Model F in two-loop order and the thermal conductivity near the superfluid transition of ^{4}He

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(Received 22 August 2005; revised manuscript received 23 December 2005; published 13 March 2006)

A nonlinear renormalization-group (RG) analysis of the thermal conductivity data near the λ line of ^{4}He is performed on the basis of the corrected dynamic RG flow equations of model F in two-loop order. Our analysis eliminates recent uncertainties in the regime $(T-T_\lambda)/T_\lambda \lesssim 10^{-6}$, which is most sensitive to the effect of the weak-scaling fixed point. This effect is found to be stronger than described previously. We also improve the Borel summation of the amplitude of the specific heat.

DOI: 10.1103/PhysRevB.73.092503 PACS number(s): 64.60.Ht, 67.40.Pm

The thermal conductivity near the superfluid transition of ^{4}He plays an important role for a quantitative test of the dynamic renormalization-group (RG) theory.[1] Unlike the scaling fixed point,[2] the weak-scaling fixed point[3] causes a nonuniversal increase of the effective amplitude $R_\lambda^{\text{eff}}(t)$ of the thermal conductivity for $t=(T-T_\lambda)/T_\lambda \lesssim 10^{-4}$ that must be described in terms of effective parameters of a nonlinear RG flow.[4] A fully quantitative description was provided by a complete two-loop calculation of R_λ^{eff} (Ref. 5) and of the RG flow equations[6] within model F^2, in combination with a Borel summation of critical statics.[7,8] Very good agreement with experimental data[9–11] was found,[12] thus confirming the reliability of the two-loop RG theory. Of particular significance is the regime $t \lesssim 10^{-6}$, which is most sensitive to the weak-scaling fixed point and that is outside the range over which a fit to the data[9] was performed. For this reason it was disappointing that a recent two-loop calculation of the RG flow equations by Folk and Moser[13] yielded a new prediction[14] for R_λ^{eff} that is in less good agreement with the data in the weak-scaling range $t \lesssim 10^{-6}$ (Fig. 1, dashed curve). In Ref. 15 the two-loop result of Ref. 13 was corrected but no new comparison with the data was performed, thus leaving unanswered the important question as to the quantitative reliability of the two-loop RG theory for the weak-scaling region. Also the analysis at higher pressures in Fig. 8 of Ref. 14 remained uncorrected.

In order to eliminate the existing uncertainties we have reexamined[16] our calculation.[6] Our corrected RG function ζ_Γ reads[16]

$$\zeta_\Gamma = \zeta_\Gamma^{(91)} + 4F^2\gamma D w x^2 y + 8\gamma^2 D^2 (w-w^*)\frac{xy}{w^2}\ln(2wy) + 8\gamma^3 D x^2 y w^{*-1}(ww^* - w - w^*)\ln\frac{w(w+2w^*)}{(w+w^*)^2}, \quad (1)$$

where $\zeta_\Gamma^{(91)}$ is our original result.[6] Equation (1) differs from Eq. (13) of Ref. 13 and from Eq. (36) of Ref. 14 but is in agreement with the corrected result of Ref. 15.

On the basis of our corrected result for ζ_Γ we have performed a nonlinear RG analysis[6] and have determined the effective dynamic parameters $w[t]=w'[t]+iw''[t]$ and $F[t]$ or $f[t]=(F[t])^2/w'[t]$. The result for saturated vapor pressure (SVP) is presented in Fig. 1 (thick curve) and in Table I which corrects our original Fig. 5(a) and the SVP part of Table II of Ref. 6. The corresponding static parameters $u[t]$ and $(\gamma[t])^{\text{expt}}$ are the same as in Table II of Ref. 6. The effect of the correction on R_λ^{eff} for $t<10^{-6}$ is significant compared to the uncorrected curve of Ref. 14 (dashed line in Fig. 1) and small but non-negligible compared to the original curve of Ref. 6 (thin line). The same statement holds for higher pressures (see below).

