Nonanalytic corrections to the specific heat of a three-dimensional Fermi liquid

Andrey V. Chubukov,[1,2] Dmitrii L. Maslov,[3] and Andrew J. Millis[4]

[1]*Department of Physics, University of Maryland, College Park, Maryland 20742-4111, USA*

[2]*Department of Physics, University of Wisconsin, 1150 University Avenue, Madison, Wisconsin 53706, USA*

[3]*Department of Physics, University of Florida, P.O. Box 118440, Gainesville, Florida 32611-8440, USA*

[4]*Department of Physics, Columbia University, 538 West 120th Street, New York, New York, 10027, USA*

We revisit the issue of the leading nonanalytic corrections to the temperature dependence of the specific heat coefficient, $\gamma(T) = C(T)/T$, for a system of interacting fermions in three dimensions. We show that the leading temperature dependence of the specific heat coefficient $\gamma(T) - \gamma(0) \propto T^3 \ln T$ comes from two physically distinct processes. The first process involves a thermal excitation of a single particle-hole pair, whose components interact via a nonanalytic dynamic vertex. The second process involves an excitation of three particle-hole pairs which interact via the analytic static fixed-point vertex. We show that the single-pair contribution is expressed via the backscattering amplitude of quasiparticles at the Fermi surface. The three-pair contribution does not have a simple expression in terms of scattering in particular directions. We clarify the relation between these results and previous literature on both 3D and 2D systems, and discuss the relation between the nonanalyticities in γ and those in spin susceptibilities.

I. INTRODUCTION

The thermodynamic properties of itinerant fermionic systems is a subject of long-standing experimental[1–3] and theoretical[4–30]interest. It is generally accepted[31,32] that the low-temperature behavior of a wide class of interacting fermionic systems is controlled by the Fermi-liquid fixed point. This implies that the leading temperature dependencies are the same as for free fermions, but with renormalized parameters. However, the first subleading corrections may differ qualitatively from those of noninteracting fermions. This phenomenon was first noticed in the context of the specific heat coefficient, $\gamma(T) = C(T)/T$. For noninteracting fermions, $\gamma(T)$ has a regular expansion in powers of T^2 about $T=0$, so the leading temperature dependence is $\gamma(T) - \gamma(T=0) \sim T^2$. For interacting fermions, $\gamma(T)$ is not an analytic function of T^2; the leading temperature dependence is instead proportional to $T^2 \ln(T)$ in three dimensions (3D) and T in 2D. The $T^2 \ln T$ term was first found by Eliashberg in a theoretical study of electrons interacting with acoustic phonons,[5] and the possibility of a nonanalytic temperature dependence of γ was subsequently (but apparently independently) inferred from measurements of γ for ^{3}He by Abel, Wheatley, and Andersen.[1] The ^{3}He measurements led to a large body of literature[4–30]which we review in detail below. We note here a few crucial (and seemingly contradictory) results. (i) In a very important paper,[15] Pethick and Carneiro showed that for a 3D Fermi liquid the $T^2 \ln T$ term in γ was a Fermi-liquid effect, associated with a combination of the multiple scattering of particle-hole pairs with small total momentum and a particular behavior of the quasiparticle interaction function $f_{\mathbf{p},\mathbf{p}+\mathbf{q}}$ (also at small q). (ii) More recently, it has been demonstrated[27] that in 2D, the prefactor of the T term in $\gamma(T) - \gamma(T=0) \sim T$ is determined entirely by the squares of charge and spin components of the *backscattering* amplitude. (By "backscattering" we mean scattering of fermions with

almost opposite momenta. Transferred momentum can be either small or near $2k_F$.) On the other hand the previous 3D work[12,15,16] found no special role of backscattering for the $T^2 \ln T$ nonanalyticity. (iii) The older work left the impression that in 3D the nonanalyticities were particular to γ and did not contribute to susceptibilities, whereas more recent work (based mainly on perturbative calculations) has demonstrated that nonanalytic corrections to the spin susceptibility occur both in 3D (Ref. 21) and in 2D.[24–30]

In this paper we clarify the relation between nonanalyticities in γ for 3D and 2D systems and make a few remarks concerning the relation between the nonanalyticities in γ and those in susceptibilities. We demonstrate that backscattering plays a special role also in 3D, in the sense that a part of the $T^2 \ln T$ term comes entirely from backscattering. This backscattering contribution evolves smoothly between 2D and 3D, and is entirely responsible for the nonanalytic part of γ for $D<3$. In $D \geqslant 3$, however, there exists another, physically distinct, contribution to the $T^2 \ln T$ term in γ. This contribution does not occur for $D<3$ and is not expressible solely in terms of the backscattering amplitude. We argue that this contribution is important for the 3D spin susceptibility as well. We also clarify the relation between the expressions for the prefactor of the $T^2 \ln T$ term via the forward-scattering interaction in the Pethick-Carneiro approach (and in subsequent analyses based on bosonization[22,23]) and via the backscattering amplitude. In particular, Pethick and Carnerio expressed the nonanalyticities in terms of the small q form of the scattering amplitude $f_{\mathbf{p},\mathbf{p}+\mathbf{q}}$ for momenta slightly displaced from the Fermi surface. As we will demonstrate explicitly, their result can be re-expressed in terms of a dynamical interaction between quasiparticles at the Fermi surface, in which backscattering plays a crucial role. At the same time, we show, in some disagreement with Refs. 12 and 19 that processes involving forward-scattering between quasiparticles at the Fermi surface do not contribute to the nonanalyt-
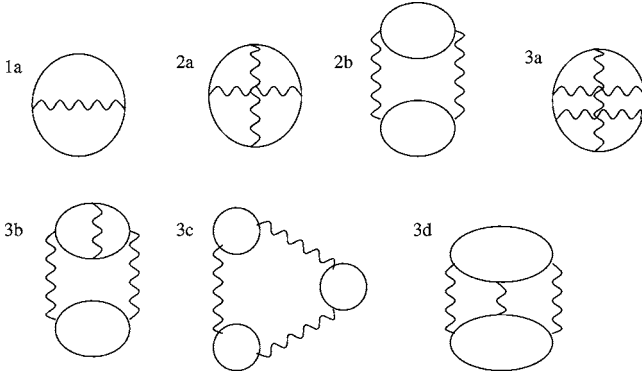
FIG. 1. Skeleton diagrams for the thermodynamic potential. The solid lines represent electron propagators and the wavy lines represent the interaction potential.

icity in γ although forward scattering does make a nonanalytic contribution to the self-energy.

The remainder of this paper is organized as follows: In Sec. II, we outline the qualitative physics underlying the results presented here. In Sec. III, we present the formalism and introduce forward-scattering and backscattering amplitudes. Sections IV and V give detailed calculations of the nonanalyticity in the entropy for the cases of three and two spatial dimensions, respectively. Section VI presents a brief discussion of the spin susceptibility. Section VII explicates the relation of our results to previous calculations of the self-energy, and Sec. VIII compares our results for the entropy to those obtained in prior work. Section IX presents a brief comparison of our results to the specific heat data for ^{3}He. Section X presents summary and conclusions. An explicit calculation of the $2k_F$ contribution to the nonanalyticity in the specific heat is presented in the Appendix.

II. QUALITATIVE PHYSICS

In this section we present a qualitative discussion of the physics underpinning our key results. Subsequent sections provide the formal justification. To compute the low-T specific heat of a Fermi liquid, one expands the thermodynamic potential (Ξ) in the number of particle-hole pairs excited above the Fermi-liquid ground state. Each excited quasiparticle interacts with the corresponding excited quasihole and with other particle-hole pairs via interaction vertices, Γ. Some low order diagrams are shown schematically in Fig. 1.

To obtain the entropy $S = -d\Xi/dT$ we must average over all excited particle-hole pairs, weighting the contributions by the derivative of the Bose distribution function $\partial n_B(\varepsilon)/\partial T$, which is odd in ε and, at low temperature T ensures that all excited electrons are near the Fermi surface. The contributions also involve the spectral function of the quasiparticle-quasihole pairs [the imaginary part of the particle-hole propagator $\Pi(\varepsilon, q)$], which is also odd in ε, and, critically, is not an analytic function of ε. For example, in $D = 3$, $\text{Im}\,\Pi(\varepsilon, q) \sim (\varepsilon/v_F q)\Theta(v_F q/\varepsilon - 1)$ with $\Theta(x)$ the step function. Therefore, only states with an odd number of excited pairs contribute to the entropy, and thus to $\gamma(T) = dS/dT$. The leading term has only one excited particle-hole pair (diagram

1a in Fig. 1) and, for momentum-independent interaction, Γ yields for the interaction-dependent part of the entropy

$$\delta S(T) \propto \Gamma \int d\varepsilon\, \varepsilon \frac{\partial n_B(\varepsilon)}{\partial T} \int \frac{d^3 q}{q}. \tag{1}$$

The momentum integral in Eq. (1) is dominated by large $q \sim k_F$, and so yields a constant, whereas the frequency integral yields $\delta S(T) \propto T$, i.e., $\gamma = \text{const}$. This is the leading Fermi-liquid result.

The subleading terms contain contributions from three, five, etc. thermally excited particle-hole pairs. For thermally allowed excitations $\varepsilon \sim T$, so each extra particle-hole pair brings an extra factor of T to the thermodynamic potential. One might therefore expect an expansion of the form $\delta S = \gamma_0 T + \alpha T^3 + \cdots$. However, the nonanalytical behavior of the particle-hole spectral function implies that this conclusion is not correct. The nonanalyticity enters in two ways. First, processes involving three pairs with the same total momentum q (see, e.g., diagram 3c of Fig. 1) contain the cube of the spectral function, and so give a term proportional to $(\varepsilon/q)^3$; the integral over q leads to a logarithmic divergence cut off at $\varepsilon \sim T$, and hence, in 3D, to a $T^3 \ln T$ contribution $\delta S(T)$. Second, the general form of the interaction vertex Γ has a term of order $(\varepsilon/q)^2$ arising from the Kramers-Kronig transform of $\text{Im}\,\Pi(\varepsilon, q)$ which, in combination with the ε/q from a single excited particle-hole pair, produces in 3D an additional $T^3 \ln T$ contribution to $\delta S(T)$ (see, e.g., diagram 2b in Fig. 1).

Note that dimension $D = 3$ is marginal in the sense that only for $D \leq 3$ the leading nonanalytic term is larger than the leading analytic $\mathcal{O}(T^3)$ correction. As will be seen, it is also marginal in the sense that for $D < 3$ only the single-pair term contributes to the nonanalyticity.

Both contributions were identified by Pethick and Carneiro.[15] They can be also clearly seen in the "paramagnon model"[10,17] of electrons coupled to overdamped spin fluctuations (although only the three-pair contributions were discussed in Refs. 10 and 17). The paramagnon part of the entropy is obtained by summing up loop diagrams with a small momentum transfer and is given by[10]

$$\delta S = \frac{3}{8\pi T^2} \int d\varepsilon \frac{\varepsilon}{\sinh^2(\varepsilon/2T)}$$

$$\times \int \frac{d^D q}{(2\pi)^D} \tan^{-1} \frac{2g\,\text{Im}\,\Pi(\varepsilon, q)}{1 + 2g\,\text{Re}\,\Pi(\varepsilon, q)}, \tag{2}$$

where g is the coupling constant in the paramagnon model (see Sec. VIII C 1 below). One obtains one $(\varepsilon/q)^3$ term from the process involving three real excited particle hole pairs by treating $\text{Re}\,\Pi$ as a constant and expanding the $\tan^{-1}$ to third order in $\text{Im}\,\Pi \sim \varepsilon/q$. In addition, one obtains another $(\varepsilon/q)^3$ term by combining one power of $\text{Im}\,\Pi \propto \varepsilon/q$ (representing a real excited particle-hole pair) with the nonanalytic term in $\text{Re}\,\Pi \propto (\varepsilon/q)^2$ (representing the nontrivial dynamic structure arising from $\text{Re}\,\Pi$).

The two processes evidently represent different physics and differ mathematically as well. The single-pair mechanism occurs already at the second-order in the interaction g

and is not specific to $D=3$. A simple extension of the $3D$ analysis to arbitrary integer D shows that the single-pair mechanism yields a non-analytic $\delta S \propto T^D$ in any dimension (with an additional $\ln T$ in odd dimensions). On the contrary, the three-pair mechanism occurs to third order in the interaction and gives rise to nonanalyticity in δS only in odd dimensions $D \geqslant 3$.

Let us consider the kinematics of these contributions in more detail. In both processes, the nonanalyticity arises from Landau damping, which for the physically relevant limit of $\varepsilon \ll q$ is dominated by fermions with momenta nearly perpendicular to $\mathbf{q}$. For a single-pair process, represented, e.g., by diagram 2b in Fig. 1, the constraint that the momenta on all four fermion lines are perpendicular to the same vector $\mathbf{q}$ implies that in two dimensions all four momenta are either nearly parallel to each other or two are nearly antiparallel to the other two. A closer analysis[27] shows that only the nearly antiparallel case contributes to the nonanalyticity in γ so the nonanalyticity is controlled by backscattering. However, it appears surprising that backscattering would play a special role in 3D. Indeed, in $D=3$ the constraint that all four momenta are perpendicular to q implies only that all four momenta lie in a common plane, so that any multiparticle excitation would seem to involve a more general scattering process of four fermions, two of which have momenta near $\mathbf{k}$ and two near $\mathbf{p}$ but with $\mathbf{k}$ and $\mathbf{p}$ perpendicular to $\mathbf{q}$ but not otherwise constrained. Indeed the three pair contributions have precisely this structure. Nevertheless we will show explicitly that in the single pair process only the case of nearly antiparallel momenta contributes.

Some physical understanding of the importance of backscattering even in 3D may be obtained by considering the problem in a slightly more general way. The discussion just given has focused on "small momenta" in the sense that it considered processes describable in terms of electron-hole pairs where the electron and the hole momenta are very close (on the scale set by k_F), and the nonanalyticity arises from the singular structure of the Landau damping $\sim \varepsilon/q$ in the small ε, q limit. However, Fermi liquids also exhibit nonanalyticities involving momenta near $2k_F$ (reflected for example in the slow decay of Friedel oscillations), and previous literature has raised the question of the contribution of $2k_F$ processes to the nonanalytic behavior.[12] To understand this issue it is useful to consider again the simple perturbative contribution shown in diagram 2b of Fig. 1, which may be written as the product of two particle-hole bubbles,

$$\Xi_{2b} \sim \int d\varepsilon \, \frac{dn_B(\varepsilon)}{dT} \int d^D q \, \mathrm{Im}\, \Pi(q,\varepsilon) \mathrm{Re}\, \Pi(q,\varepsilon). \quad (3)$$

There are two possible momentum regions which may give rise to nonanalyticities: small momentum and $q \sim 2k_F$. Consider the contribution to Ω_{2b} arising from large momentum transfer, $\mathbf{q} = (2k_F + \bar{q})\hat{q}$. The nonanalyticity arises from the "Landau-damping" structure of Π, which in 3D and for momenta near $2k_F$ is $\mathrm{Im}\, \Pi(\mathbf{q}, \varepsilon) \propto \varepsilon \Theta(|\varepsilon| - \bar{q})$ and $\mathrm{Re}\, \Pi - \mathrm{Re}\, \Pi(2k_F, 0) \sim \varepsilon^2/\bar{q}$ (see Appendix). We note that the orientation of $\mathbf{q}$ is irrelevant, so $\int d^D q$ becomes a one-dimensional integral over the scalar quantity $\bar{q}$ leaving a

logarithm for $D=3$. The $2k_F$ "Landau damping" terms involve fermions with momenta close to k_F, so that if $q \approx 2k_F$ then in each bubble one has one fermion of momentum $\mathbf{k} \approx \mathbf{q}/2$ and one with $\mathbf{k} \approx -\mathbf{q}/2$; thus the $2k_F$ contribution involves a one-dimensional process controlled by the "backscattering amplitude" $\Gamma(\mathbf{k}, -\mathbf{k}; \mathbf{k}, -\mathbf{k})$. However, we may also view the diagram for Ξ_{2b} in a different way, regarding the two fermion lines with momentum $\mathbf{k} \sim \mathbf{q}/2$ as a bubble with a small total momentum $\bar{q}$, and similarly with the two lines of momentum near $-\mathbf{q}$. Thus for the process with one real excitation and first dynamical correction to interaction, the $2k_F$ contribution may be subsumed into the small q contribution.

The total nonanalytic contribution to $\delta S(T)$ may thus be described in terms of two sorts of small momentum processes. One may be thought of in terms of $2k_F$ processes, and involves the vertex $\Gamma(\mathbf{k}, -\mathbf{k}; -\mathbf{k}, \mathbf{k})$. The other arises from consideration of small q processes and seemingly involves the effective interaction $\Gamma(\mathbf{k}, \mathbf{p}; \mathbf{k}, \mathbf{p})$ with arbitrary values of $\mathbf{k} \cdot \mathbf{p}$. Graphically, the two contributions differ only by interchanging the two outgoing momenta in the effective interaction. It is thus plausible to expect that both contributions are expressed in terms of the spin and charge components of the same antisymmetrized vertex. The vertex $\Gamma(\mathbf{k}, -\mathbf{k}; -\mathbf{k}, \mathbf{k})$ is the spin part of the backscattering amplitude. We may expect that $\Gamma(\mathbf{k}, \mathbf{p}; \mathbf{k}, \mathbf{p})$ is expressed in terms of charge component of the same amplitude. If this is the case, then $\mathbf{p}$ must be antiparallel to $\mathbf{k}$, i.e., the original small q, single-pair contribution also involves 1D process, and hence the total single-pair contribution involves 1D scattering.

For the three-pair mechanism, this argument does not hold. The kinematics of $2k_F$ processes is such that if more than two pairs are involved the result does not have a strong enough nonanalyticity. Indeed, combining three factors of $\mathrm{Im}\, \Pi(\varepsilon, 2k_F) \propto \varepsilon$, we obtain only a regular term in $\delta S(T)$. Thus the argument for the presence, in the calculation, of a purely one-dimensional scattering process fails and backscattering plays no special role.

