Quantum ether: Photons and electrons from a rotor model

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We give an example of a purely bosonic model—a rotor model on the 3D cubic lattice—whose low energy excitations behave like massless U(1) gauge bosons and massless Dirac fermions. This model can be viewed as a "quantum ether;" a medium that gives rise to both photons and electrons. It illustrates a general mechanism for the emergence of gauge bosons and fermions known as "string-net condensation." Other, more complex, string-net condensed models can have excitations that behave like gluons, quarks and other particles in the standard model. This suggests that photons, electrons and other elementary particles may have a unified origin: string-net condensation in our vacuum.

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I. INTRODUCTION

Throughout history, people have attempted to understand the universe by dividing matter into smaller and smaller pieces. This approach has proven extremely fruitful: successively smaller distance scales have revealed successively simpler and more fundamental structures. Over the last century, the fundamental building blocks of nature have been reduced from atoms to electrons, protons and neutrons, to most recently, the "elementary" particles that make up the $U(1) \times SU(2) \times SU(3)$ standard model. Today, a great deal of research is devoted to finding even more fundamental building blocks, such as superstrings.

This entire approach is based on the idea of reductionism—the idea that the fundamental nature of particles is revealed by dividing them into smaller pieces. But reductionism is not always useful or appropriate. For example, in condensed matter physics there are particles, such as phonons, that are collective excitations involving many atoms. These particles are "emergent phenomena" that cannot be meaningfully divided into smaller pieces.

This suggests alternate line of inquiry. Could the elementary particles in the standard model be analogous to phonons? That is, could they be collective modes of some "structure" that we mistake for empty space?

Recent work suggests that they might be.[5,6] This work has revealed the existence of new and exotic phases of matter whose collective excitations are gauge bosons and fermions. The microscopic degrees of freedom in these models are analogous to spins on a lattice—purely local, bosonic objects with local interactions. There is no trace of gauge boson or fermion degrees of freedom in the underlying lattice model. The gauge bosons and fermions are thus emergent phenomena—a result of the collective behavior of many spins. (In this paper, we use the term "spin" to mean any local degree of freedom. This local degree of freedom does not necessarily have anything to do with spins of electrons or other particles or even with the spatial rotation group. Indeed, in the context of high energy physics, these underlying local degrees of freedom *give rise* to electrons and their spin.)

What is the mechanism responsible for their emergence? In the same way that symmetry breaking is responsible for

the emergence of phonons,[1–4] there is a simple mechanism for the emergence of gauge bosons and fermions. In these exotic phases, the spins organize into a special pattern—a particular kind of entangled ground state, which we call a "string-net condensed" state. A string-net condensed state is a spin state where the spins organize into large stringlike objects (or more generally networks of strings) which in turn form a quantum string liquid (see Fig. 1). This kind of ground state naturally gives rise to gauge bosons and fermions. The gauge bosons correspond to fluctuations in the strings—the collective motions of the strings that fill the space.[7–10] The fermions correspond to endpoints of the strings, that is, defects in the string liquid where a string ends in empty space.[5,6,11]

What makes the string-net picture particularly compelling is that the gauge bosons and fermions naturally emerge *together*. They are just different aspects of the same underlying structure. Therefore, if we believe that the vacuum is such a string-net condensate then the presence of gauge interactions and Fermi statistics in the standard model is no longer mysterious. String-net condensation can explain what gauge bosons and fermions are, why they exist, and why they appear together.[6]

The general theory of string-net condensation was worked out in Ref. 5. One of the main results in that paper was a series of exactly soluble models realizing all possible string-net condensates. These models are quite general and can realize gauge bosons with any gauge group. However, they are also complicated when discussed in full generality, and Ref. 5 did not provide an explicit example of the most physically
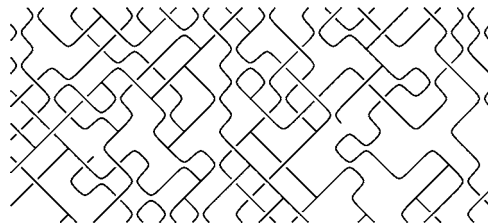

FIG. 1. A picture of quantum ether: the fluctuations of the string-nets give rise to gauge bosons (such as photons). The ends of the strings give rise to fermions (such as electrons).

relevant case—a model realizing gauge bosons and fermions in $(3+1)$ dimensions.

In this paper, we attempt to remedy this problem. We demonstrate the string-net picture of $(3+1)$D emerging gauge bosons and fermions with concrete lattice models. We describe a rotor model on the cubic lattice that produces both $U(1)$ gauge bosons and fermions. The fermions can be gapped excitations (as in an insulator) or gapless (as in a Fermi liquid). They can also behave like massless Dirac fermions. In this case, the low energy physics of the rotor model is identical to massless quantum electrodynamics (QED). The rotor model can then be viewed as a "quantum ether;" a medium that gives rise to both photons and electrons. In addition, the rotor model is closely related to $U(1)$ lattice gauge theory coupled to a Higgs field. It demonstrates that a simple modification or "twist" can change the Higgs boson into a fermion.

This is not the first lattice bosonic model with emergent gauge bosons and massless Dirac fermions. In fact, the slave boson approach has provided a number of such spin models, most of which are simpler than the model described in this paper.[12,14,15] So one may wonder, why should one bother to study this rotor model? The reason is that the rotor model has several advantages over these examples. First, the mapping between the rotor model and QED is essentially exact. Thus, the emergence of gauge bosons and fermions is on solid ground. This is in contrast with previous examples, where the emergence can only be established in a large N limit. Second, the rotor model is a special case of a general construction,[5,6] unlike the other models which were in some sense, discovered by accident.

But perhaps the most important contribution of the rotor model is conceptual. The rotor model demonstrates a simple physical picture for the emergence of gauge bosons and fermions. It reveals that gauge bosons and fermions are natural consequences of string-net condensation, just as phonons and vortices are natural consequences of boson condensation. In addition to its relevance to high energy physics, this general picture may prove useful to condensed matter physics, particularly the search for phases of matter with exotic low energy behavior. Indeed, while the specific rotor model described in this paper is unrealistic in the context of condensed matter systems, the concepts underlying this model may lead to the construction of simpler and more realistic spin models. Such models have already been discovered in the case of emergent photons coupled to charged bosons.[8,16–19]

The paper is organized as follows: we begin with a "warm-up" calculation in Sec. II—a rotor model with emergent photons and *bosonic* charges. This model is closely related to $U(1)$ lattice gauge theory coupled to a Higgs field. Then, in Sec. III, we show that the rotor model can be modified in a natural way by adding a phase factor or "twist" to a term in the Hamiltonian. This modified or "twisted" rotor model has (massive) fermionic charges. In Sec. IV, we describe further modifications which give rise to gapless Fermi liquids and massless Dirac fermions.
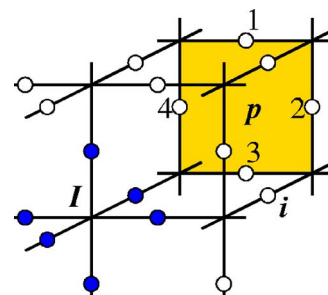

FIG. 2. (Color online) A picture of the rotor model (1). The term $Q_I = (-1)^I \Sigma_{\text{legs of } I} L_i^z$ acts on the six "legs" of I, that is, the six rotors adjacent to I, drawn above as filled dots. The term $B_p = L_1^+ L_2^- L_3^+ L_4^-$ acts on the four rotors, labeled by 1, 2, 3, 4, along the boundary of the plaquette p.

