

## Fusion rules and vortices in $p_x+ip_y$ superconductors

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The “half-quantum” vortices ( $\sigma$ ) and quasiparticles ( $\psi$ ) in a two-dimensional  $p_x+ip_y$  superconductor obey the Ising-like fusion rules  $\psi \times \psi = 1$ ,  $\sigma \times \psi = \sigma$ , and  $\sigma \times \sigma = 1 + \psi$ . We explain how the physical fusion of vortex-antivortex pairs allows us to use these rules to read out the information encoded in the topologically protected space of degenerate ground states. We comment on the potential applicability of this fact to quantum computation.

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### I. INTRODUCTION

The magic of a quantum computer<sup>1-4</sup> is that it makes a virtue out of a vice. The numerical simulation of even simple quantum systems requires exponentially large storage space and exponentially long computation times. The quantum state of a collection of  $N$  spin-1/2 particles, or qubits, requires  $2^N$  classical variables for its description, and solving the Schrödinger equation to follow its time evolution requires a conventional computer to perform up to  $2^N \times 2^N$  operations per time step. Conversely, a computer built out of quantum components would be able to exploit the massive parallelism inherent in quantum time evolution to provide fast solutions for problems that would require exponential time on conventional machines.

The difficulties to be overcome in building a quantum computer are many. The quantum system at its core must be strongly isolated from the environment so as to avoid decoherence; the unitary transformations that constitute the elementary computational steps must be performed with sufficient precision that error-correcting codes<sup>5</sup> remain effective; and, after all the quantum computation has been performed, the system must be capable of being reconnected to the outside world in such a way that the output can be read off *via* a measurement process that does not disturb the result.

A particularly appealing scheme for quantum computation exploits topologically protected macroscopic quantum states in many-body systems.<sup>6-9</sup> The first examples of such topologically ordered and protected many-body quantum states were found by Xiao-Gang Wen and Qian Niu<sup>10</sup> in the context of the fractional quantum Hall effect. Wen and Niu observed that if we place a filling-fraction  $\nu=1/n$  quantum Hall state on a genus  $g$  Riemann surface (a thought experiment only!) then there are  $n^g$  degenerate, and essentially indistinguishable, ground states. They also observed that by creating a vortex-antivortex pair in the Hall fluid and then moving the vortex around one of the homology generators of the surface they could cause the system to roll over from one degenerate state to the next. Since the degeneracy is lifted only by the exponentially suppressed tunneling of vortices around the generators, the coherence of a linear superposition of such states is *topologically protected*. Furthermore, as described

above, such superpositions can in principle be manipulated by controlling the motion of their vortex defects, and so inducing on them representations of the braid group of vortex world-lines.

For simple systems, such as the  $\nu=1/n$  Hall state on a simply connected surface, these braid-group representations are abelian, interchange and braiding giving rise only to a phase factor. This is not sufficient for a universal quantum computer.<sup>8,11</sup> We need a non-abelian representation—i.e., particles with non-abelian statistics.<sup>12</sup> There are a various two-dimensional physical systems that should, in theory, have excitations with non-abelian statistics. Those closest to being realized in practice are all associated with many-body wavefunctions that contain a Pfaffian factor. These candidates are: (i) A quantum Hall effect state seen at filling-fraction  $\nu=5/2$ ,<sup>13</sup> (ii) an anticipated, but as yet unobserved, phase of a rapidly rotating atomic Bose gas, (iii) a two-dimensional  $p_x+ip_y$  superconducting state—an example of which has possibly been observed in the layered  $\text{Sr}_2\text{RuO}_4$  superconductor.<sup>14,15</sup> While the non-abelian statistics of these Pfaffian states is still not quite adequate for the construction of a universal computer, it is still worthy of study.

The mathematical origin of the non-abelian braiding in the first two candidate systems is quite technical, involving complicated linear dependences among various Pfaffian wave functions.<sup>16</sup> The non-abelian braid statistics of the  $p_x+ip_y$  superconductor is much easier to understand physically. The mechanism has been beautifully explained by Ivanov<sup>17</sup> and further clarified by Stern *et al.*<sup>18</sup>

The essential ingredient is that the core of an Abrikosov vortex in any superconductor contains localized low-energy bound states whose existence is guaranteed by an index theorem that relates the phase winding number  $n$  of the vortex to the number of branches of low energy excitations.<sup>19</sup> The  $p_x+ip_y$  superconductor is unusual in that it can host a “half-quantum” vortex,<sup>20</sup> where the phase winding is seen by only one of the two components of the electron spin. The quasiparticles bound in the core of a two-dimensional version of such a vortex are Majorana, i.e., they are identical to their antiparticles. Further, for odd phase winding numbers, each vortex binds a zero energy mode whose field-expansion coefficient is Hermitian. In the presence of  $2N$  such vortices,

the many-body Hilbert space of zero-energy states is  $2^{N-1}$  dimensional. This number is in contrast with the naïve count of  $2^{2N}$  dimensions that would come from (wrongly) supposing that each single-particle zero-energy mode must be either occupied or unoccupied. Because these  $2^{N-1}$  independent states cannot be associated with any individual vortex, the many-body state encodes highly nonlocal information. Ivanov<sup>17</sup> demonstrated that the operation of braiding or interchanging the vortices acts on this zero-energy space in the same way that braiding acts on the  $2^{N-1}$  dimensional space of  $N$ -point conformal blocks in the level  $k=2$ ,  $SU(2)$  Wess-Zumino-Witten (WZW) model.

It is not surprising that this particular WZW model plays some role. It is known that it describes the low energy physics of the Pfaffian quantum Hall state.<sup>21</sup> It also describes the low energy physics of the Bose gas pfaffian state,<sup>22,23</sup> and is a component of the low energy effective action of the two-dimensional  $p_x+ip_y$  superconductor.<sup>24</sup>

Because the zero-energy states consist of an equal superposition of particle and hole, they are electrically neutral. They are also nonlocal. As a consequence they should be little affected by local impurities and external fields, and so be topologically protected. The vortices binding them, however, carry magnetic flux and so might be manipulable by external probes such as STM tips. These vortices are thus potential candidates for the basic building blocks of a quantum computer. The problem is that the internal state in the  $2^{N-1}$ -dimensional protected zero-energy space is so decoupled from the outside world that it is hard to see how it can be measured.

This read-out problem is not unique to non-abelian braid statistics. There is as yet no completely convincing experimental demonstration of even the abelian “anyonic” statistics in the  $\nu=1/n$  Hall state—although no theorist doubts that it is present. An experiment that can detect non-abelian statistics is even harder to perform. We need some method of accessing the non-abelian state by reading out the information in the protected space.

One approach, at least in principle, is to use the “fusion rules” of the vortices to re-connect the protected states to something measurable. The vortex fusion algebra contains, in addition to the obvious rule that phase-winding number adds, a  $\mathbb{Z}_2$  factor that is essentially that of the Ising model primary-field operators  $\mathbb{1}$ ,  $\sigma$ , and  $\psi$ ,<sup>25</sup> or of the analogous affine primary fields (of spin zero, one-half, and one, respectively) in the  $SU(2)_2$  WZW model:

$$\psi \times \psi = \mathbb{1}, \quad \sigma \times \sigma = \mathbb{1} + \psi, \quad \psi \times \sigma = \sigma. \quad (1)$$

In the superconductor it is natural to identify  $\mathbb{1}$  with the ground state and  $\psi$  with the Bogoliubov quasiparticle. The latter is conserved only mod 2 because two quasiparticles can combine to form a Cooper pair and vanish into the condensate. Hence  $\psi \times \psi \rightarrow \mathbb{1}$ . Slightly less obvious is the identification of  $\sigma$  with the odd-winding-number vortex. This comes about because a pair of vortices is associated with a single zero-energy state that can be occupied ( $\psi$ ) or unoccupied ( $\mathbb{1}$ ). Hence  $\sigma \times \sigma = \mathbb{1} + \psi$ . The third product,  $\sigma \times \psi = \sigma$ , is then determined by the associativity of the fusion-product algebra. The principal claim of this paper is that it is possible

to use this algebra to determine some of the information in the topologically protected space by physically fusing a vortex with an antivortex and seeing if they leave behind a real quasiparticle  $\psi$ .

