PHYSICAL REVIEW B **73**, 014503 (2006)

# Proximity effect and Josephson current in clean strong/weak/strong superconducting trilayers

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(Received 19 July 2005; revised manuscript received 19 October 2005; published 3 January 2006)

Recent measurements of the Josephson critical current through LSCO/LCO/LSCO thin films showed an unusually large proximity effect. Using the Bogoliubov–de Gennes equations for a tight-binding Hamiltonian we describe the proximity effect in weak links between a superconductor with critical temperature $T_c$ and one with critical temperature $T_c'$, where $T_c > T_c'$. The weak link $(N')$ is therefore a superconductor above its own critical temperature and the superconducting regions are considered to have either $s$-wave or $d$-wave symmetry. We note that the proximity effect is enhanced due to the presence of superconducting correlations in the weak link. The dc Josephson current is calculated, and we obtain a nonzero value for temperatures greater than $T_c'$ for sizes of the weak links that can be almost an order of magnitude greater than the conventional coherence length. Considering pockets of superconductivity in the $N'$ layer, we show that this can lead to an even larger effect on the Josephson critical current by effectively shortening the weak link.

DOI: 10.1103/PhysRevB.73.014503 PACS number(s): 74.45.+c, 74.50.+r

## I. INTRODUCTION

The proximity effect between a superconductor and a normal metal has been thoroughly investigated using various techniques: Ginzburg-Landau theory,[2,3] quasiclassical Green function methods,[4] Gorkov equation methods,[5–8] and tight-binding Bogoliubov–de Gennes (BdG) methods.[9–11] From an experimental point of view, one of the better suited experiments is the measurement of the Josephson critical current in weak links.[12]

In a recent experiment[1] an unusually large proximity effect is reported, and the authors argue, that it cannot be explained by the conventional proximity effect. The system used in the experiment is a $c$-axis oriented one. The $c$-axis Josephson critical current is measured through a thin film system made of doped LCO $(La_2CuO_{4-\delta})$ with $T_c = 25$ K, sandwiched between optimally doped LSCO $(La_{2-x}Sr_xCuO_4)$ with $T_c = 45$ K. The thin film is considered to be in the clean limit and because of the epitaxial growth of the films the transmission at the interfaces is close to unity and interface roughness is on the order of the lattice constant. In a particular setup, the LCO thin film used had a thickness of 100 Å. Fitting the critical current around $T_c'$ the authors extract a coherence length in the LCO film which is two orders of magnitude larger than expected. Because of this discrepancy and the observation of nonzero critical current for $T < 30$ K the authors reported this effect to be a “giant proximity effect.”

Although the Josephson junction has been thoroughly investigated in the past for both $s$-wave[13] and $d$-wave symmetries,[14–19] we feel that the calculation of the Cooper pair leaking distance in the case of clean limit and superconducting weak links needs further investigation. We are interested to observe if the leaking distance will be influenced by the finite critical temperature of the weak superconductor.

We propose the use of the numerical solutions of the BdG equations in a tight-binding formulation in order to obtain a direct calculation of the coherence length and of the Josephson critical current. In the clean limit the BdG equations are particularly easy to solve because impurity averaging is not required. This method is complementary to the quasiclassical methods used in the dirty limit, namely, the Usadel equations.[20,21]

For coherent transport in the $c$-axis direction, the properties of the Josephson current for $d$-wave superconductors will be similar to the properties of the current for $s$-wave superconductors. For planar interfaces, with $\hat{z}$ the direction perpendicular to the interfaces, the $d$-wave order parameter will have no $k_z$ dependence, $\Delta(k_x,k_y,k_z) \sim \Delta_0[\cos(k_x a) - \cos(k_y a)]$ and therefore will have properties similar to a superconductor with $s$-wave symmetry. When Fourier transforming the $\hat{x}$ and $\hat{y}$ directions, and considering an effective one-dimensional (1D) problem in the $\hat{z}$ direction, the $d$-wave order parameter will be due to an effective on-site interaction within each $a$-$b$ plane. We will calculate the Josephson current in the $c$-axis direction for a 3D $d$-wave superconductor. We will also show calculations of the Josephson critical current and the Cooper pair leaking distance for a 2D $s$-wave superconductor and for the 100 interface of a 2D $d$-wave superconductor.

The giant proximity effect is observed in underdoped cuprates, for temperatures $T > T_c'$ for which the middle layer is considered to be in the pseudogap state. Previous theoretical investigations of the giant proximity effect[22,23] considered the $N'$ layer to be comprised of pockets of superconductivity. In a recent theoretical study,[24] interstitial oxygen dopants are considered to modify locally the pairing interactions. The disordered dopants are enhancing the pairing interactions, thus increasing the size of the local gap. This was observed in recent scanning tunneling microscopy experiments[25] in $Bi_2Sr_2CaCu_2O_{8+\delta}$ which showed that the regions of enhanced superconductivity are correlated with the positions of the interstitial oxygen atoms. Because of the proximity effect, the superconducting pockets will be coupled and current will flow through percolating paths. The presence of these pockets will effectively shorten the length of the weak link and the strong external superconductors will be coupled for values of the effective length comparable with the leaking distance. The modification of the leaking distance due to the finite value of $T_c'$ will have an important influence on the effective length. Considering equally spaced areas of strong superconductivity with critical temperature $T_c$ embedded in the weak superconductor with $T_c'$ we will calculate the critical Josephson current and find its dependence on the length of the weak link and on the volume of the embedded superconducting pockets. If one considers disordered regions of strong superconductivity in the $a-b$ planes of a high-$T_c$ superconductor, then the distance between two pockets from different Cu-O planes will be normally distributed. The equally spaced pockets scenario should be the one that gives maximal Josephson current and will give insight about the influence of these pockets on the current.

This paper is organized as follows. In the next section we will present our method. While the BdG procedure on a lattice is now well known, we nonetheless include some details, as some "standard" approximations are included for clarity. The treatment of infinite surfaces will be outlined. In the third section we apply the BdG equations to a trilayer system and show results of the calculation of the order parameter, the leaking distance, and the dc Josephson current. Both the cases of $s$-wave and $d$-wave symmetries of the superconducting order parameters are considered. We find that the proximity effect can be considerably enhanced at temperatures close to (but above) the critical temperature of the weak superconductor. The presence of randomly distributed pockets of superconductivity in $N'$ enhances dramatically the Josephson critical current and leads to a "giant proximity effect."