II. A 3D ROTOR MODEL WITH EMERGENT PHOTONS AND BOSONIC CHARGES

In this section, we present a warm-up example—a rotor model with emergent photons and *bosonic* charges. This model is closely related to $U(1)$ lattice gauge theory coupled to a Higgs field. While a number of other similar models[8,16–19] have been analyzed previously, this model has the advantage of being quasiexactly soluble, and generalizing easily to the fermionic case. The string-net picture of the ground state is also particularly evident in this case.

A. The model

The model is a quantum rotor model with rotors on the links of a cubic lattice. Each rotor can be viewed as a particle moving on a circle. The position of the particle is given by the angle θ, and the angular momentum of the particle by $L^z = I\dot{\theta}$, where I is the moment of inertia. The Hamiltonian of the rotor model is given by [see Fig. 2(a)][8,16]

$$H_{\text{rotor}} = V \sum_I Q_I^2 + J \sum_i (L_i^z)^2 - g \sum_p (B_p + h.c.),$$

$$B_p = L_1^+ L_2^- L_3^+ L_4^-, \quad Q_I = (-1)^I \sum_{\text{legs of } I} L_i^z, \quad (1)$$

where $I = (I_x, I_y, I_z)$ labels the vertices, i labels the links, and p labels the plaquettes of the cubic lattice. The "legs of I" are the six links that are attached to the vertex I, and 1, 2, 3, 4 label the four links of the plaquette p (see Fig. 2). $L^{\pm} = e^{i\theta}$ are the raising/lowering operators of L^z and $(-1)^I \equiv (-1)^{I_x + I_y + I_z}$.

B. The string-net picture

In this section, we show that the rotor model exhibits string-net condensation in the regime $V \gg g \gg J > 0$. In particular, we show that the rotors organize into effective extended objects, namely "string-nets," and the ground state is a quantum liquid of these string-nets. The string-net condensed ground state has two types of excitations—string collective motions (which will correspond to photons) and string endpoints (which will correspond to charges).
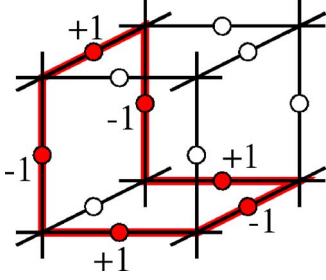
FIG. 3. (Color online) A picture of a closed string state. The rotors along the thick closed curve have angular momentum $L^z = \pm 1$, while all the other rotors (depicted as empty dots) have $L^z = 0$.

The first step is to notice that the Q_I operators commute with each other as well as the other terms in the Hamiltonian. Thus, we can label all the eigenstates of H_{rotor} by their eigenvalues under Q_I: $\{q_I\}$. The quantum number q_I, which we will call the "charge" on site I, takes values in the integers.

Clearly the lowest energy charge configuration is the charge-0 configuration $q_I = 0$. Other charge configurations cost an additional energy V. Hence, in the limit $V \gg J, g$, the low energy physics is completely contained in the charge-0 sector.

We would like to enumerate the states in the charge-0 sector. It is natural to describe these states in terms of extended objects, in particular, string-nets. The simplest state has $L_i^z = 0$ for every rotor. In the string-net language, we think of this state as the vacuum. Another state can be obtained by alternately increasing or decreasing L^z by 1 along a loop (see Fig. 3). We think of this state as the vacuum with a single (closed) string. Other states can be constructed by repeating this process. One finds that the most general charge-0 state can have many strings and that the strings can overlap to form networks of strings or "string-nets."

What is the action of the Hamiltonian (1) on these charge-0 states? If we think about the states using the string-net language, we see that the J-term penalizes strings for being long. It therefore corresponds to a string tension. On the other hand, the effect of the B_p operator is to either create a small loop of small string (if applied to the vacuum) or to deform the existing strings (if applied to a more complicated state). Thus the g-term generates string "hopping" or string fluctuations. One can think of it as a string kinetic energy.

There are two regimes to consider. When $J \gg g$, the string tension dominates and the ground state contains almost no strings. The ground state is a "normal" state. When $g \gg J$, the string kinetic energy dominates and the ground state is a superposition of many large closed strings.[7,8] The ground state is thus "string-net condensed." We expect a quantum phase transition between these two states at some J/g of order 1.

We expect that the excitations above the ground state have very different properties in these two regimes. In the string-net condensed case, there are two types of excitations: low energy excitations with charge-0 and high energy excitations with nonzero charge.

The low energy excitations can be constructed from linear combinations of string-net states. These excitations can be

thought of as collective motions of closed strings.

The high energy, charged, excitations can be constructed from linear combinations of string-net configurations with *open* strings. For example, if one takes the vacuum state $L^z = 0$ and alternately increases and decreases L^z along an open path P, the resulting state has nonzero charge at the two ends of P. This is true quite generally; nonzero charge configurations are made up of open string-nets, with the charge located at the endpoints of the open strings.

We can think of the string endpoints as point defects in the condensate. The higher energy excitations are composed out of these point defects. The defects carry a quantum number—charge—which is measured by the operator Q_I.

C. Equivalence with U(1) gauge theory

In this section, we examine the rotor model more quantitatively. We show that the low energy physics of the rotor model (1) is described by compact U(1) gauge theory coupled to infinitely massive charges. The two types of excitations discussed in the previous section have a natural interpretation in terms of the gauge theory. The collective motions of the strings give rise to gapless excitations and behave like photons, while the point defects (the endpoints of the strings) are gapped and behave like charged particles.

In the gauge theory language, the strength of the interaction between the photons (the gauge field) and the charges is characterized by a "fine structure constant" α. The "fine structure constant" also characterizes the strength of quantum fluctuations of the gauge field. When $\alpha \gg 1$, the quantum fluctuations are so large that the gauge theory is in the confining phase and there are no gauge bosons at low energies. The gauge bosons (such as photons) can exist at low energies only in the deconfined phase when α is small.

In the following, we will show that the "fine structure constant" α of the gauge theory is of order $\sqrt{J/g}$. The "speed of light" c is of order $\sqrt{gJa^2}$, where a is the lattice constant. This result provides insight into the phase transition between the normal and string-net condensed states; the normal state corresponds to the large α confining phase of the gauge theory, while the string-net condensed state corresponds to the small α deconfined phase. The string-net condensation transition is the usual confinement-deconfinement transition from gauge theory.

