Theory of diamagnetic response of the vortex liquid phase of two-dimensional superconductors

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We calculate the magnetization of a two-dimensional superconductor in a perpendicular magnetic field near its Kosterlitz-Thouless transition and at lower temperatures. We find that the critical behavior is more complex than assumed in the literature and that, in particular, the critical magnetization is *not* field independent as naive scaling predicts. In the low-temperature phase we find a substantial fluctuation renormalization of the Abrikosov mean-field result. We compare our analysis with the data on the cuprates.

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I. INTRODUCTION

The study of fluctuation superconductivity received a tremendous impetus from the discovery of the cuprate, high-temperature superconductors.[1] It was quickly realized that these were materials with large Ginzburg parameters with the additional feature that many of them are also highly two dimensional. More recently, interest has focussed on the systematics of their superconducting properties with doping and it has been found that this anisotropy is further enhanced with underdoping.[2] Conversely, the extent to which the anomalous properties of the underdoped compounds in the "pseudogap" region can be attributed to superconducting fluctuations is a question of considerable interest.[3–9]

In this context, we report here a study of the magnetization of a two-dimensional superconductor, or more precisely of a stack of decoupled two-dimensional superconducting layers, in a perpendicular magnetic field H. By combining Kosterlitz-Thouless (KT) renormalization-group flows and explicit computations for plasmas, we find the field and temperature dependence of the magnetization density, $M(H,T)$, for temperatures T near to or below the Kosterlitz-Thouless transition temperature T_{KT} at fields $H_{c1} \ll H \ll H_{c2}$. These results are interesting on three immediate fronts. First, the rough magnitude of $M(H,T)$ and the trends exhibited in Fig. 1 are in accord with much of the data on the strongly layered cuprates in the field regime where a single layer theory is expected to apply. For example, this holds for the recent data of Wang and collaborators[4] which have demonstrated the existence of substantial fluctuation magnetization in the same region of the phase diagram of BSCCO which was previously observed to exhibit a large Nernst signal.[5] Second, we find that M is *not* independent of H at criticality, as assumed in earlier analysis of the "crossing point" phenomenon in studies of the cuprates.[10] Instead, the critical magnetization follows

$$M \approx -\frac{k_B T_{KT}}{d\phi_0} \ln\left(\gamma_1 \ln \frac{\phi_0}{\mu_0 H a_0^2 \gamma_2}\right), \tag{1}$$

where d is the distance between layers, ϕ_0 is the flux quantum, H is the magnetic-field component perpendicular to the layers, and a_0 is the microscopic short-distance cutoff length (γ_1 and γ_2 are constants). Third, we find that below T_{KT}, M exhibits a low-field growth indicative of the expulsion of vortices from the KT phase,

$$M \approx -\frac{\pi\rho_s(T)}{2d\phi_0}\left(1 - \frac{2k_B T}{\pi\rho_s(T)}\right) \ln \frac{\phi_0}{\mu_0 H a_0^2 \gamma_3}, \tag{2}$$

where ρ_s is the two-dimensional (2D) superfluid density, related to the in-plane penetration depth via $\rho_s = d\phi_0^2/(4\pi^2\lambda^2\mu_0)$ (γ_3 is a constant that vanishes as T approaches T_{KT}, see below). Note that this expression gives a simple relation between the superfluid density and the derivative of the magnetization with respect to the logarithm of the field, that can be used to determine when the system is indeed behaving as a stack of essentially decoupled two-dimensional Kosterlitz-Thouless films.

These results and the limits of their applicability to actual layered superconductors are derived in Sec. II, with some details relegated to the Appendix. In Sec. III we consider experimental signatures of our analysis and the existing experimental situation. We close with a brief summary and some open questions.

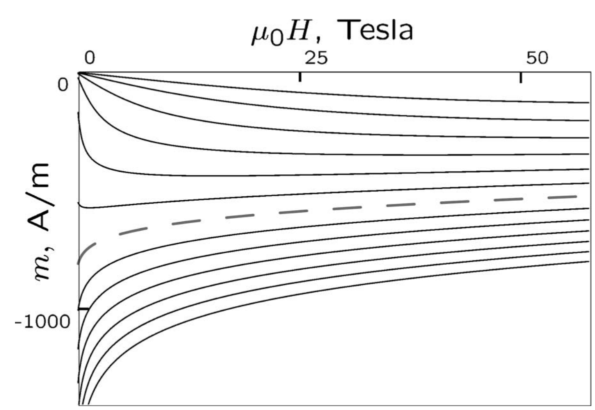

FIG. 1. Field dependence of magnetization for a range of temperatures between $T=77$ K and $T=83$ K, with the dashed curve corresponding to $T=T_{KT}=80$ K (others are spaced in 0.5 K increments). Here we take $E_c=\pi\rho_{s0}(T)$, which then implies $k_B T_{KT} \approx 1.13\rho_{s0}(T_{KT})$. We chose $a_0=30$ A, $d=15$ A, so that $k_B T_{KT}/(d\phi_0) \approx 350$ A/m and $Ba_0^2/\phi_0 \approx 0.04$ at 10 T. In fact, over the entire field range up to 50 T this last ratio remains <1, which suggests that although our results may become insufficient at these high fields, they can be of some qualitative use. Finally, we model the experimentally observed temperature dependence of the bare superfluid stiffness by $\rho_{s0}(T)/\rho_{s0}(0) = 1 - T/120$ K (see, e.g., Ref. 15).