A principal aim of the present paper is to present a more rigorous analysis substantiating the qualitative arguments given above. In this analysis the effects of single-pair and three-pair processes are separated and the contribution of each to the nonanalyticities in γ is determined in two and three dimensions.

III. FORMALISM

We calculate the specific heat coefficient $\gamma(T) = C/T$ by evaluating the difference between the thermodynamic potential, Ξ, and its value at $T=0$. The Luttinger-Ward expression for Ξ is[33]

$$\Xi = -\mathrm{Tr}(\ln[G_0^{-1} - \Sigma] + \Sigma G) + \Xi_{\mathrm{skel}} \quad (4)$$

with Σ the exact self-energy (regarded as a functional of the full Green function G) and Ξ_{skel} is the usual skeleton diagram expansion for all interaction corrections to Ξ that are not accounted for by the first two terms. The trace is taken over space, time, and spin variables.

It is not necessary to obtain the nonanalytic contributions the self-energy Σ in order to evaluate the nonanalytic contributions to Ξ. For what appear to be historical reasons associated with the fact that nonanalytic contributions to Σ were a focus of many earlier studies, the previous literature evaluates the $\mathrm{Tr}\,\ln[G_0^{-1}-\Sigma]$ separately from the other contributions, thereby effectively adding and subtracting contributions arising from nonanalyticities in Σ. However, the stationarity of Ξ with respect to variations of Σ implies that it is in fact sufficient to consider only the skeleton diagram Ξ_{skel}, which may be evaluated using the leading renormalized quasiparticle Green function $G_{\mathrm{qp}}=Z^{-1}[\omega-v_F^*(k-k_F)]^{-1}$, where Z is the renormalization factor and v_F^* is the renormalized Fermi velocity. We emphasize that, in physical terms, this corresponds to expanding Ξ in the number of particle-hole pairs thermally excited above the ground state, but assuming that these interact according to the $T=0$ Landau Fermi liquid fixed-point Hamiltonian.

A further subtlety arises: even if the Green function takes the fixed-point form, the $2k_F$ particle-hole bubble will have a nonanalytic temperature dependence. However, as we discussed in the previous section, singular $2k_F$ scattering can be re-expressed as small momentum scattering, and is fully accounted for if we restrict to small q but consider wavy lines in Fig. 1 as antisymmetrized interactions, i.e., as fixed-point vertices. The small-q diagrams have nonanalyticities arising from the Landau damping $\varepsilon/v_F q$ which has negligible temperature dependence. Once the restriction to small q is accomplished, we may calculate the leading low-T behavior of $\Xi(T)-\Xi(0)$ by focusing only on the T-dependence arising from the difference between $T\Sigma_{\omega_n}$ and $\int d\omega/2\pi$, and neglecting any explicit temperature dependence of G, Σ or interaction vertices. These latter contributions give only analytic, $\mathcal{O}(T^2)$ corrections to γ. The Appendix confirms this argument by presenting an explicit calculation of the contribution to S arising from $2k_F$ processes.

One can employ two strategies to calculate nonanalytic terms in $\Xi(T)$. The first one is to evaluate the thermodynamic potential directly in Matsubara frequencies ε_m. This strategy was adopted in a recent study of 2D systems.[27] A drawback of this approach is that it does not distinguish between the real and imaginary parts of $\Pi(\varepsilon,q)$, and the physical picture of excited particle-hole pairs does not arise. The second—and more intuitive—approach, adopted in this paper, is to work in real frequencies, when $\mathrm{Im}\,\Pi \propto \varepsilon$ and $\mathrm{Re}\,\Pi \propto \varepsilon^2$ have different physical meanings; the former describes real particle-hole pairs with given momentum and energy, whereas the latter describes the interaction arising from virtual pairs.

To evaluate nonanalytic terms in Ξ for a generic Fermi liquid we need two quantities. The first one is the propagator $\mathcal{P}_{\mathrm{ph}}(\varepsilon,q;\mathbf{n}_k)$, describing a particle-hole pair with small total momentum q and small energy ε, respectively, and with the particle and hole momenta near the Fermi surface and in direction $\mathbf{n}_k$. The second one is the fully renormalized, particle-hole *reducible* vertex $\Gamma_{\alpha\beta;\gamma\delta}(\varepsilon,q;\mathbf{n}_k,\mathbf{n}_p)$ describing scattering of one particle-hole pair state into another.

For small q and ε, $\mathcal{P}_{\mathrm{ph}}$ may be written as a sum of two terms: $\mathcal{P}_{\mathrm{ph}}=\mathcal{P}_0+\mathcal{P}$. The analytic part $\mathcal{P}_0$ is a function of ε

and q^2 and contains contributions from virtual processes involving states both near and far from the Fermi level. The nonanalytic part, denoted by $\mathcal{P}$, is determined solely by the properties of Fermi surface states and depends only on $z=v_F^*q/\varepsilon$. Expanding the product of two quasiparticle propagators near the Fermi surface, we obtain

$$\mathcal{P}(z;\mathbf{n}_k) = \frac{1}{S_D}\frac{z^{-1}}{z^{-1}-\mathbf{n}_k\cdot\mathbf{n}_q} \qquad (5)$$

with S_D being the surface area of the sphere of unit radius in D dimensions. We absorb the factor of inverse velocity and the factors of quasiparticle weight into the definition of Γ, and we have specialized to rotationally invariant systems. The frequency-dependent part of the polarization bubble $\Pi(\varepsilon,q)$ is obtained by averaging $\mathcal{P}(\varepsilon,q;\mathbf{n}_k)$ over $\mathbf{n}_k$:

$$\Pi(z) = \int d\mathbf{n}_k \mathcal{P}(z;\mathbf{n}_k).$$

In 3D,

$$\mathrm{Re}\,\Pi(z) = \frac{1}{2z}\ln\left|\frac{z+1}{z-1}\right|, \qquad (6)$$

$$\mathrm{Im}\,\Pi(z) = -\frac{\pi}{2z}\Theta(|z|-1). \qquad (7)$$

In the perturbation theory, when the interaction depends only on the momentum transfer but not on the incoming momenta, one needs to know only the angle-averaged bubble. However, the interaction in a generic Fermi liquid is described by a vertex $\Gamma_{\alpha\beta;\gamma\delta}(\varepsilon,q;\mathbf{n}_k,\mathbf{n}_p)$, which depends not only on the momentum transfer q but also on the relative directions of the incoming momenta $\mathbf{n}_k$ and $\mathbf{n}_p$. Therefore we need the $\mathbf{n}_k$-dependent propagator $\mathcal{P}(\varepsilon,q;\mathbf{n}_k)$. In evaluating Ξ, we will then have to perform angular integrations of the convolutions of $\mathcal{P}$ and Γ.

The fixed-point vertex $\Gamma_{\alpha\beta;\gamma\delta}(\varepsilon,q;\mathbf{n}_k,\mathbf{n}_p)$ is an (anti)symmetrized sum of all scattering processes between fermions *at the Fermi surface*. We may neglect any analytic dependence of Γ on ε and q separately, and consider Γ to be a function only of z. The fixed point vertex is also a tensor in the spin space. It is convenient to express this tensor as a sum of charge and spin components

$$\Gamma_{\alpha\beta;\gamma\delta}(z;\mathbf{n}_k,\mathbf{n}_p) = \delta_{\alpha\gamma}\delta_{\beta\delta}\Gamma_c(z;\mathbf{n}_k,\mathbf{n}_p) + \sigma_{\alpha\gamma}\cdot\sigma_{\beta\delta}\Gamma_s(z;\mathbf{n}_k,\mathbf{n}_p),$$
$$(8)$$

where c and s refer to the charge and spin sectors, respectively. In a model with bare fermion-fermion interaction $U(q)\rho(q)\rho(-q)$ to first order in the interaction we have

$$\Gamma_{\alpha\beta;\gamma\delta}(z;\mathbf{n}_k,\mathbf{n}_p) = \delta_{\alpha\gamma}\delta_{\beta\delta}u(0) - \delta_{\alpha\delta}\delta_{\beta\gamma}u(k_F|\mathbf{n}_k-\mathbf{n}_p|) \quad (9)$$

and thus

$$\Gamma_c(z;\mathbf{n}_k,\mathbf{n}_p) = u(0) - (1/2)u(k_F|\mathbf{n}_k-\mathbf{n}_p|) \qquad (10)$$

$$\Gamma_s(z;\mathbf{n}_k,\mathbf{n}_p) = -(1/2)u(k_F|\mathbf{n}_k-\mathbf{n}_p|), \qquad (11)$$
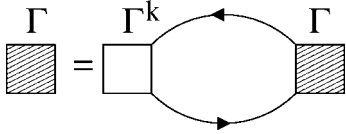
where $u(q) \equiv U(q)k_F^2/(\pi^2 v_F^*)$.

FIG. 2. Graphical representation of Eq. (12).

Pethick and Carneiro[15] showed that the charge and spin components of Γ contribute independently to the nonanalyticity in Ξ, i.e., $\Xi_{NA} = \Sigma_a w_a \Xi_a$, where $a = c, s$ and $w_c = 1$, $w_s = 3$ [for SU(2) fermions]. We explicitly verified and confirmed their result, which indeed follows simply from an elementary consideration of the possible distributions of charge and spin vertices along fermionic loops. To make our presentation more compact, we will consider only the charge component of Γ and omit the index c. The spin component will be restored in the final results.

The forward-scattering and backscattering processes in these notations correspond to $\Gamma(z; \mathbf{n}_p, \mathbf{n}_p)$ and $\Gamma(z, \mathbf{n}_p, -\mathbf{n}_p)$, respectively. In the first process, all four fermionic momenta are almost equal; in the second one, the two incoming and the two outgoing momenta are almost antiparallel.

The vertex $\Gamma(z; \mathbf{n}_p, \mathbf{n}_k)$ satisfies the integral equation

$$\Gamma(z; \mathbf{n}_k, \mathbf{n}_p) = \Gamma^k(\mathbf{n}_k \cdot \mathbf{n}_p) + \int d\mathbf{n}_l \Gamma^k(\mathbf{n}_k \cdot \mathbf{n}_l) \mathcal{P}(z; \mathbf{n}_l) \Gamma(z; \mathbf{n}_l, \mathbf{n}_p),$$
(12)

where the integral is over the area of the unit sphere in D dimensions (see Fig. 2). Virtual processes which contribute to $\mathcal{P}_0$ are absorbed into the "bare" vertex $\Gamma^k(\mathbf{n}_k \cdot \mathbf{n}_p) = \Gamma(\infty; \mathbf{n}_l, \mathbf{n}_p)$. The notation Γ^k, borrowed from Ref. 4, comes from the observation that $\mathcal{P}$ vanishes at $z = \infty$, i.e., at $\varepsilon / v_F^* q \rightarrow 0$. We recall that Γ^k coincides, up to the Z-factor, with the quasiparticle scattering amplitude. The scattering amplitude differs from the Landau function, which is related to $\Gamma^\omega(\mathbf{n}_k \cdot \mathbf{n}_p) = \Gamma(z = 0, \mathbf{n}_k, \mathbf{n}_p)$. The scattering amplitude may be expanded in spherical (or, in two dimensions, circular) harmonics as

$$\Gamma^k(\mathbf{n}_k, \mathbf{n}_p) = \Gamma^k(\mathbf{n}_k \cdot \mathbf{n}_p) = \sum_L \bar{\Gamma}_L P_L(\mathbf{n}_k \cdot \mathbf{n}_p).$$
(13)

Here P_L are Legendre polynomials in $D = 3$ and cosines in $D = 2$. The partial amplitudes, $\bar{\Gamma}_L$, of Γ^k are related to the partial amplitudes, Γ_L, of Γ^ω via

$$\bar{\Gamma}_L = \frac{\Gamma_L}{1 + \Gamma_L(2L + 1)^{-1}}$$

and

$$\bar{\Gamma}_L = \frac{\Gamma_L}{1 + \Gamma_0(2 - \delta_{L,0})^{-1}}$$

in 3D and 2D, respectively.

To generate an expansion in a number of particle-hole pairs we need an auxiliary vertex, $\bar{\Gamma}(z; \mathbf{n}_k, \mathbf{n}_p)$, which satisfies the same equation as in Eq. (12) but with Re $\mathcal{P}(z; \mathbf{n}_l)$ instead of full $\mathcal{P}(z; \mathbf{n}_l)$
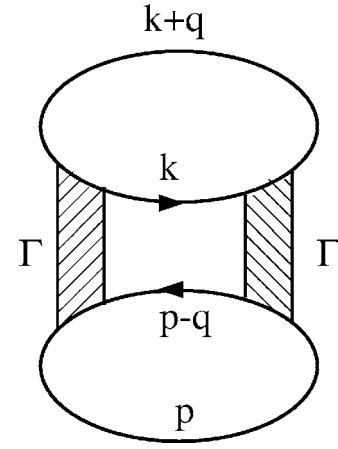

FIG. 3. Diagram 2b of Fig. 1 for the thermodynamic potential with fully dressed vertices $\Gamma(\mathbf{q}, \mathbf{k} - \mathbf{p})$, instead of the bare interaction $U(q)$.

$$\bar{\Gamma}(z; \mathbf{n}_k, \mathbf{n}_p) = \Gamma^k(\mathbf{n}_k \cdot \mathbf{n}_p)$$
$$+ \int d\mathbf{n}_l \Gamma^k(\mathbf{n}_k \cdot \mathbf{n}_l) \text{Re } \mathcal{P}(z; \mathbf{n}_l) \bar{\Gamma}(z; \mathbf{n}_l, \mathbf{n}_p).$$
(14)

The leading low-T contributions to the entropy involve particle-hole pairs excited above the Fermi surface and can be mathematically described by combinations of Im $\mathcal{P}$, representing the excited particle-hole pairs, and vertices $\bar{\Gamma}$, describing the interactions between these pairs. The diagrammatic expansion of the thermodynamic potential in series of particle-hole pairs is obtained from the skeleton diagrams of Fig. 1 by replacing the wavy lines by fully dressed interaction vertices. For example, diagram 2b of Fig. 1 is replaced by the diagram in Fig. 3. An important new feature of this expansion is that the vertices now depend not only on the momentum transfer ($\mathbf{q}$) but also on the incoming momenta ($\mathbf{k}$ and $\mathbf{p}$). Consideration of diagrams involving excitations and multiple scattering of pairs with total energy ε and momentum q leads to an expression for the nonanalytic contribution to the entropy per unit volume $\delta S_{NA} = -(1/V) d\Xi_{skel}/dT$ of the form

$$\delta S_{NA} = \int \frac{d\varepsilon}{\pi} \int \frac{d^D q}{(2\pi)^D} \frac{\varepsilon}{4T^2 \sinh^2\left(\frac{\varepsilon}{2T}\right)} \Phi(\varepsilon, q), \quad (15)$$

where

$$\Phi(\varepsilon, q) = \Phi(z) = \sum_l \frac{(-1)^{l+1}}{2l}$$
$$\times \left[\prod_{j=1}^{l} \int d\mathbf{n}_{p_j} \text{Im } \mathcal{P}(z, \mathbf{n}_{p_j}) \Lambda(z, \{\mathbf{n}_{p_j}\}) \right]. \quad (16)$$

Here the factors of Im $\mathcal{P}$ represent the excited particle-hole pairs and the factors of Λ [determined by combinations of the $\bar{\Gamma}$ from Eq. (12)] describe the interaction between them.

We note that because Im $\mathcal{P}$ is odd in ε whereas $\bar{\Gamma}$ is even, only terms involving odd powers of Im $\mathcal{P}$, i.e., odd number of pairs, contribute to δS_{NA}. To extract the nonanalytic behavior it is convenient to introduce $x = \varepsilon / 2T$ and rewrite Eq. (15) in dimension D as

$$\delta S_{\mathrm{NA}} = \frac{2^{D+1} S_D T^D}{(2\pi)^D (v_F^*)^D} \int_0^\infty \frac{dx}{\pi} \frac{x^{D+1}}{\sinh^2 x} \times \int_1^{v_F^* k_F / Tx} z^{D-1} dz \Phi(z).$$

(17)

Here, we used the fact that Im $\mathcal{P}(z, \mathbf{n}_p) = 0$ if $|z| < 1$.

IV. NONANALYTICITY IN SPECIFIC HEAT IN DIMENSION $D = 3$

A. Overview

This section presents results for the nonanalytic entropy contribution $\delta S_{\mathrm{NA}} \propto T^3 \ln T$ for a three-dimensional Fermi liquid. After an overview we present results based on expanding Eqs. (12) and (17) in the number of physical excited particle-hole pairs. We show that δS_{NA} can be expressed in terms of series of partial amplitudes Γ_L; although a compact, useful closed-form expression exists only for a part of δS_{NA} corresponding to a single-pair mechanism. In the last subsection we consider a toy model in which the expansion in particle-hole pairs can be carried out to all orders.

The key point of the calculation is this: the requirement of odd powers means that in Eq. (16) only terms with odd j, i.e., odd powers of Im $\mathcal{P}(z, \mathbf{n}_{p_j})$ contribute to $\Phi(z)$. At large z, Im $\mathcal{P}$ is of order z^{-1}; while $\Lambda = A + Bz^{-2} + O(z^{-4})$ is finite at $z^{-1} = 0$. The leading large-z contribution is thus $\mathcal{O}(z^{-1})$; the integral in Eq. (17) is dominated by the upper limit and gives a contribution to the z-integral $\sim T^{-2} + \mathrm{const}$; the net result is contributions of order T and T^3 to the entropy which are not of interest here. The subleading contribution come from the $j = 3$ term in Eq. (16) and from the z^{-2} term in Λ combined with the $j = 1$ term in Eq. (16). Both give contributions of order z^{-3} to $\Phi(z)$.

Expanding $\Phi(z)$ asymptotically for large argument yields $\Phi(z)$ at large z as

$$\Phi(z \to \infty) = \frac{\Phi_1}{z} + \frac{\Phi_3}{z^3} + \cdots.$$

(18)

Subtracting off the leading term, inserting the result in Eq. (17), and integrating gives for the entropy per unit volume

$$\delta S_{\mathrm{NA}} = \frac{4\pi^3}{5} \Phi_3 \left(\frac{T}{v_F^*} \right)^3 \ln \left(\frac{v_F^* k_F}{T} \right).$$

(19)

The problem is therefore to calculate Φ_3. It is convenient to split Φ_3 into contributions from processes with one and with three excited real particle-hole pairs, i.e.,

$$\Phi_3 = \Phi_3^{(1)} + \Phi_3^{(3)};$$

and to discuss the contributions separately.