The simplest way to derive these results is to map (1) onto a lattice gauge theory Hamiltonian. Consider the following Hamiltonian for compact U(1) gauge field coupled to a charge-1 Higgs field $e^{i\phi}$,

$$H_{\text{gauge}} = V \sum_I n_I^2 + J \sum_{\langle IJ \rangle} E_{IJ}^2 - 2g \sum_{\langle IJKL \rangle} \cos(A_{IJ} + A_{JK} + A_{KL}$$
$$+ A_{LI}) - 2t \sum_{\langle IJ \rangle} \cos(\phi_I - \phi_J - A_{IJ}). \quad (2)$$

Here, $\langle IJ \rangle$, $\langle IJKL \rangle$ is an alternative notation for links i, and plaquettes p in the cubic lattice. The operator E_{IJ} is the integer valued electric field, canonically conjugate to the vector potential: $[A_{IJ}, E_{IJ}] = i$. Similarly, n_I is the occupation number operator canonically conjugate to the Higgs field: $[\phi_I, n_I] = i$.

In lattice gauge theory, one is interested in the properties of this Hamiltonian within the gauge invariant subspace where Gauss' law holds,

$$\sum_{\langle IJ \rangle} E_{IJ} = n_J. \tag{3}$$

Our claim is that within this gauge invariant subspace, H_{gauge} is mathematically equivalent to H_{rotor}. To see this, note that the electric field operators and the occupation number operators all commute with each other. So the lattice gauge theory model has a complete basis consisting of states $|\{e_{IJ}, n_I\}\rangle$ with electric field e_{IJ}, and occupation number n_I,

$$E_{IJ}|\{e_{IJ}, n_I\}\rangle = e_{IJ}|\{e_{IJ}, n_I\}\rangle,$$

$$n_I|\{e_{IJ}, n_I\}\rangle = n_I|\{e_{IJ}, n_I\}\rangle.$$

Similarly, the rotor model H_{rotor} has a complete basis consisting of states $|\{l_i^z\}\rangle$ with angular momentum l_I^z.

On can map the rotor model Hilbert space onto the gauge invariant subspace of the lattice gauge theory by mapping the basis state $|\{l_i^z\}\rangle$ onto the basis state $|\{e_{IJ}, n_I\}\rangle$ given by

$$e_{IJ} = (-1)^I \cdot l_i^z,$$

$$n_I = (-1)^I \sum_{\text{legs of } I} l_i^z. \tag{4}$$

(Here i is the link connecting sites I and J.) Clearly, this correspondence maps the operators n_I and E_{IJ} onto the operators Q_I and $(-1)^I \cdot L_i^z$. One can also show that this correspondence maps $\cos(A_{IJ} + A_{JK} + A_{KL} + A_{LI})$ onto $1/2(B_p + h.c.)$ and $\cos(\phi_I - \phi_J - A_{IJ})$ onto $\cos(\theta_i)$. Examining the two Hamiltonians, we conclude that the gauge theory *in the special case $t=0$* can be mapped onto the rotor model.

The physics of the rotor model (1) is therefore completely equivalent to a U(1) gauge field coupled to bosonic charges. The fine structure constant α and speed of light c can be computed using the lattice gauge theory Hamiltonian (2).[7,8] As for the mass of the charges, the rotor model (1) corresponds to the special case where $t=0$, the case where the charges are infinitely massive and have no dynamics.

D. Adding a charge hopping term

In the previous section, we discovered that the point defect excitations have an infinite mass. This is not surprising, since the operator Q_I commutes with the Hamiltonian (1), and hence charge configurations are completely static.

We would like to modify H_{rotor} so that the point defects (or "charges") do have dynamics. To do this, we need to add a term, H_{hop}, to the Hamiltonian that does not commute with Q_I. Such a term will make the charges hop from site to site. We expect that any H_{hop} will give rise to the small qualitative behavior (e.g., same particle statistics). We therefore consider the simplest hopping term,

$$H_{\text{hop}} = -2t \sum_i \cos(\theta_i) = -t \sum_i (L_i^+ + L_i^-).$$

It is easy to see that this term makes the charges hop from site to site just as in a nearest neighbor tight binding model.

In fact, if we examine the mapping, Eq. (4), we see that H_{hop} corresponds exactly with the nearest neighbor hopping term in lattice gauge theory, $\cos(\phi_I - \phi_J - A_{IJ})$. Therefore, the low energy physics of the rotor Hamiltonian $(H_{\text{rotor}} + H_{\text{hop}})$ is equivalent to that of a U(1) gauge field coupled to charges with *finite* mass.

E. The statistics of the U(1) charges

In this section, we compute the statistics of the charges and show that they are bosons. The main purpose of this computation is to facilitate comparison with the "twisted" rotor model. [Indeed, the fact that the charges are bosons follows immediately from the exact mapping to lattice gauge theory, (4).]

It is easiest to perform the computation in the case $J=0$. In this case, B_p commutes with the Hamiltonian $(H_{\text{rotor}} + H_{\text{hop}})$. We can therefore divide the Hilbert space into different sectors $\{\theta_p\}$ corresponding to different flux configurations $B_p = e^{i\theta_p}$. A complete basis for each sector can be obtained by listing all the different charge configurations $\{q_I\}$. The action of the Hamiltonian $(H_{\text{rotor}} + H_{\text{hop}})$ on a charge configuration $\{q_I\}$ is simple. The first part of the Hamiltonian, H_{rotor}, does not affect the charge configuration at all, while the second part, H_{hop} acts in two ways: it either creates two charges at neighboring sites, or it makes a charge hop from one site to another.

Thus, within each sector, $(H_{\text{rotor}} + H_{\text{hop}})$ is simply a hopping Hamiltonian on the cubic lattice. The hopping operators $L_i^{\pm}$ make the charges hop from site I to site J, where I, J are the two endpoints of i. To compute the statistics of the charges, we use the statistical hopping operator algebra described in Ref. 11. We note that the hopping operators satisfy the algebra

$$L_i^+ L_j^- L_k^+ = L_k^+ L_j^- L_i^+ \tag{5}$$

for any i, j, k incident to some vertex I. According to Ref. 11, this is the bosonic hopping operator algebra. We conclude that the charged particles are bosons.

III. A 3D ROTOR MODEL WITH EMERGENT PHOTONS AND FERMIONIC CHARGES

In this section, we present another rotor model which is very closely related to the model from the previous section, (1). The Hamiltonian only differs from H_{rotor} by a "twist"—an additional phase factor in the B_p term. Despite the apparent similarity, we will see that this "twisted" rotor model gives rise to emergent photons and *fermionic* charges.

The relationship between the twisted and untwisted models is a special case of a general relationship between fermionic and bosonic string-net condensates.[5,6] It is part of a systematic and general construction, unlike previous examples of emergent photons and fermions,[7,12,13] which were in some sense discovered by accident.

A. The twisted string operator

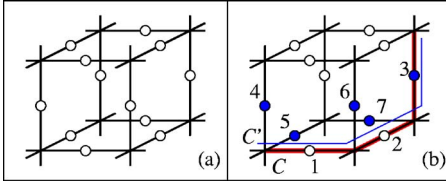In order to motivate the twisted rotor model, we first discuss the "twisted string operator."