II. THEORY

A. Definition of the problem

The average magnetization density of a superconductor in an external field $\mathbf{H}$ is

$$\mu_0\mathbf{M} = \mathbf{B}(\mathbf{H}) - \mu_0\mathbf{H}, \tag{3}$$

where $\mathbf{B}(\mathbf{H})$ is the average (uniform) magnetic field, computed via

$$\frac{\partial f_{scm}(\mathbf{B})}{\partial \mathbf{B}} = \mathbf{H}, \tag{4}$$

where $f_{scm}(\mathbf{B})$ is the total free-energy density of superconducting matter coupled to fluctuating magnetic fields with average field fixed at $\mathbf{B}$. Let us define

$$f_{sc} = f_{scm} - \frac{\mathbf{B}^2}{2\mu_0}, \text{ then} \tag{5}$$

$$\frac{\partial f_{sc}}{\partial \mathbf{B}} = \frac{\partial f_{scm}(\mathbf{B})}{\partial \mathbf{B}} - \frac{\mathbf{B}}{\mu_0} = -\mathbf{M}. \tag{6}$$

In type-II materials and for $H \gg H_{c1}$ field energy is well-approximated by its uniform value, and therefore, following the equation above, the magnetization is to be computed from the free energy of a charged superfluid in a uniform *external* magnetic field.

We consider a stack of decoupled two-dimensional layers. Hence we will replace the three-dimensional free-energy density $f_{sc}(\mathbf{B})$ by $(1/d)f(B)$ where $f(B)$ is the free-energy density of a single layer in the presence of a magnetic induction B, now restricted to be perpendicular to the layers, whose effect will be to impose a density B/ϕ_0 of field-induced vortices.

B. Coulomb gas formulation

To compute f and thence M we shall resort to the standard[11] mapping of the two-dimensional vortex problem onto a two-component Coulomb plasma whose Hamiltonian is given by (see the Appendix)

$$H = N_T E_{c0} + \pi\rho_{s0}\sum_{i<j} p_i p_j \ln\frac{r_{ij}^2}{a_0^2} + H_{\mathrm{B}}. \tag{7}$$

The number of vortices of charge $p_i = \pm 1$ is $N_\pm$. The total number of vortices is $N_{\mathrm{T}} = N_+ + N_-$ and is allowed to fluctuate by the addition and removal of neutral vortex-antivortex pairs, but the net charge $Q = N_+ - N_- = L^2 B/\phi_0$ is constrained by the field B. E_{c0} is the bare vortex core energy, a_0 is the bare short distance cutoff (e.g., vortex core radius), ρ_{s0} is the bare superfluid stiffness (inverse dielectric constant in the plasma language), L^2 is the area of the system. A uniform background density of charge to make the system neutral is necessary to insure a proper thermodynamic limit. The total Coulomb interaction potential between the vortices and this background and between the background and itself is in the constant H_B that depends on B and on the size and shape of the system, but not on the vortex configuration.

C. Renormalization

Standard Kosterlitz renormalization-group (RG) methods[11–13] can be generalized to the present, non-neutral situation. For convenience, the bare superfluid stiffness and core energy are captured by the bare couplings $x_0 = 1 - \pi\rho_{s0}/2k_B T$ and $y_0 = 2\pi e^{-E_{c0}/k_B T}$, which we will renormalize. In addition to these couplings we shall be interested in computing the free-energy density, so we keep track of the configuration-independent term $\mathcal{C}$ in the renormalized Hamiltonian, generated as degrees of freedom are integrated out, i.e., at any intermediate step in the RG process the partition function is $Z = e^{-L^2\mathcal{C}/a^2}\mathrm{Tr}e^{-\beta H}$, where a is the renormalized cutoff. Note $\mathcal{C}$ is defined to be dimensionless. We also define the dimensionless density of field-induced vortices (number per renormalized cutoff area) as $n = Ba^2/\phi_0$; its bare value is $n_0 = Ba_0^2/\phi_0$.

Provided the total number of vortices and antivortices per cutoff area is small compared to one, the following differential equations describe the renormalization upon increasing the cutoff to $a = a_0 b$ and integrating out neutral vortex-antivortex pairs with spacing less than a:

$$\frac{dy}{d\ln b} = 2xy, \tag{8}$$

$$\frac{dx}{d\ln b} = 2y^2, \tag{9}$$

$$\frac{d\mathcal{C}}{d\ln b} = 2\mathcal{C} - \frac{y^2}{2\pi}, \tag{10}$$

$$\frac{dn}{d\ln b} = 2n. \tag{11}$$

The last line shows the trivial renormalization of the number of field induced vortices per cutoff area. A somewhat unexpected fact about these equations is that the presence of field induced vortices leaves the zero-field flow equations[13] [Eqs. (8)–(10)] unaffected (also see Sec. IV). As we shall see, this simplifies the calculation of the magnetization.

By straightforward integration one obtains, with

$$c \equiv |x_0^2 - y_0^2|^{1/2} \sim \sqrt{|T - T_{KT}|},$$

the solutions

$$x_<(b) = -c\,\mathrm{cotanh}(\Theta_< + 2\mathrm{c}\ln\mathrm{b}),$$

$$y_<(b) = c\,\mathrm{cosech}(\Theta_< + 2\mathrm{c}\ln\mathrm{b}),$$

$$\mathcal{C}(b) = b^2\mathcal{C}(0) - b^2\int_0^{\log b}\frac{y_<^2(b')}{2\pi b'^2}d\ln b',$$

$$n(b) = n_0 b^2 \tag{12}$$

with $\cosh\Theta_< = -x_0/y_0$ for $T < T_{KT}$, and the solutions

$$x_>(b) = c\,\mathrm{cotan}(\Theta_> - 2\mathrm{c}\ln\mathrm{b}),$$

$$y_>(b) = c\,\mathrm{cosec}(\Theta_> - 2\mathrm{c}\ln\mathrm{b}),$$

$$\mathcal{C}(b) = b^2\mathcal{C}(0) - b^2\int_0^{\log b} \frac{y_>^2(b')}{2\pi b'^2} d\ln b',$$

$$n(b) = n_0 b^2 \tag{13}$$

with $\cos\Theta_> = x_0/y_0$ for $T > T_{KT}$.

