Chaos in glassy systems from a Thouless-Anderson-Palmer perspective

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We discuss level crossing of the free energy of Thouless-Anderson-Palmer (TAP) solutions under variations of the external parameters such as magnetic field or temperature in mean-field spin-glass models that exhibit one-step replica symmetry breaking (1RSB). We study the problem through a generalized complexity that describes the density of TAP solutions at a given value of the free energy and a given value of the extensive quantity conjugate to the external parameter. Depending on the properties of the generalized complexity, variations of the external parameter by any finite amount can induce level crossing between groups of TAP states whose free-energies are extensively different. In models with 1RSB, this means strong chaos with respect to the perturbation. The linear response induced by extensive level crossing is self-averaging and its value matches precisely the disorder average of the second moment of thermal fluctuations between low-lying, almost degenerate TAP states. We present an analytical recipe to compute the generalized complexity and test the scenario on the spherical multi-p spin models under variation of the temperature.

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I. INTRODUCTION

Mean-field spin-glass models like the Sherrington-Kirkpatrick model or the p-spin models are known to have a very complicated phase space with many metastable states.[1] An important physical consequence is that physical observables fluctuate not only within each equilibrium state but also among different equilibrium states whose free energies are sufficiently low and close to each other. The presence of many states leads also to the so-called chaos problem, i.e., the question of whether equilibrium states at different values of the external parameters such as magnetic field or temperature are correlated or not.[2] In the present paper, we discuss the problem from a Thouless-Anderson-Palmer (TAP) perspective. The states are usually identified with solutions of the TAP equations[3] for the local average magnetization m_i of the ith spin. In the Sherrington-Kirkpatrick (SK) model, they read

$$m_i = \tanh\left[\beta\left(\sum_j J_{ij}m_j - \beta(1-q)m_i + H\right)\right] \quad \forall \ i=1,\ldots,N, \tag{1}$$

where q is the self-overlap of TAP configurations $q \equiv \sum_i m_i^2/N$ and J_{ij} are quenched random coupling with zero means and variance $\overline{J_{ij}^2}=1/N$. If a given TAP solution has a nonvanishing Hessian it can be continued analytically upon a change of the external parameters H and β. We will say that two states at different external parameter coincides if one is the analytical continuation of the other. A question related to chaos is to know whether the equilibrium TAP states at a given values of, say, magnetic field h are the same (in the sense of the analytical continuation) of those at a different value of h. If this is the case, chaos is certainly not present. Instead, if the states are not the same, we derive chaos provided we assume that different states are not correlated; this is certainly the case in one-step replica symmetry-breaking (1RSB) models,[1] e.g., the spherical p-spin model.[4] Indeed the states in 1RSB models are minimally correlated, in particular in the zero-magnetic field, the overlap between two equilibrium configurations is zero if they belong to different states and is equal to the self-overlap if they belong to the same state. In systems with full replica symmetry breaking (FRSB), like the Sherrington-Kirkpatrick model, there is a range of possible values of the overlap and the connection between analytical continuation of the TAP solutions and chaos can be more subtle. In 1RSB systems, the TAP states with free energies below the threshold values have a nonvanishing Hessian, therefore, each of them can be analytically continued upon changing the external parameters. It follows that the equilibrium states at the new value of the external parameters must have been already present as some TAP states at the old values and they can be identified by tracking the evolution of the old TAP states through the variation of the parameters. We show that to describe the evolution of the TAP states, we must consider a generalized complexity which represents the density of TAP states at a given value of free energy (per spin) f and of an extensive quantity (per spin) $y=Y/N$, where N is the number of spins, conjugate to the external parameter h_y to be varied, i.e., magnetization for the magnetic field and entropy for the temperature.

In general, at fixed values of the external parameters, the typical states with a given value of f have a definite value of y but there are also TAP states with the same f and different values of y although with lower complexity than the typical ones. Thus, in general, the function $\Sigma(f,y)$ is nontrivial. Assuming the existence of this function, we draw the following conclusions. We prove that variation of the external parameters by any finite amount induces *extensive* level crossing of the free energies of the TAP state. Thus, the equilibrium TAP states at different values of the external parameter are different. Furthermore, from the function $\Sigma(f,y)$, we can compute the induced interstate linear response; it turns out to be self-averaging and its value matches precisely the value predicted by the analysis of thermal fluctuations through the fluctuation dissipation theorem (FDT). We present an analytical recipe to compute the generalized complexity and present explicit calculation for some specific cases. In particular, we show the existence of the function $\Sigma(f,m)$ for generic FRSB and 1RSB models. We also consider the entropy-free-energy function $\Sigma(f,s)$ (related to the behavior under temperature changes) in 1RSB spherical p-spin models. We show that this function exists for spherical models with multiple p-spin interactions implying chaos in temperature while its support shrinks to a single line in the (f,s) plane in the limit of a single p-spin interaction consistently with absence of chaos in temperature in this case.[4]

The problem of level crossing of TAP states has been recognized in earlier works, for instance in Refs. 5 and 6 (see Sec. II). In particular, level crossing of individual TAP states upon infinitesimal changes in the values of the magnetic field $[\delta h = O(1/\sqrt{N})]$ was observed. In the present paper, we are interested instead in the evolution of TAP states under small but *finite* changes of the external parameters, i.e., changes that induce extensive variations of the free energy. Basically, we want to know if the set of equilibrium states at a given value of the external parameters contains *as a whole* the same set of equilibrium states at different values; note that this does not exclude the possibility of some internal reshuffling of the relative weights of the states. If only the latter happens, we would have just some mild, subextensive level crossing between the states but no chaos.