B. Expansion in the number of excited particle-hole pairs

1. Single particle-hole pair

For the term involving only one excited particle-hole pair, reference to Fig. 1(a) shows that the interaction term is the quasiparticle vertex $\bar{\Gamma}$, from which we need the contribution proportional to z^{-2}. Hence

$$\Phi_3^{(1)} = \lim_{z \to \infty} \frac{z^3}{2} \int d\mathbf{n}_p \, \mathrm{Im} \, \mathcal{P}(z; \mathbf{n}_p) \Gamma(z; \mathbf{n}_p, \mathbf{n}_p)$$

(20)

with $\Gamma(z; \mathbf{n}_p)$ being the order z^{-2} term from the solution of Eq. (12). Direct inspection shows that there are two contributions to $\Gamma(z; \mathbf{n}_p)$; one, denoted $\Gamma^{(1)}$, of first order in Re $\mathcal{P}$ and another one, denoted $\Gamma^{(2)}$, of second order. These contribution are given by

$$\Gamma^{(1)}(z) = \int d\mathbf{n}_k [\Gamma^k(\mathbf{n}_p \cdot \mathbf{n}_k)]^2 \mathrm{Re} \, \mathcal{P}(z, \mathbf{n}_k),$$

(21)

$$\Gamma^{(2)}(z) = \int d\mathbf{n}_{k_1} d\mathbf{n}_{k_2} \Gamma^k(\mathbf{n}_p \cdot \mathbf{n}_{k_1}) \mathrm{Re} \, \mathcal{P}(z, \mathbf{n}_{k_1})$$
$$\times \Gamma^k(\mathbf{n}_{k_1} \cdot \mathbf{n}_{k_2}) \mathrm{Re} \, \mathcal{P}(z, \mathbf{n}_{k_2}) \Gamma^k(\mathbf{n}_{k_2} \cdot \mathbf{n}_p).$$

(22)

Accordingly, $\Phi_3^{(1)}$ is a sum of two contributions, $\Phi_3^{(1,1)}$ and $\Phi_3^{(1,2)}$. If Γ^k is angle-independent, the angular integrations in Eqs. (21) and (22) yield $\Gamma^{(1)}(z) \sim (\Gamma^k)^2 \mathrm{Re} \, \Pi(z) \propto z^{-2}$ and $\Gamma^{(2)}(z) \sim (\Gamma^k)^3 [\mathrm{Re} \, \Pi(z)]^2 \propto z^{-4}$, respectively [cf. Eq. (6)]. Hence $\Phi_3^{(1,1)} = O(1)$ and $\Phi_3^{(1,2)} \propto z^{-2} \ll \Phi_3^{(1,1)}$. However, if Γ^k is angle-dependent, $\Phi_3^{(1,1)}$ and $\Phi_3^{(1,2)}$ are of the same order. Indeed, the large-z limit of Re $\mathcal{P}(z, \mathbf{n}_k)$ in Eq. (5) is $\sim z^{-1} / (\mathbf{n}_k \cdot \mathbf{n}_q)$, which means that the product of two Re $\mathcal{P}(z, \mathbf{n}_k)$ in Eq. (22) is odd in $\mathbf{n}_{k_{1,2}}$. However, this oddness is compensated by a combination of angular harmonics of Γ^k in Eq. (22), which contains a factor of $(\mathbf{n}_{k_1} \cdot \mathbf{n}_q)(\mathbf{n}_{k_2} \cdot \mathbf{n}_q)$. As a result, the angular integrals are finite and $\Gamma^{(2)}(z) \propto (\mathrm{Re} \, \mathcal{P})^2 \propto z^{-2}$, which is of the same order as $\Gamma^{(1)}(z)$. One can readily verify that terms with three, four, etc. factors of Re $\mathcal{P}$ are irrelevant, as they scale at least as z^{-3}.

We consider the two contributions, $\Phi_3^{(1,1)}$ and $\Phi_3^{(1,2)}$, separately, beginning with the first one. A conventional way to proceed would be to expand $\Gamma(\mathbf{n}_k \cdot \mathbf{n}_p)$ in Legendre polynomials, as in (13), and evaluate the integrals order by order in Γ_L. However, it is more convenient to expand $\Gamma(\mathbf{n}_k \cdot \mathbf{n}_p)$ in powers of $\mathbf{n}_k \cdot \mathbf{n}_p$, and to write out the scalar product $\mathbf{n}_k \cdot \mathbf{n}_p$ explicitly in terms of polar angles referring to the direction of q, i.e.,

$$\Gamma^k(\mathbf{n}_k \cdot \mathbf{n}_p) = \sum_n \tilde{\Gamma}_n (\cos \theta_k \cos \theta_p + \cos \phi_k \sin \theta_k \sin \theta_p)^n.$$

(23)

In these notations, the backscattering amplitude is

$$\Gamma_{\mathrm{BS}} = \Gamma^k(\mathbf{n}_p, (-\mathbf{n}_p)) = \sum_n (-1)^n \tilde{\Gamma}_n.$$

(24)

Inserting Eq. (23) into Eq. (21) and then the result into Eq. (20), recalling that Im $\mathcal{P} = -z^{-1} \delta(z^{-1} - \mathbf{n}_p \cdot \mathbf{n}_q) / 4\pi$, exploiting

the azimuthal symmetry and the reflection symmetry about the plane $\mathbf{n}_p \cdot \mathbf{n}_q = 0$, and defining $y = \cos \theta_k$, yields

$$\Phi_3^{(1,1)} = -\frac{\pi}{4} \sum_{nm} \tilde{\Gamma}_n \tilde{\Gamma}_m I_{nm}, \tag{25}$$

where

$$I_{nm} = \lim_{z \to \infty} z^2 \int_{-1}^{1} dy J_{nm}(z,y) \mathcal{P} \frac{z^{-1}}{z^{-1} - y} \tag{26}$$

and

$$J_{nm}(z,y) = \int \frac{d\phi}{2\pi} \left(\frac{y}{z} + \cos \phi \sqrt{1 - z^{-2}} \sqrt{1 - y^2} \right)^{n+m}$$

$$= \sum_{L=0 \cdots n+m} \frac{(n+m)!}{(n+m-L)! \, L!} (\cos \phi)^{n+m-L}$$

$$\times \left(\frac{y}{z} \right)^L (\sqrt{1 - z^{-2}} \sqrt{1 - y^2})^{n+m-L}. \tag{27}$$

If $n+m = 2P$ is even, then only terms with even L contribute to the integral in Eq. (27), and J is an even function of y. In this case, the principal value integral gives a result of order z^{-2}, and only the $L=0$ term in Eq. (27) should be retained. If $n+m = 2P+1$ is odd, then J is an odd function of y. In this case the principal value integral gives a result of order z^{-1} and only the $L=1$ term is needed. The ϕ integral may then be performed, leaving

$$I_{n+m=2P} = \frac{(2P)!}{2^{2P}(P!)^2} \int dy \mathcal{P} \frac{z(1-y^2)^P}{z^{-1} - y}, \tag{28}$$

$$I_{n+m=(2P+1)} = -\frac{(2P+1)!}{2^{2P}(P!)^2} \int dy (1-y^2)^P. \tag{29}$$

Finally, performing the y integral leads to

$$I_{nm} = (-1)^{n+m}. \tag{30}$$

Substituting this into Eq. (25), we find

$$\Phi_3^{(1,1)} = -\frac{\pi}{4} \left(\sum_n (-1)^n \tilde{\Gamma}_n \right)^2 = -\frac{\pi}{4} \Gamma_{BS}^2. \tag{31}$$

We see that the contribution $\Phi_3^{(1,1)}$ is expressed solely in terms of the backscattering amplitude, i.e., it comes exclusively from 1D scattering processes. Notice that Eq. (31) can be also expressed in terms of the angular harmonics of Γ^k, introduced in Eq. (13), as

$$\Phi_3^{(1,1)} = -\frac{\pi}{4} \Gamma_{BS}^2 = -\frac{\pi}{4} \left(\sum_n (-1)^n \bar{\Gamma}_n \right)^2. \tag{32}$$

We pause here to emphasize the nontriviality of this result. As we have already mentioned, the same process with one excited particle-hole pair leads to nonanalyticity in $D = 2$. For two dimensions, however, a simple geometric argument, which we displayed in Sec. II, shows that 1D processes determine the nonanalytic part of the entropy. The straightforward extension of this argument to $D > 2$ suggests

that all relative angles are important, but Eq. (31) shows that $\Phi_3^{(1,1)}$ still comes only from 1D scattering.

We next consider the contribution involving $\Gamma^{(2)}$. Following the arguments given above we find

$$\Phi_3^{(1,2)} = -\frac{\pi}{4} \sum_{n,m,l} \tilde{\Gamma}_n \tilde{\Gamma}_m \tilde{\Gamma}_l I_{nml}. \tag{33}$$

Calculations along the same lines as above yield

$$I_{nm,l=(2P+1)} = \frac{2P+1!}{2^{2P}(P!)^2} I\left(\frac{n}{2} + p \right) I\left(\frac{m}{2} + p \right) \tag{34}$$

with

$$I(m) = \int_{-1}^{1} \frac{dy}{2} (1-y^2)^m. \tag{35}$$

The integrals may be performed in terms of Γ-functions but the final expressions seem complicated and not very useful in general. For further comparison with Ref. 15, we present here the result for $\Phi_3^{(1,2)}$, obtained by truncating the sum in Eq. (33) at $n,m,l=4$ and writing the result in terms of the angular harmonics of Γ^k,

$$\Phi_3^{(1,2)} = -\frac{\pi}{4} \left[\bar{\Gamma}_1 \bar{\Gamma}_0^2 + \frac{2}{9} \bar{\Gamma}_2 \left(2\bar{\Gamma}_0^2 + \bar{\Gamma}_0 \bar{\Gamma}_2 + \frac{4}{5} \bar{\Gamma}_2^2 \right) \right]$$

$$- \frac{\pi}{12} \left[\bar{\Gamma}_1^2 \bar{\Gamma}_2 + \frac{2}{45} \bar{\Gamma}_4 \left(4\bar{\Gamma}_1^2 + 3\bar{\Gamma}_1 \bar{\Gamma}_3 + \frac{18}{7} \bar{\Gamma}_3^2 \right) \right]. \tag{36}$$

2. Three particle-hole pairs

The second contribution to the nonanalytic term in the specific heat of a three-dimensional Fermi liquid comes from terms in Eq. (19) involving three excited particle-hole pairs (e.g., diagram 3c in Fig. 1). This contribution gives the "paramagnon" term in the specific heat studied, e.g., in Ref. 10. In this case the quasiparticle interaction Λ involves three factors of the static scattering amplitude Γ^k whereas Im $\mathcal{P}$ can be taken in the $z \to \infty$ limit, yielding

$$\Phi_3^{(3)} = \frac{1}{192} \int d\mathbf{n}_{p_1} d\mathbf{n}_{p_2} d\mathbf{n}_{p_3} \delta(\mathbf{n}_{p_1} \cdot \mathbf{n}_q) \delta(\mathbf{n}_{p_2} \cdot \mathbf{n}_q)$$

$$\times \delta(\mathbf{n}_{p_3} \cdot \mathbf{n}_q) \Gamma^k(\mathbf{n}_{p_1} \cdot \mathbf{n}_{p_2}) \Gamma^k(\mathbf{n}_{p_2} \cdot \mathbf{n}_{p_3}) \Gamma^k(\mathbf{n}_{p_3} \cdot \mathbf{n}_{p_1}).$$

Substituting Eq. (23) into the equation for $\Phi_3^{(3)}$ and evaluating the integrals, we obtain

$$\Phi_3^{(3)} = \frac{\pi^3}{48} \sum_{lmn} \tilde{\Gamma}_l \tilde{\Gamma}_m \tilde{\Gamma}_n \frac{(l+m)! \, (l+n)!}{2^{n+m+2l} \left(\frac{l+m}{2} \right)! \left(\frac{n+m}{2} \right)!}. \tag{37}$$

Truncating the sum at $l,m,n=2$ and re-expressing the result in terms of $\bar{\Gamma}_L$ [Eq. (13)], we obtain

$$\Phi_3^{(3)} = \frac{\pi^3}{48} \left[\left(\bar{\Gamma}_0 + \frac{1}{4} \bar{\Gamma}_2 \right)^3 + \frac{1}{4} \bar{\Gamma}_1^3 + \frac{27}{256} \bar{\Gamma}_2^3 \right]. \tag{38}$$

3. The final result for the nonanalytic part of the entropy in 3D

Combining all three expressions $\Phi_3^{(1,1)}$, $\Phi_3^{(1,2)}$, and $\Phi_3^{(3)}$, and restoring the sum of charge and spin components of $\bar{\Gamma}^k$, we finally obtain for the entropy per particle

$$
\delta S_{NA}(T) = -\frac{\pi^4}{5}\left(\frac{T}{v_F^* k_F}\right)^3 \ln\left(\frac{v_F^* k_F}{T}\right)(K_c + 3K_s), K_a
$$

$$
= \Gamma_{a,BS}^2 + \bar{\Gamma}_{a,1}\bar{\Gamma}_{a,0}^2 + \frac{1}{3}\bar{\Gamma}_{a,1}^2\bar{\Gamma}_{a,2}
$$

$$
- \frac{\pi^2}{12}\left[\left(\bar{\Gamma}_{a,0} + \frac{1}{4}\bar{\Gamma}_{a,2}\right)^3 + \frac{1}{4}\bar{\Gamma}_{a,1}^3 + \frac{27}{256}\bar{\Gamma}_{a,2}^3\right]
$$

$$
+ \cdots, \tag{39}
$$

where $a = c, s$, $\Gamma_{a,BS} = \Gamma_a(\infty; \mathbf{n}_k, -\mathbf{n}_k)$, and dots stand for the terms involving $\bar{\Gamma}_{a,n}$ with $n > 2$.

A comment is in order here. In the consideration above we assumed that the backscattering amplitudes $\Gamma_{a,BS}$ are temperature-independent. Strictly speaking, this is not the case as the backscattering amplitudes describe processes with total zero momentum and thus can be re-expressed via the partial components of the pairing vertex.[34] One of these partial components diverges at the pairing instability which, for a repulsive interaction, is of the Kohn-Luttinger type,[35] so our consideration is only valid at $T > T_c$. Even at these temperatures, the backscattering amplitudes acquire a logarithmic temperature dependences from the Cooper channel, and behave as

$$
\Gamma_{a,BS}(T) \propto \sum_{n=0}^{\infty} \frac{1}{(\alpha_n + \beta_n \ln T/E_F)}, \tag{40}
$$

where α_n and β_n are constants. At T_c, $\alpha_n + \beta_n \ln T/E_F = 0$ for a particular $n = n_0$. In the rest of the text, we neglect this complication and assume that the system is substantially far away from T_c that both spin and charge components of the backscattering amplitudes can be approximated by constants.

C. Ring diagram (RPA) approximation

In this subsection we present results for a toy model in which the expansion in the number of excited particle-hole pairs can be carried to all orders, in order to demonstrate explicitly that only the squares and the cubes of the partial components of the scattering amplitude, but not higher powers, determine the nonanalyticity in $S(T)$.

We consider an artificial model of fermions with spin degeneracy $N \to \infty$, coupled by a contact interaction $U(\mathbf{n}_p \cdot \mathbf{n}_k) = U_0 + U_1\mathbf{n}_p \cdot \mathbf{n}_k$. Standard techniques yield, for the large-N limit of the entropy $S_\infty = -\lim_{N \to \infty}(d\Xi(N)/dT)/N$

$$
S_\infty = S_F + \frac{S_{ring}}{N}, \tag{41}
$$

$$
S_{ring} = -\operatorname{Im}\int \frac{d\varepsilon}{\pi}\frac{\varepsilon}{4T^2 \sinh\dfrac{\varepsilon}{2T}}\int \frac{d^3q}{(2\pi)^3}
$$

$$
\times \operatorname{Tr}\ln[(1 - \Gamma(\mathbf{n}_p \cdot \mathbf{n}_k)\mathcal{P}(z, \mathbf{n}_k))], \tag{42}
$$

where $S_F = -d/dT[\operatorname{Tr}\ln G_0^{-1}]$ the free fermion entropy, the trace taken over angle ($\operatorname{Tr} = \Pi dn_k$), $\Gamma = U(\mathbf{n}_p \cdot \mathbf{n}_k)k_F^2/(\pi^2 v_F^*) = \Gamma_0 + \Gamma_1\mathbf{n}_p \cdot \mathbf{n}_k$ and

$$
\mathcal{P}_{ph}(z, \mathbf{n}_k) = -\frac{1}{4\pi}\frac{\cos(\mathbf{n}_k \cdot \mathbf{n}_q)}{\cos(\mathbf{n}_k \cdot \mathbf{n}_q) - z^{-1}}. \tag{43}
$$

Note that $\mathcal{P}_{ph}$ is the full propagator; we do not split it into analytic and nonanalytic parts. Making the changes of variables to $x = \varepsilon/(2T)$ and $z = v_F^* q/\varepsilon$, and integrating over x as before we obtain for the entropy per particle

$$
S_{ring} = -\frac{2\pi^3}{5}\left(\frac{T}{v_F^* k_F}\right)^3\int_1^{v/Tx} z^2 dz
$$

$$
\times \operatorname{Im}\ln\det[1 - \Gamma(\mathbf{n}_p \cdot \mathbf{n}_k)\mathcal{P}(z, \mathbf{n}_k)].
$$