The free-energy density of the original problem is then computed by running the RG to some scale b, where it is expressed as a sum of two parts,

$$f = \frac{\mathcal{F}(b) + k_B T\mathcal{C}(b)}{a^2}, \tag{14}$$

where $\mathcal{F}(b)$ is the free energy per renormalized cutoff area of the vortices that have not yet been integrated out. [Note: since we have chosen to rescale lengths in our RG, the free-energy density f is computed by dividing the residual free energy, $\mathcal{F}(b)$, and $\mathcal{C}(b)$, by the rescaled area, a^2.] In the zero-field problem originally treated by Kosterlitz for $T \leq T_{KT}$ one can carry out the renormalization procedure indefinitely, i.e., by setting $b=\infty$ and $\mathcal{F}(b)=0$. This is because at low temperatures the microscopic problem renormalizes onto one with no vortices and infinite screening length. Here, on the other hand, no matter what the temperature we always renormalize to a problem at strong coupling (finite screening length), and so the flow must be stopped at some finite b while Eqs. (8)–(10) are still valid. The free-energy density is fundamentally independent of b, although both $\mathcal{C} \equiv \mathcal{C}(b)$ and $\mathcal{F} \equiv \mathcal{F}(b)$ manifestly depend on it. Thus although the choice of b is implicitly governed by many considerations, in particular, the value of the magnetic field, in computing the magnetization this implicit dependence does not contribute. Finally, the observation that field induced vortices do not affect the other flows, in particular that $\mathcal{C}$ has no explicit dependence on B, simplifies the calculation of the magnetization density:

$$M = -\frac{1}{d}\frac{\partial f}{\partial B} = -\frac{1}{d\phi_0}\frac{\partial \mathcal{F}}{\partial n}, \tag{15}$$

thus reducing the problem to computing $\mathcal{F}$.

Calculations of $\mathcal{F}$ can be found in the next section. Here, we close with a qualitative discussion of different regimes of interest. The renormalized density of field-induced vortices, $n(b)$, grows under the RG flow [see Eq. (11)]. For $T \leq T_{KT}$ thermally induced vortices become increasingly dilute and the system approaches a one component plasma in the low-field limit where the flow can continue to large b. For $T > T_{KT}$, the renormalized density of thermal vortices first decreases and then increases, so that the system is always a two-component plasma with a scale-dependent charge imbalance. Our principal results for the asymptotic behavior of magnetization [Eqs. (1) and (2)] follow from the exactly known free energy of a dilute one component plasma[14] whose parameters, the charge density, core energy, and dielectric constant, are given by the RG flows above. A Debye-Hückel approximation for the free energy of the two-component plasma reproduces the correct low-field behavior for $T \leq T_{KT}$ and provides a reasonable approximation elsewhere, including $T > T_{KT}$; we shall use it to generate plots of the magnetization versus field in the vicinity of the transition.

D. Results

1. Low fields, $T \leq T_{KT}$

For $T \leq T_{KT}$ the above Kosterlitz-Thouless RG flows go to small fugacity $y \equiv y(b)$ [and $x \equiv x(b) < 0$], where the leading terms in the exact free energy are known for low density[14] (small field). Thus we run the RG to a scale $a = ba_0$, where the parameters are $n \equiv n(b) = Ba^2/\phi_0$, x and y. We stop and evaluate $\mathcal{F}$ when n, which is growing, becomes of order y. The free-energy density of the remaining vortices, to leading order in the small parameters n and y, is[14]

$$\mathcal{F} = nk_B T\left(\ln\frac{2\pi}{y} + x\ln n + \mathcal{O}(1)\right). \tag{16}$$

The first term is the (renormalized) core energy of the field-induced vortices, while the second term contains their entropy and (renormalized) interaction energy. The thermally excited vortex-antivortex pairs do not enter at this order. Thus from Eq. (15) the magnetization is

$$M = -\frac{k_B T}{d\phi_0}\left(\ln\frac{2\pi}{y} + x\ln n + \mathcal{O}(1)\right). \tag{17}$$

To start, we consider the regime of small fields below T_{KT}. For $b^2 \gg e^{2\gamma_4/c}$, where γ_4 is a constant, the renormalized parameters are computed from the Kosterlitz RG equations as [cf. Eq. (12)]

$$x \to -c \sim -\sqrt{T_{KT}-T}, \tag{18}$$

$$y \to \frac{2cy_0}{b^{2c}(c+|x_0|)}, \tag{19}$$

so the low-field magnetization is

$$M = -\frac{k_B T}{d\phi_0}\left(c\ln\frac{\phi_0}{Ba_0^2} + \ln\frac{\pi(c+|x_0|)}{cy_0} + \mathcal{O}(1)\right), \tag{20}$$

in agreement with our Eq. (2) above (with $\gamma_3 \sim c^{1/c}$). The low-field regime of validity of this expression requires $n \ll 1$, which, with the above constraint on b, translates to

$$B \ll \frac{\phi_0}{a_0^2}e^{-2\gamma_4/c}. \tag{21}$$

Observe that this condition defines a crossover length scale

$$\xi_< \sim a_0 e^{\gamma_4/c}, \tag{22}$$

which has the functional form of the correlation length above the transition, and yet is defined—like the Josephson correlation length in Goldstone phases—*below* the transition.

For higher fields and temperatures near T_{KT}, there is a crossover to the critical behavior (1), that we derive below. Note that the slope of M vs $\ln B$ at low field vanishes linearly in c as T approaches T_{KT} from below, while the crossover field below which this is the behavior vanishes exponentially.

Finally, the singular temperature dependence of the offset in Eq. (20) (second term) implies that as the transition is approached the field at which low field M vs $\ln B$ extrapolates to 0 diverges (as $\sim 1/\gamma_3$).