The density of configurations with given energy and magnetization have been already studied in the context of the random-energy model[7] and more recently Krzakala and Martin[8] (KM) studied the level crossing phenomenon in an extended version of the random energy model[7] in which each state has a random energy and a random extensive variable conjugate to an external parameter, such as temperature. Both random variables are assumed to follow Gaussian distributions. The generalized complexity we study in the present paper provides a firmer ground for their picture.

The plan of the paper is the following. In Sec. II, we review previous results related to the present paper. In Sec. III, we introduce the generalized complexity. We discuss its evolution under variations of the external parameters and explain its physical consequences. In Sec. IV, we present an explicit calculation of the evolution of the generalized complexity of a spherical multi-p spin model under variation of the temperature. At the end, we discuss our results.

II. INTRASTATE AND INTERSTATE SUSCEPTIBILITY

A well-known effect of RSB is the difference between the susceptibility inside a state and the true thermodynamical susceptibility. For example, the magnetic susceptibility inside a state α in the zero-magnetic field is given by

$$\chi_\alpha = \beta(1 - q_{EA}), \tag{2}$$

where $q_{EA} \equiv \Sigma_i m_i^2 / N$ is the Edwards-Anderson order parameter, while the actual magnetic susceptibility of the system is given, according to the Parisi solution,[1] by

$$\chi = \beta(1 - \bar{q}), \tag{3}$$

where $\bar{q}$ is the average of the overlap between replicas. De Dominicis and Young[9] shown that this is a consequence of the presence of many states, so that in the application of the fluctuation dissipation theorem, we must consider a new term in order to take into account the fluctuations of the magnetizations over different states. They assumed that the free energy of the system is given by a sum over all TAP solutions weighted with their free energy

$$F = -\frac{1}{\beta N} \ln \sum_\alpha e^{-\beta N f_\alpha}, \tag{4}$$

then the susceptibility to a change in a given external field h_y (e.g., temperature or magnetic field) reads

$$\chi_y = \frac{\partial^2}{\partial h_y^2} \frac{1}{\beta N} \ln \sum_\alpha e^{-\beta N f_\alpha} = -\left\langle \frac{\partial^2 f_\alpha}{\partial h_y^2} \right\rangle + \beta N [\langle y_\alpha^2 \rangle - \langle y_\alpha \rangle^2], \tag{5}$$

where the square brackets represent the Boltzmann average over the states

$$\langle O_\alpha \rangle = \frac{\sum_\alpha e^{-\beta N f_\alpha} O_\alpha}{\sum_\alpha e^{-\beta N f_\alpha}} \tag{6}$$

and y_α is the value on state α of the parameter conjugated to h_y (e.g., magnetization or entropy *per* spin)

$$y_\alpha = \frac{\partial f_\alpha}{\partial h_y}. \tag{7}$$

The first term is the susceptibility of a state, while the second term is the fluctuation over the states of the parameter y_α. In the case of magnetic field, the first term gives a contribution of $\beta(1-q_{EA})$, while the second term can be written as $\beta(q_{EA}-\bar{q})$ so that the correct result (3) for the susceptibility is recovered.

Another interesting feature is that the susceptibility of a given sample, defined through FDT in terms of the thermal fluctuations, is not self-averaging. This has been pointed out by Young, Bray, and Moore in Ref. 6. In that paper, they considered the magnetic susceptibility

$$\chi_J = \frac{\beta}{N} \sum_{ij} (\langle s_i s_j \rangle - \langle s_i \rangle \langle s_j \rangle), \tag{8}$$

where s_i $(i = 1, \ldots, N)$ are the spin variables. They showed that this quantity fluctuates over the samples and the non-self-averageness was interpreted as an effect of the presence of many states, with sample-dependent $O(1)$ free-energy differences between those that dominate the equilibrium measure at low temperatures. This interpretation is confirmed noting that the TAP susceptibility defined above in Eq. (5) (which is defined differently from χ_J) is indeed not self-averaging, as we show in Appendix B. In particular, its disorder variance is the same as that of χ_J computed in Ref. 6. The problem is that the total magnetization and susceptibility, derived from thermodynamic derivatives of free energy, which is itself self-averaging, should be self-averaging. Then what is the true response? Numerical studies on the finite-size SK model by exact enumeration method provided insight to this problem.[10,5] According to these obsevations,[10,5,6] the magnetization per spin $m_J(h)$ of a given sample grows in a stepwise manner under increasing magnetic field h at low temperatures (See, e.g., Fig. 2 of Ref. 6). The spacing between the steps and the height of each step varies from step to step and sample to sample and *decreases* with the system size.[5] Note that this is consistent with the fact that fluctuations are not self-averaging because the linear susceptibility defined as $\chi_J=\lim_{\delta h\to 0}\delta m_J(h)/\delta h$ is related to the fluctuations through FDT. In Ref. 6, Young, Bray, and Moore suggested that the stepwise response is due to level crossing of TAP states. Furthermore, they conjectured that the typical separation between the steps is of order $O(1/\sqrt{N})$ and that the profile converges in the thermodynamic limit to a unique limiting curve $m(h)=\lim_{N\to\infty} m_J(h)$. According to this picture at finite N, this limiting curve acquires a sample-dependent fine structure on scales $\delta h=1/\sqrt{N}$. However, this fine structure is not seen on scales $1/\sqrt{N}\ll\Delta h\ll 1$. Therefore, $m(h)$ is self-averaging and thus the linear susceptibility defined as $\chi=\lim_{\Delta h\to 0}\Delta m(h)/\Delta h$ is also self-averaging. It is also expected that χ is equal the disorder average of χ_J. Note that the scales δh and Δh used in the definitions of χ_J and χ are completely different. While δh must be chosen smaller than the typical spacing between the steps, which is likely to be of order $O(1/\sqrt{N})$, Δh can be chosen to be arbitrarily small, but fixed when the thermodynamic limit $N\to\infty$ is taken, such that on the scale Δh, the susceptibility is self-averaging. According to these results, there is level crossing of individual TAP states upon infinitesimal changes in the values of the magnetic field $[\delta h=O(1/\sqrt{N})]$. We recall, however, that in the present paper, we are interested instead in the evolution of TAP states under small but finite changes of the external parameters, i.e., changes that induce extensive variations of the free energy. We want to know if the set of equilibrium states at a given value of the external parameters coincides with the set of equilibrium states at different values but this does not exclude the possibility of some internal microscopic reshuffling of the relative weights of the states.