In subsequent sections, we will fill in the details of this process. In Sec. II we discuss the form of the zero energy solutions to the single-particle Bogoliubov-de-Gennes equation, focusing on the sign ambiguities which are the ultimate source of the particle-creation legerdemain. In Sec. III we show how tunneling between nearby vortices can be used to resolve the sign ambiguity, and how braiding followed by fusing reveals details of the protected state. Section IV is a conclusion and discussion. An Appendix describes some necessary, but technical, aspects of Berry transport in a superconducting state.

## II. VORTEX CORE STATES

The topological objects that have the properties we seek to exploit are the “half-quantum” vortices of a thin film of  $p_x+ip_y$  spin-triplet superfluid, such as <sup>3</sup>He-A, or possibly the superconductor  $Sr_2RuO_4$ . The order parameter in such a superfluid contains an angular-moment vector  $\mathbf{l}$ , a vector  $\mathbf{d}$  characterizing the Cooper-pair spin state, and an overall phase-factor  $\exp(i\chi)$ . In a half-quantum vortex the vector  $\mathbf{l}$  lies along the vortex axis and the vector  $\mathbf{d}$  lies in plane which, for convenience, we imagine to be perpendicular to  $\mathbf{l}$ . As we encircle the vortex the phase  $\chi$  increases by  $\pi$  while the  $\mathbf{d}$  vector rotates through  $180^\circ$ . Although  $\exp(i\chi)$  and  $\mathbf{d}$  separately change sign, it is their product that appears in the order parameter, and this is single-valued. The effect of the combined rotation is that the spin-up fermions see the order parameter phase wind through  $2\pi$ , while the spin-down fermions see no phase change at all. Andreev reflection off the winding phase will bind low energy spin-up quasiparticle modes in the vortex core, but there will be no spin-down quasiparticle states with energy significantly less than the gap  $|\Delta|$ . Because only one component of spin is involved in the low energy physics, we will from now on regard the fermions as being spinless. It is this spinlessness that allows the  $p_x+ip_y$  superconductor to have topological properties analogous to those of the spin polarized Pfaffian quantum Hall state.<sup>13</sup>

Consider a two-dimensional film of superfluid with its  $\mathbf{l}$  vector perpendicular to the film, and with a winding-number  $n$  vortex at the origin. We are principally interested in the case  $n = \pm 1$ , but we will keep  $n$  general for the moment, so as to bring out a distinction between vortices with  $n$  even and  $n$  odd. The Bogoliubov-de-Gennes (BdG) equation<sup>29</sup> has bound-state solutions of the form

$$\begin{pmatrix} u \\ v \end{pmatrix} = e^{i\theta} \begin{pmatrix} u_l \\ v_l \end{pmatrix} \quad (2)$$

where

$$\begin{pmatrix} u_l \\ v_l \end{pmatrix} = \begin{pmatrix} e^{i\theta(n+1)/2} [a(r)H_{l+1/2}(k_f r) + \text{c.c.}] \\ e^{-i\theta(n+1)/2} [b(r)H_{l-1/2}(k_f r) + \text{c.c.}] \end{pmatrix}. \quad (3)$$

Here  $H_l(k_f r)$  is a Hankel function of the radial co-ordinate  $r$ , the angle  $\theta$  is the polar co-ordinate, and  $k_f$  is the Fermi momentum. The coefficient functions  $a(r)$ ,  $b(r)$  are slowly varying on the length scale of  $k_f^{-1}$  and are most conveniently written in terms of a pair of auxiliary variables  $x$  and  $\theta(x)$ . These are defined in terms of the impact parameter  $r_0 \equiv l/k_f$  by

$$x = \sqrt{r^2 - r_0^2}, \quad (4)$$

$$\theta(x) = \tan^{-1}\left(\frac{x}{r_0}\right). \quad (5)$$

If we write

$$\begin{pmatrix} a(r) \\ b(r) \end{pmatrix} = \begin{pmatrix} e^{-in\theta(x)/2} \tilde{a}(x) \\ e^{in\theta(x)/2} \tilde{b}(x) \end{pmatrix} \quad (6)$$

then, in the Andreev approximation, the coefficients  $\tilde{a}(x)$  and  $\tilde{b}(x)$  obey

$$\begin{pmatrix} -iv_f \partial_x & \Delta(r) e^{in\theta(x)} \\ \Delta(r) e^{-in\theta(x)} & +iv_f \partial_x \end{pmatrix} \begin{pmatrix} \tilde{a}(x) \\ \tilde{b}(x) \end{pmatrix} = E_l \begin{pmatrix} \tilde{a}(x) \\ \tilde{b}(x) \end{pmatrix}. \quad (7)$$

The one-dimensional eigenvalue equation (7) has a physically appealing interpretation as describing the propagation of a quasiparticle along a rectilinear trajectory with  $r_0$  its distance of closest approach to the vortex centre. The variable  $x$  is the distance along the trajectory with  $x=0$  being the point of closest approach. As the quasiparticle moves it sees the local value of the order parameter  $\Delta(r(x)) \exp[in\theta(x)]$ . The Hankel function is evanescent for  $r < r_0$ , and  $\tilde{a}$  and  $\tilde{b}$  can be taken as being constant in this region. In order for the Hankel functions to combine to give the Bessel functions  $J_{\pm 1/2}(kr)$  that remain finite at  $r=0$ , we need to impose the condition that  $a$  and  $b$  (and hence  $\tilde{a}$  and  $\tilde{b}$ ) be real at  $x=0$ . This condition allows us to extend the definition of  $\tilde{a}(x)$  and  $\tilde{b}(x)$  to continuous functions on the entire real line by setting  $\tilde{a}(-x) = (\tilde{a}(x))^*$  and  $\tilde{b}(-x) = (\tilde{b}(x))^*$ . It may be verified that  $(\tilde{a}(x), \tilde{b}(x))^T$  continues to satisfy (7) in the extended domain. The one-dimensional equation is able to capture the physics of the bound states because the Andreev scattering that confines the particle/hole in the vortex core is almost exactly retro-reflective.<sup>26</sup>

Solving the one-dimensional problem shows that, for small  $l$ , the energy eigenvalue is given by  $E(l) = -l\omega_0$  where  $\omega_0$  is a frequency determined by the order parameter profile  $\Delta(r)$ . This frequency is positive if  $n > 0$  and negative if  $n < 0$ . A glance at (2) and (3) shows that single-valuedness of the BdG wave function requires  $l$  to take integer values when  $n$  is odd, and half-integer values when  $n$  is even. The spectrum may therefore be labeled by an integer  $m$  and is given by

$$E_m = \begin{cases} -\omega_0 m, & n \text{ odd} \\ -\omega_0(m + 1/2), & n \text{ even.} \end{cases} \quad (8)$$

Both cases are consistent with the  $E \leftrightarrow -E$  symmetry of the BdG eigenvalues. For odd winding numbers the spectrum contains an  $E=0$  mode. If we create  $N$  vortices there will be one of these zero modes localized in each vortex core. There will, however, always be an even number of zero modes. For an odd number of vortices, an additional zero-mode will be found on the boundary of the superfluid.<sup>24</sup>

Although the zero modes for the  $n=+1$  and  $n=-1$  vortices both have  $l=0$ , their actual angular dependence differs. For the  $n=+1$  vortex, whose circulation is in the same sense as the  $p_x+ip_y$  Cooper-pair angular momentum, we have

$$\begin{pmatrix} u_0(r, \theta) \\ v_0(r, \theta) \end{pmatrix} = \begin{pmatrix} e^{i\theta} f(r) \\ e^{-i\theta} f(r) \end{pmatrix} \quad (9)$$

For the  $n=-1$  vortex we have,

$$\begin{pmatrix} u_0(r, \theta) \\ v_0(r, \theta) \end{pmatrix} = \begin{pmatrix} f(r) \\ f(r) \end{pmatrix}. \quad (10)$$

In both cases the radial function  $f(r)$  is real and decays as  $\exp(-\Delta r/v_f)$  away from the vortex core.

