

Possible Anderson transition below two dimensions in disordered systems of noninteracting electrons

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We investigate the possibility of an Anderson transition below two dimensions in disordered systems of noninteracting electrons with symplectic symmetry. Numerical analysis of energy level statistics and conductance statistics on Sierpinski carpets with spin-orbit coupling indicates the occurrence of an Anderson transition below two dimensions.

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According to the scaling theory of Abrahams *et al.* for noninteracting electrons in disordered systems, all states are localized in two dimensions (2D) and the Anderson transition occurs only above 2D.¹ This prediction applies to systems with orthogonal symmetry, i.e., systems with time reversal and spin rotation symmetry.²⁻⁴ Accordingly, it is believed that the lower critical dimension for the Anderson transition in systems with orthogonal symmetry is

$$d_L^{(\text{orth})} = 2. \quad (1)$$

The prediction of Abrahams *et al.* does not apply to systems with symplectic symmetry, i.e., in systems with time reversal symmetry but in which spin rotation symmetry is broken by spin-orbit coupling.^{2,5} For such systems it is known that there is a transition in 2D, if the spin-orbit interaction is sufficiently strong.

The prediction of Abrahams *et al.* rests on an argument concerning the form of the β function that describes the scaling of the conductance. Reasonable assumptions concerning the asymptotic behavior of this function in the strongly metallic and localized limits, respectively, and the assumption that the β function is monotonic lead to their conclusion. As we explain below it is this latter assumption that does not hold in systems with symplectic symmetry.

In Fig. 1 we show the β function that describes the scaling of the quantity Λ^3

$$\beta(\ln \Lambda) = \frac{d \ln \Lambda}{d \ln L}. \quad (2)$$

Here Λ is the ratio of the quasi-one-dimensional localization length to the system width for electrons on a long quasi-one-dimensional system of width L . The β functions that describe the scaling of different physical quantities are expected to differ in detail, but for systems with the same dimensionality and symmetry all are expected to share some common properties. For example, a zero of the β function indicates a transition and this property should be common to all the β functions. Also the slope at the zero, which is related to the critical exponent ν describing the divergence of the localization length, should be the same for all the β functions.

The solid lines in Fig. 1 are numerical estimates of the β function (2) for systems with symplectic symmetry.^{6,7} The β function tends to $d-2$, with d being the dimensionality, in the metallic limit and becomes negative in the localized limit. In contrast to systems with orthogonal symmetry (shown schematically, see Ref. 3 for numerical result) the β functions are nonmonotonic and a fixed point exists even in 2D. It is thought that the β function for the conductance behaves similarly.^{8,9}

Looking at Fig. 1 the obvious question is, “What happens below 2D for systems with symplectic symmetry?” Does the β function change discontinuously and suddenly become monotonic? Or does it change smoothly? In the latter case, we expect the Anderson transition to persist below 2D. This is a longstanding problem.¹⁰ Our numerical results suggest that the Anderson transition persists below 2D for systems with symplectic symmetry. It is known that states in (quasi-)one-dimensional systems with symplectic symmetry are always localized, so the lower critical dimension must be at least one. This would mean that

$$1 \leq d_L^{(\text{symp})} < 2. \quad (3)$$

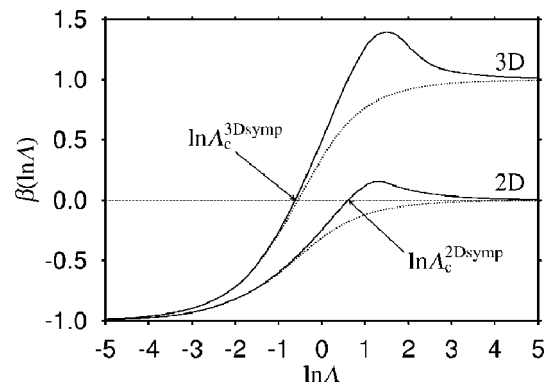


FIG. 1. The numerically estimated β functions for systems with symplectic symmetry in 2D and 3D (solid lines)^{6,7} as well as the schematic ones for systems with orthogonal symmetry (dotted lines).