To evaluate the determinant we must find the eigenvalues of the operator $1 - \Gamma\mathcal{P}$, i.e., we must solve

$$
\lambda\Lambda(z, \mathbf{n}_p) = \Lambda(z, \mathbf{n}_p) - \int d\mathbf{n}_k\Gamma(\mathbf{n}_p \cdot \mathbf{n}_k)\mathcal{P}_{ph}(z, \mathbf{n}_k)\Lambda(z, \mathbf{n}_k).
$$

$$
\tag{44}
$$

Writing

$$
\Lambda(\mathbf{n}_p) = \Lambda_0 + \Lambda_1 \cos\theta_p + \Lambda_2 \sin\theta_p(e^{i\phi} + e^{-i\phi}) \tag{45}
$$

and using Eq. (23), we find that the determinant may be written as the product $\det = D_1 \times D_2^2$ ($\operatorname{Im}\ln\det = \operatorname{Im}\ln D_1 + 2\operatorname{Im}\ln D_2$), with

$$
D_1 = (1 + \Gamma_0 I_{00})(1 + \Gamma_1 I_{11}) - \Gamma_0\Gamma_1 I_{01}^2, \tag{46}
$$

$$
D_2 = 1 + \Gamma_1\bar{I}_{11}, \tag{47}
$$

and

$$
I_{nm} = \int_{-1}^1 \frac{d(\cos\theta)}{2}\frac{(\cos\theta)^{n+m+1}}{\cos\theta - z^{-1}}, \tag{48}
$$

$$
\bar{I}_{11} = \int_{-1}^1 \frac{d(\cos\theta)}{4}\frac{\cos\theta(\sin\theta)^2}{\cos\theta - z^{-1}}. \tag{49}
$$

Evaluating I_{mn}, we find

$$
I_{00} = 1 + \frac{1}{2z}\ln\frac{1 - z^{-1}}{1 + z^{-1}} + i\frac{\pi}{2z}, \tag{50}
$$

$$
I_{01} = \frac{1}{z}I_{00}, \tag{51}
$$

$$
I_{11} = \frac{1}{3} + \frac{1}{z^2}I_{00}, \tag{52}
$$

$$\bar{I}_{11} = (I_{00} - I_{11}). \tag{53}$$

Substituting Eqs. (53) into Eqs. (46) and (47) and simplifying yields

$$D_1 = A(1 + \bar{\Gamma}_0(I_{00} - 1))(1 + \bar{\Gamma}_1(I_{11} - 1/3)) - \bar{\Gamma}_0\bar{\Gamma}_1 I_{01}^2, \tag{54}$$

$$D_2 = B + \bar{\Gamma}_1(\bar{I}_{11} - 1/3) \tag{55}$$

with $A = (1+\Gamma_0)(1+\Gamma_1/3)$, $B = 1+\Gamma_1/3$, and the reducible amplitudes $\bar{\Gamma}$ given by

$$\bar{\Gamma}_0 = \frac{\Gamma_0}{1 + \Gamma_0}, \tag{56}$$

$$\bar{\Gamma}_1 = \frac{\Gamma_1}{1 + \dfrac{\Gamma_1}{3}}. \tag{57}$$

The z integral in Eq. (44) is dominated by its upper limit. The leading term is $\mathcal{O}(z^{-1})$ and gives a renormalization of the effective mass. The next term is $\mathcal{O}(z^{-3})$ and gives the logarithm we require. Expanding in $1/z$, we obtain

$$\text{Im} \ln D_1 \rightarrow \frac{i\pi\bar{\Gamma}_0}{2z} + \frac{i\pi\bar{\Gamma}_1}{2z^3} + \frac{i\pi(\bar{\Gamma}_0^2 - 2\bar{\Gamma}_0\bar{\Gamma}_1)}{2z^3}$$
$$- \frac{i\pi^3\bar{\Gamma}_0^3}{24z^3} + \frac{i\pi\bar{\Gamma}_0^2\bar{\Gamma}_1}{2z^3}, \tag{58}$$

$$\text{Im} \ln D_2 \rightarrow \frac{i\pi\bar{\Gamma}_1}{4z} - \frac{i\pi\bar{\Gamma}_1}{4z^3} + \frac{i\pi\bar{\Gamma}_1^2}{4z^2} - \frac{i\pi\bar{\Gamma}_1^3}{192z^3}. \tag{59}$$

Combining Eqs. (58) and (59) to obtain $\text{Im} \ln \det = \text{Im} \ln D_1 + 2 \text{Im} \ln D_2$, extracting the z^{-3} term and performing the integral in Eq. (44) yields for entropy per particle

$$S_{\text{ring}} = -\frac{\pi^4}{5}\left(\frac{T}{v_F^* k_F}\right)^3$$
$$\times \ln \frac{v k_F}{T}\left((\bar{\Gamma}_0 - \bar{\Gamma}_1)^2 + \bar{\Gamma}_0^2\bar{\Gamma}_1 - \frac{\pi^2\bar{\Gamma}_0^3}{12} - \frac{\pi^2\bar{\Gamma}_1^3}{48}\right). \tag{60}$$

This expression is in agreement with our previous result, Eq. (39) once we note that to the order in N to which we work, only the charge component of Γ contributes to a nonanalytic $\delta S(T)$, and that in the RPA approximation, the dimensionless Γ coincides with the Landau interaction function (this also implies that $\bar{\Gamma}_n$ are the partial components of the scattering amplitude). Note that, as before, the terms with coefficient π^2 in Eq. (61) arise from processes with three excited particle-hole pairs while the others involve processes with one excited particle-hole pair, along with a nonanalyticity in the quasiparticle interaction.

V. NONANALYTICITY IN SPECIFIC HEAT IN DIMENSION $D=2$

A. Overview

For completeness, we present here results for the nonanalytic contribution $\delta S \propto T^2$ to the entropy a two-dimensional Fermi liquid. This contribution was found in Refs. 27 and 28 by exact evaluation of diagrams up to third order and by showing that results obtained by analysis of leading singular behavior of diagrams occurring in the general case matched the perturbative results exactly. The calculations were also performed in Matsubara frequencies. Here we show how the same result is obtained using an expansion in the number of real particle-hole pairs. In addition, we reconfirm the result that the nonanalyticity in 2D involves only backscattering processes. Unlike the 3D case, this is the only nonanalytic contribution to $S(T)$.

We begin from the two dimensional version of Eq. (17) in which the essential object is $\int_1^{v_F^* k_F/Tx} z dz \Phi_{2D}(z)$. Again we expect the leading large z behavior of Φ to be $\Phi_{2D} \sim 1/z$ giving the Fermi-liquid behavior; but the different ultraviolet behavior of the momentum integral means that the remaining terms are ultraviolet convergent, so that the z integral is just a constant and the temperature prefactor gives the nonanalytic result in 2D: $\gamma = d(\delta S_{\text{NA}})/dT \sim T$.

A further important question arises from the analytic structure of Π. In $D=2$,

$$\text{Re} \, \Pi(z) = \frac{\Theta(1 - |z|)}{\sqrt{1 - z^2}}, \tag{61}$$

$$\text{Im} \, \Pi(z) = -\frac{\text{sgn} \, z \, \Theta(|z| - 1)}{\sqrt{z^2 - 1}}. \tag{62}$$

The singularity at the boundary of the particle-hole continuum $|z| = 1$ has to be treated with care when expanding δS_{NA} in powers of $\text{Im} \, \Pi$, and indeed raises the possibility that the z integral has appreciable contributions from the neighborhood of $z=1$. In this region, the kinematics of Landau damping does not guarantee that the contributions come from fermions with momentum perpendicular to q, and therefore might not be determined solely by the backscattering amplitude.

The computation is in fact most straightforwardly carried out by rotating the ε integral in Eq. (15), leading in $D=2$ to the nonanalytic entropy per particle,

$$\delta S_{\text{NA}} = -\frac{d\Xi_{\text{NA}}(T)}{dT},$$

$$\Xi_{\text{NA}}(T) = \left(\frac{1}{2v_F^* k_F}\right)^2 T \sum_{\varepsilon_m} \varepsilon_m^2 \int_0^{v_F^* k_F/|\varepsilon_m|} dx \, x \Psi(ix), \tag{63}$$

where $x = v_F^* q/i|\varepsilon_m|$ and

$$\Psi(z) = \int \frac{dy}{\pi} \frac{\Phi(y)}{z - y} \tag{64}$$

is the function whose discontinuity across the real axis is equal to Φ from Eq. (15).

Nonanalytic contributions to Eq. (63) can only arise from the dependence of the integral over x on its upper limit, which is determined by the behavior of Ψ at a large argument. Writing

$$\Psi(z \to \infty) \to \frac{\Psi_1}{z} + \frac{\Psi_2}{z^2} + \cdots \qquad (65)$$

yields

$$\Xi_{\mathrm{NA}}(T) = \frac{\Psi_2}{4(v_F^* k_F)^2} T \sum_{\varepsilon_m} \varepsilon_m^2 \ln \frac{v_F^* k_F}{|\varepsilon_m|}. \qquad (66)$$

The frequency sum generates regular terms determined by the upper limit of the summation; these form an analytic expansion of $\Xi(T)$ in powers of T^2. However, because of the logarithm, the sum also produces a universal, i.e., independent of the upper limit, term of order T^3,

$$T \sum_{\varepsilon_m} \varepsilon_m^2 \ln \frac{v_F^* k_F}{|\varepsilon_m|} = -2\zeta(3) T^3 + \cdots, \qquad (67)$$

where $\cdots$ stands for regular terms. This universal term can be obtained either directly, by using Euler-Maclaurin summation formula, or by evaluating the frequency sum as a contour integral. Substituting Eq. (67) into Eq. (66) and differentiating over T, we obtain

$$\delta S_{\mathrm{NA}} = \frac{3\zeta(3)}{2} \Psi_2 \left(\frac{T}{v_F^* k_F} \right)^2 + \mathcal{O}(T^3). \qquad (68)$$

B. Expansion in particle-hole pairs

First, we observe that because each factor of $\mathcal{P}_{\mathrm{NA}}$ brings an extra factor of z^{-1}, the only contribution to Ψ_2 involves one real excited particle-hole pair and the leading correction to Γ, and is thus proportional to the square of the reducible interaction. Proceeding as in the previous section, we obtain the analog of Eq. (20),

$$\Psi_2^{(1)} = - \int d\theta_p \mathcal{P}(ix; \theta_p) \Gamma(ix; \theta_p, \theta_p) \qquad (69)$$

and the analog of Eq. (21)

$$\Gamma = \int d\theta_k \Gamma^k(\theta_p - \theta_k) \mathrm{Re}\, \mathcal{P}(z, \theta_k) \Gamma^k(\theta_k - \theta_p). \qquad (70)$$

Writing

$$\Gamma^k(\theta_p - \theta_k) = \sum_L \bar{\Gamma}_L \cos[L(\theta_p - \theta_k)], \qquad (71)$$

using $(\cos\theta - ix)^{-1} = -i \int d\lambda \exp[-i\lambda(\cos\theta - ix)]$ (for $x > 0$), expanding the exponential in terms of Bessel functions and using known integrals gives

$$\Psi_2^1(ix) = -\sum_{LL'} \frac{\bar{\Gamma}_L \bar{\Gamma}_{L'}}{2x^2 \sqrt{1 + x^{-2}}}$$

$$\times \left[\left(\frac{1}{x} - \sqrt{1 + x^{-2}} \right)^{L+L'} + \left(\frac{1}{x} - \sqrt{1 + x^{-2}} \right)^{L-L'} \right]^2. \qquad (72)$$

Evidently, the large-argument limit of Eq. (73) is simply

$$\Psi_2 = -\sum_{L,L'} (-1)^{L+L'} \bar{\Gamma}_L \bar{\Gamma}_{L'} = -\left(\sum_L (-1)^L \bar{\Gamma}_L \right)^2. \qquad (73)$$

We see that the nonanalytic term in the entropy is determined by the backscattering amplitude. Restoring the spin and charge components of Γ, we then obtain for the entropy per particle

$$\delta S_{\mathrm{NA}} = -\frac{3\zeta(3)}{2} \left(\frac{T}{v_F^* k_F} \right)^2 \left(\Gamma_{c,\mathrm{BS}}^2 + 3\Gamma_{s,\mathrm{BS}}^2 \right) + \mathcal{O}(T^3). \qquad (74)$$

This coincides with the result in Ref. 27.

Computation in real frequencies: For the sake of completeness, we also demonstrate how to obtain Eq. (68) by performing a computation directly in real frequencies, without rotating the ε integral in Eq. (15). Using Eq. (17) and the two-dimensional analog of Eqs. (20) and (21), we obtain in 2D for the entropy per particle

$$\delta S_{\mathrm{NA}} = \frac{4}{\pi} \left(\frac{T}{v_F^* k_F} \right)^2 \int_0^\infty dx \frac{x^3}{\sinh^2 x} \int_0^{v_F^* k_F/xT}$$

$$\times dz z \int d\mathbf{n_k} \int d\mathbf{n_p}\, \mathrm{Im}\, \Pi(z, \mathbf{n_k}) \mathrm{Re}\, \Pi(z, \mathbf{n_p})$$

$$\times [\Gamma^k(\mathbf{n_k} \cdot \mathbf{n_p})]^2. \qquad (75)$$

We know from Eq. (68) that $\delta S_{\mathrm{NA}} \propto T^2$. As T^2 is an overall factor in Eq. (75), the rest of the integral can be evaluated at $T = 0$ upon which the upper limit of z integral is set to infinity. Integrations over x and over z then decouple, and evaluating the integral we obtain

$$\delta S_{\mathrm{NA}} = \frac{6\zeta(3)}{\pi} \left(\frac{T}{v_F^* k_F} \right)^2 R, \qquad (76)$$

where

$$R = \int_0^\infty dz z \int d\mathbf{n_k} \int d\mathbf{n_p} [\Gamma^k(\mathbf{n_k} \cdot \mathbf{n_p})]^2$$

$$\times \mathrm{Im}\, \mathcal{P}(z, \mathbf{n_k}) \mathrm{Re}\, \mathcal{P}(z, \mathbf{n_p}). \qquad (77)$$

The subtlety, which we mentioned in the overview of this section, becomes clear if we assume momentarily that Γ^k is a constant. In this case, the angular integrations in Eq. (77) are performed independently, and R reduces to $[\Gamma^k]^2 \int_0^\infty dz z\, \mathrm{Im}\, \Pi(z) \mathrm{Re}\, \Pi(z)$. As Eqs. (61) and (62) show, $\mathrm{Im}\, \Pi(z)$ and $\mathrm{Re}\, \Pi(z)$ are finite on the opposite sides of the boundary of the particle-hole continuum (for $|z| > 1$ and $|z| < 1$, respectively). One might then conclude that $\delta S_{\mathrm{NA}} = 0$.

However, the confluent singularities in Re Π and Im Π require a more careful analysis. Retaining the infinitesimal $i\delta$ to Π at intermediate steps one finds

$$\int_0^\infty dzz \, \text{Im} \, \Pi(z) \text{Re} \, \Pi(z) = \int_0^\infty \frac{dzz}{z+1} \, \text{Im} \frac{1}{\sqrt{z-1+i\delta}} \text{Re} \frac{1}{\sqrt{z-1+i\delta}} = \frac{1}{2} \int_{-\infty}^\infty dx \, \text{Im} \frac{1}{\sqrt{x+i}} \text{Re} \frac{1}{\sqrt{x+i}} = -\frac{1}{4} \int_{-\infty}^\infty \frac{dy}{y^2+1} = -\frac{\pi}{4}.$$

$$(78)$$

With this in mind, we proceed with the case of an arbitrary vertex $\Gamma^k(\mathbf{n_k} \cdot \mathbf{n_p})$. As before, we expand Γ^k in harmonics $\bar{\Gamma}_L$, see Eq. (71). Substituting this expansion into Eq. (77), we obtain after simple algebra

$$R = 2 \int_0^\infty dzz \int_0^\pi d\theta_{\mathbf{k}} \int_0^\pi d\theta_{\mathbf{p}} \times \text{Im} \, \mathcal{P}(z, \theta_{\mathbf{k}}) \text{Re} \, \mathcal{P}(z, \theta_{\mathbf{p}})$$

$$\times \sum_{L,L'} \bar{\Gamma}_L \bar{\Gamma}_{L'} [\cos(L_+\theta_{\mathbf{k}})\cos(L_+\theta_{\mathbf{p}}) + (L_+ \to L_-)]$$

$$= \sum_{L,L'} \bar{\Gamma}_L \bar{\Gamma}_{L'} I_{L,L'}, \qquad (79)$$

where $L_\pm = L \pm L'$. As R turns out to depend only on whether $L_\pm$ is odd or even we can just consider the L_+ term and double the result.

Consider first contributions with odd L_+. For these contributions, as we will see, one can neglect the $i\delta$ term in $\mathcal{P}$, i.e., replace $\text{Im} \, \mathcal{P}(z, \theta_{\mathbf{k}})$ by a δ-function $\text{Im} \, \mathcal{P}(z, \theta_{\mathbf{k}}) = -(1/2z)\delta(z^{-1} - \cos \theta_{\mathbf{k}})$. Integrating then over z, we obtain

$$I_{L+L'=2P+1} = \frac{1}{\pi} \left[\int_0^{\pi/2} d\theta_{\mathbf{k}} \frac{\cos[(2P+1)\theta_{\mathbf{k}}]}{\cos \theta_{\mathbf{k}}} \right]^2. \qquad (80)$$

The integral is convergent and equal to $(-1)^P \pi/2$. Accordingly,

$$I_{L+L'=2P+1} = \frac{\pi}{4}. \qquad (81)$$

If $L+L'$ is even, the integral over z vanishes unless one keeps δ finite and takes into account that both Re $\mathcal{P}$ and Im $\mathcal{P}$ diverge as z approaches 1. This divergence come from the angular integral over a narrow range near $\theta_k = \theta_p = 0$. Accordingly, we can safely set $\theta_k = \theta_p$ outside the product Re $\mathcal{P}$ $\times$ Im $\mathcal{P}$ in Eq. (79). After simple algebra we then obtain

$$I_{L+L'=2P} = -\frac{\pi}{4}. \qquad (82)$$

Combining Eqs. (81) and (82), we obtain

$$R = -\frac{\pi}{4} \sum_{L,L'} (-1)^{L+L'} \bar{\Gamma}_L \bar{\Gamma}_{L'} = -\frac{\pi}{4} \left(\sum_L (-1)^L \bar{\Gamma}_L \right)^2. \qquad (83)$$

Substituting this result into Eq. (76) and introducing the charge and spin components of Γ, we reproduce Eq. (74).

