Phys. Soc. Jpn. **74**, 1911 (2005).
[25] N. N. Bogoliubov, Sov. Phys. JETP **34**, 41 (1958).
[26] This and the following inequalities are solved by the Mathematica package.