We have been tacitly assuming that  $\Delta$  is real. A global phase rotation  $\Delta \rightarrow e^{i\chi} \Delta$  alters each of the eigenmodes and we can choose an overall phase for the modes so that

$$\begin{pmatrix} u_l(r, \theta) \\ v_l(r, \theta) \end{pmatrix} \rightarrow \begin{pmatrix} e^{i\chi/2} u_l(r, \theta) \\ e^{-i\chi/2} v_l(r, \theta) \end{pmatrix}. \quad (11)$$

An inspection of the detailed form of the solutions shows that, with this phase choice, we retain the property that the upper component of eigenmode  $l$  is the complex conjugate of the lower component for eigenmode  $-l$ . This property does not uniquely specify the  $(u, v)^T$  vectors, however. Even after normalization, so that  $\int (|u_l|^2 + |v_l|^2) d^2x = 1$ , a choice of overall sign remains to be made.

In the vicinity of the vortex core we can use the bound states to make a mode expansion of the low energy part of the fermion field

$$\begin{pmatrix} \hat{\psi}(r, \theta) \\ \hat{\psi}^\dagger(r, \theta) \end{pmatrix} = \sum_l b_l \begin{pmatrix} u_l \\ v_l \end{pmatrix} e^{il\theta} + \text{higher-energy modes.} \quad (12)$$

The fact that the lower field component  $\hat{\psi}^\dagger$  is the Hermitian conjugate of the upper component  $\hat{\psi}$  coupled with the phase choice made in (11), enforces the hermiticity condition  $b_l = b_{-l}^\dagger$  on the annihilation and creation operator coefficients. This property is characteristic of a Majorana fermion. The bound-state quasiparticles are therefore their own antiparticles. For  $n$  odd, the zero-energy, mode coefficient obeys  $b_0 = b_0^\dagger$  and cannot be thought of as being either a creation or annihilation operator. Instead, the conditions  $\{\hat{\psi}(x), \hat{\psi}(y)\} = 0$  and  $\{\hat{\psi}^\dagger(x), \hat{\psi}^\dagger(y)\} = \delta^2(x-y)$  together with the completeness of the eigenmodes tell us that

$$b_0^2 = 1/2, \quad \{b_0, b_l\} = \{b_0, b_l^\dagger\} = 0, \quad l \neq 0. \quad (13)$$

We have taken the trouble in this section to display the bound-state wave functions in some detail. We did this to make clear the origin of the  $b_l^\dagger = b_{-l}$  Majorana condition, to make explicit our phase choices, and to point out the remaining sign ambiguity. This last point is important. For conventional fermion annihilation and creation operators, the relative sign of  $a_i$  and  $a_i^\dagger$  is fixed by the condition  $\{a, a^\dagger\} = 1$ , but the relative sign of the various  $a_i$  with respect to one another is unimportant as it can always be absorbed into the definition of the free-field modes whose coefficients they are. The relative sign of the  $b_0$ 's associated with different vortices has a physical significance that will be appreciated after we consider tunneling between nearby vortex pairs.

### III. BRAIDING AND FUSION

Suppose we start from a homogeneous ( $\Delta = \text{const.}$ ) state and adiabatically create  $N$  vortex-antivortex pairs. For ease of discussion imagine the pairs arranged so that each antivortex lies on the  $x$  axis, and the corresponding vortex lies vertically above. We label the pairs from left to right by the index  $i = 1, \dots, N$ . In the core of each vortex and antivortex there will be a zero-mode with its corresponding  $b_0$  mode operator. Up to a factor, these hermitian operators obey gamma-matrix anti-commutation relations, so it is natural to rename these  $b_0$ 's as  $\gamma_i/\sqrt{2}$  (vortex) and  $\gamma_{i+N}/\sqrt{2}$  (antivortex). These  $2N$  operators can be assembled to make  $N$  each of annihilation and creation operators

$$b_i = \frac{1}{2}(\gamma_i + i\gamma_{i+N}),$$

$$b_i^\dagger = \frac{1}{2}(\gamma_i - i\gamma_{i+N}). \quad (14)$$

These  $b_i$  and  $b_i^\dagger$ 's act on a  $2^N$ -dimensional space of degenerate ground states. This space is split into two physically equivalent  $2^{N-1}$ -dimensional spaces by a superselection rule: Although fermion number conservation is broken by the presence of the condensate, no physical process internal to the system can change an odd total fermion number into an even fermion number. Which of the two spaces we are in depends on whether the total number of particles in the system is odd or even.

The odd-even decomposition is reflected in the fusion algebra. By using the rules (1) to repeatedly fuse a  $\sigma$  with  $\sigma \times \sigma = \mathbb{1} + \psi$ , we find

$$\sigma \times \sigma \times \sigma = \sigma + \sigma,$$

$$\sigma \times \sigma \times \sigma \times \sigma = \mathbb{1} + \psi + \mathbb{1} + \psi,$$

etc. In general, for  $2N$  vortices,

$$\underbrace{\sigma \times \cdots \times \sigma}_{2N \text{ factors}} = \underbrace{(\mathbb{1} + \cdots + \mathbb{1})}_{2^{N-1} \text{ terms}} + \underbrace{(\psi + \cdots + \psi)}_{2^{N-1} \text{ terms}}. \quad (15)$$

The  $2^{N-1}$ -dimensional multiplicity spaces of the  $\mathbb{1}$  and the  $\psi$  are separately invariant under the action of the braid group.

When each vortex “ $i$ ” is brought close to its sibling antivortex “ $N+i$ ,” the  $2^N$ -fold ground state degeneracy is lifted.

Because two nearby odd-winding number vortices are effectively a vortex with an even winding number, tunneling between their two zero-energy BdG modes will cause them to realign and become two eigenmodes with some small non-zero energy  $\pm E_i$ . In the second quantized Bogoliubov Hamiltonian these two single-particle modes make one state that can either be occupied and contribute  $E_i$  to the total energy or be unoccupied and contribute energy  $-E_i$ . If we were to slowly merge each vortex with its sibling antivortex, the  $\pm E_i$  eigenmodes will split further and eventually merge with the continuum of unbound states. If the  $E_i$  state is empty, nothing remains after the vortices annihilate. If it is occupied, an unbound quasiparticle with energy of  $|\Delta|$  is left behind.

It is not easy to compute the matrix elements for the tunnel splitting, but we do not need to know them in detail. When the splitting is small we can restrict ourselves to the  $2^N$  dimensional space of nearly degenerate states, and in this space the many-body Hamiltonian must be of the form

$$\hat{H}_0 = \sum_{i=1}^N E_i \frac{1}{4i} [\gamma_{N+i}, \gamma_i] + \frac{1}{2} \sum_{i=1}^N E_i$$

$$= \frac{1}{2} \sum_{i=1}^N E_i (b_i^\dagger b_i - b_i b_i^\dagger) + \frac{1}{2} \sum_{i=1}^N E_i = \sum_{i=1}^N E_i b_i^\dagger b_i. \quad (16)$$

This is because  $(1/4i)[\gamma_{N+i}, \gamma_i]$  is the only hermitian operator that can be made solely out of  $\gamma_i$  and  $\gamma_{i+N}$ , and so the only possible operator that tunneling between vortex “ $i$ ” and “ $N+i$ ” can contribute to the Hamiltonian. Hermiticity demands that that the  $E_i$  be real. We can also assume that the  $E_i$  are positive by using this requirement to fix the relative sign ambiguity between  $\gamma_{i+N}$  and  $\gamma_i$ .

The lowest energy state  $|0\rangle$  for an even number of particles is that which is annihilated by all the  $b_i$ . In this ground state all fermions are paired, and all of the not-quite-zero-energy states are empty. If the system contains an odd number of particles, however, one will be left unpaired, and will occupy the lowest of the not-quite-zero-energy states.