C. Ring diagram (RPA) approximation in 2D

In this subsection we present the two-dimensional version of the large-N calculation given above. Our goal here is to show that only Γ_{BS}^2 determines the nonanalyticity in $S(T)$, whereas higher powers of Γ_L do not appear in the prefactor for the T^2 term in $S(T)$.

We again consider a simplified model with the effective interaction $U = U_0 + U_1 \cos(\phi_p - \phi_k)$, and spin degeneracy $N = \infty$. The two dimensional version of Eq. (44) is

$$S_{\text{ring}} = -\frac{6\zeta(3)}{\pi} \left(\frac{T}{v_F^* k_F} \right)^2$$

$$\times \int zdz \, \text{Im} \ln \det[1 - \Gamma(\phi_p - \phi_k)\mathcal{P}(z, \phi_k)], \quad (84)$$

where now $\Gamma = Uk_F/(\pi v_F^*)$ and

$$\bar{\mathcal{P}}(z, \phi_k) = -\frac{1}{2\pi} \frac{\cos \phi_k}{\cos \phi_k - z^{-1}}. \qquad (85)$$

To evaluate the determinant we again must find the eigenvalues of the operator $1 - \Gamma\bar{\mathcal{P}}$, i.e., solve

$$\lambda\Lambda(z, \mathbf{n}_p) = \Lambda(z, \mathbf{n}_p) - \int d\mathbf{n}_k \Gamma(\mathbf{n}_p \cdot \mathbf{n}_k)\bar{\mathcal{P}}(z, \mathbf{n}_k)\Lambda(z, \mathbf{n}_k).$$

$$(86)$$

We find the eigenfunctions of $1 - \Gamma\bar{\mathcal{P}}$ by making the ansatz $\Lambda = \Lambda_0 + \Lambda_1 \cos \phi + \Lambda_2 \sin \phi$. We find that $\det = D_1 D_1$ with $D_{1,2}$ given by Eqs. (46), (47) as before, but now instead of Eqs. (48) and (49) we have

$$I_{nm} = \int \frac{d\phi}{2\pi} \frac{(\cos \phi)^{n+m+1}}{\cos \phi - z^{-1}}, \qquad (87a)$$

$$\bar{I}_{11} = \int \frac{d\phi}{2\pi} \frac{(\sin \phi)^2 \cos \phi}{\cos \phi - z^{-1}}. \qquad (87b)$$

Explicitly

$$I_{00} = 1 + \frac{i}{\sqrt{z^2-1}}, \qquad (88a)$$

$$I_{01} = \frac{1}{z} + \frac{i}{z\sqrt{z^2-1}}, \qquad (88b)$$

$$I_{11} = \frac{1}{2} + \frac{1}{z^2} + \frac{i}{z^2\sqrt{z^2-1}}, \qquad (88c)$$

$$\bar{I}_{11} = \frac{1}{2} - \frac{1}{z^2} + \frac{i\sqrt{z^2-1}}{z^2}. \tag{88d}$$

Evaluating and simplifying yields

$$D_1 = (1 + \Gamma)\left(1 + \frac{\Gamma_1}{2}\right)$$

$$\times \left(1 + \frac{\bar{\Gamma}_1(1 - \bar{\Gamma}_0)}{z^2} + \frac{i\left(\bar{\Gamma}_0 + \frac{\bar{\Gamma}_1(1 - \bar{\Gamma}_0)}{z^2}\right)}{\sqrt{z^2-1}}\right), \tag{89a}$$

$$D_2 = \left(1 + \frac{\Gamma_1}{2}\right) \times \left(1 - \frac{\bar{\Gamma}_1}{z^2} + \frac{i\bar{\Gamma}_1\sqrt{z^2-1}}{z^2}\right), \tag{89b}$$

with

$$\bar{\Gamma}_0 = \frac{\Gamma_0}{1 + \Gamma_0}, \tag{90a}$$

$$\bar{\Gamma}_1 = \frac{\Gamma_1}{1 + \frac{\Gamma_1}{2}}. \tag{90b}$$

Combining $\ln D_1$ and $\ln D_2$, we find

$$\int_0^{v_F^* k_F/T} z\,dz\, \mathrm{Im} \ln \det[1 - \bar{\Gamma}\mathcal{P}] = \int_0^{v_F^* k_F/T} z\,dz\, \mathrm{Im} \ln Q(z)Q(z) = 1 - 2\frac{\bar{\Gamma}_0\bar{\Gamma}_1}{z^2} - \frac{2\bar{\Gamma}_1^2(1 - \bar{\Gamma}_0)}{z^4} - \frac{1}{\sqrt{1-z^2}}$$

$$\times \left(\bar{\Gamma}_0 + \bar{\Gamma}_1 - \frac{2\bar{\Gamma}_1\bar{\Gamma}_0}{z^2} - \bar{\Gamma}_1^2(1 - \bar{\Gamma}_0)\left(\frac{-1}{z^2} + \frac{2}{z^4}\right)\right). \tag{91}$$

The integral can be most straightforwardly evaluated by rotating the z-integral onto the imaginary axis $z \to -i\bar{z}$. Then

$$\int_0^{v_F^* k_F/T} z\,dz\, \mathrm{Im} \ln Q(z) = -\int_0^{iv_F^* k_F/T} \bar{z}\,d\bar{z}\, \mathrm{Im} \ln Q(-i\bar{z}). \tag{92}$$

As $Q(-i\bar{z})$ is real and positive, the imaginary contribution from the integral over $\bar{z}$ can only come from the upper limit. One can verify that the dependence of the integral in Eq. (92) on the upper limit comes only from the $O(1/\bar{z})$ and $O(1/\bar{z}^2)$ terms in the expansion of $\mathrm{Im} \ln Q(-i\bar{z})$ in powers of $1/\bar{z}$. Higher order terms in the expansion are ultraviolet convergent and can be safely evaluated by setting the upper limit to infinity. Expanding $\ln Q(-i\bar{z})$ to second order in $1/\bar{z}$, we obtain

$$\ln Q(-i\bar{z}) = 1 - \frac{\bar{\Gamma}_0 + \bar{\Gamma}_1}{\bar{z}} - \frac{1}{2}\frac{(\bar{\Gamma}_0 - \bar{\Gamma}_1)^2}{\bar{z}^2} + \mathcal{O}(\bar{z}^{-3}). \tag{93}$$

As in 3D, the leading term $O(\bar{z}^{-1})$ gives a renormalization of the effective mass, while the next term $O(\bar{z}^{-2})$ gives

$$-\int_0^{iv_F^* k_F/T} \bar{z}\,d\bar{z}\, \ln Q(-i\bar{z}) \to \frac{1}{2}(\bar{\Gamma}_0 - \bar{\Gamma}_1)^2\, \mathrm{Im} \ln[iv_F^* k_F/T]$$

$$= \frac{\pi}{4}(\bar{\Gamma}_0 - \bar{\Gamma}_1)^2. \tag{94}$$

Substituting Eq. (94) into Eq. (84), we reproduce Eq. (74).

VI. SPIN SUSCEPTIBILITY IN 3D

In this section we discuss the relation between nonanalyticities in the entropy and in the spin susceptibility, χ_s. Until recently, the prevailing opinion had been that the nonanalytic $T^3 \ln T$ dependence of the entropy is not paralleled by a nonanalyticity in χ_s. Crucial evidence for this view was provided by the results of Carneiro and Pethick[18] and Bèal-Monod et al.,[13] who found that the leading term in the spin susceptibility scales as T^2 in 3D. However, in an important paper Belitz et al.[21] demonstrated that the apparent analytic temperature dependence of χ_s may be misleading. They performed a perturbative calculation of the momentum-dependent spin susceptibility $\chi_s(Q, T=0)$ at small Q and found a nonanalytic at $Q^2 \ln Q$ behavior. Later, it was found[30] that the magnetic-field dependence of a nonlinear spin susceptibility parallels the Q-dependence, i.e., $\chi_s(Q=0, T=0, H) \propto H^2 \ln |H|$.

Nonanalyticity of the spin susceptibility was also found for 2D systems by Millis and Chitov[24] and, later, Chubukov and Maslov,[27] Galitski et al.,[29] and Betouras et al..[30] These authors showed that $\chi_s(T, Q, H)$ scales linearly with the largest out of the three parameters (in proper units). Furthermore, Chubukov and Maslov[27] and Galitski et al.[29] have shown that in 2D, the nonanalytic term in χ_s may be expressed in terms of the spin component of the backscattering amplitude.

These results call for a better understanding of the spin susceptibility of a 3D system. In the preceding sections, we found that the nonanalytic part of the specific heat consists of two contributions, the first one coming from the excitation of a single particle-hole pair and the second one coming from the excitation of three particle-hole pairs, and with all coefficients determined by the harmonics of the Landau function.
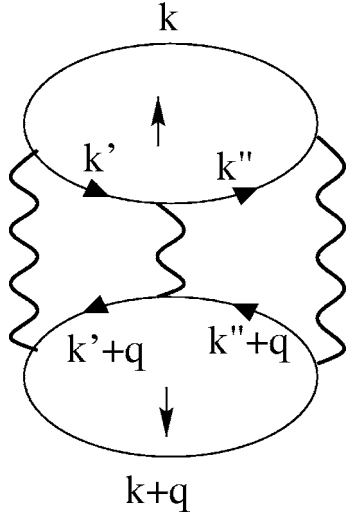
FIG. 4. A representative diagram for the thermodynamic potential, which contributes to the nonanalyticity in the spin susceptibility to third order in the interaction. Vertical arrows denote spin projections for fermions in the corresponding bubbles.

Here we ask if the nonanalytic contributions to the spin susceptibility are given in the same way. It is clear from the previous second-order calculations in 3D (Refs. 21, 24, and 27) that the single-pair contribution does contribute to the nonanalyticity in $\chi_s(Q,T,H)$, as is shown by, e.g., the second-order calculations. We show here that the three-pair terms, which were not considered in the previous literature, also contributes to the nonanalyticity of χ_s in 3D. For brevity, we refrain from a fully detailed analysis and simply indicate the origin of the effect. To this end, we consider the thermodynamic potential in a weak magnetic field perturbatively to third order in the interaction, which we take to be momentum-independent. We will be only interested in the spin effect of the field. A magnetic field (H) splits the Fermi surfaces for fermions with spins parallel and antiparallel to the field. This splitting does not affect the ε/q nonanalyticity of the polarization bubble at small q, if a particle and a hole have the same spin (in this case, a magnetic field just shifts the chemical potential), but it has a nontrivial effect on a bubble composed of a particle and a hole of opposite spins, $\Pi_{\uparrow\downarrow}(\varepsilon,q;H)$. For a spin-invariant interaction, such bubbles occur in diagram 3d of Fig. 1. Labeling the momenta as shown in the figure and integrating over fermionic momenta **k**, **k**′, and **k**″ and corresponding frequencies, we obtain the thermodynamic potential at $T=0$ and finite H in the form (see Fig. 4)

$$\Xi_{3D}(H) \sim U^3 \int d\varepsilon \int d^3q \; \mathrm{sgn}\, \varepsilon [\mathrm{Im}\, \Pi_{\uparrow\downarrow}(\varepsilon,q;H)]^3,$$

(95)

where

$$\mathrm{Im}\, \Pi_{\uparrow\downarrow}(\varepsilon,q;H) = -(\pi\varepsilon/2q)\Theta(v_F q - |\epsilon - g_L\mu_B H|)$$

and g_L and μ_B are the Lande-factor and the Bohr magneton, respectively. Note that Im Π retains its $H=0$ form but the

boundary of the continuum is shifted by the Zeeman energy. It is this shift which leads to a nonanalyticity in $\Xi(H)$. Integrating over q in Eq. (95) yields

$$\Xi(H) \sim U^3 \int d\varepsilon |\varepsilon|^3 \ln|\varepsilon - g_L\mu_B H|.$$

(96)

Consequently, the nonanalytic part of $\Xi(H)$ behaves as $H^4 \ln|H|$ and thus $\chi_s(H) \propto H^2 \ln|H|$. This contribution has the same functional form as the contribution from the spin component of the single-pair process. A similar consideration for a nonuniform field leads to a nonanalytic Q dependence of $\chi_s(Q): \chi_s(Q) \propto Q^2 \ln Q$.

Finally, at vanishing H and Q, both single-pair and three-pair processes contribute T^2 terms to the 3D spin susceptibility, which again is expressed only in terms of the Fermi-surface interaction and Fermi velocity. However, it can hardly be identified in the experiment, as there are additional, purely analytic T^2 contributions to $\chi_s(T)$ from all energies.

VII. FERMIONIC SELF-ENERGY

A. Overview

In this section we show how to obtain the nonanalytic term in the entropy and the specific heat by evaluating the fermionic self-energy first, and then using the relation between the self-energy and thermodynamic potential. We present this calculation both because historically the nonanalytic terms in the specific heat were studied by first calculating the self-energy and because, as we will show, the self-energy exhibits several nonanalyticities, not all of which contribute to the specific heat.

It has been known since the early days of the Fermi-liquid theory that at $T=0$ the subleading term in the 3D fermionic self-energy on the Fermi surface $\Sigma(\omega, k_F)$ depends on the frequency in a nonanalytic way, as $\Sigma(\omega, k_F) \propto \omega^3 \ln \omega$. Amit et al.[12] and others argued that this $\omega^3 \ln \omega$ dependence of Σ is the source for the $T^3 \ln T$ behavior of the entropy and the specific heat, as the entropy has the same scaling dimension as Σ and typical frequencies ω are generally expected to be of order T. Indeed, to second order in the interaction U, the thermodynamic potential in terms of the Matsubara self-energy is given by

$$\Xi = -\frac{1}{2}T\sum_{\varepsilon_m} \int \frac{d^D k}{(2\pi)^D}\Sigma(\omega_m,k)G_0(\omega_m,k),$$

(97)

where $G_0(\omega,k) = (i\omega_m - \epsilon_k)^{-1}$, and $G^{-1}(\omega,k) = G_0^{-1}(\omega,k) + \Sigma(\omega,k)$. If the self-energy is independent of k (which is the case for, e.g., the electron-phonon interaction), the momentum integral involves only the Green's function, and $\int d^D k G_0(\omega_m, k) \propto \mathrm{sgn}\, \omega$. Therefore, $\Xi \propto T\Sigma_{\omega_m>0}\Sigma(\omega_m)$. Evaluating the frequency sum, one obtains that the $\omega_m^3 \ln|\omega_m|$ behavior of the self-energy in 3D gives rise to a $T^3 \ln T$ term in the entropy.

In general, $\Sigma(\omega_m, k)$ depends on both ω_m and k. We will see that in the low-energy limit, the self-energy in 3D is a sum of two terms, one depending on $i\omega - \epsilon_k$, another depending separately on ω_m and ϵ_k,

$$\Sigma(\omega_m, k) = \Sigma^{(1)}(i\omega_m - \epsilon_k) + \Sigma^{(2)}(\omega_m, \epsilon_k). \qquad (98)$$

On the Fermi surface ($\epsilon_k = 0$), the two terms cannot be distinguished. However, away from the Fermi surface, their roles are different. If only the first term is present then the full Green function depends only on the variable $\omega - \varepsilon_k$ so that the locations of the poles are not changed by the interaction and thermodynamics cannot change. Alternatively speaking, the first term in Eq. (98) affects only the quasiparticle renormalization factor Z, but as is well known, the Z factor does not affect the specific heat coefficient of a Fermi liquid; hence the first term cannot contribute to the nonanalyticity in $S(T)$.

B. Explicit calculation of the fermionic self-energy in 3D

Now we proceed with an explicit evaluation of the nonanalytic part of self-energy $\Sigma(\omega_m, k)$ in 3D. To simplify the presentation, we consider the self-energy only to second order in the interaction $U(q)$, and only discuss small-angle scattering, i.e., assume $U(0) = U$, $U(2k_F) = 0$ [cf. Eqs. (10) and (11)].

As we mentioned in Sec. II, the stationarity of Ξ with respect to variations in G implies[4,16] that one can neglect any explicit temperature dependence of Σ, i.e., it suffices to include only the T-dependence arising from the difference between $T\Sigma_{\omega_m}$ and $\int d\omega/2\pi$.

The self-energy is given to leading order in U by

$$\Sigma(\omega_m, k) = U^2 T \sum_{\varepsilon_m} \int \frac{d^3 q}{(2\pi)^3} \Pi(\varepsilon_m, q) G_0(\omega_m + \varepsilon_m, \mathbf{k} + \mathbf{q}). \qquad (99)$$

The computation is tedious but straightforward. The frequency-dependent part of the polarization operator for $q \ll k_F$ is given by

$$\Pi(\varepsilon_m, q) = \frac{i\varepsilon_m}{2v_F q} \ln \frac{i\varepsilon_m + v_F q}{i\varepsilon_m - v_F q}. \qquad (100)$$

Expanding $\Pi(\varepsilon_m, q)$ in $\varepsilon_m / v_F q$, we obtain

$$\Pi(\varepsilon_m, q) = \frac{\pi |\varepsilon_m|}{2v_F q} - \left(\frac{\varepsilon_m}{v_F q} \right)^2 + \mathcal{O}(|\varepsilon_m|^3). \qquad (101)$$

Substituting Eq. (101) into Eq. (99), integrating over q, and summing over ε_m (which, to logarithmic accuracy, is equivalent to just integrating over ε_m), we find that $\Sigma(\omega_m, k)$ is given by Eq. (98) with

$$\Sigma^{(1)}(i\omega_m - \epsilon_k) = \left(\frac{mk_F}{\pi^2} U \right)^2 \frac{(i\omega - \epsilon_k)^3}{384 E_F^2} \ln \frac{E_F^2}{(i\omega - \epsilon_k)^2}. \qquad (102)$$

This contribution originates from the first, leading term in the expansion of $\Pi(\varepsilon_m, q)$ in ε_m in Eq. (101), and it comes from internal bosonic frequencies that exceed the external one, i.e., from $|\varepsilon_m| > |\omega_m|$. The second term depends on ω_m and ϵ_k separately and has contributions from both the first and second terms in Eq. (101). To logarithmic accuracy,

$$\Sigma^{(2)}(\omega_m, k) = -i \left(\frac{mUk_F}{\pi^2} \right)^2 \frac{\omega_m^3}{16 E_F^2} \ln \frac{E_F}{i\omega + \epsilon_k} + \mathcal{O}$$
$$\times \left[(i\omega_m - \epsilon_k) \ln \frac{E_F}{i\omega_m + \epsilon_k} \right] \qquad (103)$$

(see comment[36]). Note that $\Sigma^{(2)}(\omega_m, k)$ comes from the range of bosonic frequencies between $-|\omega_m|$ and $|\omega_m|$, i.e., relevant internal frequencies which are smaller than the external one.