To study this problem we have simulated noninteracting electrons on the Sierpinski carpet¹¹ with spin-orbit coupling. The Sierpinski carpet is constructed by repeating the following iteration. We start from a $b \times b$ square lattice with a central $c \times c$ square removed. This first generation or “generator” is denoted as $SC(b, c, 1)$. At each step the next generation is constructed by magnifying the lattice b times and replacing each site of the current generation with the generator. The linear size of the k th generation $SC(b, c, k)$ is $L = b^k$ and the number of sites is $N = (b^2 - c^2)^k$. The Sierpinski carpet becomes a true fractal in the mathematical sense when the generation number k tends to infinity. This true fractal is denoted as $SC(b, c)$. According to a lot of studies of thermal magnetic phase transitions, the Sierpinski carpets are useful to study phase transitions in fractal dimensionality less than two¹² (see also a recent work Ref. 13 and references therein). We have studied $SC(3, 1, k)$ and $SC(5, 1, k)$ with fixed boundary conditions imposed in both directions.

A fractal system has a fractal dimension d_f . This dimension describes how the mass of the fractal depends on the size of the fractal. The fractal dimension d_f of the Sierpinski carpet $SC(b, c)$, defined by $N = L^{d_f}$, is equal to $d_f = \ln(b^2 - c^2) / \ln b$. We find $d_f \approx 1.893$ for $SC(3, 1)$ and $d_f \approx 1.975$ for $SC(5, 1)$. A fractal system also has a spectral dimension d_s .¹⁴ This dimension describes the spectra of the low energy vibrations on the fractal. Numerical studies of the Anderson transition in bifractal systems indicate that this is the relevant dimension when discussing the Anderson transition.^{4, 15} The spectral dimension of the Sierpinski carpet has been estimated by simulating a classical random walk on the fractal.¹⁶ The values reported for $SC(3, 1)$ and $SC(5, 1)$ are, respectively, $d_s = 1.785 \pm 0.008$ and $d_s = 1.940 \pm 0.009$.

We have simulated the $SU(2)$ model,^{6, 17}

$$H = \sum_{i, \sigma} \epsilon_i c_{i\sigma}^\dagger c_{i\sigma} - \sum_{\langle i, j \rangle, \sigma, \sigma'} R(i, j)_{\sigma\sigma'} c_{i\sigma}^\dagger c_{j\sigma'}. \quad (4)$$

Here $c_{i\sigma}^\dagger (c_{i\sigma})$ denotes the creation (annihilation) operator of an electron at the site i with spin σ . This model has symplectic symmetry. There is an on-site random potential ϵ_i which is identically and independently distributed with uniform probability in the range $[-W/2, W/2]$. There is also nearest neighbor hopping between lattice sites with the hopping matrices $R(i, j)$ sampled uniformly from the group $SU(2)$. Apart from the fractal lattice the model is exactly as described in Refs. 6 and 17 and the reader should refer to these for further details.

To investigate the localization of electrons on these fractals we have examined the energy level statistics of (4). In particular, we have looked at the distribution $P(s)$ of the nearest neighbor level spacing s measured in units of the mean level spacing.¹⁸ It is known that $P(s)$ is well approximated by the Wigner surmise for the Gaussian symplectic ensemble $P_{\text{GSE}}(s) = A s^4 e^{-Bs^2}$ in the extended limit, and the Poisson distribution $P_{\text{P}}(s) = e^{-s}$ in the localized limit. We have calculated the energy eigenvalues using either LAPACK or the Lanczos method.¹⁹ An unfolding procedure has been applied to the level spacings to compensate for the variation