## II. METHOD

In order to describe the superconducting state we use the tight-binding extended Hubbard Hamiltonian:

$$\mathcal{H} = -\sum_{\langle ij\rangle\sigma} t_{ij} c_{i\sigma}^\dagger c_{j\sigma} - \mu \sum_{i\sigma} c_{i\sigma}^\dagger c_{i\sigma} + \sum_i U_i n_{i\uparrow} n_{i\downarrow} + \frac{1}{2}\sum_{\langle ij\rangle\alpha\beta} V_{ij} n_{i\alpha} n_{j\beta}, \tag{1}$$

where $t_{ij}$ is the nearest-neighbor hopping amplitude that describes the kinetic energy, $\mu$ is the chemical potential used to fix the filling of the system, $U_i$ is the on-site interaction, $V_{ij}$ is the nearest neighbor interaction, and $n_{i\sigma}=c_{i\sigma}^\dagger c_{i\sigma}$ is the density operator at site $i$ corresponding to spin $\sigma$.

The properties of this Hamiltonian have been studied previously;[26] it should be viewed as an effective Hamiltonian with which one can describe $s$-wave and $d$-wave symmetries of the superconducting order parameter. For an $s$-wave superconductor we choose an attractive on-site interaction $U_i < 0$ and no nearest-neighbor interaction $V_{ij}=0$, while for a $d$-wave superconductor we set the nearest-neighbor interaction to be attractive $V_{ij}<0$ and the on-site interaction to vanish or be repulsive. The interaction parameters $U_i$ and $V_{ij}$ are dependent on position, breaking translational invariance. This will allow us to describe interfaces between different types of materials.

Using the Hartree-Fock mean-field decomposition this Hamiltonian can be transformed into a one-particle mean-field Hamiltonian:

$$\mathcal{H} = \sum_{\langle ij\rangle\sigma} (-t_{ij} - \delta_{ij}\mu) c_{i\sigma}^\dagger c_{j\sigma} + \sum_i (\Delta_i c_{i\uparrow}^\dagger c_{i\downarrow}^\dagger + \text{H.c.}) + \sum_{\langle ij\rangle} [\Delta_{ij}(c_{i\uparrow}^\dagger c_{j\downarrow}^\dagger + c_{i\downarrow}^\dagger c_{j\uparrow}^\dagger) + \text{H.c.}]. \tag{2}$$

For planar junctions, infinite surfaces can be considered and therefore translational invariance in the direction parallel to the surface is recovered. By doing a Fourier transform of the Hamiltonian in the direction parallel to the surface we only have to solve one-dimensional inhomogeneous problems. For any point in $k$ space the problem becomes a one-dimensional inhomogeneous problem. In the case of a 2D superconductor with an infinite surface along the $\hat{y}$ direction the Hamiltonian becomes

$$\begin{aligned}\mathcal{H} = &\sum_{k_y}\sum_{\langle ij\rangle\sigma} \{-(1-\delta_{ij})t_{ij}^\perp - \delta_{ij}[\mu + 2t_i^\parallel \cos(k_y a)]\} c_{i\sigma}^\dagger(k_y) c_{j\sigma}(k_y) \\ &+ \sum_{k_y}\sum_i [\Delta_i + 2\Delta_i^\parallel \cos(k_y a)] c_{i\uparrow}^\dagger(k_y) c_{i\downarrow}^\dagger(k_y) + \text{H.c.} \\ &+ \sum_{k_y}\sum_{\langle ij\rangle} \Delta_{ij}^\perp [c_{i\uparrow}^\dagger(k_y) c_{j\downarrow}^\dagger(k_y) + c_{i\downarrow}^\dagger(k_y) c_{j\uparrow}^\dagger(k_y)] + \text{H.c.},\end{aligned} \tag{3}$$

where $i$ and $j$ are now in the direction perpendicular to the surface and $a$ is the lattice constant. $t^\perp$ and $\Delta_{ij}^\perp$ are the hopping amplitude and pair potential in the direction perpendicular to the surface and $t^\parallel$ and $\Delta_i^\parallel$ are the hopping amplitude and the pair potential in the direction parallel to the surface. The mean-field order parameters are to be calculated self-consistently:

$$\Delta_i = \frac{1}{N_y}\sum_{k_y} U_i \langle c_{i\downarrow}(k_y) c_{i\uparrow}(k_y)\rangle, \tag{4}$$

$$\Delta_i^\parallel = \frac{1}{N_y}\sum_{k_y} V_i^\parallel \langle c_{i\downarrow}(k_y) c_{i\uparrow}(k_y)\rangle \cos(k_y a), \tag{5}$$

$$\Delta_{ij}^\perp = \frac{1}{N_y}\sum_{k_y} \frac{V_{ij}^\perp}{2} [\langle c_{i\downarrow}(k_y) c_{j\uparrow}(k_y)\rangle + \langle c_{i\uparrow}(k_y) c_{j\downarrow}(k_y)\rangle], \tag{6}$$

where $\Delta_i$ is the $s$-wave order parameter, $\Delta_i^\parallel$ is the $d$-wave order parameter of a link in the direction parallel to the surface, and $\Delta_{ij}^\perp$ is the $d$-wave order parameter of a link in the direction parallel to the surface.

In the 3D $c$-axis geometry, the surface is considered to be in the $\hat{x}$-$\hat{y}$ plane. After Fourier transforming in these directions, the Hamiltonian becomes

$$\begin{aligned}\mathcal{H} = &\sum_{k_x k_y}\sum_{\langle ij\rangle\sigma} \{-(1-\delta_{ij})t_{ij}^\perp - \delta_{ij}\{\mu + 2t_i^\parallel[\cos(k_x a) + \cos(k_y a)]\}\} c_{i\sigma}^\dagger(k_x,k_y) c_{j\sigma}(k_x,k_y) + \sum_{k_x k_y}\sum_i \{\Delta_i \\ &+ 2\Delta_i^\parallel[\cos(k_x a) - \cos(k_y a)]\} c_{i\uparrow}^\dagger(k_x,k_y) c_{i\downarrow}^\dagger(k_x,k_y) + \text{H.c.}\end{aligned} \tag{7}$$

The self-consistency in the order parameters is now given by the following equations:

$$\Delta_i = \frac{1}{N_x N_y} \sum_{k_x,k_y} U_i \langle c_{i\downarrow}(k_x,k_y) c_{i\uparrow}(k_x,k_y)\rangle, \tag{8}$$

$$\Delta_i^{\parallel} = \frac{1}{N_x N_y} \sum_{k_x,k_y} V_i^{\parallel} \langle c_{i\downarrow}(k_x,k_y) c_{i\uparrow}(k_x,k_y)\rangle [\cos(k_x a) - \cos(k_y a)], \tag{9}$$

where $i$ is now taken to be the site index in the $\hat{z}$ direction. $\Delta_i$ is the on-site $s$-wave order parameter while $\Delta_i^{\parallel}$ is the $d$-wave order parameter which has components only in the $\hat{x}$ and $\hat{y}$ directions.