If, on the other hand, one is near the Kosterlitz-Thouless transition (so $c \ll 1$), the RG flows for $1 \ll b^2 \ll e^{2\gamma_4/c}$ follow

$$-x \approx y \approx \frac{1}{2\ln b}. \tag{23}$$

Here when we match at $n \approx y$, it is at $y \approx -1/\ln(Ba_0^2/\phi_0)$ and the resulting magnetization is

$$M = -\frac{k_B T}{d\phi_0}\left(\ln \ln \frac{\phi_0}{Ba_0^2} + \mathcal{O}(1)\right), \tag{24}$$

which is the same as Eq. (1). The field range where this result applies is

$$\frac{\phi_0}{a_0^2} \gg B \gg \frac{\phi_0}{\xi_<^2}, \tag{25}$$

which is an intermediate field range for T near to but below T_{KT}, and is all low fields for $T = T_{KT}$. Our method of analysis in principle also gives the functional form of the crossover scaling function between these two regimes, but we have not been able to write this function in any concise form. However, below we obtain an approximate scaling function covering both regimes.

It is worth noting that in the vicinity of T_{KT} the evaluation of $\mathcal{F}$, which we choose to do at $n \cong y$, occurs at $|x| \ll 1$, so it is the renormalized core energy [the first term in Eq. (17)] that is dominant in determining the magnetization.

2. Approximate results, all T

Building on this last observation we now consider a Debye-Hückel mean-field theory whereby the matching free energy is approximated by the contributions from the renormalized core energy and the entropy. Conveniently, this Debye-Hückel approximation correctly reproduces the asymptotically exact results for $T \leq T_{KT}$ and it will also allow us to obtain numerical results for the magnetization in all field and temperature regimes of interest. However, we must caution that these are approximate results, and the overall value of the magnetization does depend on aspects of the calculation that are neither *a priori* constrained by what we already know about these materials nor universal. These unconstrained freedoms in the approximation we present below include (i) the form of the free energy, in particular one might also include an approximation to the energy of vortex-vortex interactions, (ii) the scale at which this free energy is evaluated (matched), and (iii) the bare vortex core energy.

Specifically, we approximate the residual free-energy density as

$$\mathcal{F} = (n^+ + n^-)E_c + k_B T(n^+ \ln n^+ + n^- \ln n^-), \tag{26}$$

where n^+ and n^- are the residual vortex and antivortex densities, respectively. The field constrains the difference between these densities to be $n^+ - n^- = n$, but their mean, $\bar{n} = (n^+ + n^-)/2$, can vary and will take on the value that minimizes the free energy:

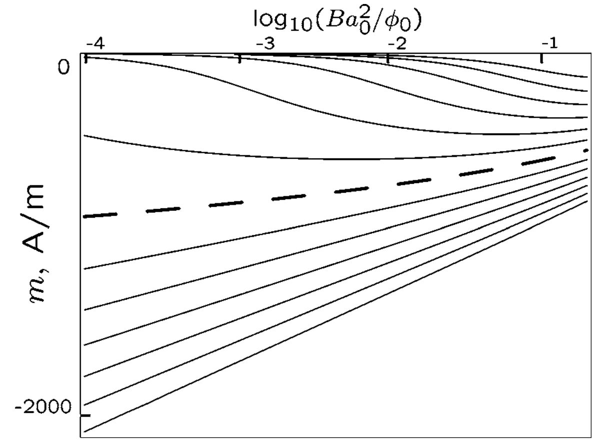

FIG. 2. Semilogarithmic plot of magnetization. Parameters are as in Fig. 1.

$$\frac{\partial \mathcal{F}}{\partial \bar{n}} = 2E_c + k_B T \ln\left(\bar{n}^2 - \frac{n^2}{4}\right) + 2k_B T = 0. \tag{27}$$

We define the matching condition by asking that the average density of vortices [as determined from Eq. (27)] be 1:

$$\bar{n} = \sqrt{\frac{n^2}{4} + \left(\frac{y}{2\pi e}\right)^2} \equiv 1 \tag{28}$$

and this is used to solve for $b(B,T)$. The choice of precisely unit density is clearly arbitrary—we make it for concreteness. With these assumptions the magnetization is given by

$$M = -\frac{k_B T}{2d\phi_0} \ln \frac{1 + \frac{n}{2}}{1 - \frac{n}{2}} \tag{29}$$

$$= -\frac{k_B T}{d\phi_0} \ln \frac{1 + \sqrt{1 - (y/2\pi e)^2}}{y/(2\pi e)} \tag{30}$$

which is plotted in Figs. 1–3.

Above T_{KT} the RG flows at asymptotically small fields are terminated at a finite fugacity and $a_0 b \sim \xi$ (the zero-field correlation length), so that as $B \to 0$ the magnetization vanishes as

$$M \approx -\frac{k_B T}{2d\phi_0^2}\xi^2 B \equiv -\frac{k_B T a_0^2}{2d\phi_0^2} e^{\pi/c} B. \tag{31}$$

This expression is consistent with an earlier linear-response result.[16]

3. Comments

Finally, three comments are in order. First, the divergence of M below T_{KT} signals the expulsion of vortices from that

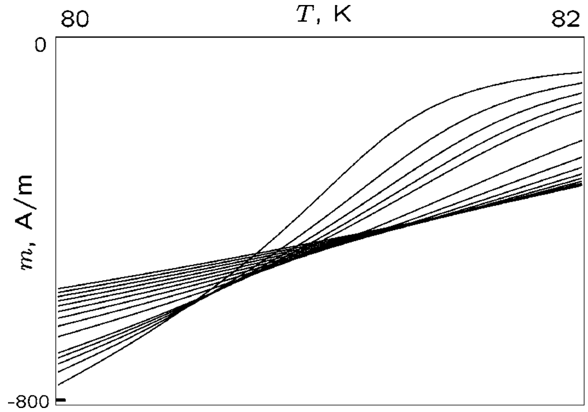

FIG. 3. Magnetization at 1, 2, 3, 4, 5, 10, 15, 20, 25, 30, 35, 40, 45, and 50 T as a function of temperature. Parameters are as in Fig. 1, in particular, T_{KT}=80 K.

phase—indeed the logarithmic divergence of the free energy of a single vortex with system size. The divergence we find *at* T_{KT} likewise signals the now weaker expulsion of vorticity; at T_{KT} the free energy of a single vortex grows with system size as $\ln(\ln L)$.