The problem of the analytical continuation of the states is in deep relationship with the existence of two susceptibilites. This connection is the starting point of our work and was emphasized by G. Parisi.[11] According to Parisi's argument, the difference between the susceptibility and the intrastate susceptibility in general implies that the equilibrium states at different values of the external parameter cannot be the same. Indeed from Eq. (3), it follows that the magnetization of the equilibrium states in the presence of a small but *finite* magnetic field h becomes

$$m \simeq \beta(1-\bar{q})h. \tag{9}$$

On the other hand, the analytical continuation of the old equilibrium states would develop a smaller magnetization $\beta(1-q_{EA})h$, where q_{EA} is the Edwards-Anderson order parameter. Therefore, the equilibrium states in the presence of a small but *finite* field h had a *nonzero* magnetization per spin

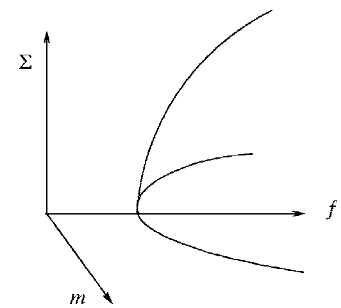

FIG. 1. Complexity of the TAP solution as a function of the free energy and magnetization per spin.

even in the *absence* of the field

$$m \simeq \beta(q_{EA}-\bar{q})h. \tag{10}$$

Therefore, the new equilibrium states cannot be the analytical continuation of the old equilibrium states.

Here an important point is that h is chosen arbitrarily small but fixed when the thermodynamic limit $N\to\infty$ is taken, i.e., h is at the scale of Δh and not of δh. In particular, at this scale, we have no problems of lack of self-averageness. As we explained in the introduction, we are interested in extensive level crossing; therefore, in the following, we are going to consider always variations in the external parameter at scale Δh. Note that this argument can be applied whenever the susceptibility to a given field h_y is different from the intrastate susceptibility, i.e., whenever the fluctuation of the conjugated parameter y, i.e., the second term in Eq. (5), is not zero. The fluctuations obviously vanish if there is only one state.

III. EXTENSIVE LEVEL CROSSINGS

The argument of Sec. II predicts the presence of metastable states with extensive nonzero magnetization in the zero field. This may appear rather counterintuitive; however, in the TAP context, their number can be computed. One can indeed show that there is an exponential number of solutions with nonzero magnetization, although with a smaller complexity with respect to the solutions with zero magnetization. In order to have a deeper look into the evolution of the phase space, we consider a generalized complexity, i.e., the logarithm of the number of TAP solutions with given values of the free energy *and* of the magnetization

$$\Sigma(f,m)=\frac{1}{N}\ln\sum_\alpha \delta(m_\alpha-m)\delta(f-f_\alpha). \tag{11}$$

We want to study the evolution of the curve $\Sigma(f,m)$ under the application of a magnetic field.

In Fig. 1, we show a schematic plot of the function $\Sigma(f,m)$ near the lower-band edge where the equilibrium states are located. Near the equilibrium states and below the critical temperature, the function can be expanded as

$$\Sigma(f,m)=\beta x f - a m^2, \tag{12}$$

where x is the Parisi parameter and a is some parameter to be determined, and we have shifted the free energy so that the equilibrium free energy in the zero field is zero. Let us emphasize that the function $\Sigma(f,m)$ is, by definition, an extensive self-averaging quantity. Now, we want to consider the evolution of the states on this curve when the small field h is switched on. The free energy of a state will be modified according to

$$f'_\alpha = f_\alpha + \frac{\partial f_\alpha}{\partial h}h + \frac{1}{2}\frac{\partial^2 f_\alpha}{\partial h^2}h^2 + O(h^3). \tag{13}$$

By definition, we have $m_\alpha = -\partial f_\alpha / \partial h$, therefore, the new free energy of the set of TAP solutions with given values of f and m is given by

$$f' = f - mh + \frac{1}{2}\frac{\partial^2 f_\alpha}{\partial h^2}h^2 + O(h^3), \tag{14}$$

their magnetization is given by

$$m' = m + \frac{\partial^2 f_\alpha}{\partial h^2}h + O(h^2). \tag{15}$$

Expressing through Eq. (12) the free energy in terms of the complexity c and the magnetic field, we get

$$f = \frac{c}{\beta x} + bm^2, \tag{16}$$

where we defined

$$b = a/\beta x. \tag{17}$$

Putting this expression into Eq. (14), we obtained the new value (after the field is switched on) of the free energy of the states that in the zero field had complexity c and magnetization m

$$f' = bm^2 - mh + \frac{1}{2}\frac{\partial^2 f_\alpha}{\partial h^2}h^2 + O(h^3) + \frac{c}{\beta x}. \tag{18}$$