Figure 1 demonstrates that the effect of the weak-scaling fixed point at SVP is stronger than described earlier.[6,13,14]

The weak-scaling effect is not restricted to SVP but is also important at higher pressures. The weak-scaling fixed point introduces a source of nonuniversality (pressure dependence) along the λ line $T_\lambda(P)$ that is expected to be part of the origin of the observed breakdown of scaling and universality of recent finite-size thermal conductivity data.[18] In order to distinguish nonuniversal *static* effects from nonuniversal *dynamic* effects within finite-size RG theory[12,19] it is

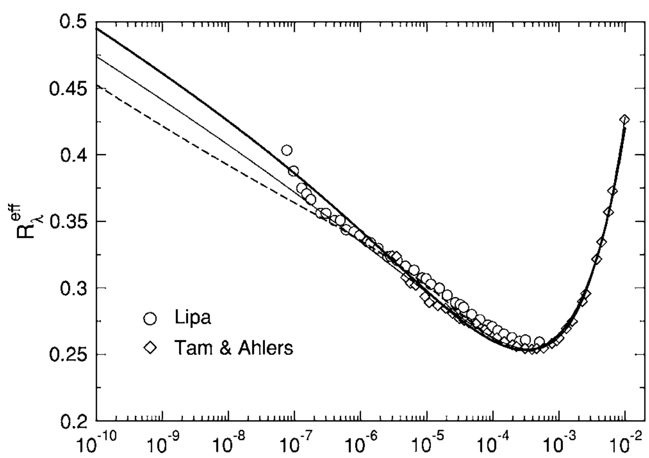

FIG. 1. Effective amplitude $R_\lambda^{\text{eff}}(t)$, Eq. (4.25) of Ref. 6, versus t at SVP. The thick solid curve is the prediction of the present paper. It is obtained by fitting $R_\lambda^{\text{eff}}(t)$ to the data of Tam and Ahlers (Ref. 9) in the range $10^{-6}<t\lesssim 10^{-2}$ using the corrected ζ_Γ of Eq. (1). For numerical values see our Table I. Dashed curve from Fig. 7 of Ref. 14. Thin curve from Fig. 5(a) of Ref. 6. The circles are the data of Lipa (Ref. 17). The increase of $R_\lambda^{\text{eff}}(t)$ for $t<10^{-4}$ is due to the weak-scaling fixed point.

TABLE I. Representative values of the amplitude $R_\lambda^{\mathrm{eff}}(t)$ shown in Fig. 1 (thick curve) and of the corresponding parameters $w'[t]$, $w''[t]$, and $F[t]$ at SVP.

$-\log_{10} t$	$w'[t]$	$w''[t]$	$F[t]$	$R_\lambda^{\mathrm{eff}}(t)$
2.0	0.6365	0.3185	0.6277	0.4206
3.0	0.6179	0.3415	0.7894	0.2650
4.0	0.4831	0.2141	0.7275	0.2601
6.0	0.2982	0.0650	0.5503	0.3430
9.0	0.1871	0.0161	0.4186	0.4614

imperative to correctly identify not only the dynamic RG parameters along the λ line on the basis of the corrected ζ_Γ function (1) but also to determine the pressure-dependent static RG parameters $u[t]$ and $(\gamma[t])^{\mathrm{expt}}$ as accurately as possible. This requires us to reexamine the static amplitude function $F_+(u)$ of the specific heat above T_λ (Refs. 8 and 20) on the basis of recent achievements within critical statics.[21,22]