Comparing Eqs. (102) and (103), we see that for a generic ratio of ϵ_k / ω_m and, in particular, for a particle on the Fermi surface, where $\epsilon_k = 0$, both contributions behave as $\Sigma(\omega_m, k_F) \propto \omega_m^3 \ln |\omega_m|$; this result is in agreement with Ref. 12. However, only $\Sigma^{(2)}(\omega_m, k)$ actually gives rise to the $T^4 \ln T$ term in Ξ, and hence to the $T^3 \ln T$ terms in the entropy and the specific heat. Indeed, the contribution of $\Sigma^{(1)}(\omega_m, k)$ to Ξ is

$$T \sum_{\omega_m} \int d\epsilon_k \Sigma^{(1)}(\omega_m, k) G_0(\omega_m, k) \propto T \sum_{\omega_m} \int_{-E_F}^{E_F} d\epsilon_k (i\omega_m$$
$$- \epsilon_k)^2 \ln \frac{E_F^2}{(i\omega_m - \epsilon_k)^2} = 2T \sum_{\omega_m} \mathrm{Re} \int_0^{E_F - i\omega_m} d\zeta \zeta^2 \ln \frac{E_F^2}{\zeta^2},$$

$$\zeta \equiv \epsilon_k - i\omega_m. \qquad (104)$$

The resulting integral is determined by the upper cutoff of the integration and is an analytic function of ω_m, hence giving only an analytic contribution to the thermodynamic potential. On the other hand, the convolution of $\Sigma^{(2)}(\omega_m, k)$ with the Green's function yields

$$T \sum_{\omega_m} \int d\epsilon_k \Sigma^{(2)}(\omega_m, k) G_0(\omega, k) \propto T \sum_{\omega_m} \omega_m^3$$
$$\times \int \frac{d\epsilon_k}{\epsilon_k - i\omega_m} \ln \frac{E_F}{\epsilon_k + i\omega_m}. \qquad (105)$$

To logarithmic accuracy, the integral over ϵ_k gives

$$\int \frac{d\epsilon_k}{\epsilon_k - i\omega_m} \ln \frac{E_F}{\epsilon_k + i\omega_m} = \int \frac{d\epsilon_k \epsilon_k}{\epsilon_k^2 + \omega_m^2} \ln \frac{E_F}{\epsilon_k + i\omega_m}$$
$$+ i\omega_m \int \frac{d\epsilon_k}{\epsilon_k^2 + \omega_m^2} \ln \frac{E_F}{\epsilon_k + i\omega_m}$$
$$= i\pi \, \mathrm{sgn} \, \omega_m \left[\int_{|\omega_m|}^{E_F} \frac{d\epsilon_k}{\epsilon_k} + \ln \frac{E_F}{|\omega_m|} \right]$$
$$= 2i\pi \, \mathrm{sgn} \, \omega_m \ln \frac{E_F}{|\omega_m|} \qquad (106)$$

Substituting this into Eq. (97), evaluating the frequency sum using

$$T \sum_{\omega_m} \omega_m^3 \ln \frac{E_F}{|\omega_m|} = \frac{2\pi^3}{15} T^4 \ln \frac{E_F}{T}, \qquad (107)$$

and differentiating over T, we obtain for the entropy per particle

$$S(T) = -\frac{\pi^4}{5}\left(\frac{mUk_F}{\pi^2}\right)^2\left(\frac{T}{v_F^* k_F}\right)^3 \ln\frac{v_F^* k_F}{T} \qquad (108)$$

This coincides with Eq. (39) expanded to second order, and evaluated at $U(0)=U$, $U(2k_F)=0$ (in this approximation, $K_c = mk_F U/\pi^2$, and $K_s=0$). We also verified that the contributions to $\Sigma^{(2)}$ from ε_m/q and ε_m^2/q^2 terms in Π yield equal contributions to Ξ. This is consistent with our earlier observation that non-analytic term in Ξ comes from the cross-product of the $O(\varepsilon_m)$ and $O(\varepsilon_m^2)$ terms in Π.

We see therefore that the nonanalytic term in $S(T)$ can indeed be obtained by evaluating the self-energy first and then convoluting it with the fermionic Green's function. However, only a part of the self-energy actually contributes to the nonanalyticity in $S(T)$. The term in the self-energy that depends on $i\omega_m - \epsilon_k$ gives a renormalization of the quasiparticle weight and does not lead to a nonanalyticity in $S(T)$.

It is instructive to comment here on the role of forward scattering. In 2D, forward scattering is special in the sense that its contribution to the self-energy is comparable to that from backscattering;[27] both are of order $\varepsilon_m^2 \ln|\varepsilon_m|$. In fact, the 2D analog of $\Sigma^{(1)}(\varepsilon_m,k)$ comes entirely from forward scattering. In 3D, forward scattering is much less effective because of phase space restrictions. One can show that in this case $\Sigma^{(1)}(\varepsilon_m,k)$, evaluated to all orders in the perturbation theory in the dimensionless coupling u, is not restricted to forward scattering but rather comes from a wide range of scattering angles. As we will discuss in the next section, there still exists $\omega^3 \log \omega$ contribution to $\Sigma(\omega,k_F)$ specific to forward scattering in 3D, but this contribution is exponentially small for a weak interaction.

VIII. COMPARISON TO EARLIER STUDIES

As we said in the Introduction, the analysis of the $T^3 \ln T$ term in the specific heat of a 3D Fermi liquid has a long history. In this section we compare our findings with the existing literature on the subject. We begin with reviewing the $T^3 \ln T$ nonanalyticities for the cases of electron-phonon and fermion-paramagnon interactions, and then proceed with the generic Fermi liquid case.

A. Electron-phonon interaction

The electron-phonon coupling leads to nonanalyticities in the entropy; in fact, the nonanalytic correction to the Fermi-liquid fixed point was discovered first by Eliashberg[5] in a study of acoustic phonons coupled to fermions via a deformation potential. Integrating out the phonons, one obtains a new contribution to the electron-electron interaction. For optical phonons, or for acoustic phonons with a piezoelectric coupling, this new interaction is frequency dependent on the scale of the Debye frequency, but at low frequencies it just generates an additional contribution to the electron vertex Γ^k, so the results we have presented in previous sections carry over directly.

The case of an acoustic phonon-deformation potential, studied by Eliashberg, presents a new feature: the effective interaction itself contains a nontrivial ε^2/q^2 term which leads

to a new non-analytic contribution to entropy in $D=3$ (but not in $D=2$). We discuss this case in more detail. Perturbation theory in the electron-phonon interaction is controlled by a small Migdal parameter—the ratio of the sound velocity c to the Fermi velocity v_F. To leading order in c^2/v_F^2, the thermodynamic potential can be expressed as[5,37]

$$\Xi_{e\text{-ph}} = \frac{1}{2}\int\frac{d\varepsilon}{\pi}n_B(\varepsilon/T)\int d^3q\,\text{Im}[\ln D^{-1}(\varepsilon,q)], \quad (109)$$

where D^{-1} is the inverse phonon propagator given by

$$D^{-1}(\varepsilon,q) = D_0^{-1}(\varepsilon,q) + 2g_{\text{ph}}^2 c^2 q^2 \Pi(\varepsilon,q), \qquad (110)$$

with bare propagator

$$D_0^{-1}(\varepsilon,q) = -\varepsilon^2 + c^2 q^2. \qquad (111)$$

Here g_{ph} is the dimensionless effective electron-phonon coupling. In a typical metal, $g_{\text{ph}} \sim 1$[4]. The expression for the thermodynamic potential $\Xi_{e\text{-ph}}$ is similar to the RPA expression for electron-electron interaction, cf. Eq. (42), but there are two important differences. First, the bare propagator D_0 describes gapless excitations. Second, the second term of Eq. (110) contains an extra factor q^2. This factor guarantees that the $q=0$ phonon mode does not affect the electrons ("Adler principle"). These two differences combine to preserve the $T^2 \log T$ form the nonanalytic correction to γ in 3D. One can rearrange the Im ln… in Eq. (109) by factoring out purely real quantities to get

$$\text{Im}\ln D^{-1} = \text{Im}\ln\left[1 - \frac{\varepsilon^2}{(c^*)^2 q^2} + g_*^2\Pi(\varepsilon,q)\right], \quad (112)$$

where now $(c^*)^2 = c^2 + 2g_{\text{ph}}^2\,\text{Re}\,\Pi(q,0)$ is the sound velocity renormalized by the analytic terms in Π, and g_* similarly is the renormalized coupling, $g_*^2 = 2g_{\text{ph}}^2(c/c^*)^2$. In analogy to the previous section, we now expand in powers of Im Π. As before, the term with one real particle-hole pair (i.e., of first order in Im Π) combines with the leading ε^2/q^2 term in the interaction vertex to give a $T^3 \ln T$ nonanalyticity. In addition, however, another $T^3 \ln T$ emerges from the combination of Im Π and $\sim \varepsilon^2/c^{*2}q^2$ term from the bare phonon propagator. Such a term does not exist for the electron-electron interaction. A simple analysis shows that this new contribution is larger by the inverse Migdal parameter $(v_F/c)^2$ than the contribution from the interaction vertex and hence is the dominant phonon $T^3 \ln T$ contribution to the specific heat.[38]

Note that the sign of the phonon $T^3 \ln T$ term is opposite to that of the electron one, i.e., $d\gamma(T)/dT$ is positive for electron-phonon contribution and negative for the electron-electron contribution.

In $D=2$ (when both electrons and phonons are two-dimensional) the q dependence of the phonon propagator remains the same, but the integration measure changes, and g_{ph}^2 term does not give rise to a nonanalyticity in Ξ. Only the "conventional" term of order g_{ph}^4 gives rise to nonanalyticity. From this perspective, the electron-phonon problem in 2D does not differ qualitatively from the electron-electron interaction. In the case of mixed dimensionality, e.g., planes of electrons embedded into a 3D elastic continuum, more complex situations are possible.

One may also extract the "phonon" $T^3 \ln T$ term in the entropy from the nonanalytic form of the electron self-energy for the electron-phonon interaction $\Sigma_{\mathrm{ph}}(\omega_m) \sim \omega_m^3 \ln|\omega_m|$,[4] in analogy with Eq. (97). We reiterate that a direct relation between the nonanalyticities in the self-energy and entropy exists only for the case when the self-energy depends only on frequency. This point will be important for the discussion of the zero-sound mode contribution to the entropy in Sec. VIII C 1.

Finally, we note that the large phonon-induced nonanalyticity in the entropy is not paralleled by a similar term in the spin susceptibility. This comes about because phonons contribute directly to the charge vertex, and only indirectly to the spin vertex, at the subleading order in the Migdal parameter. The basic reasoning for this was given a long time ago by Prange and Kadanoff,[39] who showed that in the Migdal approximation, phonon contribution to the thermodynamic potential is confined to the Fermi surface and does not change appreciably when the Fermi surface position changes. As a result, an applied magnetic field, which shifts spin up and spin down Fermi surfaces, does not change the phonon contribution to the thermodynamic potential, to leading order in the Migdal approximation. Mathematically, this can be seen as follows: (i) The static spin susceptibility $\chi_s(q)$ is given by the particle-hole bubble. (ii) For free fermions, $\chi_s(q)$ comes from the states away from Fermi surface, if one integrates first over fermionic dispersion ϵ_k, while the contribution from the states near the Fermi surface vanishes because the poles of the integrand as a function of dispersion ϵ_k reside in the same half-plane. (iii) The renormalization of the spin susceptibility due to electron-phonon interaction occurs via self-energy insertions into the spin-polarization bubble. As the self-energy does not depend on the electron momentum, these insertions still leave the two poles of the integral over ϵ_k in the same half-plane, hence the integral over low-energies still vanishes.

The analysis beyond leading order in the Migdal parameter requires further analysis, not given here. However, for comparable electron-phonon and electron-electron couplings, the phonon renormalization of χ_s is of the same order as the purely electron one.

B. Paramagnon model

In the paramagnon model,[10] the fermion-fermion interaction is approximated by an interaction between fermions and overdamped long-wavelength spin fluctuations (paramagnons). The model has been extensively used to describe itinerant electrons near a ferromagnetic instability. While the analysis is almost identical to the analyses we have already presented, the possibility of tuning the model through a ferromagnetic quantum phase transition raises a new issue which requires discussion.

The RPA thermodynamic potential per unit volume in the paramagnon theory has the same form as Eq. (42) but with slightly different normalization factors; we write it explicitly here for convenience:

$$\Xi = \Xi_0 + \frac{3}{2} \int \frac{d\varepsilon}{\pi} n_B(\varepsilon/T) \int \frac{d^D q}{(2\pi)^D} \ln[D_S^{-1}(\varepsilon, q)]. \tag{113}$$

Here Ξ_0 is the thermodynamic potential of the noninteracting fermions and $D_S^{-1}(\varepsilon, q) = 1 + 2g\mathcal{P}_{\mathrm{ph}}(\varepsilon, q)$ is the inverse propagator of spin fluctuations, g is the spin-fermion coupling, and $\mathcal{P}_{\mathrm{ph}}$ is the full polarization bubble for noninteracting fermions. This model is therefore equivalent to the previously studied RPA case, but with only the $L=0$ spin channel interaction retained, so that the determinant [cf. Eq. (44)] may be written down immediately. However, one extra piece of physics becomes important. This theory can describe a ferromagnetic instability, and indeed an original motivation for the model was the study of effects of long wavelength spin fluctuations, which may be expected to be important in nearly ferromagnetic materials. Near a ferromagnetic instability, the dependence of the *analytic* part of Π on q becomes important. Decomposing Π into the analytic $(\text{const}+q^2)$ and nonanalytic (Π_{NA}) parts, we have

$$D_S^{-1}(\varepsilon, q) = (1 + B) + \xi_0^2 q^2 + 2g\Pi_{\mathrm{NA}}(\varepsilon, q) \tag{114}$$

with $B = 2g\Pi(0,0) \to -1$ as the ferromagnetic transition is approached. We now proceed as before expanding in Im Π_{NA} (representing real particle-hole pairs) and Re Π_{NA} (representing the nonanalytic part of the dynamical interaction) The expansion contains inverse powers of $1 + B + \xi_0^2 q^2$, so that the power counting of the momentum integrals changes for $q > \xi = \xi_0/\sqrt{1+B}$.[40] The nonanalyticities we have discussed in this paper arise only from the Fermi-liquid regime $T/v_F < q < \xi^{-1}$. As criticality is approached, ξ diverges and the temperature window in which the $T^3 \ln T$ term may be observed becomes vanishingly small. It is thus somewhat misleading to state that the $T^3 \ln T$ nonanalyticities become the critical nonanalyticities; rather, they should be regarded as a property of the Fermi-liquid regime only, which is the "quantum disordered regime" of the ferromagnetic quantum phase transition.

With this proviso, the previously given analysis can be applied directly. We note here that the original work[10,17] focused on the three-pair contribution, obtained by treating Re Π_{NA} as a constant. In 3D one finds

$$\delta S^{(3)}(T) = \frac{\pi^6}{20} \left(\frac{B}{\Lambda}\right)^3 \left(\frac{T}{v_F^* k_F}\right)^3 \ln \frac{\Lambda v_F^* k_F}{T}, \tag{115}$$

with $B = -2g\bar{\Pi}(0,0)$ and $\Lambda = (\xi_0/\xi)^2 = 1 + B$. This expression coincides with Eq. (39), if one considers only spin contribution, neglects the backscattering term in K_s and all $\bar{\Gamma}$ except for $\bar{\Gamma}_{s,0}$, and identify B with the Landau parameter $\Gamma_{s,0}$, such that $\bar{\Gamma}_{s,0} = B/(1+B)$. Note that the narrowness of the "Fermi-liquid" momentum regime appears here as a dependence of the upper cutoff of the logarithm. Incorporating the cutoff into the analytical T^3 terms, while mathematically justified, obscures these physics.

The single-pair contribution, overlooked in the paramagnon literature,[10,17] arises (as previously discussed) by including the nonanalytic momentum dependence of Re Π and is

$$\delta S^{(1)} = -\frac{3\pi^4}{5}\left(\frac{B}{\Lambda}\right)^2\left(\frac{T}{v_F^* k_F}\right)^3 \ln\frac{\Lambda v_F^* k_F}{T}. \qquad (116)$$

This expression also coincides with Eq. (39), if one again considers only the spin contribution, but now takes into account only backscattering term in K_s, neglect all $\bar{\Gamma}$ except for $\bar{\Gamma}_{s,0}$, and again identify B with the Landau parameter $\Gamma_{s,0}$. Comparing the two nonanalytic contributions to $\delta S(T)$, we find that for negative B, i.e., positive g, they have the same sign and become equal at $B \approx -0.55$. Further away from the ferromagnetic instability (at smaller $1+B$), the single-pair contribution dominates, while closer to the instability, the three-pair contribution is larger.

The same situation holds in 2D. Away from criticality, there is only one nonanalytic contribution to $S(T)$, from $\Pi^2(\varepsilon_m,q)$, i.e, from a single-pair excitation with a frequency-dependent interaction. It yields

$$\delta S(T) \propto \left(\frac{B}{\Lambda}\right)^2\left(\frac{T}{v_F^* k_F}\right)^2. \qquad (117)$$

Equation (117) only applies for $T < v_F/\xi$; again as criticality is approached the window of validity closes.