The new equilibrium states are those that minimize f'. First of all, we note that the minimum with respect to c is obtained for $c=0$, this is consistent with the fact that the equilibrium states under any circumstance below the critical temperature should always have zero complexity, and therefore, the zero-field TAP states that are candidates to become equilibrium states in a field must must have zero complexity. Thus, we are interested in the evolution of the equilibrium states along the zero-complexity line $f=bm^2$. In order to minimize f' with respect to m, we note that the third term in Eq. (18), in principle, depends on m; but for values of m of order $O(h)$, this variation is basically a third order effect, therefore, at second order in h it can be considered as a constant. Then we obtain

$$\frac{df'}{dm} = 2bm - h = 0 \to m = \frac{h}{2b}. \tag{19}$$

Thus, the equilibrium states in the presence of a field are the states that had a nonzero magnetization $m=h/2b$ in the zero field and the evolution of the TAP states is driven by *extensive* level crossing, indeed the free energy difference between these states was $\Delta f = h^2/4b$ in zero field while it becomes negative $\Delta f = -h^2/4b$ in the presence of a field. This is the same result obtained above: in the presence of a field, the TAP solutions with lowest free energy are *not* the continuation of the TAP solutions with the lowest free energy in the zero field. Accordingly, the magnetization is given by

$$m' = \frac{h}{2b} - \frac{d^2 f}{dh^2}h + O(h^2), \tag{20}$$

and the full linear susceptibility is given by

$$\chi = \frac{1}{2b} - \frac{d^2 f}{dh^2}. \tag{21}$$

In 1RSB models, TAP states with extensive difference in the free energy have zero overlap with respect to each other;[1] therefore, they are totally uncorrelated. Thus, extensive level crossing automatically means strong chaos in the 1RSB systems.

The basic assumption of this derivation is the existence of the zero-complexity curve $f=bm^2$, which follows from the existence of the function $\Sigma(f,m)$. Once the existence of this function is assumed, the nontrivial result is that the evolution of the TAP states under a change in the magnetic field is driven by extensive level crossing. As such, the previous derivation can be extended to any couple (h_y, y) representing an external field and its conjugated extensive variable, e.g., temperature and entropy, provided the zero-complexity curve $f=by^2$ exists. This assumption is equivalent to the assumption of Sec. II that the parameter y_α fluctuates over the states. The connection with the result of Sec. II can be established also at a quantitative level by showing that the two expressions for the susceptibility Eqs. (21) and (5) are equivalent. In order to do that, we introduce the function

$$\Phi(\lambda_y) = \frac{1}{N}\ln\sum_\alpha e^{-\beta N f_\alpha + \lambda_y N y_\alpha}. \tag{22}$$

This is a summation over all TAP states with a weight which depends also on the value of $Y=Ny$; when $\lambda_y=0$, it reduces to the Boltzmann weight such that $\Phi(0)$ is minus the free energy. From the definition, it follows that:

$$\left.\frac{\partial^2 \Phi}{\partial \lambda_y^2}\right|_{\lambda_y=0} = N(\langle y_\alpha^2\rangle - \langle y_\alpha\rangle^2). \tag{23}$$

On the other hand, using the generalized complexity through Eq. (16), we can write

$$\Phi(\lambda) = \max_{c,y}\left[c - \beta b y^2 - \frac{c}{x} + \lambda_y y\right]. \tag{24}$$

Again, the maximum is at $c=0$ and the maximization with respect to y gives

$$\frac{\partial \Phi}{\partial \lambda} = \langle y\rangle_{\lambda_y} = \frac{\lambda_y}{2b\beta}, \tag{25}$$

which is linear with respect to λ_y. Using Eqs. (25) and (23), we get

$$\beta \left.\frac{\partial^2\Phi}{\partial\lambda}\right|_{\lambda_y=0} = \frac{1}{2b}. \tag{26}$$

This equation together with Eq. (23) proves the equivalence between Eqs. (21) and (5) for the susceptibility, which can be written as

$$\chi_y = \chi_{y\alpha} + \beta \left.\frac{\partial^2\Phi}{\partial\lambda_y^2}\right|_{\lambda_y=0}, \tag{27}$$

where the first term is the generalized susceptibility inside a state, e.g., the specific heat if h_y is the temperature and y is the entropy. Notice that we do not need to compute the intrastate susceptibility to infer the picture, it is sufficient to check the existence of the zero-complexity line.

In Appendix A, we report the general method to compute the function $\Phi(\lambda)$ for a generic model. In particular, in the case of the magnetic field, we can show that the second derivative of $\Phi(\lambda)$ has the correct value needed to recover the right TAP susceptibility in either FRSB and 1RSB models

$$\left.\frac{\partial^2\Phi}{\partial\lambda_m^2}\right|_{\lambda_m=0} = q_{EA} - \bar{q}. \tag{28}$$

Note that the derivation of this section assumes that the zero-complexity curve $f=bm^2$ is a self-averaging smooth function. Of course, at any finite N, this curve is actually made of points; therefore, on sufficiently small m scale (i.e., scales that go to zero with some proper power of $1/N$), we expect it to have rapid sample-to-sample fluctuations around its sample-independent average. These fluctuations and the corresponding lack of self-averaging in the right-hand side (rhs) of Eq. (23) are irrelevant at the much larger scales, which we consider and to which the derivation of the present section applies.