In Refs. 6, 13, and 14 the amplitude function $F_+(u)$ was used in the form

$$F_+(u) = -2 - 16u(1 + 7.59u), \tag{2}$$

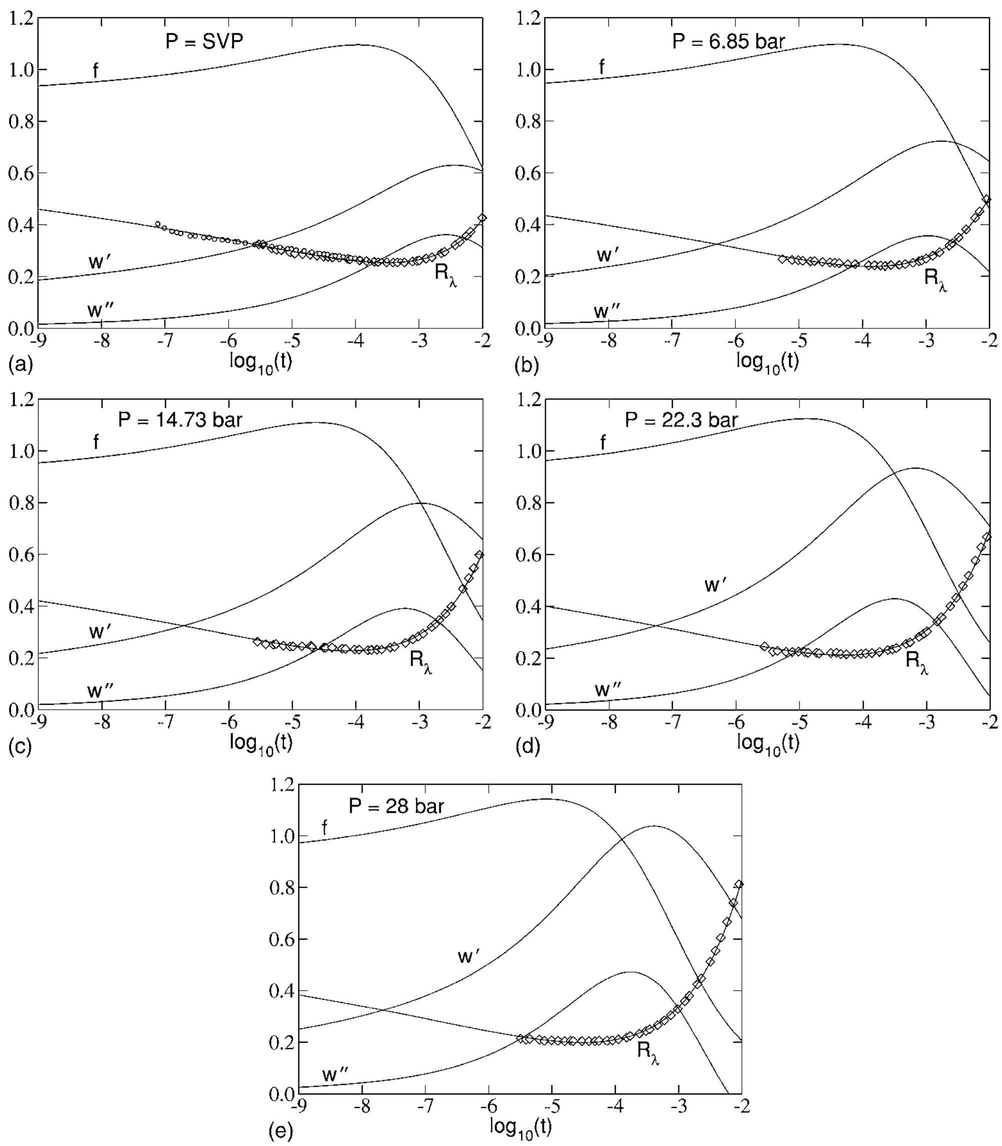

FIG. 2. Effective parameters $w'[t]$, $w''[t]$, and $f[t]=(F[t])^2/w'[t]$ of model F as obtained by integrating Eqs. (4.17) and (4.18) of Ref. 6 and by fitting the effective amplitude $R_\lambda^{\mathrm{theor}}(t)$, Eq. (4.25) of Ref. 6, to the data of Tam and Ahlers (Ref. 9) in the range $10^{-6} < t \leqslant 10^{-2}$. (a) SVP, (b) 6.85 bars, (c) 14.73 bars, (d) 22.3 bars, and (e) 28.0 bars. The static coupling $u[t]$ employed in this fit is the solution of the RG flow equation (4.7) of Ref. 6 with the initial values determined from a fit to the specific heat using Eq. (4.9) of Ref. 6. Instead of the solution $\gamma[t]$ of Eq. (4.8) of Ref. 6 the effective coupling $(\gamma[t])^{\mathrm{expt}}$ is employed which is derived from Eq. (D2) of Ref. 6. In all calculations the improved representation of our Eq. (3) for $F_+(u)$ is employed instead of Eq. (4.10) of Ref. 6. For numerical values related to (a) and (e) see our Table II. The circles in (a) are the data of Lipa (Ref. 17).

TABLE II. Effective parameters $u[t]$, $(\gamma[t])^{\text{expt}}$, $w'[t]$, $w''[t]$, $F[t]$, and $f[t]=(F[t])^2/w'[t]$ as well as the effective amplitude $R_\lambda^{\text{eff}}(t)$ for SVP and 28 bars shown in Figs. 2(a) and 2(e).