FIG. 4. (Color online) (a) A projection of the 3D cubic lattice onto the 2D plane. (b) An example of a curve C with a framing C'. The links along the string are labeled 1,2,3. The links which cross C' are the filled dots labeled by 3,4,5,6,7. The corresponding twisted string operator $W^{\text{tw}}(C)$ is given by $L_1^+ L_2^- L_3^+ (-1)^{L_3^z + L_4^z + L_5^z + L_6^z + L_7^z}$.

We begin with the original "untwisted" rotor model H_{rotor}. Notice that the operator $B_p = L_1^+ L_2^- L_3^+ L_4^-$ is a special case of a general string operator $W(C)$ that can be associated with any curve $C = i_1 i_2 i_3 \cdots i_n$ in the cubic lattice

$$W(C) = L_{i_1}^+ L_{i_2}^- L_{i_3}^+ \cdots L_{i_n}^- \qquad (6)$$

B_p corresponds to the case where C is the boundary of a plaquette ∂p.

This "untwisted" string operator has an important property. When C is a closed curve, $W(C)$ commutes with all charge operators and all other closed string operators,

$$[Q_I, W(C)] = [W(C'), W(C)] = 0. \qquad (7)$$

This property is essential for the mapping to lattice gauge theory (4) to hold. It implies that the $W(C)$ can be simultaneously diagonalized within a given charge sector. The simultaneous eigenstates can then be interpreted as different flux configurations, and the $W(C)$ can be interpreted as Wilson loop operators measuring the flux through a curve C.

It turns out that there is another string operator—which we call the "twisted string operator"—that also has this property. The twisted string operator $\widetilde{W}(C)$ is similar to the usual string operator $W(C)$ except that it contains an additional phase factor that depends on rotors on the "legs" of C, that is, rotors on the links incident to vertices I_k along the curve C.

To define $\widetilde{W}(C)$, one must first choose a projection of the 3D lattice onto a 2D plane. Any projection will work; in this paper, we choose the particular projection illustrated in Fig. 4(a). The twisted string operator is then defined by

$$\widetilde{W}(C) = (L_{i_1}^+ L_{i_2}^- \cdots)(-1)^{\Sigma_{\text{legs of } C} n_i^c L_i^z}. \qquad (8)$$

Here, n_i^c is the number of times that the link i crosses the curve C', where C' is some "framing" of C. A "framing" is a curve C' drawn next to C, but not exactly on it [see Fig. 4(b)]. The choice of framing, C', is not unique, so whenever we discuss $\widetilde{W}$, we will need to specify a particular choice of framing.

One can check directly that both of the equalities in (7) clstrid hold for the twisted string $\widetilde{W}$. Alternatively, one can refer to the general argument given in Ref. 5.
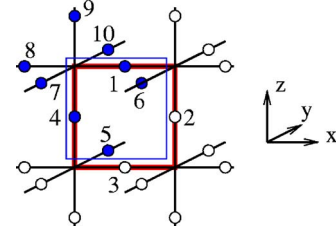

FIG. 5. (Color online) A picture of the $\widetilde{B}_p$ term in the twisted rotor model (9). Just like any other twisted string operator, the term $\widetilde{B}_p = L_1^+ L_2^- L_3^+ L_4^- (-1)^{L_1^z + L_4^z + L_5^z + L_6^z + L_7^z + L_8^z + L_9^z + L_{10}^z}$ acts on the four rotors, labeled by 1,2,3,4, along the boundary of the plaquette p, and the 8 rotors, labeled by 1,4,5,6,7,8,9,10, that cross the framing curve.

B. The twisted rotor model

The twisted model is obtained by modifying the B_p term in (1). Instead of using a B_p term based on the usual string operator, that is, $B_p = W(\partial p)$, we use a modified B_p term based on the twisted string operator: $\widetilde{B}_p = \widetilde{W}(\partial p)$. We use a framing obtained by taking the plaquette boundary ∂p and shifting it up and to the left (see Fig. 5). The result is the twisted rotor Hamiltonian

$$H_{\text{rotor}}^{\text{tw}} = V \sum_I Q_I^2 + J \sum_i (L_i^z)^2 - \sum_p g(\widetilde{B}_p + \text{h.c.}),$$

$$Q_I = (-)^I \sum_{\text{legs of } I} L_i^z,$$

$$\widetilde{B}_p = L_1^+ L_2^- L_3^+ L_4^- (-1)^{L_1^z + L_4^z + L_5^z + L_6^z + L_7^z + L_8^z + L_9^z + L_{10}^z}, \qquad (9)$$

where the above explicit definition of $\widetilde{B}_p$ applies to plaquettes in the xz plane (see Fig. 5). [The definition for plaquettes in the xy and yz planes is similar and can be obtained using $\widetilde{B}_p = \widetilde{W}(\partial p)$ together with the above framing convention.]

Note that that Eq. (9) is very similar to the original rotor model (1) (and hence also very similar to lattice gauge theory). The $\widetilde{B}_p$ term alternately increases and decreases L^z along the plaquette boundary ∂p, just like the B_p term in the model (1). The only difference is that here the amplitude for this process is not always 1. The amplitude is ± 1, depending on the rotors on the legs of the plaquette boundary.

C. Physical properties of the twisted rotor model

The twisted rotor model can be analyzed in the same way as the untwisted model (Sec. II B). The charge operators Q_I commute with the Hamiltonian, so all the states are associated with a charge configuration $\{q_I\}$. Low energy states have charge-0 and can be thought of as string-net states. They can be enumerated in the same way as in the untwisted model. Just as the untwisted closed-string states could be obtained by repeatedly applying different B_p's to the vacuum state (the state with $L_I^z = 0$), the twisted closed-string states can be obtained by repeatedly applying different $\widetilde{B}_p$'s to the vacuum state. Just as before, the J term can be interpreted as a string

tension, and the g term as a string kinetic energy. When $g \gg J$, string-net condensation occurs and the resulting ground state has two types of excitations: low energy string collective motions, and higher energy defects in the condensate.

More quantitatively, one can show that the low energy physics of the twisted rotor model is described by a compact U(1) gauge field coupled to infinitely massive charges—just like the untwisted model. In fact, the twisted and untwisted rotor Hamiltonians (9) and (1)are completely equivalent and can be mapped onto one another.

To see this, note that the operators Q_I and $\widetilde{B}_p$ commute with each other and can therefore be simultaneously diagonalized. Let the simultaneous eigenstates be denoted $|\{q_I, \theta_p\}, \mathrm{tw}\rangle$ where

$$Q_I|\{q_I, \theta_p\}, \mathrm{tw}\rangle = q_I|\{q_I, \theta_p\}, \mathrm{tw}\rangle,$$

$$\widetilde{B}_p|\{q_I, \theta_p\}, \mathrm{tw}\rangle = e^{ip}|\{q_I, \theta_p\}, \mathrm{tw}\rangle,$$

and the q_I, θ_p are integers and real numbers, respectively. The states $|\{q_I, \theta_p\}, \mathrm{tw}\rangle$ form a complete basis for the twisted rotor model. Similarly, we can simultaneously diagonalize Q_I and B_p, to construct a complete basis $|\{q_I, \theta_p\}\rangle$ for the untwisted model.