IV. SPHERICAL *p*-SPIN MODELS

In this section, we show that the picture of Sec. III applies to 1RSB spherical p-spin models with single[4] and multiple p-spin interactions.[12] In particular, the presence of chaos in temperature can be unequivocally associated to the behavior of the zero-complexity line as a function of the free energy and of the entropy. Following Ref. 13, we consider the Hamiltonian

$$H = -\sum_{i_1<\cdots<i_p}^{N} J_{i_1\cdots i_p}\sigma_{i_1}\cdots\sigma_{i_p} - \epsilon\sum_{l_1<\cdots<l_r}^{N} K_{l_1\cdots l_r}\sigma_{l_1}\cdots\sigma_{l_r}, \tag{29}$$

where the spins σ_i are subject to the spherical constraint $\Sigma_i\sigma_i^2=N$, and the Gaussian random couplings $J_{i_1\cdots i_p}$ and $K_{l_1\cdots l_r}$ have variance $p!/2N^{p-1}$ and $r!/2N^{r-1}$. The $p+r$ spherical models may display a nontrivial thermodynamic behavior when $p\geq 3$ and $r=2$: in that case, there is a transition from a 1RSB thermodynamic phase (low ϵ) to a FRSB phase (large ϵ).[12] On the contrary, if both p and r are strictly larger than 2, the model is expected to have a normal 1RSB thermodynamic behavior. This is the case we will analyze. In particular, we have studied numerically the case $p=3$ and $r=4$. The TAP free energy density is[13]

$$\begin{aligned}\beta f_{TAP} = &-\frac{\beta}{N}\sum_{i_1<\cdots<i_p}^{N} J_{i_1\cdots i_p}m_{i_1}\cdots m_{i_p}\\ &-\epsilon\frac{\beta}{N}\sum_{l_1<\cdots<l_r}^{N} K_{l_1\cdots l_r}m_{l_1}\cdots m_{l_r} - \frac{1}{2}\ln(1-q)\\ &-\frac{\beta^2}{4}[(p-1)q^p - pq^{p-1}+1]\\ &-\epsilon^2\frac{\beta^2}{4}[(r-1)q^r - rq^{r-1}+1],\end{aligned} \tag{30}$$

where $m_i=\langle\sigma_i\rangle$ are the local magnetizations, and q is the self-overlap of a state, $q=\Sigma_i m_i^2/N$. In the case of the single p-spin interaction,[4,14,15] it is straightforward to see that there is no chaos in temperature. Indeed by writing $m_i=q^{1/2}\hat{s}_i$, where $\hat{s}_i$ is the vector of the angular variables normalized to one, we see that the TAP equations for the angular variables do not depend on the temperature; therefore, the ordering of the states does not change in temperature. The decomposition of the free energy in the angular part and the overlap part breaks down if the model has more than a single p-spin interaction, and this could lead to chaos in temperature. In particular, in Ref. 16, the dynamical evolution under temperature changes of the TAP states was considered between the dynamical and the critical temperature. We note that with some modification, the present picture of extensive level crossing can be extended also in the region of temperatures where the complexity of the equilibrium states is finite. On the other hand, chaos in temperature in the $p+r$ model below T_c can be proven considering the free-energy shift between two real replicas forced to have a given value of the overlap.[17] Here we want to show that this result can be recovered through the study of the entropic zero-complexity line.

The computation of the complexity of the model (30) can be done through standard methods like those sketched in Sec. III and was presented (up to order ϵ^2) in Ref. 13. The complexity at fixed value of the free energy can be obtained by extremizing the following effective action with respect to the parameters B, T, q, and u

$$\begin{aligned}\hat{S} = &\beta u[g(q)+\epsilon^2 h(q) - f] + (B^2-T^2)\left[\frac{1}{4}p(p-1)\beta^2 q^{p-2}\right.\\ &\left.+\epsilon^2\frac{1}{4}r(r-1)\beta^2 q^{r-2}\right] - \frac{1}{2}\ln\left(\frac{1}{2}\beta^2 pq^{p-2} + \frac{1}{2}\epsilon^2\beta^2 rq^{r-2}\right)\\ &-\ln T + \frac{1}{4}\beta^2 u^2(q^p+\epsilon^2 q^r) - \frac{1}{2} + \frac{1}{4}\beta^2 B^2(pq^{p-2}+\epsilon^2 rq^{r-2})\\ &+\beta(B+T)[A(q)+\epsilon^2 C(q)] + \frac{1}{2}\beta^2 uB(pq^{p-1}+\epsilon^2 rq^{r-1}),\end{aligned} \tag{31}$$

and where we used the following definitions:

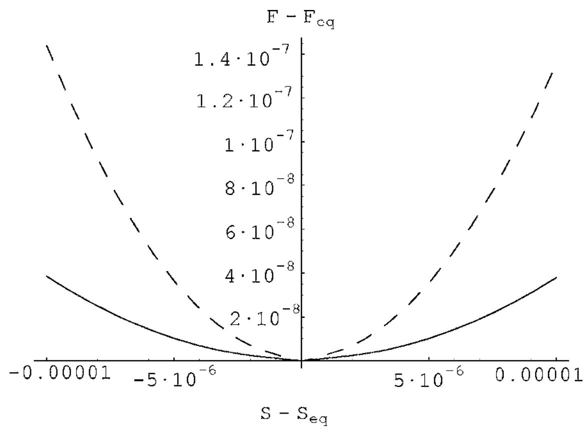

FIG. 2. Zero-complexity line of the free energy as a function of the entropy for the 3+4 spherical model with $\epsilon=.1$ (dashed line) and $\epsilon=.2$ (continuous line), the second derivative of $f(s)$ in $s=s_{eq}$ diverges as $1/\epsilon^2$ for $\epsilon \to 0$, in this limit the model has a single p-spin interaction and chaos in temperature disappears.