P (bars)	$-\log_{10} t$	$u[t]$	$(\gamma[t])^{\text{expt}}$	$w'[t]$	$w''[t]$	$F[t]$	$f[t]$	$R_\lambda^{\text{eff}}(t)$
SVP	1.0	0.031088	0.2074					
SVP	2.0	0.034518	0.2948	0.6074	0.3118	0.6130	0.6187	0.4209
SVP	3.0	0.035690	0.2433	0.5981	0.3391	0.7767	1.0086	0.2649
SVP	4.0	0.036048	0.2061	0.4720	0.2150	0.7189	1.0948	0.2602
SVP	5.0	0.036153	0.1806	0.3655	0.1188	0.6227	1.0609	0.2976
SVP	6.0	0.036184	0.1621	0.2948	0.0662	0.5469	1.0148	0.3427
SVP	7.0	0.036193	0.1478	0.2471	0.0389	0.4918	0.9790	0.3855
SVP	8.0	0.036195	0.1363	0.2127	0.0244	0.4504	0.9539	0.4245
SVP	9.0	0.036196	0.1269	0.1864	0.0164	0.4177	0.9363	0.4603
SVP	10.0	0.036196	0.1189	0.1653	0.0118	0.3907	0.9234	0.4940
28	1.0	0.016548	0.2521					
28	2.0	0.026377	0.3409	0.6783	−0.0711	0.3738	0.2060	0.8288
28	3.0	0.032542	0.2884	1.0028	0.3304	0.7730	0.5959	0.3368
28	4.0	0.035039	0.2395	0.9647	0.4608	0.9906	1.0172	0.2096
28	5.0	0.035851	0.2049	0.7081	0.2942	0.8992	1.1420	0.2056
28	6.0	0.036096	0.1803	0.5038	0.1515	0.7474	1.1088	0.2432
28	7.0	0.036167	0.1620	0.3793	0.0777	0.6314	1.0510	0.2920
28	8.0	0.036188	0.1478	0.3025	0.0426	0.5513	1.0048	0.3398
28	9.0	0.036194	0.1363	0.2511	0.0256	0.4942	0.9724	0.3835
28	10.0	0.036196	0.1269	0.2141	0.0168	0.4510	0.9500	0.4235