Note that the matrix elements of $\widetilde{B}_p$, Q_I in the twisted basis are the same as those of B_p, Q_I in the untwisted basis. Furthermore the two sets of operators, $(Q_I, \widetilde{B}_p, L_i^z)$ and (Q_I, B_p, L_i^z), satisfy the same commutation relations. Thus, L_I^z must also have the same matrix elements in the two models. We conclude that the two Hamiltonians, (9) and (1), have exactly the same matrix elements (in their respective bases). The two models are therefore equivalent.

D. Adding charge dynamics

As before, we would like to modify the Hamiltonian $H_{\mathrm{rotor}}^{\mathrm{tw}}$ so that the point defect excitations (or "charges") have a finite mass. To accomplish this, we need to add a term $H_{\mathrm{hop}}^{\mathrm{tw}}$ that does not commute with Q_I. Such a term will make the charges hop to site.

We expect that any hopping term will give rise to the same qualitative behavior (e.g., same particle statistics). However, the resulting Hamiltonian is easier to analyze if we pick a hopping term which commutes with $\widetilde{B}_p$. While in the untwisted case the simplest hopping term happened to have this property, in the case at hand we have to work a little harder.

We notice that the untwisted hopping term H_{hop} is simply $-t(W(i) + W(i)^\dagger)$ where $W(i)$ is the shortest open string operator (the string consists of a single link i). By analogy we choose the twisted hopping term to be $-t(\widetilde{W}(i) + \widetilde{W}(i)^\dagger)$ with a framing convention given by drawing the framing curve just down and to the right of the link i. The result is (see Fig. 6)

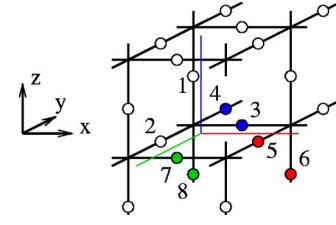$$H_{\mathrm{hop}}^{\mathrm{tw}} = -t\sum_i (\widetilde{L}_i^+ + \widetilde{L}_i^-), \tag{10}$$

where

FIG. 6. (Color online) A picture of the hopping term $H_{\mathrm{hop}}^{\mathrm{tw}} = -t\Sigma_i(\widetilde{L}_i^+ + \widetilde{L}_i^-)$. The $\widetilde{L}_i^\pm$ are defined as twisted string operators $\widetilde{W}(i)$, with the framing curve drawn just below and to the right of i, as shown above. The result is $\widetilde{L}_i^\pm = L_3^\pm(-1)^{L_5^z + L_6^z}$ when i is the link labeled 3 pointing in the x direction, $\widetilde{L}_i^\pm = L_2^\pm(-1)^{L_7^z + L_8^z}$, when i is the link labeled 2 pointing in the y direction, and $\widetilde{L}_i^\pm = L_1^\pm(-1)^{L_3^z + L_4^z}$ when i is the link labeled 1 pointing in the z direction.

$$\widetilde{L}_i^\pm = \begin{cases} L_i^\pm(-1)^{L_5^z + L_6^z}, & \text{if } i \parallel x \\ L_i^\pm(-1)^{L_7^z + L_8^z}, & \text{if } i \parallel y \\ L_i^\pm(-1)^{L_3^z + L_4^z}, & \text{if } i \parallel z. \end{cases} \tag{11}$$

(One can check that $\widetilde{L}_i^\pm$ does in fact commute with $\widetilde{B}_p$—either by explicit calculation or using the general properties of twisted string operators.[5])

Note that the term $H_{\mathrm{hop}}^{\mathrm{tw}}$ causes the charges to hop to neighboring sites. The low energy physics of $(H_{\mathrm{rotor}}^{\mathrm{tw}} + H_{\mathrm{hop}}^{\mathrm{tw}})$ is therefore described by a compact U(1) gauge field coupled to charges with *finite* mass. However, these charges are not Higgs bosons as in the untwisted case. Indeed, with the addition of charge dynamics, the two Hamiltonians $(H_{\mathrm{rotor}}^{\mathrm{tw}} + H_{\mathrm{hop}}^{\mathrm{tw}})$, $(H_{\mathrm{rotor}} + H_{\mathrm{hop}})$, are no longer equivalent. The two models are only equivalent in the special case where $t=0$ and the charges have infinite mass. As soon as the charges can hop, the mapping between the two models breaks down, as does the mapping to lattice gauge theory. The reason for this is that the charged particles in $(H_{\mathrm{rotor}}^{\mathrm{tw}} + H_{\mathrm{hop}}^{\mathrm{tw}})$ are not bosons, but *fermions*, as we will see in the next section.

E. The statistics of the U(1) charges

In this section, we compute the statistics of the charges in the twisted rotor model. We use the same technique as in Sec. II E.

As before, we perform the computation in the case $J=0$. Since B_p^{tw} commutes with the Hamiltonian $(H_{\mathrm{rotor}}^{\mathrm{tw}} + H_{\mathrm{hop}}^{\mathrm{tw}})$, we can divide the Hilbert space into different sectors $\{\theta_p\}$ corresponding to different flux configurations $B_p^{\mathrm{tw}} = e^{i\theta_p}$. The states within each sector can be labeled by their charge configuration $\{q_I\}$. The action of the Hamiltonian $(H_{\mathrm{rotor}}^{\mathrm{tw}} + H_{\mathrm{hop}}^{\mathrm{tw}})$ on a charge configuration $\{q_I\}$ is simple. The first part of the Hamiltonian, $H_{\mathrm{rotor}}^{\mathrm{tw}}$, does not affect the charge configuration at all, while the second part, $H_{\mathrm{hop}}^{\mathrm{tw}}$ acts in two ways: it either creates two charges at neighboring sites, or it makes a charge hop from one site to another.

Thus, within each sector, $(H_{\mathrm{rotor}}^{\mathrm{tw}} + H_{\mathrm{hop}}^{\mathrm{tw}})$ is simply a hopping Hamiltonian on the cubic lattice. The hopping operators $\widetilde{L}_i^\pm$ make the charges hop from site I to site J, where I, J are the two endpoints of i. To compute the statistics of the

charges, we use the statistical hopping operator algebra,[11] just as in the untwisted case. We find that the hopping operators satisfy the same algebra but with a − sign. That is,

$$\tilde{L}_i^+ \tilde{L}_j^- \tilde{L}_k^+ = (-1) \cdot \tilde{L}_k^+ \tilde{L}_j^- \tilde{L}_i^+ \qquad (12)$$

for any i, j, k incident to some vertex I. This sign makes all the difference. The above algebra is not the bosonic hopping operator algebra, but rather the *fermionic* hopping algebra. We conclude that in the twisted rotor model the charged particles are fermions.