We follow the standard procedure[27] of introducing a canonical transformation of the electron operators:

$$c_{i\uparrow}(k_x,k_y) = \sum_n u_n^i(k_x,k_y)\gamma_{n\uparrow} + v_n^{i*}(k_x,k_y)\gamma_{n\downarrow}^{\dagger}, \tag{10}$$

$$c_{i\downarrow}(k_x,k_y) = \sum_n u_n^i(k_x,k_y)\gamma_{n\downarrow} - v_n^{i*}(k_x,k_y)\gamma_{n\uparrow}^{\dagger}. \tag{11}$$

This transformation will diagonalize the Hamiltonian and one obtains the BdG equations for each pair of momentum vectors $k_x, k_y$:

$$\begin{pmatrix} H^0(k_x,k_y) & \Delta(k_x,k_y) \\ \Delta^*(k_x,k_y) & -H^0(k_x,k_y) \end{pmatrix} \begin{pmatrix} u(k_x,k_y) \\ v(k_x,k_y) \end{pmatrix} = \epsilon(k_x,k_y) \begin{pmatrix} u(k_x,k_y) \\ v(k_x,k_y) \end{pmatrix}. \tag{12}$$

These equations describe the quasiparticle states in inhomogeneous superconductors. The BdG equations are equivalent to an eigenvalue problem with parameters that require self-consistent calculation. We start with an initial guess for the order parameter profile and we diagonalize the resulting Hamiltonian. In our infinite-surface setup, we need to diagonalize a one-dimensional Hamiltonian for every point in momentum space. Using the self-consistency Eqs. (4)–(6) we recalculate the order parameter profile. The solution is obtained when the difference in the order parameters between two steps is smaller than a desired accuracy.

The self-consistent calculation of the order parameter ensures that the order parameter in the "normal metal" region has knowledge about the pair potential in this layer. If the initial guess is a step function, i.e. the order parameter in the middle region is zero, after one iteration the pair amplitude will become nonzero because of the proximity effect. In the case $U'=0$, the order parameter will remain zero: $\Delta \sim U_i' \langle c_{i\uparrow} c_{i\downarrow}\rangle$, while for the case $U'<0$ the new order parameter has a finite value throughout the layer. If we were to fix the order parameter in the superconducting regions (we do not), the $U'=0$ solution would need only one iteration to converge.

The BdG formalism allows us to calculate the dc current in the absence of applied voltages. In the tight-binding formulation the current operator is

$$J_{ij} = \sum_{\sigma} t_{ij}(c_{i\sigma}^{\dagger} c_{j\sigma} - c_{j\sigma}^{\dagger} c_{i\sigma}). \tag{13}$$

The expectation value of the current will be nonzero only if the order parameters in the two superconducting layers have different phases. For the mean-field Hamiltonian with $s$-wave order parameters one gets for the continuity equation:

$$\begin{aligned} \left\langle \frac{\partial n_i}{\partial t} \right\rangle &= \langle [H, n_i] \rangle \\ &= \sum_{\sigma} t_{ij}(\langle c_{i\sigma}^{\dagger} c_{j\sigma}\rangle - \langle c_{j\sigma}^{\dagger} c_{i\sigma}\rangle) + \langle c_{i\downarrow} c_{i\uparrow}\rangle \Delta_i^{\star} - \langle c_{i\uparrow}^{\dagger} c_{i\downarrow}^{\dagger}\rangle \Delta_i. \end{aligned} \tag{14}$$

If the order parameter is calculated self-consistently $\Delta_i \sim \langle c_{i\downarrow} c_{i\uparrow}\rangle$, then we recover the continuity equation, $\langle \partial n_i / \partial t\rangle = \langle J_{ij}\rangle$. Otherwise, if the order parameters are not calculated self-consistently but set to a desired value (the case of hard boundary) then the last two terms in Eq. (14) can be seen as current source terms.

In our calculation the coherence factors, $u_k$ and $v_k$, are complex numbers and they will give the magnitude and the phase of the order parameters. The magnitude of the order

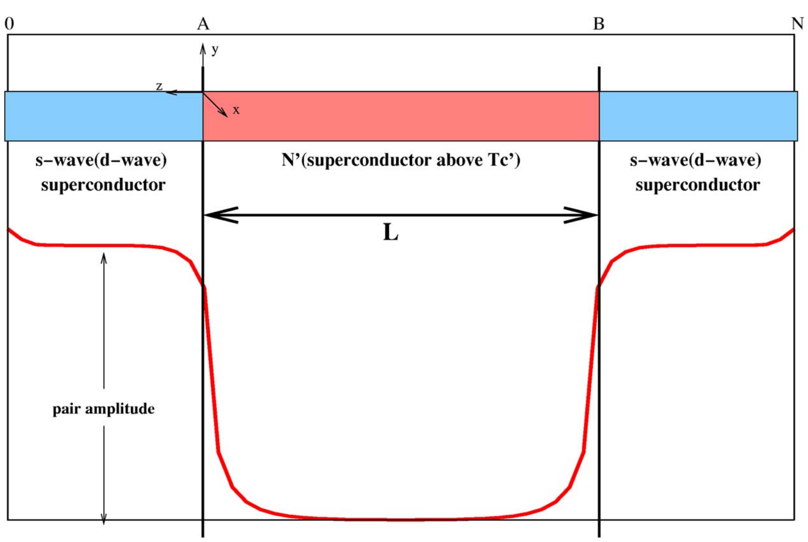


FIG. 1. (Color online) The pair amplitude profile through a Josephson junction. The pair amplitude is shown for $U=-3t$ and $U'=0$. The regions 0-$A$ and $B$-$N$ are superconducting, while the region $A$-$B$ is a superconductor above its critical temperature. The interfaces are considered to have perfect transmission and the whole system is in the clean limit.