$$g(q) = -\frac{1}{2\beta}\ln(1-q) - \frac{\beta}{4}[(p-1)q^p - pq^{p-1} + 1], \quad (32)$$

$$h(q) = -\frac{\beta}{4}[(r-1)q^r - rq^{r-1} + 1], \quad (33)$$

$$\frac{\partial g}{\partial m_i} = A(q)m_i, \quad (34)$$

$$\frac{\partial h}{\partial m_i} = C(q)m_i. \quad (35)$$

In order to compute the complexity at a given value of the free energy f and of the entropy s we must add to (31) a term $\lambda_s s - \lambda_s s(q,\beta)$ and extremize with respect to λ_s. The function $s(q,\beta)$ is the complexity of a given solution which can be obtained from Eq. (30)

$$s(q,\beta) = -\frac{df_{TAP}}{dT} = \frac{1}{2}\ln(1-q) - \frac{\beta^2}{4}[(p-1)q^p - pq^{p-1} + 1] - \epsilon^2\frac{\beta^2}{4}[(r-1)q^r - rq^{r-1} + 1]. \quad (36)$$

The corresponding saddle point (SP) equations can be solved numerically. As noted in Ref. 13, there are two solutions of the saddle point equations, one that is Becchi-Rouet-Stora-Tyutin (BRST) symmetric and another that is not. The lower band edge is described by the BRST solution. Numerically, we start from this solution and consider the complexity of states with entropy different from the equilibrium one. Solving the SP equations with respect to B, T, q, u, λ_s and with the constraint that the complexity is zero yields the zero-complexity curve.

In Fig. 2, we plot the entropic zero-complexity line for a 3+4 model at temperature $T=.35<T_c$ and at values $\epsilon=.1$ and $\epsilon=.2$. Numerically, the second derivative of $f(s)$ in $s=s_{eq}$ diverges as $1/\epsilon^2$ for $\epsilon \to 0$. In this limit, the angular variables can be factorized and the entropy of the states is determined by their free energy; correspondingly, the two branches of the zero-complexity curve join on a single line $s=s_{typ}(f)$, which is the typical complexity of the states with free energy f. Note that since the divergence is proportional to ϵ^2 it is consistent to consider the action (31) which is valid at $O(\epsilon^2)$. From the existence of the zero-complexity curve follows that the dominant TAP states at different temperatures are different. This implies chaos in temperature because in a 1RSB system different states have vanishing mutual overlap. In this context, the disappearance of chaos in the limit $\epsilon \to 0$ is determined by the divergence of the second derivative of the zero-complexity line.

V. DISCUSSION

Our approach applies to all situations in which TAP states at a given value of some external parameter h_y (e.g., temperature or magnetic field) can be continued analytically at different values of h_y. If this is the case, it must be possible to characterize the set of states at a given value of h_y from the knowledge of the equilibrium states at another value of h_y. We have shown that this can be done by studying the generalized complexity $\Sigma(f,y)$ and in particular the zero-complexity line $\Sigma(f,y)=0$. These arguments apply to 1RSB models because the Hessian of the equilibrium TAP states is nonvanishing and the states can be continued analytically. In 1RSB models, we could also establish a connection between level crossings and chaos. In this context, absence of chaos with respect to the external parameter h_y (magnetic field or temperature) appears when the support of the function $\Sigma(f,y)$ (y is the parameter conjugated to h_y) shrinks to a single line in the (f,y) plane, otherwise chaos is present. Thus, our results provide firmer grounds for the phenomenological picture proposed by Krzakala and Martin in Ref. 8.

The application of our approach to FRSB is complicated by the fact that the equilibrium TAP states are marginal so, in principle, we cannot be sure that they can be continued. Nevertheless, one could study the zero-complexity line $\Sigma(f,y)=0$. In the case of magnetic field, this curve certainly exists and has the correct slope $q_{EA}-\bar{q}$. It would be interesting to check the existence of the zero-complexity line for the entropy. This is a further motivation to obtain the quenched solution for the complexity in the FRSB model. Provided the zero-complexity lines exist in the FRSB model for a given perturbation, another problem is to assess the stability of the corresponding states; we suspect that they are not marginal. However, in FRSB, the connection bewteen analytical continuation and chaos is less clear. Let us mention that the equilibrium states of the FRSB spherical $p+r$ model[12] have highly nonchaotic correlations with respect to temperature changes,[18] although they are marginal at any temperature. In this respect, it would be interesting to check the existence or not of the entropic zero-complexity curve of the FRSB spherical model, which is the only FRSB model known to be nonchaotic in temperature[18] at variance with the SK model.[19]

Starting from the function $\Sigma(f,y)$, we obtained the total linear susceptibility using the level crossing argument. Indeed to obtain a description of the evolution of the states at first order in h_y, it was sufficient to consider $\Sigma(f,y)$. To obtain the next order, we must consider the complexity $\Sigma(f,y,\chi_y)$, where χ_y is the intrastate susceptibility associated to the field h_y. Then the associated zero-complexity line $f=f(y,\chi_y)$ must be used in Eq. (18). Extremizing with respect to y and χ_y, we can obtain the value of the third derivative of the TAP free energy with respect to the external field h_y. Higher orders are obtained in the same way; in general, to obtain the kth derivative of the TAP free energy, we need $\Sigma(f,y,\chi_y,\ldots,\chi_y^{(k-1)})$, i.e., the complexity as a function of the intrastate susceptibilities up to order $k-1$.