Now we consider how we can manipulate the occupation numbers of the not-quite-zero modes. Following Ivanov,<sup>17</sup> we adiabatically transport vortex  $i$  around vortex  $j$  and bring it back to its original position. In this process the local phase  $\chi$  seen by each vortex will increment by  $\pm 2\pi$ , and so cause the phase factors  $e^{i\chi/2}$  [see (11)] in the zero modes of vortex  $i$  and  $j$  to change sign. The field operators  $\hat{\psi}(x)$ ,  $\hat{\psi}^\dagger(x)$  are indifferent to the choice of modes in which we expand them, and must be unchanged by the braiding process. The sign change of the mode vector  $(u_0, v_0)$  must therefore be compensated by a change in the sign of the mode coefficients  $\gamma_i$  and  $\gamma_j$ . Consequently

$$\hat{H}_0 = \cdots + E_i \frac{1}{4i} [\gamma_{N+i}, \gamma_i] + \cdots + E_j \frac{1}{4j} [\gamma_{N+j}, \gamma_j] + \cdots$$

$$+ \frac{1}{2} E_i + \cdots + \frac{1}{2} E_j + \cdots$$

is changed to

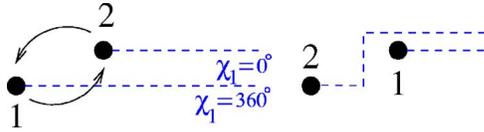


FIG. 1. (Color online) Phase changes due to the braid operation  $T_1$ : Before the braiding the phase  $\chi_2(1)$  seen by vortex 1 due to vortex 2 is just more than  $180^\circ$ , and the phase  $\chi_1(2)$  seen by vortex 2 due to vortex 1 is just more than  $0^\circ$ . After the braiding the phase  $\chi_1(2)$  at vortex 2 due to vortex 1 is just more than  $180^\circ$ , but because vortex 1 remains below the displaced vortex-2 branch cut, the phase  $\chi_2(1)$  is now just more than  $360^\circ$ . Consequently  $\gamma_1$  is replaced by  $\gamma_2$ , but  $\gamma_2$  is replaced by  $-\gamma_1$ .

$$\hat{H}_0^{\text{new}} = \cdots - E_i \frac{1}{4i} [\gamma_{N+i}, \gamma_i] + \cdots - E_j \frac{1}{4j} [\gamma_{N+j}, \gamma_j] + \cdots \\ + \frac{1}{2} E_i + \cdots + \frac{1}{2} E_j + \cdots$$

In addition to this explicit monodromy in  $\hat{H}_0$ , the state  $|0\rangle$  might acquire a non-abelian, holonomy from Berry transport. It was argued by Stern *et al.*,<sup>18</sup> that, for the choice of phases in  $(u_0, v_0)$ , the state  $|0\rangle$  is at most multiplied by an overall Berry phase, and so essentially returns to itself after the braiding process. We agree with this conclusion, although we find the discussion in Ref. 18 unnecessarily involved. We, therefore, provide our own derivation of this key fact in the Appendix.

After the braiding the state  $|0\rangle$  remains an eigenvector of all the  $(1/4i)[\gamma_{N+k}, \gamma_k]$  with eigenvalue  $-1/2$ . Because of the sign changes in  $\hat{H}_0$ , however, it is no longer the lowest energy state of  $\hat{H}_0^{\text{new}}$ . It is instead an excited state with energy  $E_i + E_j$ , corresponding to the two not-quite-zero modes of the  $i$  and  $j$ th pairs being occupied. If we now adiabatically fuse the  $i$ th vortex with its sibling antivortex we recover the uniform state together with a quasiparticle  $\psi$ . The same is true for the  $j$ th pair.

We now suppose we make an anti-clockwise interchange  $T_i$  of the  $i$  and  $i+1$ th vortices. In order to follow what happens, we must first fix the sign ambiguity between the  $\gamma_i$ . We therefore draw “branch cuts” to the right of each vortex and parallel to the  $x$  axis and set the  $\chi_i$  produced by the vortex equal to zero immediately above the cut. The local phase  $\chi(i)$  seen by vortex  $i$  is then the sum of the  $\sum_{j \neq i} \chi_j$ ’s of the other vortices and anti-vortices. Under the interchange, and keeping track of the explicit monodromy of the local  $\chi_i$ ’s, we find that the zero mode of the  $i+1$ th vortex replaces that of the  $i$ th while the  $i$ th zero mode replaces *minus* that of the  $i+1$ th. Thus (see Fig. 1 and Ref. 17)

$$T_i: \begin{cases} \gamma_i \rightarrow \gamma_{i+1} \\ \gamma_{i+1} \rightarrow -\gamma_i \end{cases} \quad (17)$$

Since the geometric arrangement of the vortices is unchanged, the Hamiltonian becomes

$$\hat{H}_0^{\text{new}} = \cdots + E_i \frac{1}{4i} [\gamma_{N+i}, \gamma_{i+1}] + \cdots - E_{i+1} \frac{1}{4i} [\gamma_{N+i+1}, \gamma_i] \\ + \cdots + \frac{1}{2} E_i + \cdots + \frac{1}{2} E_{i+1} + \cdots$$

The crucial effect is not so much the sign change but that the Berry transport of  $|0\rangle$  preserves its property that it is the state killed by the  $b_k$  of the *original*  $(1/4i)[\gamma_{N+k}, \gamma_k]$ . Consequently  $|0\rangle$  is no longer an eigenstate of the new Hamiltonian, but is instead a linear superposition of eigenstates. The outcome of fusing vortex  $i$  with its antivortex is no longer certain. Same is true for the outcome of fusing vortex  $i+1$  with its antivortex. In fact, it will be shown later that we have constructed an entangled state.

The sign change in  $\gamma_i \rightarrow -\gamma_j$  does have significance, however, as it ensures that the result of braiding is to take  $\gamma_i \rightarrow \gamma'_i$  with

$$\gamma'_i = \gamma_j O_{ji} = U \gamma_i U^{-1},$$

where  $O_{ij}$  is an element of  $SO(2N)$ , as opposed to  $O(2N)$ . Here  $U$  is a spin-representation matrix that would act on the  $2^N$  dimensional space of degenerate ground states were we to make the braid group act by holonomy on the states, instead of by explicit monodromy on the Hamiltonian. The  $O(2N)$  spin representation is irreducible, but under restriction to  $SO(2N)$  it decomposes into two irreducible components, these being the spaces of odd or even fermion number. For the elementary braiding operation  $T_i$ , Ivanov showed<sup>17</sup> that the relevant unitary operator  $U(T_i)$  can be taken to be

$$\tau_i = \frac{1}{\sqrt{2}} (1 + \gamma_{i+1} \gamma_i), \quad (18)$$

as this is unitary and obeys

$$\tau_i \gamma_i \tau_i^{-1} = \gamma_{i+1},$$

$$\tau_i \gamma_{i+1} \tau_i^{-1} = -\gamma_i.$$

The operation of taking vortex  $i$  completely around vortex  $i+1$  and back to its starting point is, therefore

$$\tau_i^2 \gamma_i \tau_i^{-2} = -\gamma_i,$$

$$\tau_i^2 \gamma_{i+1} \tau_i^{-2} = -\gamma_{i+1}.$$

For the rest of this section we will take the holonomy point of view—i.e., the Hamiltonian will be kept fixed as  $\hat{H}_0 = \sum_i E_i b_i^\dagger b_i$ , and a braiding  $T$  will act on the state by  $U(T)$ .

We now consider the effect of various braid group generators on the occupation number basis states

$$|n_1, \dots, n_N\rangle \equiv (b_1^\dagger)^{n_1} \cdots (b_N^\dagger)^{n_N} |0\rangle. \quad (19)$$

When we expand out the generator  $\tau_i$  in terms of the annihilation and creation operators, we find that

$$\tau_i = \frac{1}{\sqrt{2}} (1 + b_{i+1} b_i + b_{i+1}^\dagger b_i + b_{i+1} b_i^\dagger + b_{i+1}^\dagger b_i^\dagger), \quad (20)$$

and so

$$\begin{aligned} \tau_i |n_1, \dots, n_i, n_{i+1}, \dots, n_N\rangle = & \frac{1}{\sqrt{2}} \{ |n_1, \dots, n_i, n_{i+1}, \dots, n_N\rangle + |n_1, \dots, (n_i - 1), (n_{i+1} - 1), \dots, n_N\rangle + |n_1, \dots, (n_i - 1), (n_{i+1} + 1), \dots, n_N\rangle \\ & - |n_1, \dots, (n_i + 1), (n_{i+1} - 1), \dots, n_N\rangle - |n_1, \dots, (n_i + 1), (n_{i+1} + 1), \dots, n_N\rangle \}. \end{aligned} \quad (21)$$

Here we understand that when the  $\pm 1$  takes the occupation number  $n_j$  out of the set  $\{0,1\}$  the illegal state is to be replaced by zero.