Second, the divergence of $M(B)$ at low B is a correct statement about the *magnetic induction* dependence of the free energy of a single layer in the limit of infinite penetration depth. It does not imply the divergence of an actual three-dimensional magnetization density. For a stack of planes, the self-consistency implicit in Eq. (6) keeps $M(\mu_0 H)$ from blowing up—instead we get the Meissner phase ($B=0$) at sufficiently low applied fields. Our results are valid in the experimentally relevant regime $M \ll H$ (equivalently $H \gg H_{c1}$) where this distinction is not important.

Finally, for $T \neq T_{KT}$ as we have already remarked $B^* \approx \phi_0/\xi_<^2$ separates the small field behavior from the critical $\ln\ln 1/B$ dependence at larger fields. Interestingly, the magnetization displays a noticeable temperature dependence even at these larger fields. It is noteworthy that this dependence is opposite to that implied by the explicit prefactor of temperature [e.g., in Eq. (30)], which by itself predicts an unintuitive enhanced diamagnetism at higher temperatures. To understand this important detail consider raising T in the vicinity of T_{KT}. This leads to an increase in y, which in turn gets strongly amplified by the singularity at $y=0$ of the functional dependence $M \sim \ln 1/y$. This variation always overwhelms the contribution from the prefactor of $k_B T$, so the combined effect is a reduction of $|M|$ upon raising the temperature, as expected.

E. Dimensional crossovers in layered and quantum systems

In sufficiently anisotropic materials we expect that various weak interlayer couplings present will only affect the physics at long length scales and can therefore be treated to simply restrict the domain of applicability of the purely two-dimensional treatment above. As the application of the magnetic field introduces a new length scale, the intervortex separation or the magnetic length, this restriction means our two-dimensional behavior will cross over to three-dimensional behavior near a low crossover field. There are two basic types of interlayer interactions here: Josephson couplings induced by charge fluctuations transverse to the layers, and electromagnetic couplings between currents in different layers mediated by the fluctuating magnetic field. We consider their effects in turn.

The Josephson coupling, J, will (i) induce true three-dimensional long-range order in the low-temperature phase in zero field and alter the universality class of the transition to that of the 3D *XY* model, (ii) shift the transition temperature upwards to T_c, and (iii) cause the vortices to crystallize three dimensionally at any temperature below T_c for fields less than the melting field $B_m(T)$. While this is a problem that has been studied in some detail[17] we can get a feeling for the scales involved by considering renormalization of Ja^2, the Josephson coupling per cutoff area, near the (decoupled layer) Kosterlitz-Thouless fixed point,

$$\frac{\partial(Ja^2)}{\partial \ln b} = \left(2 - \frac{k_B T}{2\pi\rho_s}\right)(Ja^2). \tag{32}$$

From this equation we can obtain the scaling of both the shift of the critical temperature in zero field, δT_c, and the field scale, B_J, marking the crossover between two- and three-dimensional behaviors. This is done by evaluating the length scale at which the renormalized value of Josephson coupling is comparable to the in-plane stiffness, $Ja^2 \approx \rho_s$ (which we shall set to its critical value $\rho_s = 2k_B T_{KT}/\pi$, since its renormalization is comparatively less important than that of Ja^2). The upward shift of the critical temperature due to the Josephson coupling, δT_c, can be estimated by identifying this scale with the zero-field correlation length, giving[18–20]

$$\frac{\delta T_c}{T_{KT}} = \frac{(7\pi\gamma_5/8)^2}{\ln^2 \dfrac{k_B T_{KT}}{\gamma_6 J_0 a_0^2}}, \tag{33}$$

where γ_5 is a nonuniversal constant related to the behavior of c near T_{KT} via $\gamma_5 c \approx \sqrt{(T-T_{KT})/T_{KT}}$, γ_6 is another constant of order 1, and J_0 is the bare Josephson coupling per unit area. Similarly, by substituting the magnetic length in place of ξ, we find for the crossover field at T_{KT}

$$B_J = \frac{\phi_0}{a_0^2}\left(\frac{J_0 a_0^2}{k_B T_{KT}}\right)^{8/7}. \tag{34}$$

The case of electromagnetic interactions alone is a little muddier. The situation in finite magnetic fields is that the vortices in different layers now experience an attraction and thus can crystallize three dimensionally even when a single layer is a vortex liquid. Following Glazman and Koshelev[21] one can use the elastic theory of such a crystal to identify a field scale, B_{cr}, at which crossover to two-dimensional behavior takes place. In the limit of zero Josephson coupling our estimate reads

$$B_{cr} \sim \phi_0/\lambda^2, \tag{35}$$

up to (potentially large) factors of order $\ln \lambda/d$. Ideally, one would have liked to complement this estimate with an RG based analysis starting in the vortex liquid (as in the previous paragraph). Unfortunately, we have not managed such an analysis, partly because the impact of these magnetic interlayer couplings on the zero-field transition does not appear to be a settled problem. The interlayer interactions are also logarithmic and thus are marginal operators at face value. This has led to assertions that the actual transition is still in the Kosterlitz-Thouless universality class.[1,22] However, a renormalization-group analysis by Timm[23] finds flows at variance with this scenario.