In the present paper, we focused on the evolution of density of TAP states under the variation of external parameters over a small but finite range Δh. As discussed in Sec. II, if we go down to a scale of order $\delta h \sim O(1/\sqrt{N})$, we will observe individual level crossings whose characters are strongly non-self-averaging. Presumably this is relevant for problems of *heterogenous* thermal fluctuations and responses at mesoscopic scales,[20–22] some of which have now become accessible experimentally. Further investigation of the intermediate scales between δh and Δh will be interesting in this respect.[23]

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APPENDIX A

In this appendix, we show how to compute the function $\Phi(\lambda)$, which is basically the Legendre transform of the zero-complexity curve $f=by^2$. In the following, we assume that there exists a local function $y(m_i,q_{EA})$ such that the parameter y can be expressed as $y_\alpha=\Sigma_i y(m_i^\alpha,q_{\alpha\alpha})/N$. This includes the case of the magnetization and of the entropy. The computation of the function $\Phi(\lambda_y)$ can be done following standard techniques for computing averages over TAP solutions, we present the result in the case of the SK model and skip the details of the derivation, which are largely described in the literature (see, e.g., Refs. 24 and 25). In order to further simplify the presentation, we report the expression of $\Phi(\lambda,0)$ defined as

$$\Phi(\lambda_y,0)=\frac{1}{N}\ln\rho, \quad \rho\equiv\sum_\alpha e^{\lambda_y N y_\alpha}, \tag{37}$$

the quenched disorder average of $\Phi(\lambda_y,0)$ can be computed through the replica method

$$\overline{\Phi(\lambda_y,0)}=\lim_{n\to 0}\frac{1}{n}\ln\overline{\rho^n}. \tag{38}$$

Using the supersymmetric formulation of Ref. 24, the disorder average of $\overline{\rho^n}$ can be expressed as an integral over eight macroscopic bosonic and fermionic variables $\Theta\equiv\{r_{ab},t_{ab},q_{ab},\lambda_{ab},\bar{\rho}_{ab},\rho_{ab},\bar{\mu}_{ab},\mu_{ab}\}$

$$\overline{\rho^n}=\int d\Theta\,\exp[N\Sigma_1^{(n)}+N\Sigma_2^{(n)}], \tag{39}$$

where the action is specified by

$$\Sigma_1^{(n)}=-\lambda_{ab}q_{ab}-\frac{r_{ab}^2}{2\beta^2}+\frac{t_{ab}^2}{2\beta^2}+\bar{\mu}_{ab}\mu_{ab}+2\bar{\mu}_{ab}\rho_{ab}+\bar{\rho}_{ab}\rho_{ab} \tag{40}$$

and

$$\Sigma_2^{(n)}=\ln\Bigg[\int\prod_a dm_a dx_a d\psi_a d\bar{\psi}_a\exp\Bigg[x_a\phi_1(q_{aa},m_a)+\bar{\psi}_a\psi_a\phi_2(q_{aa},m_a)+\frac{q_{ab}\beta^2x_ax_b}{2}+r_{ab}m_ax_b+t_{ab}\bar{\psi}_a\psi_b+\lambda_{ab}m_am_b+-\mu_{ab}\beta m_a\psi_b-\bar{\psi}_a m_b\bar{\rho}_{ab}\beta-\rho_{ab}\beta x_a\psi_b-\bar{\psi}_a x_b\bar{\mu}_{ab}\beta+\lambda_y y(m_a,q_{aa})\Bigg]\Bigg] \tag{41}$$

and the functions $\phi_1(q,m)$ and $\phi_2(q,m)$ are given by

$$\phi_1(q,m)=\beta^2(1-q)m+\tanh^{-1}(m), \tag{42}$$

$$\phi_2(q,m)=\beta^2(1-q)+\frac{1}{1-m^2}. \tag{43}$$

Note that the only modification with respect to the standard computation (i.e., $\lambda_y=0$) is in the presence of the term $\lambda_y y(m_a,q_{aa})$ in the integral in $\Sigma_2^{(n)}$. The second derivative of $\Phi(\lambda_y,0)$ at $\lambda_y=0$ is given by

$$n\overline{\frac{\partial^2\Phi}{\partial\lambda_y^2}}=\left\langle\frac{\partial^2\Sigma_2^{(n)}}{\partial\lambda_y^2}\right\rangle+N\left[\left\langle\left(\frac{\partial\Sigma_2^{(n)}}{\partial\lambda_y}\right)^2\right\rangle-\left\langle\frac{\partial\Sigma_2^{(n)}}{\partial\lambda_y}\right\rangle^2\right], \tag{44}$$

where the square brackets mean average with respect to the action Eq. (39) and

$$\frac{\partial\Sigma_2^{(n)}}{\partial\lambda_y}=\left\langle\left\langle\sum_a y(m_a,q_{aa})\right\rangle\right\rangle \tag{45}$$

$$\frac{\partial^2\Sigma_2^{(n)}}{\partial\lambda_y^2}=\left\langle\left\langle\sum_{ab} y(m_a,q_{aa})y(m_b,q_{bb})\right\rangle\right\rangle-\left\langle\left\langle\sum_a y(m_a,q_{aa})\right\rangle\right\rangle^2, \tag{46}$$