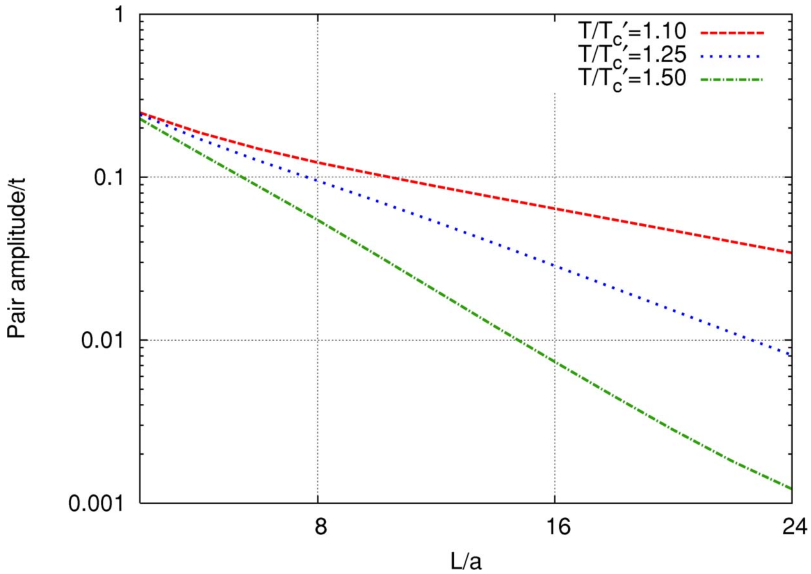


FIG. 2. (Color online) The pair amplitude at $L/2$ as a function of $L$ for different temperatures above $T_c'$ for the $SN'S$ system with $U'=-2t$ and $U=-4t$.

parameters is calculated self-consistently and after each iteration the phase of the external layers is set to a desired value. For the calculation of the dc Josephson current we restrict the phase of the two superconductors to a desired phase difference, while for the weak link, the phase is calculated self-consistently.

## III. RESULTS FOR $SN'S$

For the $SN'S$ (strong s-wave/weak superconducting/strong s-wave) trilayers the interactions are only on-site attractive interactions. The value of the parameter $U_i$ will set the magnitude of the order parameter throughout the sample. We consider the following setup (Fig. 1), $U_i=U$ for $0<i<A$ and $B<i<N$, while $U_i=U'$ for $A<i<B$. In this particular case $V_{ij}$ is vanishing, because we ignore the $d$-wave symmetry. The value of $U'$ is chosen so that $|U'|<|U|$, allowing us to describe the $N'$ material with a lower critical temperature $T_c'$. For temperatures greater than $T_c'$ the $a$-$b$ region cannot sustain superconductivity by itself. The order parameter will leak from the stronger superconductors, and the characteristic length is called the "leaking" distance.

Figure 1 shows the order parameter profile for $U=-3t$, $U'=-2t$ and $T=0.21t$. Note that $T_c'=0.205t$ for the weak superconductor, while $T_c=0.459t$ for the strong one. Similar to the $T_c'=0$ case the order parameter has an exponential dependence on distance away from the interface, $\Delta \sim \Delta_0 \exp(-x/\xi)$. This is true only for temperatures much larger than $T_c'$ and for distances from the interface greater than the coherence length of the stronger superconductors. The coefficient of the exponential decay is given by the leaking distance, $\xi$. In the normal metal case ($T_c'=0$ K) the clean limit leaking distance is inverse proportional to the temperature:

$$\xi = \frac{\hbar v_F}{k_B T}. \tag{15}$$

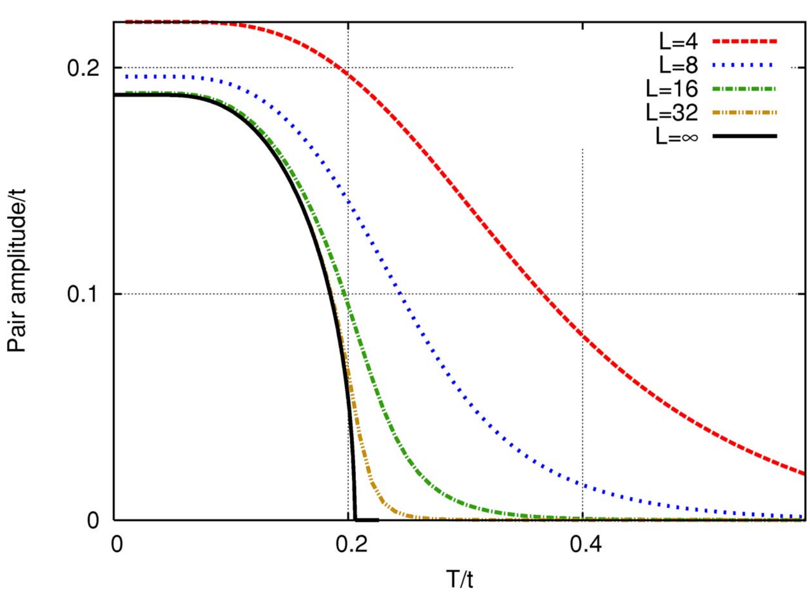


FIG. 3. (Color online) The pair amplitude at $L/2$ as a function of temperature for different $L$ for the $SN'S$ system with $U'=-2t$ and $U=-4t$.

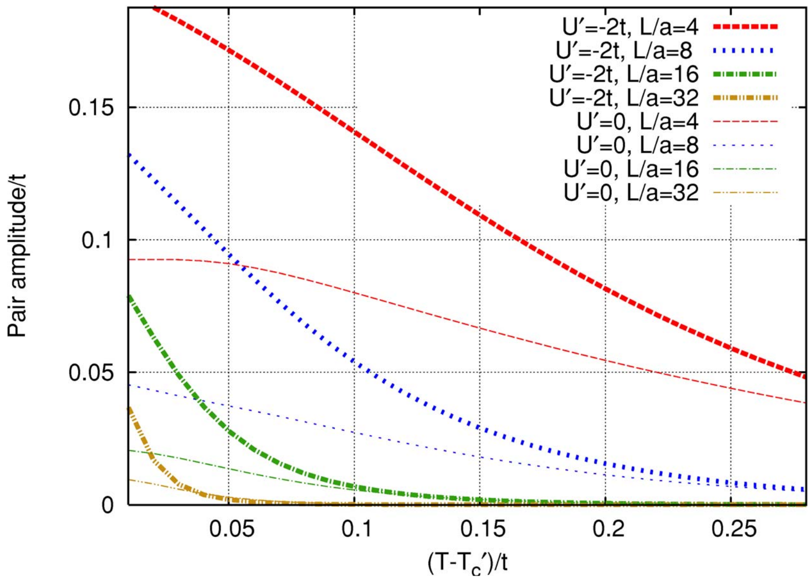


FIG. 4. (Color online) The pair amplitude at $L/2$ as a function of relative temperature for different $L$ for the $SN'S$ system for $U=-4t$, $U'=0$ and $U=-4t$, $U'=-2t$.