We next define  $T_i^{(0)}$  to be the operation of interchanging vortex  $i$  with its sibling antivortex. (A vortex and an antivortex being distinguishable, this operation does not return the system to its original configuration and hence cannot be considered as a braiding operation.) The corresponding unitary operator is

$$\tau_i^{(0)} = \frac{1}{\sqrt{2}}(1 + \gamma_{i+N}\gamma_i) = \frac{1}{\sqrt{2}}(1 + i(b_i^\dagger b_i - b_i b_i^\dagger)). \quad (22)$$

From this, we find

$$\tau_i^{(0)} |n_1, \dots, n_i, \dots, n_N\rangle = e^{i\pi n_i/4} |n_1, \dots, n_i, \dots, n_N\rangle. \quad (23)$$

One can also consider an operation involving interchanging vortex  $i$  with the antivortex of its partner,  $i+1$ , to the right. We will call this operation  $T_i^{(1)}$ . The corresponding operator is

$$\tau_i^{(1)} = \frac{1}{\sqrt{2}}(1 - ib_{i+1}b_i + ib_{i+1}^\dagger b_i - ib_{i+1}b_i^\dagger + ib_{i+1}^\dagger b_i^\dagger), \quad (24)$$

and this acts as

$$\begin{aligned} \tau_i^{(1)} |n_1, \dots, n_i, n_{i+1}, \dots, n_N\rangle = & \frac{1}{\sqrt{2}} \{ |n_1, \dots, n_i, n_{i+1}, \dots, n_N\rangle - i |n_1, \dots, (n_i - 1), (n_{i+1} - 1), \dots, n_N\rangle + i |n_1, \dots, (n_i - 1), (n_{i+1} \\ & + 1), \dots, n_N\rangle + i |n_1, \dots, (n_i + 1), (n_{i+1} - 1), \dots, n_N\rangle - i |n_1, \dots, (n_i + 1), (n_{i+1} + 1), \dots, n_N\rangle \}. \end{aligned} \quad (25)$$

A strategy for accessing the protected information is now apparent. The wave function of the protected state  $|\Psi\rangle$  belonging to the even (or odd) fermion number sector is specified by the  $2^{N-1}$  complex numbers forming the its components in the occupation-number basis  $|n_1, \dots, n_N\rangle$ . Provided that we can have access to multiple copies of  $|\Psi\rangle$ , we can repeatedly fuse the vortex-antivortex pairs and so estimate the probability of a particular pattern  $(n_1, \dots, n_N)$  of relict particles. In this way we obtain the numbers  $|\langle\Psi|n_1, \dots, n_N\rangle|^2$ . We lose all relative phase information in this process, however. This loss occurs not only because we are finding probabilities, but also because the different occupation-number states have unpredictably different tunneling energies, and so time evolution will scramble their relative phases during the fusion process.

Not all is lost, however, (21) and (25) seem to suggest that we can use direct fusion combined with controlled braiding  $\tau_i$  and interchange  $\tau_i^{(1)}$  to find all three of the numbers

$$|A_{01}|^2 \equiv |\langle\Psi|n_1, \dots, 0, 1, \dots, n_N\rangle|^2,$$

$$|A_{10}|^2 \equiv |\langle\Psi|n_1, \dots, 1, 0, \dots, n_N\rangle|^2,$$

$$\begin{aligned} |A_{10} + A_{10}|^2 \equiv & |\langle\Psi|n_1, \dots, 0, 1, \dots, n_N\rangle \\ & + \langle\Psi|n_1, \dots, 1, 0, \dots, n_N\rangle|^2, \end{aligned}$$

$$\begin{aligned} |A_{01} + iA_{10}|^2 \equiv & |\langle\Psi|n_1, \dots, 0, 1, \dots, n_N\rangle \\ & + i\langle\Psi|n_1, \dots, 1, 0, \dots, n_N\rangle|^2. \end{aligned}$$

If this is possible, we would be able to obtain all relative phases of  $\langle\Psi|n_1, \dots, n_N\rangle$ 's, for given  $|z_1|^2$ ,  $|z_2|^2$ ,  $|z_1 + z_2|^2$ , and  $|z_1 + iz_2|^2$ , it is possible to recover the complex numbers  $z_1$  and  $z_2$  up to a common phase factor.

This is not quite the case, however. A vortex-antivortex interchange would leave one vortex-vortex pair and one antivortex-antivortex pair. These pairs are problematic, because the excited state of this pair would not lead to an unbound Bogoliubov quasiparticle excitation, but rather a bound excited state of a ‘‘double half-quantum’’ vortex (or antivortex). This means that fusing such pairs would not give us a result that we can read out. In short,  $|z_1 + iz_2|^2$  is not available to us.

Nevertheless, there is still much we can figure out about relative phases of  $\langle\Psi|n_1, \dots, n_N\rangle$ 's. For  $N$  complex numbers,  $z_i$ , if one knows all their absolute values and distance between each other,  $|z_i - z_j|$ , for all  $i, j$  (or equivalently  $|z_i + z_j|$ ), the geometric configuration of  $z_i$ 's are determined *rigidly*. By this we mean that we have determined  $z_i$ 's up to overall rotation around the origin and reflection with respect to the real axis (that is, complex conjugation). It needs to be noted that knowing  $|z_i \pm z_j|$  for all  $i, j$  actually overdetermines  $z_i$ 's. So it turns out that even though direct fusion combined with

controlled braiding  $\tau_i$  only gives us a subset of  $\{|z_i \pm z_j\rangle\}$ , it is still sufficient for us to determine  $\langle \Psi | n_1, \dots, n_N \rangle$ 's up to complex conjugation and overall phase. The exact procedure will be given in the Appendix. (One needs to be note that if there is a superposition of even total occupation number states and odd total occupation number states, there is no way to obtain phase relation between them, which reflects the fact that, as Ivanov pointed out,<sup>17</sup> the superconducting Hamiltonian creates or destroys electrons only in pairs.)

#### IV. CONCLUSIONS AND OPEN QUESTIONS

Half-quantum vortices in the  $p_x+ip_y$  superconductors provide, at least in principle, a way to generate and manipulate entangled states in a topologically protected Hilbert space. We have shown that by physically fusing vortices with appropriate antivortices it is possible to reconnect the protected space to the rest of the Hilbert space and so read out the information encoded there.

A number of questions remain, however: (i) We have assumed that the fusion process is slow enough that the  $E_i$  bound state merges adiabatically with the continuum. What happens if the process is too fast? Can we find a tractable model for the annihilation process that would allow us to determine how adiabatic it has to be in order not to lose information? (ii) Our picture of generating and moving vortices is at present only a thought experiment. The necessary half-quantum vortices have not even been detected in any real system. If they can be found, can we come up with some practical device (a configuration of STM probes, current sources, drains, gates, etc.) that can create, guide, and monitor the vortices? Since no experiment is likely to be able to detect a single quasi-particle, we would have arrange for a continuous operation and measure currents; (iii) Perhaps the most interesting question is whether we take the insights developed from the relatively simple picture of braiding and fusion in the superconductor and extend them to the other candidate systems with non-abelian statistics.

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#### APPENDIX A: BERRY PHASES AND BOGOLIUBOV TRANSFORMATIONS

In this Appendix we review the formal algebraic aspects of Bogoliubov transformations, and their implications for the computation of Berry phases for BCS states.

##### 1. BCS ground state

Suppose that  $H_{ij}$  is an  $N$ -by- $N$  matrix representing a one-particle Hamiltonian. When we include the effect of a superconducting condensate, the second-quantized Bogoliubov Hamiltonian becomes

$$\begin{aligned} \hat{H}_{\text{Bogoliubov}} &= a_i^\dagger H_{ij} a_j + \frac{1}{2} \Delta_{ij} a_i^\dagger a_j^\dagger + \frac{1}{2} \Delta_{ij}^\dagger a_i a_j \\ &= \frac{1}{2} (a_i^\dagger \ a_i) \begin{pmatrix} H_{ij} & \Delta_{ij} \\ \Delta_{ij}^\dagger & -H_{ij}^T \end{pmatrix} \begin{pmatrix} a_j \\ a_j^\dagger \end{pmatrix} + \frac{1}{2} \text{tr} H. \end{aligned} \quad (\text{A1})$$

Here  $a_i^\dagger$  and  $a_i$  are fermion creation and annihilation operators, the gap function  $\Delta_{ij}$  is a skew symmetric matrix, and  $H^T$  denotes the transpose of the Hermitian matrix  $H$ . For a continuum superconductor, the index “ $i$ ” should be understood to incorporate both the space co-ordinate  $x$ , and the spin index. A sum over  $i$ , therefore, implies both an integral over real space and a sum over spin components.