One might expect that, by analogy with the untwisted case, there is a mapping to fermionic lattice gauge theory. However, no such mapping exists. The reason is that, at the lattice scale, these fermions behave differently from the usual fermions in a tight binding model.

There are two key differences. First, in a standard tight binding model of spinless fermions, each site can be occupied by 0 or 1 fermion. However, in our case the charge Q_I can be any integer from $-\infty$ to ∞. Thus, a given site can be occupied by arbitrarily many fermions. Second, in a standard tight binding model, the fermions are totally noninteracting in the zero coupling limit $J=0$. However, in our case the the fermions do interact, even in this limit. For example, the hopping term H_{hop} allows positive and negatively charged fermions to annihilate each other if they occupy neighboring sites.

Neither of these differences should affect the physics of the twisted rotor model in the low energy limit, if V is large. In that case, the low energy physics of the rotor model is equivalent to an insulator; it is described by two species of gapped fermions with opposite charges coupled to a U(1) gauge field.

However, these microscopic differences could be important when V becomes small enough that a Fermi surface develops. Unfortunately, in order to construct massless Dirac fermions, we need to consider that limit. Therefore, in order to make our calculation well-controlled, we need to modify the twisted rotor model so that the fermions are equivalent to tight binding model fermions, *even at the lattice scale*. This is the subject of the next section.

IV. ENGINEERING MASSLESS DIRAC FERMIONS

In this section, we modify the twisted rotor model ($H_{\text{rotor}}^{\text{tw}} + H_{\text{hop}}^{\text{tw}}$) so that the fermions are exactly equivalent to tight binding model fermions. We then show that this more well-controlled model can give rise to massless Dirac fermions. It can also give rise to Fermi liquid behavior.

A. The model

There are many ways to make the fermions in ($H_{\text{rotor}}^{\text{tw}} + H_{\text{hop}}^{\text{tw}}$) behave like the usual fermions from a tight binding model. Here we will only describe one of them. First, we divide the vertices of the cubic lattice into 3 classes: "*eee* vertices," "*ooo* vertices," and "mixed vertices." A vertex is called an *eee* vertex if the three components $I = (I_x, I_y, I_z)$ are all even, an *ooo* vertex if the three components $I = (I_x, I_y, I_z)$

are all odd, and a mixed vertex otherwise. We then modify the V term in (1) to

$$V\left(\sum_{eee} Q_I(Q_I-1) + \sum_{ooo} Q_I(Q_I+1) + \sum_{\text{mixed}} Q_I^2 \right).$$

Throughout our discussion we will take the $V \rightarrow +\infty$ limit. In this limit, Q_I must be 0, 1 on the *eee* vertices, 0, −1 on the *ooo* vertices, and 0 on the mixed vertices. In other words, the U(1) charges can only live on the *eee* or *ooo* vertices, and these two sublattices can only contain unit positive and negative charges respectively.

Next, we modify the hopping Hamiltonian $H_{\text{hop}}^{\text{tw}}$ to

$$-t \sum_{\langle ij \rangle, \text{coll.}} (\tilde{L}_i^+ \tilde{L}_j^- + \tilde{L}_i^- \tilde{L}_j^+), \qquad (13)$$

where the sum runs over pairs of neighboring links $\langle ij \rangle$ which are collinear and can be joined together to form a single line segment of length 2. This hopping term has the property that charges only move within the *eee* and *ooo* sublattices.

Putting this all together, and including a term which will correspond to a chemical potential μ, we arrive at the following rotor model:

$$
\begin{aligned}
H_{\text{t.b.}} = J\sum_i (L_i^z)^2 &- g\sum_p (\tilde{B}_p + \text{h.c.}) - 2\mu \sum_i L_i^z \\
&+ V\left(\sum_{eee} Q_I(Q_I-1) + \sum_{ooo} Q_I(Q_I+1) + \sum_{\text{mixed}} Q_I^2 \right) \\
&- t \sum_{\langle ij \rangle, \text{coll.}} (\tilde{L}_i^+ \tilde{L}_j^- + \tilde{L}_i^- \tilde{L}_j^+)).
\end{aligned}
\qquad (14)
$$

B. Solving the model

We begin by considering the case where $t, J=0$. In this case $H_{\text{t.b.}}$ is exactly soluble. Indeed, the basis states $|\{q_I, \theta_p\}, \text{tw}|$ are eigenstates of $H_{\text{t.b.}}$ with an energy

$$
\begin{aligned}
E = V\left(\sum_{eee} q_I(q_I-1) + \sum_{ooo} q_I(q_I+1) + \sum_{\text{mixed}} q_I^2 \right) \\
- \mu \sum_I (-1)^I q_I - 2g\sum_p \cos(\theta_p).
\end{aligned}
\qquad (15)
$$

As we mentioned earlier, we will take the $V \rightarrow \infty$ limit. In this limit, the only low energy states are those with $q_I=0,1$ in the *eee* sublattice, $q_I=0,-1$ in the *ooo* sublattice and $q_I=0$ everywhere else. If we restrict to this low energy subspace, the energy E is simply given by

$$E = -\mu N - 2g\sum_p \cos(\theta_p), \qquad (16)$$

where $N = \sum_{eee} q_I - \sum_{ooo} q_I$ is the total number of charged particles.

Now, consider the case $t \neq 0$. The t term generates nearest neighbor hopping in the *eee* and *ooo* sublattices. The action of the hopping Hamiltonian (13) within the low energy subspace can be written as

$$\sum_{eee,\boldsymbol{\nu}} (\widetilde{t}_{I(I+2\boldsymbol{\nu})} + h.c.) + \sum_{ooo,\boldsymbol{\nu}} (\widetilde{t}_{I(I+2\boldsymbol{\nu})} + h.c.)$$

where $\boldsymbol{\nu}$ runs over the unit vectors $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$ and the $\widetilde{t}_{I(I+2\boldsymbol{\nu})}$ are hopping operators that make the particles hop $I + 2\boldsymbol{\nu}$ to site I. They are defined by

$$\widetilde{t}_{I(I+2\boldsymbol{\nu})} = -t \cdot P \widetilde{L}_i^{\pm} \widetilde{L}_j^{\mp} P, \tag{17}$$

where P is the projection onto the low energy subspace, i is the link connecting $I, I+\boldsymbol{\nu}$, j is the link connecting $I+\boldsymbol{\nu}, I + 2\boldsymbol{\nu}$, and we use the upper (lower) signs for the *eee* (*ooo*) sublattices.

The low energy effective Hamiltonian can then be written as

$$H_{\text{t.b.,eff}} = \sum_{eee,\boldsymbol{\nu}} (\widetilde{t}_{I(I+2\boldsymbol{\nu})} + h.c.) + \sum_{ooo,\boldsymbol{\nu}} (\widetilde{t}_{I(I+2\boldsymbol{\nu})} + h.c.) - \mu N$$

$$- 2g \sum_{p} \cos(\theta_p). \tag{18}$$

This Hamiltonian describes two types of hardcore charged particles with charges ± 1 hopping on two different sublattices. We can completely characterize this Hamiltonian by investigating two physical properties: the statistics of the charges, and the gauge flux through each plaquette p.