For the case of a weak superconductor, the relevant temperature scale is $T$-$T_c'$. Plotting the order parameter versus distance from the interface on a semilog scale (Fig. 2) for different temperatures, we can extract the leaking distance. As expected from the $T_c'=0$ K case, the leaking distance is decreasing with increasing temperature.

If we plot the order parameter as a function of temperature for different lengths ($L$) of the weak link (Fig. 3), we can observe two main effects. First, at $T=0$ K, the proximity effect will modify the order parameter at $L/2$. For lengths smaller than ten lattice constants this effect is important. Because the $N'$ layer is superconducting at $T=0$ K, the main length scale in this layer is the superconducting coherence length $\xi=\hbar v_F/\Delta$. The second effect is observed at temperatures close to $T_c'$. If the $N'$ layer was not connected to the superconducting layers, then, according to the mean-field behavior, the order parameter would vanish at $T_c'$. For temperatures higher than $T_c'$ the $N'$ layer cannot sustain superconductivity by itself. It is only in the presence of the $S$ layers, that the order parameter at $L/2$ has nonzero values. Note that *the length L for which we obtain nonzero values of the order parameter above $T_c'$ is much larger than the value of the length beyond which effects are unobservable at* $T=0$ K. In Fig. 4, we compare the order parameters at $L/2$ for two cases: $U'=0$ and $U'=-2t$. We observe that the order parameter for the case $U'=-2t$ is larger and that close to $T_c'$ the discrepancy is enhanced. This is a clear indication that the Cooper pair leaking distance is larger in the case of a nonzero $T_c'$.

In order to investigate further the dependence of the leaking distance on the magnitude of the superconducting correlations in the $N'$ layer, in Fig. 5, we summarize the extracted leaking distance obtained for different parameters. The $T_c'=0$ K line (dashed) is inversely proportional to the temperature, as expected. For $T_c'>0$ K the leaking distance is diverging at $T_c'$; this, of course leads to a giant proximity effect near these temperatures, as the figure visually demonstrates. Another feature of the calculation is that for any given

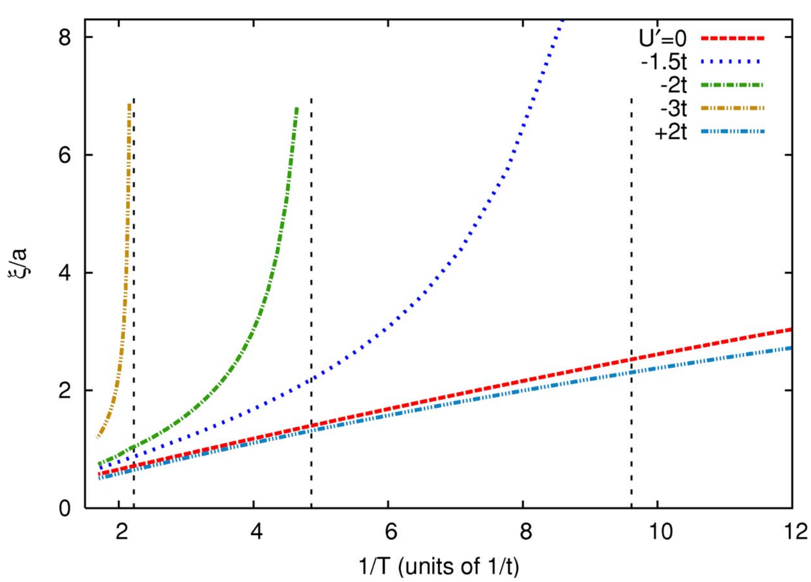


FIG. 5. (Color online) The leaking distance as a function of $1/T$ for different interaction parameters $U'$ for the $SN'S$ system with $U=-4t$. The vertical dashed lines represent the inverse of the critical temperatures for the corresponding $U'$ parameters: $T_c(U'=-1.5t)=0.104t$, $T_c(U'=-2t)=0.205t$, and $T_c(U'=-3t)=0.46t$.

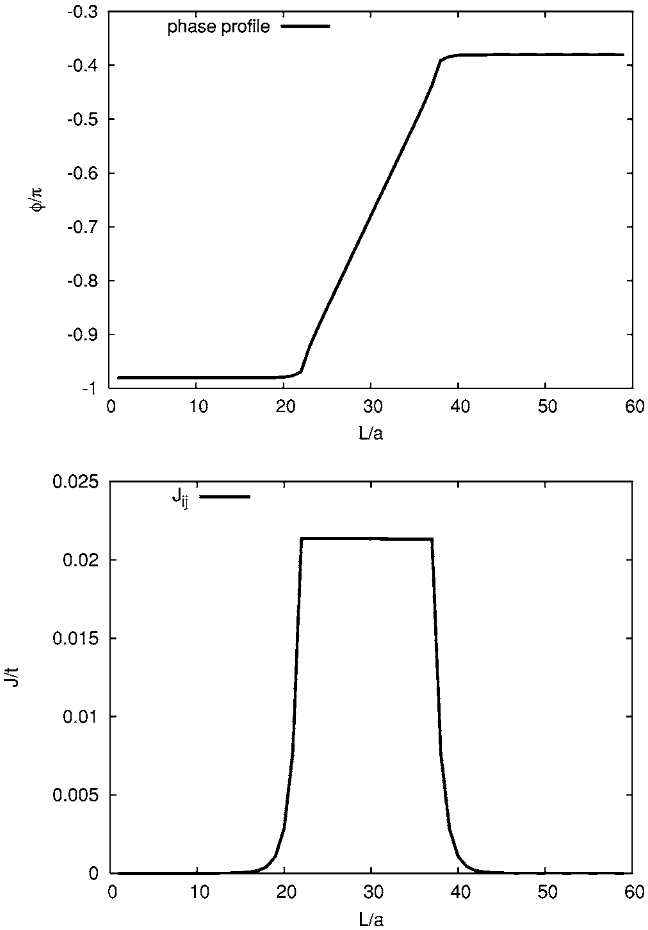


FIG. 6. (a) Phase profile and (b) dc Josephson current as a function of position. The phase is calculated self-consistently only in the middle layer and the continuity equation is satisfied only in this layer.

temperature, a higher $T_c'$ will result in a larger leaking distance. For repulsive on-site interactions, $U'=+2t$, the leaking distance is even smaller than in the normal metal case. This is a clear demonstration of the fact that interactions in the $N'$ layer will influence the way Cooper pairs leak from the superconducting side. Such "feedback" will not be captured in calculations that are not self-consistent.