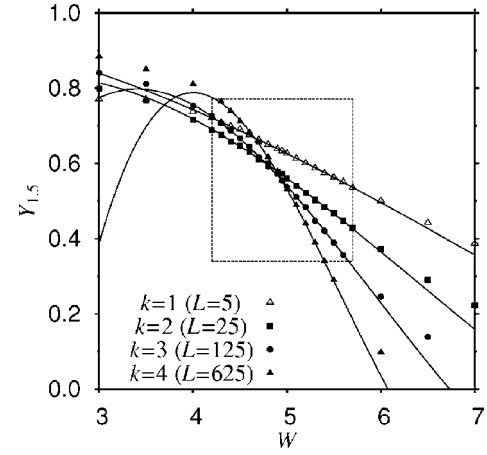


FIG. 2. $Y_{1,5}$ vs W for $SC(5, 1, k)$ ($k=1, 2, 3, 4$). The solid lines are the best fits of (7) to the data inside the box (shown by dashed lines). Outside the box the fit deviates from the numerical data because the expansion parameters $\psi L^{1/\nu}$ and w are no longer small.

with energy of the ensemble average density of states. For a quantitative analysis of $P(s)$, we define

$$Y_{s_0} = \frac{\int_0^{s_0} ds P(s) - \int_0^{s_0} ds P_{\text{P}}(s)}{\int_0^{s_0} ds P_{\text{GSE}}(s) - \int_0^{s_0} ds P_{\text{P}}(s)}. \quad (5)$$

In the extended limit $Y_{s_0} = 1$, and in the localized limit $Y_{s_0} = 0$. Note that since each energy level has a twofold Kramers' degeneracy we consider only distinct eigenvalues when calculating the spacing distribution.

We have simulated lattices $SC(5, 1, k)$ up to $k=4$. Calculation of higher generations is not possible at present because the CPU time required is too large. For the first generation $k=1$ the number of eigenvalues is small. Therefore, we restricted our analysis to a single pair of consecutive levels per sample, the smallest positive eigenvalue and the largest negative eigenvalue. When we choose the pair of levels in this special way, the probability $\tilde{P}(s)$ of a level spacing with a value s is equal to $sP(s)$. Therefore we multiply $\tilde{P}(s)$ by $1/s$ to obtain level spacing distribution $P(s)$. We simulated 1 000 000 samples for $k=1$. For higher generations the number of eigenvalues is greater so we reverted to the usual procedure of analyzing a sequence of consecutive levels. The number of samples was reduced accordingly. We used levels in the interval $E = [-0.3, 0.3]$. We simulated 15 000, 600, and 10 samples for $k=2, 3$, and 4 , respectively. The total number of the level spacings for each of the (k, W) pair is thus of order 10^5 to 10^6 .

Figure 2 shows $Y_{1,5}$ as a function of disorder W for $SC(5, 1, k)$. There are large corrections to scaling in the first generation. The data for the higher generations $k=2, 3$, and 4 indicate that an Anderson transition occurs on $SC(5, 1)$ around $W \approx 5$. When disorder is weak enough, $Y_{1,5}$ increases with k for $k \geq 2$ indicating a delocalized phase. When disorder

TABLE I. The details of the scaling analyses for SC(5, 1, k). Data with W in [4.2, 5.7] have been used. The range of Y_{s_0} has also been restricted as indicated. Here, N_p is the number of parameters, N_d the number of data, Q the goodness of fit probability, and Y_c the critical value of Y_{s_0} . The precision of the fitted parameters is expressed as 95% confidence intervals.

s_0	Y_{s_0}	n_R, n_I, m_R, m_I	N_p	N_d	Q	W_c	Y_c	ν	y
0.5	[0.60,0.93]	4,1,2,0	11	63	0.3	4.99 ± 0.02	0.791 ± 0.009	3.50 ± 0.11	-0.80 ± 0.09
1.5	[0.34,0.77]	3,1,2,0	10	63	0.7	4.98 ± 0.03	0.536 ± 0.013	3.44 ± 0.15	-0.77 ± 0.11

der is strong enough, $Y_{1.5}$ decreases as k increases indicating a localized phase.