The scales B_J and B_{cr} mark the rough boundary between two- and three-dimensional physics. At T_{KT} the low-field three-dimensional state is crystalline, and the high-field two-dimensional dependence of M crosses over to fairly standard (but with renormalized parameters) low-field behavior in the Abrikosov vortex lattice. At higher temperature (at the three-dimensional T_c), in the presence of Josephson couplings, the crossover is instead to the $M\sim-\sqrt{H}$ behavior expected at low field in the 3D *XY* critical regime.

Finally, one additional crossover, this time at large fields, is possible when the thermal phase transitions take place in proximity to a quantum phase transition out of the superconducting state—as may be germane in the case of the underdoped cuprates. Standard scaling, when applied to magnetization of the $2+1$-dimensional quantum critical theory predicts $M\sim-\sqrt{H}$, with crossover to this behavior taking place at sufficiently large fields where quantum fluctuations of the order parameter are important. The transition between the forms derived in this paper and this regime would be a striking signature of such fluctuations.

F. Prior work

Early experiments on the most anisotropic cuprates saw a near crossing point for the curves $M(T)$ taken at varying fields B. This is equivalent to the statement that they found a temperature at which the magnetization was field independent. This was interpreted in the scaling framework[2] as evidence for two-dimensional critical behavior with the one-parameter scaling form,

$$f(t,B) = \frac{1}{\xi^2}\tilde{f}(B\xi^2), \tag{36}$$

for the free-energy density. As we have shown in this paper, the scaling is not so simple at the Kosterlitz-Thouless transition, with its two marginal operators, and magnetization instead has a weak double-logarithmic dependence on the field. Aside from this general scaling argument, there are three types of prior computations that we are aware of (not listed in chronological order):

First, Gaussian fluctuations for the Ginzburg-Landau theory in $d=2$, yield a field-independent magnetization at the Gaussian (mean field) transition temperature.[24] This is consistent with the absence of any marginal operators at this unstable fixed point, but inconsistent with the true low-field critical scaling of M, as demonstrated by our calculation.

Second, the so called "lowest Landau level" approximation has been used by Tesanovic and others to study strong amplitude fluctuations at high fields near mean field H_{c2} (see, e.g., Refs. 25 and 26). Calculations of this kind have no overlapping regime of validity with ones presented above for the low-field regime. It would be worthwhile to see if the two sets of results can together capture the behavior over the entire range of fields of interest.

Finally, Bulaevskii, *et al.*[27] have considered the effects of thermal fluctuations on the magnetization of an Abrikosov lattice in a layered superconductor with the Josephson coupling being the dominant source of the three-dimensional order. Treating the phonons of the lattice to quadratic order they compute the entropic correction to the Ginzburg-Landau-Abrikosov free energy. At low fields $B \ll B_{cr}$ they find a field-independent correction to the leading (Abrikosov) logarithm. For $B \gg B_{cr}$ they report a correction which is itself a logarithm and leads to an expression identical to our Eq. (2). However, the Lindemann estimate for the melting field of the three-dimensional vortex lattice in a layered superconductor is, in fact, comparable to B_{cr} in the regime of weak interlayer Josephson coupling and temperatures near T_{KT} of interest here. Moreover, since even the 2D quasi-long-range crystalline order is absent near T_{KT}, the method employed by Bulaevskii *et al.* is not really valid in the relevant regime of the logarithmic dependence where the system is a vortex liquid, rather than a crystal. Nevertheless, at sufficiently low fields $B \ll \phi_0/\xi_<^2$ and $T < T_{KT}$ their result and ours agree, a sign that the vortex liquid is locally quite similar to the crystal for these parameters. At higher fields and at T_{KT}, however, their reasoning breaks down as our calculation explicitly demonstrates.

These and other results were nicely summarized and extended by Koshelev.[24] Indeed, this paper contains an expression for the magnetization [Eqs. (51) and (55) taken together] that is claimed to apply the vortex liquid state and uses results from the theory of the one-component plasma and thus, *prima facie*, anticipates our work. However, it appears that the basic physics, e.g., in our Eq. (2), that at low fields the logarithmic field derivative should vanish as T_{KT} is approached from below is missed in this work. Also, as this work explicitly ignores thermally excited vortex-antivortex pairs it cannot account for the critical field dependence of magnetization in Eq. (1).

III. EXPERIMENTS

We now turn to the existing data on the cuprates, the systems that have motivated this work, and some suggestions for experimental tests of the theory.

Many of the cuprates are highly two dimensional and appear to become increasingly so with underdoping. The two most commonly used diagnostics of anisotropy are the resistivity and superfluid density tensors. Using either of these in materials such as BSSCO-2212 we arrive at estimates of the anisotropy of order $10^{-4}-10^{-6}$ between the *ab* (Cu-O) planes and the c axis. A somewhat more direct measure of the anisotropy can be obtained from observations of the c-axis plasma resonance due to interlayer Josephson coupling. From the results of Ref. 28 for Josephson coupling per area $J_0 \sim 10^{-8}\ \mathrm{J/m^2}$ we obtain

$$\frac{J_0 a_0^2}{k_B T_c} \approx 10^{-6}. \tag{37}$$

For such anisotropies and other parameters appropriate to the cuprates, e.g., as in the caption of Fig. 1, we find $\delta T_c/T_{KT} \approx 0.02$, $B_J \approx 0.0002$ T, and $B_{cr} \approx 0.004$ T using the estimates derived in Sec. II E. This indicates that our two-dimensional theory should give a useful account of the superconducting fluctuations for a reasonable range of fields.