The proximity effect can be observed either by growing superconducting thin films on top of normal metals and measuring the critical temperature of the system, or by forming a Josephson junction and measuring the Josephson current through the weak link. If the two superconducting sides are not coupled then there is no Josephson current. As we bring the superconducting sides closer to one another, the proximity effect will influence the value of the order parameter in the $N'$ layer. A nonzero value of the order parameter throughout the whole system will result in a nonzero value of the Josephson current.

The BdG equations are well suited for calculating the dc Josephson current. In order to have current between the two superconducting sides, the order parameters in the two sides have to have different phases. In our calculation we fix the phases of the order parameter on the $S$ layers, and our self-consistent calculation will give the magnitude and the phase of the order parameter in the $N'$ side. The results of such a calculation are shown in Figs. 6(a) and 6(b). The phase of the order parameter in $N'$ will vary continuously from $\phi_{left}$ to $\phi_{right}$ and the dc Josephson current will be constant throughout the layer. An interesting case is the one where $\Delta\phi=\pi$, for which there is a phase-slip point at $L/2$. Right at the phase slip the order parameter vanishes. In order to extract only the proximity effect from the current calculation, we need to find the phase difference for which the current is maximal. For a point contact Josephson junction the current has the following behavior:[28]

$$J = J_m \sin(\Delta\phi), \tag{16}$$

while for a long junction it deviates from the sinusoidal behavior.[29]

We calculate the dc Josephson current for different lengths of the weak link and for different temperatures for $U'=-1.5t$ and $U=-3t$. The results are summarized in Fig. 7. When the two superconducting layers ($S$) are close together

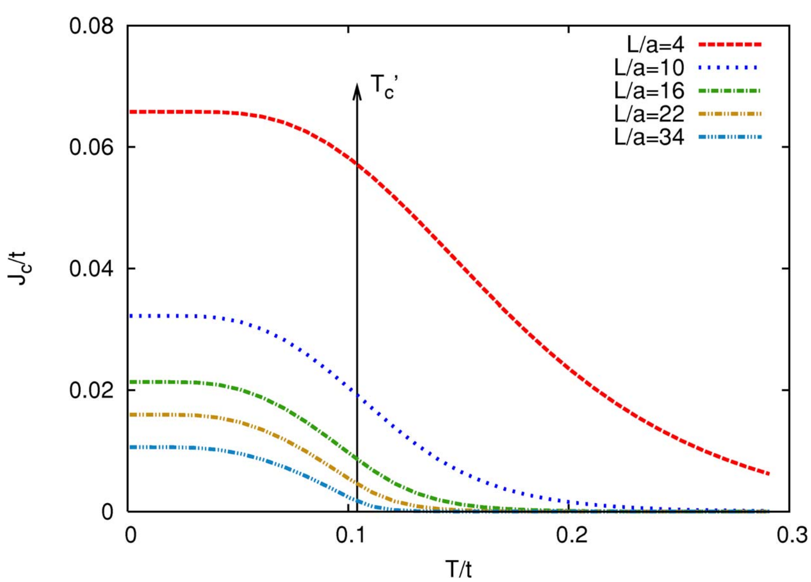


FIG. 7. (Color online) The dc Josephson current in the middle layer as a function of temperature for different lengths $L$ of the weak link for the $SN'S$ system with $U=-3t$ and $U'=-1.5t$. The arrow represents the critical temperature of the middle layer, $T_c(U'=-1.5t)=0.104t$.

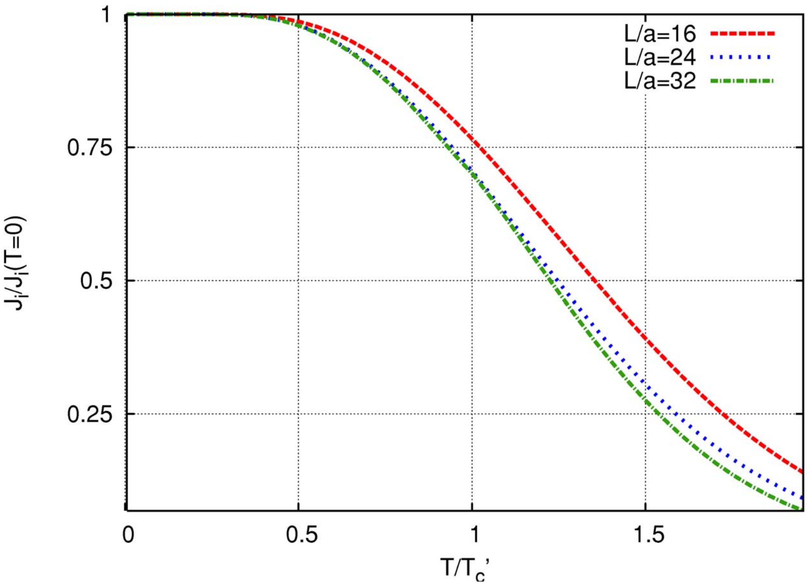


FIG. 8. (Color online) The dc Josephson current in the middle layer as a function of temperature for different lengths $L$ of the weak link for the $SN'S$ system with $U=-4t$ and $U'=-2t$. Equally spaced areas of superconductivity in the $N'$ layer are considered. The percent volume of the pockets of superconductivity with $U=-4t$ is $p=0.2$.

the proximity effect modifies the magnitude of the order parameter at $L/2$ in the $N'$ layer. A large value of the order parameter will give a large value for the current. As $L$ increases the order parameter decreases exponentially. This results in a decay of the current as a function of $L$. The main result is that the behavior of the Josephson current as a function of $L$ and $T$ reflects the existence of a leaking distance larger than the one expected from a normal metal. For $U'=-1.5t$ and $T=0.125t$, the normal metal gives a leaking distance of $\xi_0 \sim 2a$ while the self-consistently calculated one gives a value of $\xi \sim 7a$. This is seen in Fig. 7, where for $L=16a$ the current is nonzero for temperatures close to but greater than $T_c'$, and it has a linear dependence on temperature near $T_c'$.