Further analysis of the level statistics on SC(5,1) is based on the assumption that finite size scaling applies to fractals in the same way as to systems with integral dimension. Accordingly the statistic Y_{s_0} should, in the absence of any corrections to scaling, obey the single parameter scaling law

$$Y_{s_0} = f_{\pm}(L/\xi) = F_0(\psi L^{1/\nu}). \quad (6)$$

Here ξ is the localization (correlation) length, ν the critical exponent for the divergence of ξ , and ψ a smooth function of disorder W that crosses zero linearly at the critical disorder W_c . The subscript \pm distinguishes the scaling function in the delocalized and localized phases. It is, however, clear from Fig. 2 that corrections to scaling are not negligible. Taking account of corrections up to the first order in an irrelevant variable ϕL^y we have²⁰⁻²²

$$Y_{s_0} = F_0(\psi L^{1/\nu}) + \phi L^y F_1(\psi L^{1/\nu}). \quad (7)$$

For the purpose of fitting, the scaling functions F_0 and F_1 are approximated by a power series up to the order n_R and n_I in $\psi L^{1/\nu}$. The functions ψ and ϕ are also approximated by expansions in terms of dimensionless disorder $w = (W_c - W)/W_c$ up to the order m_R and m_I . The best fit to data is determined by minimizing the χ^2 statistics and the precision of the parameters are determined using the Monte Carlo method.²³

The best fit is shown in Fig. 2 and the results of the scaling analyses for $s_0=0.5$ and $s_0=1.5$ are tabulated in Table I. The estimates of the critical disorder W_c and the critical

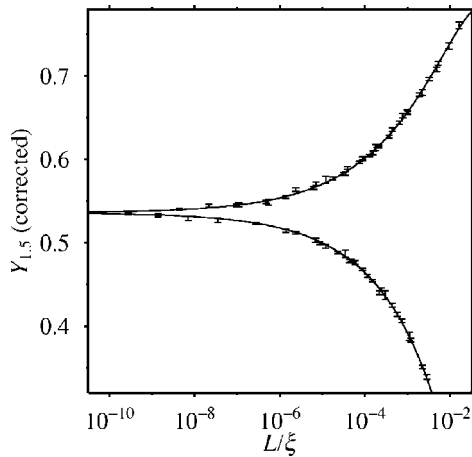


FIG. 3. The numerical data $Y_{1.5}$ after subtraction of the corrections to scaling plotted as function of the ratio L/ξ .

exponent ν for $s_0=0.5$ and $s_0=1.5$ are consistent as required. Our estimate $\nu=3.4 \pm 0.2$ for SC(5,1) is clearly different from the value $\nu=2.746 \pm 0.009$ in 2D,⁶ reflecting the difference of the dimensionality. To exhibit single parameter scaling the data are replotted, after subtraction of the corrections due to an irrelevant variable in Fig. 3. The two branches correspond to the delocalized and localized phases.

To complement our study of the level statistics, we have also analyzed statistics of the Landauer conductance g (in units of e^2/h).^{21,22,24} Two perfect leads of width $L=5^k$ are attached to the Sierpinski carpet SC(5, 1, k) and the recursive Green's function method is used to calculate the conductance.²⁵ The hopping matrices in the direction transverse to the current are set to the unit matrix. As a scaling quantity, we choose the typical conductance

$$g_{\text{typ}} = e^{\langle \ln g \rangle}. \quad (8)$$

Here $\langle \rangle$ means the ensemble average. We have set the Fermi energy to $E=0$ and have accumulated 3 000 samples for $k=2, 3$ and from 150 to 300 samples for $k=4$. Figure 4 shows the typical conductance g_{typ} as a function of W for SC(5, 1, k). These data also indicate the occurrence of an Anderson transition at about $W \approx 5$.

We have also analyzed energy level statistics and conductance statistics on SC(3, 1, k) up to the fourth generation. We did not find any evidence for a delocalized phase. We cannot conclude from this that SC(3,1) is below the lower critical dimension because the SU(2) model has significant randomness even when the on-site random potential is zero.