which was derived in Ref. 8 by means of a Borel summation.At that time, however, the perturbation expression for $F_+(u)$ within the minimal subtraction scheme was incomplete because of unknown higher-order terms of the RG function $B(u)$. In the meantime complete five-loop expressions of $B(u)$ and $F_+(u)$ have been derived.[21] While the Borel-resummed fixed-point value $B(u^*)=1.0053$ (Ref. 21) provides an excellent justification for our earlier approximation[6] $B(u)\approx 1$ this is not the case for the amplitude function $F_+(u)$. It was found[21] that the perturbation series of $F_+(u)$ is not well behaved in the sense that the first four coefficients of the series do not have alternating signs which then prevented the authors from performing a reliable Borel summation of $F_+(u)$. Thus the reliability of Eq. (2) is still an open problem. The function $F_+(u)$ enters the theoretical expression for the specific heat above T_λ (Ref. 20) and therefore affects the effective static couplings $u[t]$, $\gamma[t]$, and $(\gamma[t])^{\text{expt}}$ via Eqs. (4.9) and (D2) of Ref. 6. Furthermore the function $F_+(u)$ enters the expression for R_λ^{eff} via Eq. (4.24) of Ref. 6.

Here we circumvent the problem of a direct Borel summation of $F_+(u)$. We shall make use of a very recent four-loop calculation of the amplitude function $F_-(u)$ of the specific heat below T_λ (Ref. 22) and of reliable Borel summations[22,23] of $u^*F_-(u^*)$ and of the difference $u^*[F_-(u^*)-F_+(u^*)]$ at the fixed point $u^*=0.0362$. The results are[22,23] $u^*[F_-(u^*)-F_+(u^*)]=0.461$ and $u^*F_-(u^*)=0.383$. Taking the difference of these values we obtain $F_+(u^*)=-2.16$ instead of $F_+(u^*)=-2.74$ of Eq. (2) which improves the fixed-point value of F_+ by about 20%. Correspondingly we use the representation

$$F_+(u) = -2 - 16u(1-20.0u), \tag{3}$$

where the last term is determined such that Eq. (3) yields $F_+(u^*)=-2.16$ at the fixed point. Equation (3) significantly improves the earlier[6,8] approximation, Eq. (2).

We have determined both the effects of the corrected function ζ_Γ, Eq. (1), and of the improved function $F_+(u)$, Eq. (3), on $R_\lambda^{\text{eff}}(t)$ and on the RG parameters for several pressures along the λ line. The results are presented in Fig. 2 and Table II which correct and improve our original Figs. 3 and 5 and Table II of Ref. 6 as well as Fig. 8 of Ref. 14. The values of $R_\lambda^{\text{eff}}(t)$ for $t<10^{-6}$ are slightly higher than predicted previously. The changes of $F_+(u)$ and of $u[t]$ and $(\gamma[t])^{\text{expt}}$ in the nonasymptotic region are non-negligible within a highly accurate RG theory of the bulk[9,17] and finite-size[24] specific heat along the λ line. The changes of $w'[t]$ and $F[t]$ are small but non-negligible. The reduced value of $w''[t]$ in the background region $t\gtrsim 10^{-3}$ is now closer to our earlier estimate.[25] The slightly negative value of w'' at the highest pressure in the region $t\gtrsim 10^{-2}$ indicates the necessity of complementing the minimally renormalized RG theory by additional nonasymptotic contributions rather than enforcing[14,26] a positive value of w'' in the fitting procedure. This imaginary part w'' of w, however, has very little effect on R_λ^{eff} and on other observable quantities whose analytic expressions depend only weakly on w''. This implies that all earlier results and conclusions with regard to the experimentally observable transport coefficients in the region $t>10^{-6}$ based on our two-loop model-F flow equations remain essentially unchanged.[27] Nevertheless, the determination of the RG parameters $u[t]$, $(\gamma[t])^{\text{expt}}$, $w'[t]$, $w''[t]$, and $F[t]$ presented in this paper is relevant to a fully quantitative description of static[24] and dynamic[18,28] finite-size effects as well as of boundary effects[29] above and below the λ line $T_{\lambda}(P)$ by means of RG theory within model F.[12,19,30]

I thank A. Kriner and M. Töpler for carrying out the numerical computations of the figures and tables, and R. Folk for information on the analysis in Figs. 7 and 8 of Ref. 14. Support by DLR is also gratefully acknowledged.

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