The many-body Hamiltonian is diagonalized by means of a Bogoliubov transformation. To construct this transformation we begin by solving the single-particle Bogoliubov-de Gennes (BdG) eigenvalue problem

$$\begin{pmatrix} H & \Delta \\ \Delta^\dagger & -H^T \end{pmatrix} \begin{pmatrix} u_\alpha \\ v_\alpha \end{pmatrix} = E_\alpha \begin{pmatrix} u_\alpha \\ v_\alpha \end{pmatrix}. \quad (\text{A2})$$

Here  $u_\alpha$  and  $v_\alpha$  are  $N$ -dimensional column vectors, which we take to be normalized so that  $|u_\alpha|^2 + |v_\alpha|^2 = 1$ . If we explicitly display the column-vector index  $i$  they become matrices  $u_{i\alpha}$  and  $v_{i\alpha}$ . Taking the complex conjugate of (A2) tells us that

$$\begin{pmatrix} H & \Delta \\ \Delta^\dagger & -H^T \end{pmatrix} \begin{pmatrix} v_\alpha^* \\ u_\alpha^* \end{pmatrix} = -E_\alpha \begin{pmatrix} v_\alpha^* \\ u_\alpha^* \end{pmatrix}, \quad (\text{A3})$$

and so the BdG eigenvalues come in  $\pm$  pairs. We will always take  $E_\alpha$  to be the positive eigenvalue.

We now set

$$\begin{aligned} b_\alpha &= u_{i\alpha} b_\alpha + v_{i\alpha}^* b_\alpha^\dagger \\ a_i^\dagger &= v_{i\alpha} b_\alpha + u_{i\alpha}^* b_\alpha^\dagger. \end{aligned} \quad (\text{A4})$$

The mutual orthonormality and completeness of the eigenvectors  $(u_\alpha, v_\alpha)^T$  ensures that the  $b_\alpha, b_\alpha^\dagger$  have the same anti-commutation relations as the  $a_i, a_i^\dagger$ . In terms of the  $b_\alpha, b_\alpha^\dagger$ , the second-quantized Hamiltonian becomes

$$\hat{H}_{\text{Bogoliubov}} = \sum_{\alpha=1}^N E_\alpha b_\alpha^\dagger b_\alpha - \frac{1}{2} \sum_{\alpha=1}^N E_\alpha + \frac{1}{2} \sum_{i=1}^N E_i^{(0)}. \quad (\text{A5})$$

Here the  $E_i^{(0)}$  are the eigenvalues of  $H$ . These can be of either sign.

If all the  $E_\alpha$  are *strictly* positive, the new ground state is nondegenerate and is the unique state  $|0\rangle_b$  annihilated by all the  $b_\alpha$ . If we could find a unitary operator  $U$  such that

$$\begin{aligned} b_\alpha &= a_i u_{i\alpha}^* + a_i^\dagger v_{i\alpha}^* = U a_\alpha U^{-1} \\ b_\alpha^\dagger &= a_i^\dagger u_{i\alpha} + a_i v_{i\alpha} = U a_\alpha^\dagger U^{-1} \end{aligned} \quad (\text{A6})$$

then we would have  $|0\rangle_b = U|0\rangle_a$ , where  $|0\rangle_a$  is the no-particle vacuum state. It is not easy to find a closed-form expression for  $U$ , however. An alternative strategy for obtaining  $|0\rangle_b$  begins by noting that if that the matrix  $u_{i\alpha}$  is invertible then the condition  $b_i|0\rangle_b = 0$  is equivalent to

$$(a_i + a_k^\dagger v_{k\alpha}^* (u^{*-1})_{\alpha i}) |0\rangle_b = 0, i = 1, \dots, N. \quad (\text{A7})$$

We, therefore, introduce the skew-symmetric matrix

$$S_{ij} = v_{i\alpha}^* (u^*)_{\alpha j}^{-1} \quad (\text{A8})$$

and observe that

$$\exp\left\{\frac{1}{2}a_i^\dagger a_j^\dagger S_{ij}\right\} a_k \exp\left\{-\frac{1}{2}a_i^\dagger a_j^\dagger S_{ij}\right\} = a_k + a_i^\dagger S_{ik}. \quad (\text{A9})$$

From this we conclude that

$$|0\rangle_b = N \exp\left\{\frac{1}{2}a_i^\dagger a_j^\dagger S_{ij}\right\} |0\rangle_a \quad (\text{A10})$$

where  $|0\rangle_a$  is the original-no particle state. Equation (A10) explicitly displays the superconducting ground state as a coherent superposition of Cooper-pair states, and allows us to identify  $S_{ij}$  with the (unnormalized) pair wave function.

The normalization factor  $\mathcal{N}$  is found from

$$\langle S_1 | S_2 \rangle = \det^{1/2}(I + S_1^\dagger S_2), \quad (\text{A11})$$

where

$$|S\rangle = \exp\left\{\frac{1}{2}a_i^\dagger a_j^\dagger S_{ij}\right\} |0\rangle_a, \quad (\text{A12})$$

to be

$$\mathcal{N} = \det^{-1/4}(I + S^\dagger S). \quad (\text{A13})$$

Because the group of Bogoliubov transformations on  $a_i, a_i^\dagger, i=1, \dots, N$  is  $\text{SO}(2N)$ , and because the subgroup  $U(N) \simeq \text{Sp}(2N, \mathbb{R}) \cap \text{SO}(2N)$  of transformations that mix the  $a$ 's only with themselves (i.e., not with the  $a^\dagger$ 's), preserves the no-particle vacuum  $|0\rangle_a$ , the set of physically distinct ground states is parameterized by the symmetric space  $\text{SO}(2N)/U(N)$ . As a check of this assertion, observe that

$$\dim \text{SO}(2N) - \dim U(N) = N(2N-1) - N^2 = N(N-1), \quad (\text{A14})$$

which is the number of independent real parameters in the complex skew-symmetric matrix  $S_{ij}$ . These  $S_{ij}$  for  $i < j$  serve as complex co-ordinates on all but a set of measure zero in the manifold of possible ground states.

## 2. Clifford algebra and Lie [SO(2N)]

We can make the  $\text{SO}(2N)$  character of the Bogoliubov transformations manifest by introducing a set of  $2N$  Dirac gamma operators. These are related to the fermion annihilation and creation operators by

$$\begin{aligned} \gamma_i &= (a_i + a_i^\dagger) \\ \gamma_{i+N} &= i(a_i^\dagger - a_i). \end{aligned} \quad (\text{A15})$$

The  $\gamma_i$  are Hermitian, and obey the Clifford algebra

$$\{\gamma_i, \gamma_j\} \equiv \gamma_i \gamma_j + \gamma_j \gamma_i = 2\delta_{ij}. \quad (\text{A16})$$

The Hamiltonian can be rewritten in terms of the  $\gamma_i$  as

$$\hat{H}_{\text{Bogoliubov}} = \frac{1}{2} \sum_{i,j=1}^{2N} h_{ij} \Gamma_{ij} + \frac{1}{2} \text{tr } H \quad (\text{A17})$$

where

$$\Gamma_{ij} = \frac{1}{4i} [\gamma_i, \gamma_j] \quad (\text{A18})$$

are the spinor generators of the Lie algebra of  $\text{SO}(2N)$ , and the matrix  $h_{ij}$  has entries

$$h_{ij} = \begin{pmatrix} -\Im H - \Im \Delta & -\Re H + \Re \Delta \\ \Re H + \Re \Delta & -\Im H + \Im \Delta \end{pmatrix}_{ij}. \quad (\text{A19})$$