where the double brackets mean average performed with respect to the integrand in the definition of $\Sigma_2^{(n)}$. The previous averages must be evaluated at $\lambda_y=0$. The action (39) can be evaluated through a saddle-point method. Note that, in general, to evaluate the the second term in Eq. (44), we need to study the Hessian of the saddle point which, in general, is very complicated. However, in the case of magnetic field perturbation in the zero field, we have $y(m_a,q_{aa})=m_a$, and $\partial\Sigma_2^{(n)}/\partial\lambda_m$ at $\lambda_m=0$ is identically zero for symmetry reasons; therefore, only the first term survives and we don't need to compute the Hessian of the SP. Thus, only the first term in the rhs of Eq. (46) contributes to the second derivative of $\Phi(\lambda_y,0)$ and we recover the result

$$\left.\frac{\partial^2\Phi}{\partial\lambda_m^2}\right|_{\lambda_m=0} = \lim_{n\to 0}\frac{1}{n}\left\langle\left\langle\sum_{ab} m_a m_b\right\rangle\right\rangle = q_{EA}-\bar{q}, \quad (47)$$

where we have used that SP equations $q_{ab}=\langle\langle\Sigma_{ab}m_a m_b\rangle\rangle$.

APPENDIX B

In this appendix, we show how to compute the sample-to-sample fluctuation of the TAP susceptibility Eq. (5) following the similar computation for the true thermodynamic susceptibility. The first term is the intrastate susceptibility and does not fluctuate with the disorder, analytically this is a consequence of the fact that it is a single replica quantity.[1,6] The second term is the fluctuation of the total magnetization over all TAP solutions $N^{-1}[\Sigma_{ij}\langle m_i m_j\rangle - \langle m_i\rangle\langle m_j\rangle]$, in order to check if it is self-averaging, we compute the average of its square. The computation can be done along the lines of the same replica computation of the thermodynamic susceptibility fluctuations.[6] The objects one needs to compute are averages of the form $\overline{\langle m_{i,1}m_{j,1}\rangle_{TAP}\langle m_{i,2}\rangle_{TAP}\langle m_{j,3}\rangle_{TAP}}$, where 1, 2, 3 are different replicas with the same realization of the disorder where the square brackets mean summation over all TAP states with the Boltzmann weight. Introducing source fields λ_i in the definition of $\rho\equiv\Sigma_\alpha \exp[-\beta f_\alpha+\lambda_i m_{i,\alpha}]$ this can be written as

$$\overline{\langle m_{i,1}m_{j,1}\rangle_{TAP}\langle m_{i,2}\rangle_{TAP}\langle m_{j,3}\rangle_{TAP}} = \overline{\left(\frac{\partial^4}{\partial\lambda_{i,1}\partial\lambda_{j,1}\partial\lambda_{i,2}\partial\lambda_{i,3}}\rho_1\rho_2\rho_3\right)\rho_1^{-1}\rho_2^{-1}\rho_3^{-1}}. \quad (48)$$

Now we multiply the quantity in the above disorder average by a factor ρ^n and divide the whole average by $\overline{\rho^n}$; taking the limit $n\to 0$, the result does not change; therefore, we can write

$$\overline{\langle m_{i,1}m_{j,1}\rangle_{TAP}\langle m_{i,2}\rangle_{TAP}\langle m_{j,3}\rangle_{TAP}} = \lim_{n\to 0}\frac{\partial^4}{\partial\lambda_{i,1}\partial\lambda_{j,1}\partial\lambda_{i,2}\partial\lambda_{i,3}}\ln\overline{\rho^n}. \quad (49)$$

The expression of $\overline{\rho^n}$ in presence of the source field can be computed as in Appendix A, the result is

$$\overline{\rho^n} = \int d\Theta \exp[N\Sigma_1^{(n)}+N\Sigma_2^{(n)}]\langle\langle e^{\lambda_{i,1}m_{i,1}+\lambda_{i,3}m_{i,3}}\rangle\rangle \times\langle\langle e^{\lambda_{j,1}m_{j,1}+\lambda_{j,2}m_{j,2}}\rangle\rangle, \quad (50)$$

the derivative is

$$\overline{\langle m_{i,1}m_{j,1}\rangle_{TAP}\langle m_{i,2}\rangle_{TAP}\langle m_{j,3}\rangle_{TAP}} = \lim_{n\to 0}\langle\langle\langle m_{i,1}m_{i,3}\rangle\rangle\langle\langle m_{j,1}m_{j,2}\rangle\rangle\rangle, \quad (51)$$

where the meaning of the double square brackets and of the square bracket is the same as in Appendix A. In the thermodynamic limit, these quantities can be averaged by the saddle-point method, in particular using the saddle-point equation with respect to λ_{ab}, we get

$$\overline{\langle m_{i,1}m_{j,1}\rangle_{TAP}\langle m_{i,2}\rangle_{TAP}\langle m_{j,3}\rangle_{TAP}} = q_{13}q_{12}. \quad (52)$$

This must be summed over the different SP, instead we can evaluate on a single SP the same object under all possible permutations of the replica indices

$$\overline{\langle m_{i,1}m_{j,1}\rangle_{TAP}\langle m_{i,2}\rangle_{TAP}\langle m_{j,3}\rangle_{TAP}} = \frac{1}{n(n-1)(n-2)}\sum_{(a,b,c)} q_{ab}q_{ac}. \quad (53)$$

All the various terms can be evaluated with this method and at the end it turns out that the rhs of Eq. (5) is not self-averaging. Furthermore, as shown in Ref. 6, at the lower band edge, the matrix q_{ab}, of the TAP computation coincides with the Parisi solution, and one can show that its disorder variance is equal to that of the thermodynamic susceptibility computed in Ref. 6.

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