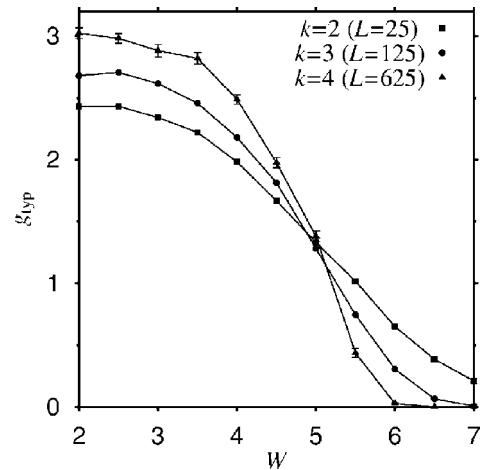


FIG. 4. The typical conductance g_{typ} vs W for SC(5,1, k) ($k=2, 3, 4$). The lines are a guide to the eye only.

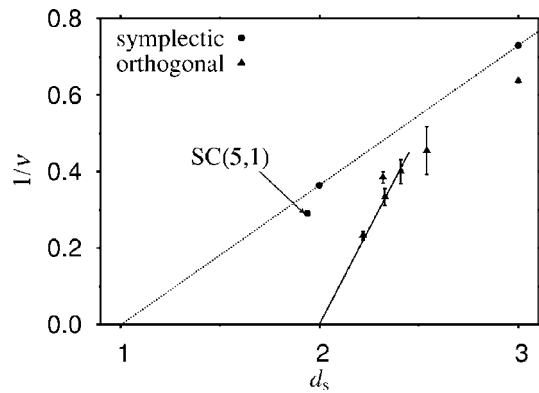


FIG. 5. $1/\nu$ vs d_s for systems with orthogonal and symplectic symmetries. The values for 2D and 3D symplectic systems are from Refs. 6 and 7, respectively. The estimates for systems with orthogonal symmetry with dimension $2 < d_s < 3$ are from Ref. 4 and that in 3D is from Ref. 20. (The exponent ν for fractals with $2 < d_s < 3$ was also estimated in Ref. 15.) The solid line indicates $\nu=1/(d_s-2)$.

In Fig. 5 we have plotted the available results for the critical exponent for systems with orthogonal and symplectic symmetries. For 2D and 3D system with symplectic symmetry the critical exponent are estimated to be $\nu=2.746\pm 0.009$ in 2D (Ref. 6) and $\nu=1.37\pm 0.02$ in three-dimension (3D).⁷ For systems with orthogonal symmetry we have used the estimates reported in Refs. 4 and 20. In the figure we plot $1/\nu$ as a function of d_s .

We recall that the slope of the β function at the fixed point is equal to $1/\nu$. Assuming that the β function changes con-

tinuously as the dimensionality changes, then just above the lower critical dimension the fixed point should correspond to the maximum of the β function. We therefore expect the critical exponent ν to diverge at the lower critical dimension. This is consistent with the theoretical analysis of the nonlinear σ model with orthogonal symmetry where it was found that $\nu=1/\epsilon$ for $\epsilon=d-2\ll 1$.²⁶ The numerical data for orthogonal systems shown in Fig. 5 are consistent with this.

For systems with symplectic symmetry fewer data are available.^{6,7} When only the estimates for ν in 2D and 3D are considered, it can leave the impression (cf. the dotted line in Fig. 5) that the lower critical dimension is 1 in this case. However, our estimate of the critical exponent on SC(5,1) is well below the dotted line, suggesting that the lower critical dimension is closer to 2 than to 1.

To confirm the occurrence of the Anderson transition below 2D for systems of noninteracting electrons in disordered systems with symplectic symmetry, and to determine more precisely the lower critical dimension, further numerical studies on a variety of fractals over a larger range of generations are needed. Analysis of fractals with d_s intermediate between the values for SC(3,1) and SC(5,1) might be particularly helpful. The nature of the possible delocalized phase also warrants further study.

Recently electron transport through systems with fractal perimeter was studied experimentally.²⁷ Electron transport through fractal structures can be an interesting topic for further theoretical and experimental investigation.

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