As shown in previous attempts to explain the "giant proximity effect," the presence of pockets of superconductivity in the $N'$ layer will greatly enhance the current through the system, even for long weak links. Coupled with the enhancement of the leaking distance around $T_c'$ the presence of the superconducting pockets will effectively decrease the length of the weak link. We consider equally spaced superconducting areas with on-site interactions of strength $U=-4t$ embedded in the weak link with interaction strength $U=-2t$. In Fig. 8 we show the Josephson current for different lengths of the weak link and with superconducting pockets occupying a volume percentage $p=0.2$ of the weak link. The size of the considered pockets is one lattice site. The effect on the Josephson current is drastic—the current has nonzero values well above $T_c'$. We also notice that for this volume of embedded superconductivity, the current has a weak dependence on the length of the junction.

The strength of the coupling between the exterior superconductors will be given by the volume of these pockets. This is seen in Fig. 9, where we plot the Josephson current for a weak link of length $L=40a$ as a function temperature for different percent volumes of strong superconducting pockets embedded in the weak superconductor. As expected, for $p=0.0$ the junction is too long to couple the strong su-

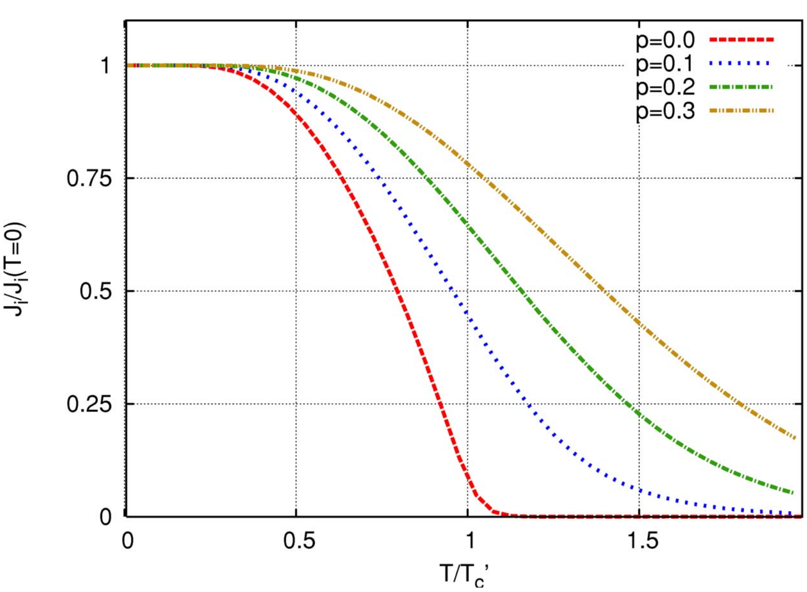


FIG. 9. (Color online) The dc Josephson current in the middle layer as a function of temperature for different percent volumes of embedded superconductivity in $N'$ for the $SN'S$ system with $U=-4t$ and $U'=-2t$. The length $L$ of the weak link is $L=40a$.

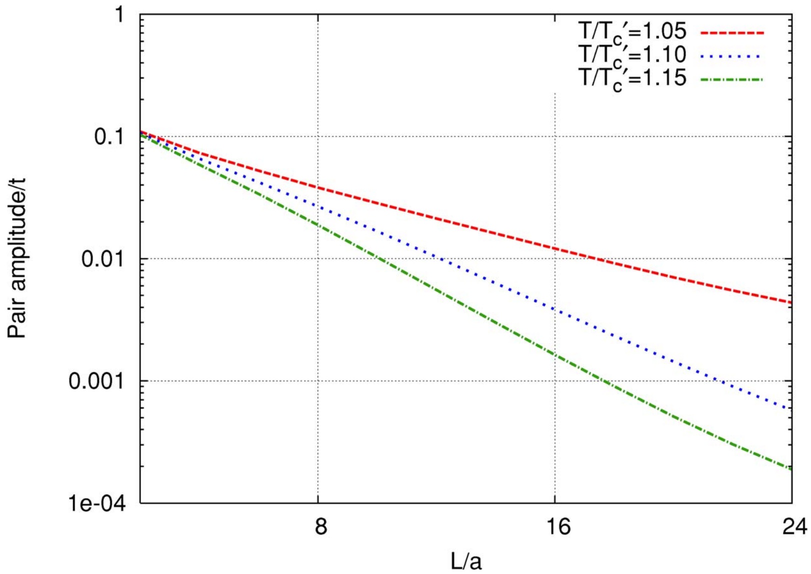


FIG. 10. (Color online) $d$-wave order parameter at $L/2$ as a function $L$ for different temperatures—100 $d$-wave case with $V=-4t$ and $V'=-2t$.

perconductors and the current vanishes above but very close to $T_c'$. Increasing $p$, the junction will effectively shorten and the two exterior superconductors will be coupled well above $T_c'$.

## IV. RESULTS FOR $DN'D$

The $d$-wave symmetry of the order parameter is attained if we consider nearest-neighbor interactions $V_{ii+\delta}=-V_i$ and vanishing or repulsive on-site interactions. The $d$-wave order parameter is calculated in the following way:

$$\Delta_d(i) = \tfrac{1}{4}[\Delta_x(i) + \Delta_{-x}(i) - \Delta_y(i) - \Delta_{-y}(i)], \tag{17}$$

where $\Delta_x$ describes superconducting correlations in the $\hat{x}$ direction. In a similar manner, as detailed in the $SN'S$ case, we set up $V_i$ so that $V_i=V$ for $0<i<A$ and $B<i<N$, while $V_i=V'$ for $A<i<B$. This will allow us to describe a weak link with a nonzero critical temperature.