C. Fermi Liquid

The nonanalytic term in the entropy of a 3D Fermi Liquid arising from a generic fermion-fermion interaction was considered in a number of publications in 1960s and 1970s.[6–8,12,14–16] Recently, the $T^3 \ln T$ term in $S(T)$ was reproduced via multidimensional bosonization.[22,23]

1. Zero-sound mode

The very early studies[6–8] focused on the nonanalytic contribution due to the interaction with zero sound. As this theme returns in later work (see, e.g., Ref. 19), it is worthwhile to summarize the status of the problem here. The idea that the interaction with zero sound may lead to a nonanalyticity in the specific heat was put forward by Anderson.[6] Following this suggestion, Balian and Fredkin[7] considered a phenomenological model which treated the interaction with zero sound in analogy with the electron-phonon interaction. However, a nonphysical choice of the interaction vertex (it remained nonzero at $q \to 0$, and thus did not satisfy the Adler principle) led Balian and Fredkin to the conclusion that the nonanalyticity in the self-energy was very strong ($\omega_m \ln|\omega_m|$). Later, Engelsberg and Platzman[8] pointed out that the zero-sound vertex vanishes as q for $q \to 0$, as it does for the deformation-potential-type interaction with phonons [cf. Eqs. (109) and (110)]. They obtained an $\omega_m^3 \ln|\omega_m|$ nonanalyticity in the self-energy at the Fermi surface, with the prefactor proportional to the difference of the zero-sound and the Fermi velocities $c_{zs} - v_F > 0$. In 3D, this difference is exponentially small for weak interactions u: $c_{zs} - v_F \propto e^{-1/u}$. In 2D, the analogous analysis yields an $\omega_m^2 \ln|\omega_m|$ term in the self-energy, with a prefactor of order u^2.

In this and subsequent publications, it was *assumed* that the zero-sound nonanalyticity in the self-energy automatically implies a nonanalyticity in the specific heat. Indeed, if

one treats zero sound as a free bosonic mode, a phase space argument suggests that the corresponding contribution to $C(T)$ is of order T^2.[19] However, two of us and our collaborators have shown recently[27,28] that the nonanalyticity of the self-energy due to interaction with zero-sound does not lead to a nonanalyticity in the thermodynamics.

To be more precise, the zero-sound contributions to the electron self-energy come with prefactors which scale as powers of the difference between the zero-sound velocity c_{ZS} and the fermi velocity v_F. This difference $c_{ZS} - v_F$ becomes finite only when the perturbation theory is summed up to all orders, so that a putative "zero sound contribution" to the specific heat would be nonperturbative in the coupling constant. However, it was shown in Refs. 27 and 28 that the prefactor for the $C(T) \propto T^2$ in 2D is expandable in regular series of the coupling constant, i.e., there are no *nonperturbative* contributions to the nonanalytic term in $C(T)$. An alternative derivation of the same result was provided by Catelani and Aleiner,[42] who argued away the forward-scattering (and thus the zero-sound) contribution to thermodynamics on the basis of gauge invariance of the Fermi-liquid kinetic equation.

The absence of a zero-sound contribution to $C(T)$ may be easily seen in calculations of the thermodynamic potential performed in the Matsubara formalism. In this formalism poles do not explicitly appear, and as seen for example in the "RPA" analyses of previous sections or in Refs. 27 and 28, one may simply expand in powers of the coupling constant. However, in the real-frequency formalism the calculation is slightly tricky because two physical effects contribute: first, a zero sound singularity exists in the density response function, and gives rise to a contribution to $C(T) \propto T^2$ (see, Ref. 19 and Appendix E of Ref. 27). Second, the presence of this singularity leads to a modification of the particle-hole pair excitation spectrum near the upper boundary of the particle-hole continuum. This gives rise to an additional term in $C(T) \propto T^2$, which precisely cancels the "pure" zero-sound contribution.

On a more technical level, we come back to the argument that a direct relation between the nonanalyticities in the self-energy and entropy exists only if fermions interact with slow boson modes, so that the self energy is approximately k-independent ("local"). As the zero-sound velocity is necessarily larger than the Fermi velocity, this relation does not have to hold for this case. The breakdown of this relation becomes evident on examination of the real-frequency analog of Eq. (97),

$$\delta S = \frac{2}{\pi}\frac{\partial}{\partial T}\left[\frac{1}{T}\int\frac{d^Dk}{(2\pi)^D}\int_{-\infty}^{\infty}d\omega\,\omega\frac{\partial n_F}{\partial\omega}\{\text{Re}\,\Sigma(\omega,k)\text{Im}\,G_0(\omega,k) \right.$$
$$\left. + \text{Im}\,\Sigma(\omega,k)\text{Re}\,G_0(\omega,k)\}\right], \qquad (118)$$

where $G_0(\omega,k) = (\omega - \epsilon_k + i0^+)$ and n_F is the Fermi function. In a local theory, the k integral of the $\text{Re}\,\Sigma\text{Im}\,G_0$ term in Eq. (118) vanishes, and only $\text{Re}\,\Sigma$ contributes to δS. In a nonlocal theory, both the $\text{Re}\,\Sigma$ and the $\text{Im}\,\Sigma$ terms contribute. An explicit computation shows[27,28] that the zero-sound contribu-

tions from the two terms in Eq. (118) cancel each other in 2D. Although we have not performed such a calculation in 3D, we believe that the result will be the same as in 2D, i.e., the zero-sound contribution vanishes. This also follows from the simple argument that in the RPA (ring) approximation in 3D, the entropy for the interaction with zero-sound is given by Eq. (42), in which the argument of the logarithm is replaced by the inverse effective interaction $U_{\text{eff}}(\varepsilon, q)$. Near the zero-sound pole, $U_{\text{eff}}(\varepsilon, q) \propto e^{-1/u} q^2 / (\varepsilon^2 - c_{\text{zs}}^2 q^2)$ and the zero-sound contribution to the entropy coincides with that of a free bosonic mode. Accordingly, the entropy is analytic and scales as T.[34,43]

2. Fermion-fermion interaction

Detailed calculations of the self-energy and $S(T)$ in generic models of interacting fermions have been performed by Amit, Kane and Wagner[12] and Pethick and Carneiro,[15] and we compare our results with theirs.

Amit *et al.* computed the self-energy near the mass shell, expressed the result in terms of Landau parameters keeping only the partial amplitudes with $L = 0, 1$, and then used the resulting expression for the self-energy to compute the $T^3 \ln T$ correction to the entropy obtaining a result somewhat similar to the one presented here. There are however three qualitative distinctions between their results and ours. First, Amit *et al.* argued that the nonanalytical $\omega_m^3 \ln |\omega_m|$ term in the self-energy comes only from the Landau damping term (ε/q) in the polarization bubble. As we have shown, there are actually two contributions, one from the Landau damping (ε_m/q) term in $\Pi(m, q)$ (considered by Amit *et al.*) and another from ε_m^2/q^2 (overlooked by Amit). Second, Amit *et al.* argued that the nonanalytical self-energy at the mass shell partly comes from the interaction with zero sound. We showed in the previous subsection that this interaction does not contribute to the $T^3 \ln T$ term in $S(T)$. Third, Amit *et al.* argued that $2k_F$ scattering does not contribute to thermodynamic potential to second order in the interaction. We found that the contributions from small momentum scattering and from $2k_F$ scattering are identical at this order [up to prefactors $U(0)$ and $U(2k_F)$]. We also note that the final result of Amit *et al.* for $\delta S(T)$ [Eqs. (IV.17) from Ref. 12 and (VI.8) and (VII.18) from Ref. 12] is not identical to the result we presented. Their combination of A_0 and A_1 does not reduce to the square of the backscattering amplitude, and their overall factor in $\delta S(T)$ and the relative prefactor for the terms with $\pi^2/12$ in K_s in Eq. (39) are different from ours.

Our result for the entropy fully agrees with that of Carneiro and Pethick,[15] if we use the same approximation as they did, namely, neglecting all Landau coefficients with $L > 2$. Within this approximation, our Eq. (39) coincides with Eqs. (22) and (A19) in Ref. 15. In particular, to second order in the scattering amplitude, their prefactor for the entropy is $A_0^2 + A_1^2 + A_2^2 + 2A_0A_2 - 2A_2A_1 - 2A_0A_1$ [the last term is missing in Eq. (A19) of Ref. 15, but this is obviously a misprint]. This combination is nothing but the square of the backscattering amplitude $A_{\text{BS}}^2 = (A_0 - A_1 + A_2)^2$, which is the first term in our prefactor K in Eq. (39). From this perspective, our new result is the observation that the sum of all bilinear

products of partial components of A reduces to the square of the backscattering amplitude.

It is frequently stated in the literature that Pethick and Carneiro considered forward scattering between Landau quasiparticles, apparently in disagreement with our result that forward scattering does not give rise to a nonanalyticity in Ξ. There is in fact no contradiction because the forward scattering considered by Pethick and Carneiro involves small angle scattering between particles slightly away from the Fermi surface, whereas we consider scattering of particles at the fermi surface.

To see this explicitly, we note that Pethick and Carneiro used the Fermi liquid relation between the entropy and the thermal correction to the quasiparticle energy

$$\delta S(T) = \sum_{\mathbf{p}} \Delta\epsilon_{\mathbf{p}}(T) \frac{\partial n_{\mathbf{p}}^0}{\partial T}, \qquad (119)$$

where $n_{\mathbf{p}}^0$ is Fermi function (a sum over spins is implicit), expressed $\Delta\epsilon_{\mathbf{p}}(T)$ in terms of the Landau function

$$\Delta\epsilon_{\mathbf{p}}(T) = \sum f_{\mathbf{p},\mathbf{p}+\mathbf{q}} n_{\mathbf{p}+\mathbf{q}}^0, \qquad (120)$$

and argued that when f is expanded in powers of $\mathbf{p} \cdot \mathbf{q}$, the expansion contains a quadratic term $(\mathbf{p} \cdot \mathbf{q})^2$. This quadratic term gives rise to a $T^3 \ln T$ contribution to $\Delta S(T)$.

It is important to realize that the Landau function $f_{\mathbf{p},\mathbf{p}+\mathbf{q}}$ describes the interaction between physical quasiparticles, for which $\omega = v_F^*(|p| - p_F)$. Accordingly, $f_{\mathbf{p},\mathbf{p}+\mathbf{q}}$ coincides, up to an overall factor, with the fully renormalized vertex $\Gamma[(\mathbf{p}, \omega), (p+q, \omega+\varepsilon); (\mathbf{p}'\omega'), (\mathbf{p}'+\mathbf{q}, \omega'+\varepsilon)]$ (where the first two pairs of arguments corresponds to the initial states and the last two to the final ones), taken at $\mathbf{p} = \mathbf{p}', \omega' = \omega$ and $\varepsilon = \epsilon_{\mathbf{p}+\mathbf{q}} - \epsilon_{\mathbf{p}} \approx v_F \mathbf{n_p} \cdot \mathbf{q}$, where $\mathbf{n_p}$ is a unit vector along $\mathbf{p}$. In fact, the $(\mathbf{p} \cdot \mathbf{q})^2$ term in $f_{\mathbf{p},\mathbf{p}+\mathbf{q}}$ describes how the interaction between quasiparticles evolves when they move away from the Fermi surface. A similar consideration has recently been applied to the analysis of nonanalytic terms in the spin susceptibility.[29]

The full irreducible vertex

$$\Gamma[(\mathbf{p}, \omega), (p+q, \omega+v_F\mathbf{n_p}\cdot\mathbf{q}); (\mathbf{p}'\omega'),$$
$$(\mathbf{p}'+\mathbf{q}, \omega'+v_F\mathbf{n_p}\cdot\mathbf{q})]\big|_{\varepsilon=v_F\mathbf{n_p}\cdot\mathbf{q}} = \Gamma_{\mathbf{p},\mathbf{p}'}(\theta_0),$$
$$(121)$$

where θ_0 is the angle between $\mathbf{p}$ and $\mathbf{q}$, satisfies an integral equation, graphically shown in Fig. 1 of Ref. 15. The quadratic dependence on $\cos\theta_0$, which eventually leads to the $T^3 \ln T$ term in $S(T)$, comes from a virtual particle-hole bubble, composed from quasiparticles at finite energy, i.e., away from the Fermi surface. All analytic corrections are absorbed into the bare vertex $\Gamma_{\mathbf{p},\mathbf{p}'}^0(\cos\theta_0)$, in which one can then safely set $\cos\theta_0 = 0$. Explicitly,

$$\Gamma_{\mathbf{p},\mathbf{p}'}(\theta_0) = \Gamma_{p,p'}^0 + A \int d^3p_1 \Gamma_{\mathbf{p},\mathbf{p}_1}^0 G_{\mathbf{p}_1} G_{\mathbf{p}_1+\mathbf{q}} \Gamma_{\mathbf{p}_1,\mathbf{p}'}(q, \theta_0),$$
$$(122)$$

where A is a constant prefactor. The bare vertex describes the interaction between physical fermions at the Fermi surface

and coincides, up to an overall factor, with $\Gamma^k(\mathbf{n}_p \cdot \mathbf{n}_{p'})$ from Eq. (13). However, $\Gamma_{\mathbf{p},\mathbf{p}}(\theta) \propto f_{\mathbf{p},\mathbf{p}+\mathbf{q}}$ is only obtained by solving the integral Eq. (122).

Pethick and Carneiro solved Eq. (122) keeping only $L = 0..2$ partial components of Γ^k and found contributions to the $\cos^2 \theta_0$ term in $\Gamma_{\mathbf{p},\mathbf{p}}(\theta_0)$ of the second and third orders in $\Gamma^0_{\mathbf{p},\mathbf{p}'}$. The second-order contribution and part of the third-order corresponds physically to processes with one excited particle-hole pair, while the rest of the third-order contribution describes the process with three excited particle-hole pairs.

One can easily demonstrate, starting from Eq. (122), that the second-order contribution is indeed expressed in terms of the backscattering amplitude, as we found in Eq. (39). Applying one iteration to Eq. (122) and integrating over $\epsilon_{\mathbf{p}_1}$ and over frequency, we obtain

$$\Gamma_{p,p}(\theta_0) \propto \cos \theta_0 \int \frac{d\Omega_\theta}{4\pi} \frac{\Gamma^0_{p,p_1} \Gamma^0_{p_1,p}}{\cos \theta_0 - \cos \theta}, \qquad (123)$$

where Ω_θ is a solid angle and θ is the angle between $\mathbf{p}_1$ and $\mathbf{q}$. Substituting an expansion of $\Gamma^0_{\mathbf{p},\mathbf{p}_1} = \Gamma^0(\theta - \theta_0)$

$$\Gamma^0_{\theta-\theta_0} = \sum_n \bar{\Gamma}_n \cos^n(\theta - \theta_0), \qquad (124)$$

into Eq. (123), expanding the integrand to order $\cos^2 \theta_0$ and evaluating the prefactors in the same way as in Sec. IV, we obtain after rather straightforward algebra that

$$\Gamma_{p,p}(\theta_0) \propto \cos^2 \theta_0 \sum_{n,m} \bar{\Gamma}_n \bar{\Gamma}_m (-1)^{n+m} = \cos^2 \theta_0 \Gamma^2_{\mathrm{BS}}.$$
$$(125)$$

This fully agrees with Eq. (39).

The same analysis can also be applied to the 2D case. The only difference is that in 2D the T^2 term in $S(T)$ comes from all angles θ_0, rather than from a specific range of small θ_0. Still, one can obtain an integral equation for $\Gamma_{\mathbf{p},\mathbf{p}'}(\theta_0)$, similar to Eq. (122) and solve it by the same method as in Sec. V.

IX. COMPARISON TO EXPERIMENT

In this section we present a brief comparison of the results to measurements of the $T^2 \ln T$ term in the specific heat coefficient $\gamma(T) = C(T)/T$ in ^{3}He (Ref. 2) and in several heavy-fermion materials.[3,44] The ^{3}He data were analyzed using a formula equivalent to our Eq. (39) but retaining only $L = 0, 1$ harmonics in charge and spin channels. The values of $\Gamma_{c,0}$, $\Gamma_{c,1}$, and $\Gamma_{s,0}$ were taken from independent measurements of the compressibility, effective mass, and spin susceptibility, respectively, and $\Gamma_{s,1}$ was chosen as a free parameter to fit the observed $T^2 \ln T$ term in $\gamma(T)$. At zero pressure, $\Gamma_{c,0} = 9.15$, $\Gamma_{c,1} = 5.27$, $\Gamma_{s,0} = -0.700$, and $\Gamma_{s,1} = -0.55$.[2] The use of Eq. (13) then gives $\Gamma_{\mathrm{BS},c} = -1.01$ and $\Gamma_{\mathrm{BS},s} = -1.66$, which yields for the combination of backscattering amplitudes entering Eq. (39), $\Gamma^2_{\mathrm{BS},c} + 3\Gamma^2_{\mathrm{BS},s} = 9.28$. Estimating the rest of the terms in $K_c + 3K_s$ (which are cubic in the amplitudes) in the same approximation, we obtain (K_c

$+ 3K_s)_{\mathrm{cubic}} = 20.05$. Thus in ^{3}He at ambient pressure it appears that the backscattering contribution is about a third of the total result. At higher pressures, the relative importance of the backscattering contribution diminishes to about a quarter of the total result. This suggests that in ^{3}He the "paramagnon" ($\Gamma^3_{s,0}$) contribution is quite important, although the lack of information about higher Landau parameters means that this conclusion must be regarded as provisional.

Similar analysis of the data for UPt$_3$ was performed in Ref. 44. In this case, however, only the $L = 0$ harmonics in Eq. (39) were retained. The value of $\Gamma_{s,0}$ extracted from the magnitude of the $T^2 \ln T$ term in $\gamma(T)$ does not agree with the value obtained from the enhancement of the spin susceptibility. Given this controversy, we refrain from estimating the relative importance of the backscattering contribution to $\gamma(T)$ for this system.

X. SUMMARY

We have revisited the old subject of nonanalytic contributions to the entropy (or specific heat coefficient) in light of recent progress in our understanding the structure of the leading corrections to Fermi-liquid theory. We have clarified the connection to the Fermi-liquid fixed point by showing that the coefficients of the nonanalytic terms may be expressed in terms of partial harmonics of the fully reducible static scattering amplitude $\Gamma^k(\mathbf{n}_p, \mathbf{n}_{p'})$ with both momenta taken on the Fermi surface. We have presented a formalism which allows us to distinguish between excited particle-hole pairs and their (possibly nonanalytic) interaction.