To determine the statistics of the charges, we use the statistical hopping operator algebra.[11] We notice that the hopping operators satisfy the relation

$$\widetilde{t}_{IL} \widetilde{t}_{KI} \widetilde{t}_{IJ} = (-1) \cdot \widetilde{t}_{IJ} \widetilde{t}_{KI} \widetilde{t}_{IL} \tag{19}$$

for any J, K, L adjacent to I. According to Ref. 11, this is the fermionic hopping operator algebra, so the particles are fermions.

To determine the flux that the particles see, we compute the product of hopping operators around a 2×2 plaquette $\langle IJKL \rangle$ in the *eee* or *ooo* sublattice. We find that (up to some signs having to do with orientation conventions),

$$\widetilde{t}_{IJ} \widetilde{t}_{JK} \widetilde{t}_{KL} \widetilde{t}_{LI} = (-t)^4 \cdot e^{\sum_{p \in \langle IJKL \rangle} i \theta_p} n_I (1 - n_J)(1 - n_K)(1 - n_L). \tag{20}$$

Here the sum on the right-hand side runs over the four plaquettes in the original lattice that are contained in the (doubled) plaquette $\langle IJKL \rangle$. Also, $n_I = (-1)^I Q_I$ denotes the occupation number of site I. This relation implies that the particles see a flux θ_p through each plaquette p.

The hopping operators $\widetilde{t}_{IJ}$ are completely characterized by the two algebraic relations (19) and (20). Any collection of hardcore hopping operators that satisfy these relations will

give rise to a Hamiltonian equivalent to (18). A particularly simple collection of hopping operators satisfying these relations can be constructed from fermionic operators c_I with

$$\{c_I, c_J\} = 0, \quad \{c_I, c_J^\dagger\} = \delta_{IJ}.$$

The hopping operators are given by

$$\widetilde{t}_{I(I+2\boldsymbol{\nu})} = -t \cdot e^{\pm i(A_{I(I+\boldsymbol{\nu})} + A_{(I+\boldsymbol{\nu})(I+2\boldsymbol{\nu})})} c_I^\dagger c_{I+2\boldsymbol{\nu}}, \tag{21}$$

where the upper (lower) sign applies to the *eee* (*ooo*) sublattice and $e^{iA_{IJ}}$ are phases defined on each link satisfying

$$e^{iA_{I_1 I_n}} e^{iA_{I_n I_{n-1}}} \cdots e^{iA_{I_3 I_2}} e^{iA_{I_2 I_1}} = e^{\sum_{p \in C} i \theta_p}$$

for any closed curve C. In other words, A is a vector potential for the flux configuration θ_p.

Rewriting the low energy effective Hamiltonian in terms of these fermionic operators gives

$$H_{\text{t.b.,eff}} = \sum_{eee,\boldsymbol{\nu}} (-t \cdot e^{i(A_{I(I+\boldsymbol{\nu})} + A_{(I+\boldsymbol{\nu})(I+2\boldsymbol{\nu})})} c_I^\dagger c_{I+2\boldsymbol{\nu}} + h.c.)$$

$$+ \sum_{ooo,\boldsymbol{\nu}} (-t \cdot e^{-i(A_{I(I+\boldsymbol{\nu})} + A_{(I+\boldsymbol{\nu})(I+2\boldsymbol{\nu})})} c_I^\dagger c_{I+2\boldsymbol{\nu}} + h.c.)$$

$$- \mu \sum_{eee} c_I^\dagger c_I - \mu \sum_{ooo} c_I^\dagger c_I - 2g \sum_{p} \cos(\theta_p). \tag{22}$$

Note that this is nothing other than the standard tight binding Hamiltonian for fermions hopping in fixed flux configurations θ_p. If we now allow $J \neq 0$, then the flux configurations acquire dynamics. The Hamiltonian (22) becomes modified by the addition of an electric energy term $J \sum_{\langle IJ \rangle} E_{IJ}^2$, where E_{IJ} is canonically conjugate to A_{IJ}: $[A_{IJ}, E_{IJ}] = i$. This term is precisely what one needs for compact lattice U(1) gauge theory. We conclude that in the general case $J, t \neq 0$, the twisted rotor model (14) is mathematically equivalent to a tight binding model with ± 1 charged fermions coupled to a compact U(1) gauge field.

C. Fermi liquid behavior

The Hamiltonian $H_{\text{t.b.,eff}}$ naturally gives rise to Fermi liquid behavior. This is easiest to understand in the regime $g \gg |J|, |t|$. In this case, the minimum energy flux configuration is the configuration $\theta_p = 0$. The flux θ_p fluctuates about this minima giving rise to photonlike excitations. The fermion dispersion in the background flux configuration is given by

$$E(\boldsymbol{k}) = -2t(\cos(2k_x a) + \cos(2k_y a) + \cos(2k_z a)) - \mu. \tag{23}$$

The ground state is obtained by filling all the negative energy levels. Clearly, the model behaves differently for different values of chemical potential μ. If $\mu < -6t$, then there are no negative energy levels. All the energy levels are empty in the ground state, and the system is an insulator. In this case, the rotor model (14) describes gapped fermions with a gap $\Delta = -6t - \mu$ coupled to a compact U(1) gauge field. The fermions come in two species (corresponding to the two sublattices, *eee* and *ooo*) with charges $+1$ and -1, respectively. Hence, just like the original twisted rotor model ($H_{\text{rotor}}^{\text{tw}}$

$+H_{\text{hop}}^{\text{tw}}$), the physics is equivalent to that of a U(1) gauge field coupled to two species of gapped fermions with opposite charges.

On the other hand, if $\mu > -6t$, then the result is a Fermi liquid or more precisely two Fermi liquids corresponding to the two species of fermions. These two Fermi liquids are both coupled to a compact U(1) gauge field.

D. Massless Dirac fermions

The model (22) can also give rise to massless Dirac fermions. Indeed, massless Dirac fermions can occur whenever the fermion band structure has nodal points. This happens naturally in a π-flux configuration where each plaquette contains a flux of π.

In our case, the relevant plaquettes are not plaquettes p in the original cubic lattice, but rather "doubled" 2×2 plaquettes in the eee and ooo sublattices. One way to ensure that the flux through these "doubled" plaquettes is π, is to introduce a spatial dependence into the parameter t. That is replace t by $t \cdot \chi_{IJ}$ where

$$\chi_{I,I+2x} = 1, \quad \chi_{I,I+2y} = (-1)^{I_x/2},$$

$$\chi_{I,I+2z} = (-1)^{(I_x+I_y)/2} \tag{24}$$

on the eee sublattice and

$$\chi_{I,I+2x} = 1, \quad \chi_{I,I+2y} = (-1)^{(I_x-1)/2},$$

$$\chi_{I,I+2z} = (-1)^{(I_x-1+I_y-1)/2} \tag{25}$$

on the ooo sublattice. The result is that (20) becomes modified so that each fermion sees an effective flux of $\Sigma_{p \in \langle IJKL \rangle} \theta_p + \pi$ through each doubled plaquette $\langle IJKL \rangle$.