Here  $\Re Z \equiv X$  and  $\Im Z \equiv Y$  denote the real and imaginary parts of  $Z = X + iY$ , and the vector space has been partitioned so that  $i=1, \dots, N$  is the first block and  $i=N+1, \dots, 2N$  is the second. This rewriting reveals that a general Bogoliubov Hamiltonian is, up to an additive constant, an element of the Lie algebra of  $\text{SO}(2N)$ . The Bogoliubov transformation that diagonalizes  $H$  is, therefore, the operation of conjugating the Lie algebra element  $\hat{H}$  into the Cartan sub-algebra. This we can take to be spanned by the commuting set of operators

$$\frac{1}{4i} [\gamma_{N+i}, \gamma_i] = \frac{1}{2} (a_i^\dagger a_i - a_i a_i^\dagger), \quad i = 1, \dots, N. \quad (\text{A20})$$

After conjugation,

$$\hat{H}_{\text{Bogoliubov}} \rightarrow U \hat{H}_{\text{Bogoliubov}} U^{-1}, \quad (\text{A21})$$

with

$$U = \exp\left\{\frac{i}{2} \sum_{i,j} \theta_{ij} \Gamma_{ij}\right\} \quad (\text{A22})$$

for some parameters  $\theta_{ij}$ , the Hamiltonian becomes

$$\begin{aligned} \hat{H}_{\text{Bogoliubov}} &\rightarrow \sum_{i=1}^N E_\alpha \frac{1}{4i} [\gamma_{N+\alpha}, \gamma_\alpha] + \frac{1}{2} \sum_{i=1}^N E_i^{(0)} \\ &= \frac{1}{2} \sum_{i=1}^N E_\alpha (a_\alpha^\dagger a_\alpha - a_\alpha a_\alpha^\dagger) + \frac{1}{2} \sum_{i=1}^N E_i^{(0)} \\ &= \sum_{\alpha=1}^N E_\alpha a_\alpha^\dagger a_\alpha - \frac{1}{2} \sum_{\alpha} E_\alpha + \frac{1}{2} \sum_i E_i^{(0)}, \end{aligned} \quad (\text{A23})$$

which is the same as (A5). Again it is convenient to regard the energies  $E_\alpha$  as being positive, even though the  $E_i^{(0)}$ , the eigenvalues of  $H$ , can have either sign. The additive constant is then the ground-state energy of superconducting system.

## 3. Zero modes

The gamma-operator language is particularly useful when there are zero modes. Because of the  $\pm E$  symmetry, any zero energy eigenvectors of the BdG Hamiltonian must come in pairs. Suppose that  $(u_0, v_0)^T$  becomes degenerate with its negative energy sibling  $(v_0^*, u_0^*)^T$ . Then we can write this pair of eigenvectors' contribution to the mode expansion as

$$\begin{aligned} \begin{pmatrix} a_i \\ a_i^\dagger \end{pmatrix} &= \begin{pmatrix} u_{i0} \\ v_{i0} \end{pmatrix} b_0 + \begin{pmatrix} v_{i0}^* \\ u_{i0}^* \end{pmatrix} b_0^\dagger + \dots \\ &= \begin{pmatrix} U_{i0} \\ V_{i0} \end{pmatrix} \frac{\gamma_0}{\sqrt{2}} + \begin{pmatrix} U_{iN} \\ V_{iN} \end{pmatrix} \frac{\gamma_N}{\sqrt{2}} + \dots, \end{aligned} \quad (\text{A24})$$

where

$$\begin{aligned} \begin{pmatrix} U_{i0} \\ V_{i0} \end{pmatrix} &= \frac{1}{\sqrt{2}} \left[ \begin{pmatrix} u_{i0} \\ v_{i0} \end{pmatrix} + \begin{pmatrix} v_{i0}^* \\ u_{i0}^* \end{pmatrix} \right], \\ \begin{pmatrix} U_{iN} \\ V_{iN} \end{pmatrix} &= \frac{i}{\sqrt{2}} \left[ \begin{pmatrix} u_{i0} \\ v_{i0} \end{pmatrix} - \begin{pmatrix} v_{i0}^* \\ u_{i0}^* \end{pmatrix} \right], \end{aligned} \quad (\text{A25})$$

and

$$\gamma_0 = (b_0 + b_0^\dagger), \quad \gamma_N = i(b_0^\dagger - b_0). \quad (\text{A26})$$

The column vectors  $(U_0, V_0)^T$  and  $(U_N, V_N)^T$  both have the feature that  $U^* = V$ . This anti-linear up-down symmetry is characteristic of the localized zero modes in the vortex cores.

#### 4. Berry connection

We need to compute the Berry connection  $iA = \langle \tilde{S} | d | \tilde{S} \rangle$ , where  $|\tilde{S}\rangle = \mathcal{N} | S \rangle$  is the normalized ground state. To do this we exploit the fact that the un-normalized states  $|S\rangle$ , being functions only of the  $S_{ij}$ , and not of the  $S_{ij}^*$ , defines a holomorphic line-bundle over the Kähler manifold  $\text{SO}(2N)/\text{U}(N)$ . We can therefore read off the one-form connection  $iA$  from derivatives of the Kähler potential

$$\ln \mathcal{N} = -\frac{1}{4} \ln \det(I + S^\dagger S) \quad (\text{A27})$$

as  $iA = \bar{\partial} \ln \mathcal{N} - \partial \ln \mathcal{N}$ .<sup>27</sup> If we express the parameters  $S_{ij}$  in terms of the normalized Bogoliubov eigenvectors  $(u, v)^T$ , we find that

$$\begin{aligned} iA &= \sum_{i < j} \left( \frac{\partial \ln \mathcal{N}}{\partial S_{ij}^*} dS_{ij}^* - \frac{\partial \ln \mathcal{N}}{\partial S_{ij}} dS_{ij} \right) = \frac{1}{2} \sum_{\alpha=1}^N (v_\alpha u_\alpha) d \begin{pmatrix} v_\alpha^* \\ u_\alpha^* \end{pmatrix} \\ &\quad + \frac{i}{2} d \{ \text{Arg}(\det u) \}. \end{aligned} \quad (\text{A28})$$

The expression (A28) has a simple interpretation. From (A3) we see that the column vectors  $(v_\alpha^*, u_\alpha^*)^T$  are the negative-energy eigenstates of the one-particle Bogoliubov-de-Gennes Hamiltonian

$$H_{\text{BdG}} = \begin{pmatrix} H & \Delta \\ \Delta^\dagger & -H^T \end{pmatrix}. \quad (\text{A29})$$

If this were a Dirac fermion problem, we would fill the Dirac sea consisting of these negative energy states. The Berry phase of the vacuum would then be the sum of the Berry phases of the occupied states. The first term in (A28) is precisely one-half of this sum. The factor of one-half compensates for the artificial doubling of the degrees of freedom in passing from  $H$  to  $H_{\text{BdG}}$ . The second term in (A28) is a total derivative, and reflects a choice of gauge.

We next compute the Berry connection for an excited state

$$|\alpha_1, \dots, \alpha_n\rangle = b_{\alpha_1}^\dagger \dots b_{\alpha_n}^\dagger |\tilde{S}\rangle, \quad (\text{A30})$$

by using

$$\begin{aligned} db_{\alpha}^\dagger &= (u_{i\beta} b_{\beta} + v_{i\beta} b_{\beta}^\dagger) dv_{i\alpha} + (v_{i\beta} b_{\beta} + u_{i\beta} b_{\beta}^\dagger) du_{i\alpha} \\ &= (v_{i\beta}^* dv_{i\alpha} + u_{i\beta}^* du_{i\alpha}) b_{\beta}^\dagger + (u_{i\beta} dv_{i\alpha} + v_{i\beta} du_{i\alpha}) b_{\beta}. \end{aligned} \quad (\text{A31})$$

When the state  $|\alpha_1, \dots, \alpha_n\rangle$  is nondegenerate, we are interested only in the diagonal  $\beta = \alpha$  term

$$d b_{\alpha}^\dagger |_{\text{diag}} = (v_{i\alpha}^* dv_{i\alpha} + u_{i\alpha}^* du_{i\alpha}) b_{\alpha}^\dagger \quad (\text{no sum on } \alpha), \quad (\text{A32})$$

and we find

$$\begin{aligned} iA &= \langle \alpha_1, \dots, \alpha_n | d | \alpha_1, \dots, \alpha_n \rangle \\ &= \langle \tilde{S} | b_{\alpha_n} \dots b_{\alpha_1} d (b_{\alpha_1}^\dagger \dots b_{\alpha_n}^\dagger | \tilde{S} \rangle) \\ &= \langle \tilde{S} | d | \tilde{S} \rangle + \langle \tilde{S} | b_{\alpha_n} \dots b_{\alpha_1} d (b_{\alpha_1}^\dagger \dots b_{\alpha_n}^\dagger) | \tilde{S} \rangle \\ &= \langle \tilde{S} | d | \tilde{S} \rangle + \sum_{m=1}^n (u_{\alpha_m}^* v_{\alpha_m}^*) d \begin{pmatrix} u_{\alpha_m} \\ v_{\alpha_m} \end{pmatrix}, \end{aligned} \quad (\text{A33})$$

which is the sum of the many-body ground-state Berry connection and the Berry connections of the individual one-particle excited states. Observe that there is no factor of “1/2” in the contribution of these occupied excited states.