The scale for magnetization effects is set by $k_B T_{KT}/d\phi_0$ which is ≈ 350 A/m for our parameters. The dimensionless factors multiplying it in our expressions are not too different from unity. This order of magnitude estimate is consistent with experimental observations.[4,29,30] The trends in the temperature and field dependence of the magnetization we show in our figures here are mostly in good qualitative agreement with those seen in the experiments.[4,29,30] The most detailed published investigation is that by Kogan and collaborators[29] (see also Ref. 30) who were inspired by the predictions of Ref. 27. They reported evidence that $M \sim \ln H$ with a coefficient that changes sign at T_{KT}. While this claim below T_{KT} is consistent with our analysis, this is not so above T_{KT} where the functional form is no longer a logarithm. This suggests that a reexamination of that regime is in order.

Finally, we record the salient results of our analysis that can be tested against careful measurements:

(i) A logarithmic variation of the magnetization with field in the low-field two-dimensional vortex liquid below T_{KT}, with a coefficient set by the superfluid density and the temperature. Specifically,

$$\frac{\partial M}{\partial \log H} \approx \frac{\pi \rho_s(T)}{2 d \phi_0}\left(1 - \frac{2 k_B T}{\pi \rho_s(T)}\right).$$

Thus the theory predicts this simple relation between these directly measurable quantities, that can be checked in any sufficiently two-dimensional material where this regime should exist.

(ii) The low-field measurements of M can be used to *define* an effective H_{c2} by extrapolating M vs $\ln H$ (which starts off a straight line) to zero (see, e.g., Ref. 4). Contrary to the standard, mean-field, behavior of $H_{c2} \to 0$ near transition, our results predict that the thus defined H_{c2} *diverges* (as $\sim 1/\gamma_3$) near the transition, although too close to the transition the procedure breaks down as the size of the linear regime shrinks.

(iii) The double logarithmic variation of M in Eq. (1) at T_{KT} or at intermediate fields away from T_{KT}. Optimistically, one might hope for a direct fit to this functional form. This field dependence also implies that in an M vs $\ln H$ plot, the curves for $T < T_{KT}$, though straight in the low-field regime, should exhibit an upward curvature at larger fields (see Fig. 2).

(iv) Correspondingly, plots of $M(T)$ at different fields should exhibit a systematic drift of the "crossing point." This is clear from plotting our results, as in Fig. 3, which explicitly shows the downward creep of the crossing point towards T_{KT} from above as the field is decreased within the two-dimensional vortex liquid regime. This trend definitely appears to be there in the recent BSSCO data of Wang, *et al.*[4] for underdoped and optimally doped samples.

IV. SUMMARY AND FUTURE DIRECTIONS

The mechanism of superconductivity in the cuprates remains one of the outstanding puzzles of the physics of correlated electrons. Nevertheless, the proposition that aspects of their finite temperature behavior can be understood as consequences of sizeable thermal fluctuations of the superconducting order parameter has gained support in recent work. In this work we have examined the effects of such fluctuations in the two-dimensional limit near the Kosterlitz-Thouless transition and presented an asymptotically exact calculation of the magnetization in this vortex liquid state. The preliminary comparison with the highly anisotropic cuprates such as BSSCO is encouraging.

On the theoretical side, it would be very useful to extend our calculation to higher fields, e.g., by keeping terms of higher order in vortex density in deriving flow Eqs. (8)–(10). The high-field behavior is one aspect of experimental magnetization data that does not appear to be captured well by our calculation. Strong signatures of superconducting fluctuations are also present in the Nernst coefficient. Indeed, Ref. 4 has reported that the magnetization and the Nernst signal track each other. It would be desirable to have a theory of the Nernst effect near the Kosterlitz-Thouless transition. We hope to report on this in the near future.

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APPENDIX: SUPERCONDUCTOR IN A FIELD TO COULOMB PLASMA

For completeness, we sketch the mapping between a superconductor in a transverse field and a non-neutral Coulomb plasma.[11] The Hamiltonian of a two-dimensional superconductor with a uniform magnetic induction $\mu_0 H_{c1} \ll B \ll \mu_0 H_{c2}$ can be approximated as

$$H_{\mathrm{SC}} = \int d^2x \frac{\rho_{s0}}{2}\left|\nabla\theta - \frac{2e}{\hbar c}\mathbf{A}\right|^2, \tag{A1}$$

where θ, $\mathbf{A}$, and $B = \hat{\mathbf{z}} \cdot \nabla \times \mathbf{A}$ are the usual phase, gauge, and perpendicular magnetic fields. We also impose the constraint that the net vortex charge density matches B. Next, we explicitly separate the phase field into its longitudinal and transverse components by introducing a spin-wave field ϕ and vortex charge density field n and Fourier-transform the Hamiltonian

$$H_{\mathrm{SC}} = L^2 \sum_{\mathbf{q}\neq\mathbf{0}} h_{\mathbf{q}}, \tag{A2}$$

$$h_{\mathbf{q}} = \frac{\rho_{s0}}{2}\left| \mathbf{q}\phi_{\mathbf{q}} + n_{\mathbf{q}}\frac{\mathbf{z}\times\mathbf{q}}{q^2}\right|^2 \tag{A3}$$

$$= \frac{\rho_{s0}}{2}|\mathbf{q}\phi_{\mathbf{q}}|^2 + \frac{\rho_{s0}}{2}\left| n_{\mathbf{q}}\frac{\mathbf{z}\times\mathbf{q}}{q^2}\right|^2. \tag{A4}$$

We then ignore the spin-wave part as it decouples from the rest of the problem. Transforming back into real space we arrive at the Hamiltonian of a non-neutral two-component Coulomb gas,

$$H_{\mathrm{CG}} = \frac{\pi\rho_{s0}}{4}\int d^2\mathbf{x}d^2\mathbf{y}\,\delta n(\mathbf{x})\ln\left(\frac{(\mathbf{x}-\mathbf{y})^2}{a_0^2}\right)\delta n(\mathbf{y}), \tag{A5}$$

where $\delta n(\mathbf{y}) = n(\mathbf{y}) - B/\phi_0$. The range of integration above excludes the infinite self-interaction $\mathbf{x}=\mathbf{y}$ of each vortex. One last remaining ingredient is the bare core energy E_{c0} determined by the physics omitted in this derivation, it is usually taken as the energy cost of suppressing the order parameter inside the vortex core. So, finally, the Hamiltonian of a superconductor in the external field is written $H_{\mathrm{SC}} = H_{\mathrm{CG}} + E_{c0}N_T$, which is a continuum version of Eq. (7).