We have found two classes of contributions to the specific heat nonanalyticity. The first one, pertinent to both two- and three-dimensional Fermi liquids, arises from an excitation of a single-particle pair above the ground state combined with a nonanalyticity in the quasiparticle interaction vertex. The second one, pertinent only to Fermi liquids in dimension $D \geq 3$, arises from the excitation of three particle-hole pairs, which interact via the analytic fixed-point interaction. In $D < 3$ the first sort of nonanalyticity is determined solely by the backscattering amplitude, but in $D \geq 3$ additional contributions may arise.

The fact that the nonanalyticity in the entropy involves the spin component of the scattering amplitude suggests that the same physics is responsible for the nonanalyticities in the susceptibility which have been discussed extensively elsewhere.[21,25,27,28] We have presented simple calculations which confirm this suggestion. We thus conclude that in $D = 3$ the nonanalytic momentum and magnetic field dependencies of χ_s, reported earlier, involve terms up to third order in the exact quasiparticle scattering amplitude, and the quadratic term is the square of the spin component of the backscattering amplitude. From this perspective, the nonanalyticities in the spin susceptibility are the same nonanalyticities which have been studied in the context of specific heat for many years.

APPENDIX: $2k_F$ CONTRIBUTION TO NONANALYTICITY IN THE SPECIFIC HEAT

We argued in Sec. II that one can re-express the $2k_F$ contribution to the thermodynamic potential Ξ as an effectively small q contribution. For completeness, however, we show in this Appendix how to evaluate the $2k_F$ contribution to Ξ directly. The technical difficulty of this approach is that one needs to know the polarization bubble near $2k_F$ at finite frequency *and* at finite temperature. For simplicity, we limit our analysis to the second-order perturbation theory.

We first obtain the asymptotic form of $\Pi(\varepsilon_m, q, T)$ and then use it in the calculation of the thermodynamic potential. It is convenient to separate $\Pi(\varepsilon_m, q, T)$ into a $T=0$ part and the T-dependent part as

$$\Pi(\varepsilon_m, q, T) = \Pi_0 + \Pi_T,$$

where

$$\Pi_0 \equiv \Pi(\varepsilon_m, q, 0)$$
$$= \int \frac{d^3k}{(2\pi)^3} \Theta(-\epsilon_{\mathbf{k}}) \left[\frac{1}{i\varepsilon_m - \epsilon_{\mathbf{k+q}} + \epsilon_{\mathbf{k}}} - \frac{1}{i\varepsilon_m - \epsilon_{\mathbf{k}} + \epsilon_{\mathbf{k-q}}} \right],$$

$$\Pi_T \equiv \Pi(\varepsilon_m, q, T) - \Pi(\varepsilon_m, q, 0)$$
$$= \int \frac{d^3k}{(2\pi)^3} [n_F(\epsilon_{\mathbf{k}}) - \Theta(-\epsilon_{\mathbf{k}})]$$
$$\times \left[\frac{1}{i\varepsilon_m - \epsilon_{\mathbf{k+q}} + \epsilon_{\mathbf{k}}} - \frac{1}{i\varepsilon_m - \epsilon_{\mathbf{k}} + \epsilon_{\mathbf{k-q}}} \right],$$

and $n_F(\epsilon_{\mathbf{k}}) = 1/[\exp(\epsilon_{\mathbf{k}}/T) + 1]$ is the Fermi function. Consider the $T=0$ part first. Expanding fermionic dispersions near $q \approx 2k_F$ yields

$$\epsilon_{\mathbf{k\pm q}} = -\epsilon_k + v_F \bar{q} + 2E_F \alpha^2,$$

where $\bar{q} = q - 2k_F \ll 2k_F$ and $\alpha \ll 1$ is the angle between $-\mathbf{q}$ and $\mathbf{k}$ (for $\epsilon_{\mathbf{k+q}}$) and between $\mathbf{q}$ and $\mathbf{k}$ (for $\epsilon_{\mathbf{k-q}}$). We assume that the k integral is confined to the Fermi surface, and replace $\int d^3k/(2\pi)^3 \cdots \rightarrow (\nu_F/2) \int d\epsilon_k \int d\alpha \alpha \cdots$, where $\nu_F = mk_F/2\pi^2$ is the density of states at the Fermi level per one spin orientation. Integrating over ϵ_k, we arrive at

$$\Pi_0 = \frac{\nu_F}{16} \int_0^1 dx \ln[(\bar{Q} + x)^2 + \bar{\Omega}^2],$$

where

$$\bar{\Omega}_m \equiv \varepsilon_m/E_F, \quad \bar{Q} \equiv v_F \bar{q}/E_F, \quad \text{and} \quad x \equiv 2\alpha^2.$$

Performing an elementary integration, we obtain

$$\Pi_0 = \frac{\nu_F}{16} \times \left[\pi|\bar{\Omega}_m| - 2|\bar{\Omega}_m| \arctan \frac{\bar{Q}}{|\bar{\Omega}_m|} - \bar{Q} \ln(\bar{\Omega}_m^2 + \bar{Q}^2) \right]. \tag{A1}$$

Notice that since we restricted the momentum integration to the vicinity of the Fermi surface, our expression does not contain a regular part of Π, which comes from all energies. However, in what follows we will need only a nonanalytic part of Π.

At $\varepsilon_m = 0$, Eq. (A1) reproduces the static Kohn anomaly

$$\Pi_0 \Big|_{\varepsilon_m = 0} = -\frac{\nu_F}{8} \bar{Q} \ln|\bar{Q}|. \tag{A2}$$

At finite ε_m and $q = 2k_F$, i.e., $\bar{Q} = 0$,

$$\Pi_0 \Big|_{q = 2k_F} = -\frac{\nu_F}{16} |\bar{\Omega}_m|. \tag{A3}$$

This expression implies that the spectral weight of particle-hole pairs with the center-of-mass momentum near $2k_F$ is proportional to the excitation frequency.

For the T-dependent part, we find after integrating over α,

$$\Pi_T = -\frac{\nu_F}{8} \int_0^\infty d\xi \, n_F(\xi E_F) \ln \frac{\bar{\Omega}_m^2 + (2\xi + \bar{Q})^2}{\bar{\Omega}_m^2 + (2\xi - \bar{Q})^2}, \tag{A4}$$

where $\xi = \epsilon_k/E_F$.

The $2k_F$ contribution to the thermodynamic potential, Ξ_{2k_F}, is given by

$$\Xi_{2k_F} = -3NU_{2k_F}^2 T \times \sum_{\varepsilon_m} \int_{-1}^1 d\bar{Q}[\Pi_0^2 + 2\Pi_0\Pi_T + \Pi_T^2], \tag{A5}$$

where N is the number density. The choice of the cutoff in the $\bar{Q}$ integral does not affect the result to logarithmic accuracy. The rest of the calculation is simplified by making two observations. First, the first term ($|\bar{\Omega}_m|$) in Eq. (A1) is independent of $\bar{Q}$, while the rest of the terms in this equation as well as Π_T are odd in $\bar{Q}$. The square of the first term in Eq. (A1) does not lead to a nonanalyticity in Ξ, whereas the cross-products of this term with the rest of Π is odd in $\bar{Q}$ and thus vanish upon integration. Therefore, the first term can then be safely eliminated from Eq. (A1) which can then be written as

$$\Pi_0 = -\frac{\nu_F}{16} \times \left[2|\bar{\Omega}_m| \arctan \frac{\bar{Q}}{|\bar{\Omega}_m|} + \bar{Q} \ln(\bar{\Omega}_m^2 + \bar{Q}^2) \right]. \tag{A6}$$

Second, typical ξ in Eq. (A4) are of order $\bar{T} \equiv T/E_F$, while the $\ln T$ dependence of the thermodynamic potential comes from the region $\bar{Q}^2, \bar{\Omega}_m^2 \gg \bar{T}^2$. The integral in Eq. (A4) can then be expanded in powers of ξ, and a simple analysis shows that one needs to keep only terms of order ξ and of order ξ^3. With this simplification, Π_T becomes

$$\Pi_T = -\frac{\pi^2 \nu_F}{12}\left[\bar{T}^2 \frac{\bar{Q}}{\bar{Q}^2 + \bar{\Omega}_m^2} - \frac{16}{5}\pi^2 \bar{T}^4 \bar{Q} \frac{\bar{\Omega}_m^2 - \bar{Q}^2/3}{(\bar{Q}^2 + \bar{\Omega}_m^2)^3}\right].$$

(A7)

There are four distinct $T^4 \ln T$ contributions to Ξ_{2k_F}. The first one comes from Π_0^2 and is obtained by forming a cross-product of the two terms in Eq. (A6). In the limit of $|\bar{Q}| \gg |\bar{\Omega}_m|$,

$$\Pi_0 \approx -\frac{\nu_F}{16}[2\bar{Q}\ln|\bar{Q}| + \pi|\bar{\Omega}_m|],$$

(A8)

and this combination results in a nonanalytic term in Ξ_{2k_F}

$$\Xi_{2k_F} \propto T\sum_{\bar{\Omega}_m} |\bar{\Omega}_m|^3 \int_{|\bar{\Omega}_m|} \frac{d\bar{Q}}{\bar{Q}} \propto T^4 \ln \frac{E_F}{T}.$$

(A9)

Contrary to the $q=0$ contribution, however, the explicit T-dependence of Π [entering via the second and third term in Eq. (A5)] gives three more $T^4 \ln T$ contributions in the thermodynamic potential. The second and third contributions come from cross-products of the T^2 and T^4 terms in Eq. (A4) with Π_0. The logarithmic terms here come, respectively, from $\int d\bar{Q}/\bar{Q} \to \ln \bar{\Omega}_m \to \ln T$ and from $T\sum_{\Omega_m} 1/|\Omega_m| \to \ln T$. The fourth contribution comes from the square of the T^2 term in Eq. (A4). Each of these four contributions is of order $T^4 \ln T$, such that

$$\Xi_{2k_F} = -\frac{3\pi^4}{8}(\nu_F U(2k_F))^2 NT\left(\frac{T}{v_F k_F}\right)^3 \ln \frac{E_F}{T}\sum_{i=1}^{4} \gamma_i.$$

(A10)

Using

$$T\sum_{\Omega_m} |\Omega_m|^3 \ln|\Omega_m| \to -\frac{2\pi^3}{15}T^4 \ln \frac{E_F}{T},$$

$$T^3 \sum_{\Omega_m} |\Omega_m|\ln|\Omega_m| \to \frac{\pi}{3}T^4 \ln \frac{E_F}{T},$$

$$T^4 \sum_{\Omega_m} \frac{1}{|\Omega_m|}\frac{1}{\pi}T^4 \ln \frac{E_F}{T},$$

(A11)

we find

$$\gamma_1 = -\frac{8}{45}, \quad \gamma_2 = -\frac{8}{9}, \quad \gamma_3 = \frac{14}{45}, \quad \gamma_4 = \frac{2}{9}.$$ (A12)

Collecting all contributions, we finally obtain

$$\Xi_{2k_F} = \frac{\pi^4}{5}(\nu_F U(2k_F))^2 NT\left(\frac{T}{v_F k_F}\right)^3 \ln \frac{E_F}{T}.$$

(A13)

It is instructive to compare this result with a small q contribution to Ξ. We recall that $\Pi(\varepsilon_m, q)$ for $|\varepsilon_m| \ll q \ll k_F$ is given by Eq. (101). The logarithmic singularity in Ξ is obtained by taking the cross-product of the $|\varepsilon_m|/q$ and $(|\varepsilon_m|/q)^2$ terms in Eq. (101)

$$\Xi_0 \propto T\sum_{\varepsilon_m} |\varepsilon_m|^3 \int dq q^2/q^3 \propto T^4 \ln \frac{E_F}{T}.$$

Calculating the prefactor, we obtain

$$\Xi_{q=0} = \frac{\pi^4}{5}(\nu_F U(0))^2 NT\left(\frac{T}{v_F k_F}\right)^3 \ln \frac{E_F}{T}.$$

This almost coincides with Eq. (A13), the only difference is between the prefactors $U^2(0)$ and $U^2(2k_F)$.

---

[1] W. R. Abel, A. C. Anderson, W. C. Black, and J. C. Wheatley, Phys. Rev. **147**, 111 (1966).

[2] D. S. Greywall, Phys. Rev. B **27**, 2747 (1983), and references therein.

[3] G. R. Stewart, Rev. Mod. Phys. **56**, 755 (1984).

[4] A. A. Abrikosov, L. P. Gorkov, and I. E. Dzyaloshinski, *Methods of Quantum Field Theory in Statistical Physics* (Dover, New York, 1963); E. M. Lifshitz and L. P. Pitaevski, *Statistical Physics* (Pergamon, Oxford, 1980).

[5] G. M. Eliashberg, Sov. Phys. JETP **11**, 696 (1960); G. M. Eliashberg, Sov. Phys. JETP **16**, 780 (1963).

[6] P. W. Anderson, Physics (Long Island City, N.Y.) **2**, 1 (1965).

[7] R. Balian and D. R. Fredkin, Phys. Rev. Lett. **15**, 480 (1965).

[8] S. Engelsberg and P. M. Platzman, Phys. Rev. **148**, 103 (1965).

[9] N. F. Berk and J. R. Schrieffer, Phys. Rev. Lett. **17**, 433 (1966).

[10] S. Doniach and S. Engelsberg, Phys. Rev. Lett. **17**, 750 (1966).

[11] W. F. Brinkman and S. Engelsberg, Phys. Rev. **169**, 417 (1968).

[12] D. J. Amit, J. W. Kane, and H. Wagner, Phys. Rev. **175**, 313 (1968).

[13] M. T. Bèal-Monod, S.-K. Ma, and D. R. Fredkin, Phys. Rev. Lett. **20**, 929 (1968).

[14] E. Riedel, Z. Phys. **210**, 403 (1967).

[15] C. J. Pethick and G. M. Carneiro, Phys. Rev. A **7**, 304 (1973).

[16] C. J. Pethick and G. M. Carneiro, Phys. Rev. B **11**, 1106 (1975).

[17] A. I. Larkin and V. I. Mel'nikov, Sov. Phys. JETP **20**, 173 (1975).

[18] G. M. Carneiro and C. J. Pethick, Phys. Rev. B **16**, 1933 (1977).

[19] D. Coffey and K. S. Bedell, Phys. Rev. Lett. **71**, 1043 (1993).

[20] M. A. Baranov, M. Yu. Kagan, and M. S. Mar'enko, JETP Lett. **58**, 709 (1993).

[21] D. Belitz, T. R. Kirkpatrick, and T. Vojta, Phys. Rev. B **55**, 9452 (1997).

[22] A. Houghton and J. B. Marston, Phys. Rev. B **48**, 7790 (1993).

[23] A. H. CastroNeto and E. H. Fradkin, Phys. Rev. Lett. **72**, 1393 (1994); Phys. Rev. B **49**, 10877 (1994).

[24] G. Y. Chitov and A. J. Millis, Phys. Rev. Lett. **86**, 5337 (2001); Phys. Rev. B **64**, 054414 (2001).

[25] A. V. Chubukov and D. L. Maslov, Phys. Rev. B **68**, 155113 (2003); **69**, 121102(R) (2004).

[26] V. M. Galitski and S. Das Sarma, Phys. Rev. B **70**, 035111 (2004).

[27] A. V. Chubukov, D. L. Maslov, S. Gangadharaiah, and L. I. Glazman, Phys. Rev. B **71**, 205112 (2005).

[28] A. V. Chubukov, D. L. Maslov, S. Gangadharaiah, and L. I. Glazman, Phys. Rev. Lett. **95**, 026402 (2005); S. Gangadharaiah, D. L. Maslov, A. V. Chubukov, and L. I. Glazman, *ibid.* **94**, 156407 (2005).

[29] A. Chubukov, V. Galitski, and S. Das Sarma, Phys. Rev. B **71**, 201302(R) (2005).

[30] J. Betouras, D. Efremov, and A. Chubukov, cond-mat/0506083 (unpublished).

[31] A. A. Abrikosov, L. P. Gorkov, and I. E. Dzyaloshinskii, *Methods of Quantum Field Theory in Statistical Physics* (Dover, New York, 1963).

[32] D. Pines and P. Nozieres, *Theory of Fermi Liquids* (Benjamin, Reading, 1966).

[33] J. M. Luttinger and J. C. Ward, Phys. Rev. **118**, 1417 (1960).

[34] I. Aleiner and K. Efetov (private communication).

[35] W. Kohn and J. M. Luttinger, Phys. Rev. Lett. **15**, 524 (1965).

[36] The $\ln E_F/(i\omega+\epsilon_k)$ term in Eq. (103) is the sum of the $\ln E_F/|\epsilon_k|$ contribution from the ε_m^2/q^2 term in $\Pi(\varepsilon_m,q)$ and the $\ln|\epsilon_k|/(i\omega_m-\epsilon_k)$ contribution from the $|\varepsilon_m|/q$ term in $\Pi(\varepsilon_m,q)$.

[37] J. Bardeen and M. Stephen, Phys. Rev. **136**, A1485 (1964).

[38] Note that at not extremely small T, this $T^3 \ln T$ contribution is still smaller than the purely phonon T^3 contribution to the specific heat by a factor of $g_{ph}^2 c/v_F \ln \omega_D/T$, where ω_D is the Debye frequency.

[39] R. E. Prange and L. P. Kadanoff, Phys. Rev. **134**, A566 (1964).

[40] Recent calculations demonstrate (Refs. 21, 25, and 41) that the momentum dependence of $D_S(q,0)$ is by itself nonanalytic in q and approximating it by q^2 is only justified at some distance away from the criticality.

[41] A. V. Chubukov, C. Pèpin, and J. Rech, Phys. Rev. Lett. **92**, 147003 (2004).

[42] G. Catelani and I. L. Aleiner, JETP **100**, 331 (2005).

[43] Note that in 2D, a similar consideration leads to a nonanalytic T^2 contribution which is of the same order as the nonanalyticity from electron-electron interaction. In this situation, an additional analysis is required to eliminate the zero-sound contribution (Refs. 27 and 28).

[44] A. de Visser, A. Menovsky, and J. J. M. Franse, Physica B & C **147**, 81 (1987).