The non-abelian Berry connection of a set of degenerate many-body states is computed in the same manner. It will include a diagonal term from the reference state  $|\tilde{S}\rangle$  and a sum of nondiagonal terms of the form

$$iA_{\beta\alpha} = (u_{\beta}^* v_{\beta}^*) d \begin{pmatrix} u_{\alpha} \\ v_{\alpha} \end{pmatrix}. \quad (\text{A34})$$

In the case of the vortices, the states of interest are the exponentially localized core states. Because of this localization only the overlap of each core state with itself has any chance of providing a nonzero term in the connection, but it is readily verified that with the phase choices made in the text, all these contributions are zero. The only “Berry phase” produced by the vortex braiding is the overall diagonal Berry phase associated with the Magnus force<sup>28</sup>

#### APPENDIX B: DETERMINING PHASE RELATION AMONG COEFFICIENTS OF OCCUPATION NUMBER BASIS

Let us consider how many vortex interchange steps would be needed in order to *rigidly*—this rigidity being defined in the last page of the Sec. III—determine geometric configuration on the complex plane of all  $\langle \Psi | n_1, \dots, n_N \rangle$ 's, with  $|\Psi\rangle$  having a definite parity in total occupation number.

In the case  $N=2$  one can easily see from

$$\tau_1(A_{00}|00\rangle + A_{11}|11\rangle) = \frac{1}{\sqrt{2}} \{ (A_{00} + A_{11})|00\rangle - (A_{00} - A_{11})|11\rangle \},$$

$$\tau_1(A_{01}|01\rangle + A_{10}|10\rangle) = \frac{1}{\sqrt{2}}\{(A_{01} + A_{10})|01\rangle - (A_{01} - A_{10})|10\rangle\} \quad (\text{B1})$$

that the fusion following the vortex interchange process gives us, up to the sign, phase difference between  $A_{00}$  and  $A_{11}$  (and likewise between  $A_{01}$  and  $A_{10}$ ). Given that we already know  $|A_{00}|$  and  $|A_{11}|$ , this is sufficient to rigidly determine on the complex plane the configuration of  $A_{00}$  and  $A_{11}$ . Same can be said for  $A_{01}$  and  $A_{10}$ .

It is instructive to work out the next simplest case  $N=3$ , where  $|\Psi\rangle = A_{000}|000\rangle + A_{110}|110\rangle + A_{011}|011\rangle + A_{101}|101\rangle$ . From  $N=2$  case, one can see that  $\tau_1$  rigidly determines the configuration of  $A_{000}$  and  $A_{110}$  on the complex plane. It also rigidly determines the configuration of  $A_{011}$  and  $A_{101}$  on the complex plane. However any phase relation between the former and latter remains completely unknown at this point. But then from

$$\tau_2|\Psi\rangle = \frac{1}{\sqrt{2}}\{(A_{000} + A_{011})|000\rangle - (A_{101} - A_{110})|110\rangle - (A_{000} - A_{011})|011\rangle + (A_{101} + A_{110})|101\rangle\}, \quad (\text{B2})$$

one can see that after fusion following the implementation of  $\tau_2$ , the configuration of both the  $A_{000}, A_{011}$  pair and the  $A_{101}, A_{110}$  pair would be determined rigidly with respect to the origin. We now have a rigid configuration of all four coefficients— $A_{000}, A_{011}, A_{101}$ , and  $A_{110}$ —on the complex plane; they have been determined up to complex conjugation and overall phase.

The  $N=3$  case gives us ideas about how to make use of a recursion argument for the general case in figuring out the phase relation. One can assume that for  $N=m$  there is some process consisting of fusion of vortex-antivortex pairs and applications of  $\tau_k$ 's (where  $k \leq m-1$ ) which give us the rigid configuration on the complex plane of the coefficient of states with even total occupation number. The same process would also give us the rigid configuration on the complex plane of the coefficient of states with odd total occupation number as well. (We have seen that this holds true for  $m=2$ .) Now note that the even (or odd) total occupation number sector of the Hilbert space in  $N=m+1$  case can be divided into the following two classes:

$$|n_1, \dots, n_m, 0\rangle, \quad (\text{B3})$$

$$|n'_1, \dots, n'_m, 1\rangle. \quad (\text{B4})$$

The operations that were used in the procedure we have applied for obtaining all phase relation in the  $N=m$  case does not affect  $n_{m+1}$ , and so by applying these operations we would obtain the rigid configuration for the coefficients of states belonging to (B3). Same can be said for the coefficients of the states of belonging to (B4) (though  $|n_1, \dots, n_m\rangle$  and  $|n'_1, \dots, n'_m\rangle$  have opposite parity in total occupation number). Now all we need to do is to figure out the rigid configuration of two pairs of coefficients, each of which consists of one coefficient for one of the states belonging to (B3) and one coefficient for one of the states belonging to (B4). So for  $|\Psi\rangle = \sum A_{n_1, \dots, n_m, n_{m+1}} |n_1, \dots, n_m, n_{m+1}\rangle$ , (note that summation is restricted to even, or odd, total occupation number)

$$\begin{aligned} \tau_m|\Psi\rangle &= \tau_m\left(\sum A_{n_1, \dots, n_m, 0} |n_1, \dots, n_m, 0\rangle + \sum A_{n'_1, \dots, n'_m, 1} |n'_1, \dots, n'_m, 1\rangle\right) = \frac{1}{\sqrt{2}}\left\{\sum (A_{n_1, \dots, n_{m-1}, 0, 0} + A_{n_1, \dots, n_{m-1}, 1, 1}) |n_1, \dots, n_{m-1}, 0, 0\rangle \right. \\ &\quad - (A_{n_1, \dots, n_{m-1}, 0, 0} - A_{n_1, \dots, n_{m-1}, 1, 1}) |n_1, \dots, n_{m-1}, 1, 1\rangle\left.\right\} + \frac{1}{\sqrt{2}}\left\{\sum (A_{n'_1, \dots, n'_{m-1}, 0, 1} + A_{n'_1, \dots, n'_{m-1}, 1, 0}) |n'_1, \dots, n'_{m-1}, 0, 1\rangle \right. \\ &\quad \left. - (A_{n'_1, \dots, n'_{m-1}, 0, 1} - A_{n'_1, \dots, n'_{m-1}, 1, 0}) |n'_1, \dots, n'_{m-1}, 1, 0\rangle\right\}. \quad (\text{B5}) \end{aligned}$$

One can easily see that all pairs belonging to  $(A_{n_1, \dots, n_{m-1}, 0, 0}, A_{n_1, \dots, n_{m-1}, 1, 1})$  or  $(A_{n'_1, \dots, n'_{m-1}, 0, 1}, A_{n'_1, \dots, n'_{m-1}, 1, 0})$  now have rigid configuration. It is clear that we now have rigid configuration for  $\{A_{n_1, \dots, n_{m-1}, n_m, n_{m+1}}\}$ . Also we can see that in this scheme of figuring out the rigid configuration of  $2^{N-1}$  coefficients of even (or odd) fermion sector in the occupation-number basis,  $N-1$  vortex interchange steps are needed

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