[1] G. Blatter, M. V. Feigel'man, V. B. Geshkenbein, A. I. Larkin, and V. M. Vinokur, Rev. Mod. Phys. **66**, 1125 (1994).
[2] *Phase Transition Approach to High Temperature Superconductivity*, edited by T. Schneider and J. M. Singer (Imperial College Press, London, 2000).
[3] J. Orenstein and A. J. Millis, Science **288**, 468 (2000).
[4] Y. Wang, L. Li, M. J. Naughton, G. Gu, and N. P. Ong, Phys. Rev. Lett. **95**, 247002 (2005).
[5] Y. Wang, Z. A. Xu, T. Kakeshita, S. Uchida, S. Ono, Y. Ando, and N. P. Ong, Phys. Rev. B **64**, 224519 (2001).
[6] Y. Wang, N. P. Ong, Z. A. Xu, T. Kakeshita, S. Uchida, D. A. Bonn, R. Liang, and W. N. Hardy, Phys. Rev. Lett. **88**, 257003 (2002).
[7] I. Ussishkin, S. L. Sondhi, and D. A. Huse, Phys. Rev. Lett. **89**, 287001 (2002); I. Ussishkin and S. L. Sondhi, Int. J. Mod. Phys. B **18**, 3315 (2004).
[8] M. Randeria, N. Trivedi, A. Moreo, and R. T. Scalettar, Phys. Rev. Lett. **69**, 2001 (1992); M. Randeria, in *Models and Phenomenology for Conventional and High-Temperature Superconductors*, edited by G. Iadonisi, J. R. Schrieffer, and M. L. Chiofalo (IOS Press, Amsterdam, 1998), p. 53.
[9] V. J. Emery and S. A. Kivelson, Nature (London) **374**, 434 (1995).
[10] P. H. Kes, C. J. van der Beek, M. P. Maley, M. E. McHenry, D. A. Huse, M. J. V. Menken, and A. A. Menovsky, Phys. Rev. Lett. **67**, 2383 (1991); J. Hofer, T. Schneider, J. M. Singer, M. Willemin, H. Keller, T. Sasagawa, K. Kishio, K. Conder, and J. Karpinski, Phys. Rev. B **62**, 631 (2000), and references therein; see also references in Refs. 2 and 24.
[11] P. Minnhagen, Rev. Mod. Phys. **59**, 1001 (1987).
[12] P. W. Anderson and G. Yuval, J. Phys. C **4**, 607 (1971).
[13] J. M. Kosterlitz, J. Phys. C **7**, 1046 (1974). Equations (8)–(10) in Sec. II were derived by generalizing the Appendix of this paper by allowing for a nonequal number of indistinguishable positive and negative charges, and modifying the combinatorics involved in the process. In the end, however, different factors cancelled out yielding zero-field (neutral) flow equations.
[14] B. Jancovici, Phys. Rev. Lett. **46**, 386 (1981).
[15] S. F. Lee, D. C. Morgan, R. J. Ormeno, D. Broun, R. A. Doyle, J. R. Waldram, and K. Kadowaki, Phys. Rev. Lett. **77**, 735 (1996).
[16] B. I. Halperin and D. R. Nelson, J. Low Temp. Phys. **36**, 599 (1979).
[17] A. E. Koshelev, Phys. Rev. B **56**, 11201 (1997).
[18] See, e.g., L. I. Glazman and A. E. Koshelev, Sov. Phys. JETP **70**, 774 (1990).
[19] J. R. Clem, Phys. Rev. B **43**, 7837 (1991).
[20] H. Fangohr, A. E. Koshelev, and M. J. W. Dodgson, Phys. Rev. B **67**, 174508 (2003).
[21] L. I. Glazman and A. E. Koshelev, Phys. Rev. B **43**, 2835 (1991).
[22] C. S. O'Hern, T. C. Lubensky, and J. Toner, Phys. Rev. Lett. **83**, 2745 (1999).
[23] C. Timm, Phys. Rev. B **52**, 9751 (1995).
[24] A. E. Koshelev, Phys. Rev. B **50**, 506 (1994).
[25] Z. Tesanovic, L. Xing, L. Bulaevskii, Q. Li, and M. Suenaga, Phys. Rev. Lett. **69**, 3563 (1992).
[26] D. Li and B. Rosenstein, Phys. Rev. Lett. **86**, 3618 (2001).
[27] L. N. Bulaevskii, M. Ledvij, and V. G. Kogan, Phys. Rev. Lett. **68**, 3773 (1992).
[28] S. P. Bayrakci, O. K. C. Tsui, N. P. Ong, K. Kishio, and S. Watauchi, Europhys. Lett. **46**, 68 (1999).
[29] V. G. Kogan, M. Ledvij, A. Yu. Simonov, J. H. Cho, and D. C. Johnston, Phys. Rev. Lett. **70**, 1870 (1993).
[30] J. C. Martinez, P. J. E. M. van der Linden, L. N. Bulaevskii, S. Brongersma, A. Koshelev, J. A. A. J. Perenboom, A. A. Menovsky, and P. H. Kes, Phys. Rev. Lett. **72**, 3614 (1994); J. C. Martinez, S. H. Brongersma, A. Koshelev, B. Ivlev, P. H. Kes, R. P. Griessen, D. G. de Groot, Z. Tarnavski, and A. A. Menovsky, Phys. Rev. Lett. **69**, 2276 (1992).