The minimum energy flux configuration is still $\theta_p = 0$, but now the fermions see an effective flux of π through each doubled plaquette. The fermion dispersion in this background is given by 2 pairs of degenerate bands with energies

$$E(\mathbf{k}) = \pm t \sqrt{\cos^2(2k_x a) + \cos^2(2k_y a) + \cos^2(2k_z a)} - \mu \tag{26}$$

and Brilluoin zone, $(-\pi/4a, \pi/4a) \times (-\pi/4a, \pi/4a) \times (-\pi/2a, \pi/2a)$. The ground state of $H_{\text{t.b.,eff}}$ is again obtained by filling all the levels with negative energies. If $\mu = 0$, then the Fermi surface consists of 2 nodal points with linear dispersion, at $\mathbf{k} = (\pi/4a, \pi/4a, \pm \pi/4a)$. The low energy fermionic excitations are concentrated near these 2 nodes. Together, the two nodes give rise to one four-component Dirac fermion in the continuum limit. Since the bands are doubly degenerate, and there are two species of fermions (corresponding to the eee and ooo sublattice), there are a total of four massless four component Dirac fermions.

We conclude that the low energy physics of the rotor model $H_{\text{t.b.}}$ with $\mu = 0$, $V \to \infty$, $g \gg J, t$, and t given by the above formula, is equivalent to QED with four species of massless Dirac fermions. Thus, $H_{\text{t.b.}}$ is a local bosonic model that gives rise to both photons and massless fermions. It can be viewed as a realization of a quantum ether—an ether that produces not only light, but also electrons!

We would like to point out that the speed of light $c \sim \sqrt{gJ}a$ and the speed of the massless Dirac fermion $c_e \sim ta$ are not the same in our rotor model. However, we can tune the value of t to make $c = c_e$. In this case the low energy effective theory will have Lorentz invariance. The reason that we do not get a Lorentz invariant low energy effective theory naturally is that our approach is based on Hamiltonian formalism where space and time are treated very differently. If one uses a path integral formalism and discretizes space and time in the same way, Lorentz invariance can emerge naturally in a low energy effective theory.

Another difficulty with the above model is that the emergent electrons are massless. However, this problem can also be overcome: when bosonic models contain emergent non-Abelian gauge bosons, the models can produce electrons with a *finite* mass and such a mass can be much less than the cut-off scale without fine tuning any parameters.[13]

V. CONCLUSION

In this paper we have explicitly demonstrated the string-net picture of $(3+1)$D emerging gauge bosons and fermions. We presented a rotor model on the cubic lattice with a string-net condensed ground state, and excitations which behave just like massless photons and massless Dirac fermions. We saw that the model was closely related to U(1) lattice gauge theory coupled to a Higgs field. The only real difference was a phase factor or "twist" in the Hamiltonian. Surprisingly, this "twist" was enough to turn the Higgs boson into a fermion.

This model is part of a much more general construction[5] that can produce gauge bosons with any gauge group. In particular, one should be able to construct a model that gives rise to SU(3) gauge bosons and massless Dirac fermions, that is, QCD. These models could have applications to lattice gauge theory. They may make it possible to simulate QED with electrons and QCD with quarks without ever introducing quark or electron degrees of freedom on the sites. Only the link variables (representing the gauge field) would be necessary in such a simulation.

From a high energy point of view, the string-net picture of the vacuum is quite appealing. It explains why the standard model looks the way it does, that is, why nature chooses such peculiar things as gauge bosons and fermions to describe itself. In addition, it unifies the mysterious gauge symmetries and anticommuting fields into a single underlying structure: a string-net condensate.

But can we actually construct a string-condensed local bosonic model that produces the entire standard model? We are close, but not quite there. In terms of elementary particles we can produce photons, gluons, leptons and quarks, but we do not know how to produce neutrinos or SU (2) gauge bosons.

The problem with the neutrinos and the SU (2) gauge bosons is the famous chiral-fermion problem.[20] Neutrinos are chiral fermions and the SU (2) gauge bosons couple chirally to other fermions. At the moment, we do not know how to obtain chiral fermions and chiral gauge theories from *any* local lattice model, much less a local bosonic model.

From a condensed matter point of view, the above model shows how a simple spin system can give rise to emergent fermions and gauge bosons. Extended objects can give rise to both of these phenomena, as long as their condensate has an appropriate "twist." This may provide intuition in the search for new and exotic phases of matter—phases beyond the scope of Landau's theory of symmetry breaking. The discovery of a real material with string-net condensation would represent a breakthrough, particularly a material containing excitations which behave just like the photons and electrons in our vacuum.

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[1] L. D. Landau, Phys. Z. Sowjetunion **11**, 26 (1937).

[2] L. D. Landau and E. M. Lifschitz, *Statistical Physics—Course of Theoretical Physics* (Pergamon, London, 1958), Vol. 5.

[3] Y. Nambu, Phys. Rev. Lett. **4**, 380 (1960).

[4] J. Goldstone, Nuovo Cimento **19**, 154 (1961).

[5] M. A. Levin and X.-G. Wen, Phys. Rev. B **71**, 045110 (2005).

[6] M. A. Levin and X.-G. Wen, Rev. Mod. Phys. **77**, 871 (2005).

[7] X.-G. Wen, *Quantum Field Theory of Many-Body Systems—From the Origin of Sound to an Origin of Light and Electrons* (Oxford University Press, Oxford, 2004).

[8] X.-G. Wen, Phys. Rev. B **68**, 115413 (2003).

[9] J. Kogut and L. Susskind, Phys. Rev. D **11**, 395 (1975).

[10] T. Banks, R. Myerson, and J. B. Kogut, Nucl. Phys. B **129**, 493 (1977).

[11] M. Levin and X.-G. Wen, Phys. Rev. B **67**, 245316 (2003).

[12] X.-G. Wen, Phys. Rev. Lett. **88**, 011602 (2002).

[13] X.-G. Wen, Phys. Rev. D **68**, 065003 (2003).

[14] J. B. Marston and I. Affleck, Phys. Rev. B **39**, 11538 (1989).

[15] G. Baskaran and P. W. Anderson, Phys. Rev. B **37**, R580 (1988).

[16] O. I. Motrunich and T. Senthil, Phys. Rev. Lett. **89**, 277004 (2002).

[17] R. Moessner and S. L. Sondhi, Phys. Rev. B **68**, 184512 (2003).

[18] M. Hermele, M. P. A. Fisher, and L. Balents, Phys. Rev. B **69**, 064404 (2004).

[19] E. Ardonne, P. Fendley, and E. Fradkin, Ann. Phys. **310**, 493 (2004).

[20] M. Lüscher, hep-th/0102028 (2001).