For the 100 interface (between the $a$-$b$ planes of a high-$T_c$ superconductor), the dependence of the order parameter is very similar to the $s$-wave case. Figure 10 shows the semilogarithm plot of the order parameter as a function of distance from the interface for different temperatures. Again, we can observe the exponential decay and define the leaking distance $\xi$. The dependence of the order parameter on temperature for different lengths of the weak link is shown in Fig. 11 and the two manifestations of the proximity effect are seen. First at $T=0$ K the order parameter is modified if $L/2$ is of the order of the superconducting coherence length in the $N'$ layer. Secondly, above $T_c'$ the order parameter decays with increasing temperature but has a nonzero value even if $L/2$ is greater than the conventional leaking distance defined by the $T_c'=0$ K case. The self-consistently calculated leaking distance is shown in Fig. 12. Similar to the $s$-wave case it diverges at $T_c'$ and, for the same temperature, larger interactions in the weak superconductor will increase the leaking distance.

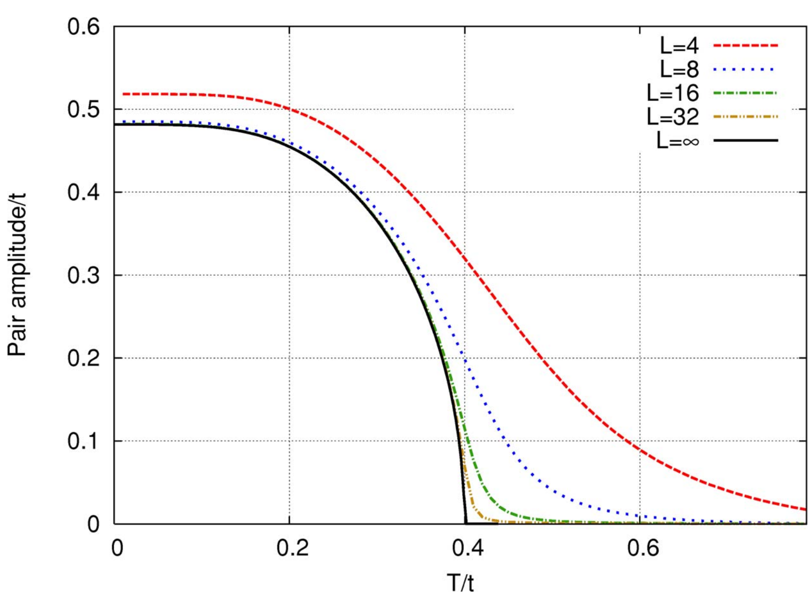


FIG. 11. (Color online) $d$-wave order parameter at $L/2$ as a function of $T$ for different lengths—100 $d$-wave case with $V=-4t$ and $V'=-2t$.

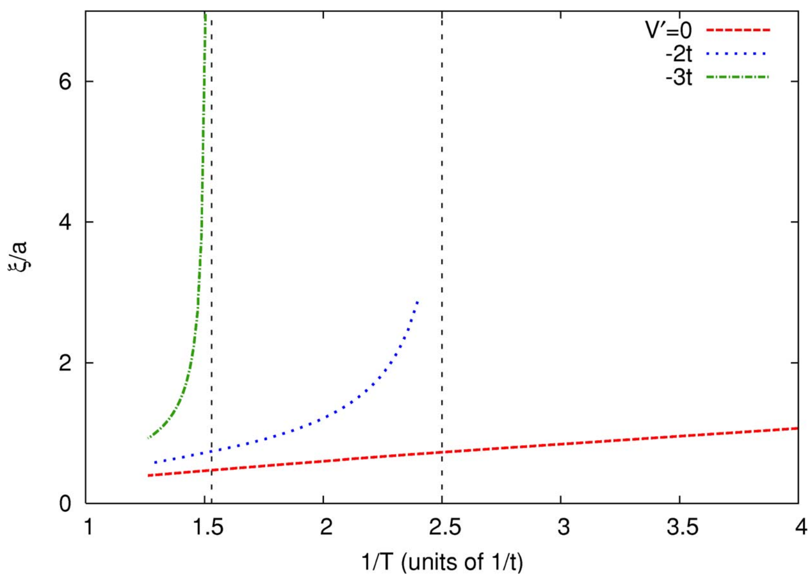


FIG. 12. (Color online) Leaking distance as a function of inverse temperature for different interaction strengths—100 $d$-wave case with $V=-4t$. The vertical dashed lines represent the inverse of the critical temperatures for the corresponding $V'$ parameters: $T_c(V'=-2t)=0.4t$, $T_c(V'=-3t)=0.67t$.

The coherent transport in the $c$-axis direction will be described by the hopping amplitude in the $\hat{z}$ direction, $t^{\perp}=0.5t^{\parallel}$. Figure 13 shows the Josephson critical current as a function of temperature for different lengths of the weak superconducting layer. The general behavior is similar to the one of the $s$-wave junction. The $d$-wave order parameter has no $k_z$ dependence, and thus in the $\hat{z}$ direction it is an effective on-site order parameter. For short weak links the current does not vanish abruptly above $T'_c$ but rather has a smooth dependence on temperature. This dependence on temperature above $T'_c$ shows that the length of the weak link is comparable to the leaking distance in this layer. The increase of the leaking distance due to the finite $T'_c$ cannot explain by itself the observed "giant proximity effect." It is only the conjunction with the presence of disordered pockets of superconductivity in the weak link that makes this effect possible. The calculation of the Josephson current in the presence of the disordered pockets from the previous section stands also for the $c$-axis geometry. The extra dimension will only affect the necessary volume of superconductivity needed to observe a "giant proximity effect."

## V. SUMMARY

In summary, using a tight-binding formulation of the extended Hubbard Hamiltonian, we solve the BdG equations for a system composed of three layers: two superconducting layers (either $s$ wave or $d$ wave), and a weaker superconductor sandwiched in between. We examined the proximity effect induced by the exterior superconducting layers in the "normal metal" interior layer. We observed that, in agreement with previous calculations, the order parameter has an exponential decay behavior, the characteristic decay length being the leaking distance. In both $s$-wave and $d$-wave cases the leaking distance is only dependent on the properties of the $N'$ layer. For $T'_c=0$ K (normal metal) for both $s$-wave and $d$-wave symmetries the leaking distance is inversely proportional to the temperature. If $T'_c>0$ K, the leaking distance diverges at $T'_c$ and at the same temperature larger attractive interactions in the middle layer will increase the leaking distance. Essentially, the BdG formalism provides a means for the normal layer to feel pairing fluctuations above its critical temperature, $T'_c$. These are not spontaneous, in that they arise

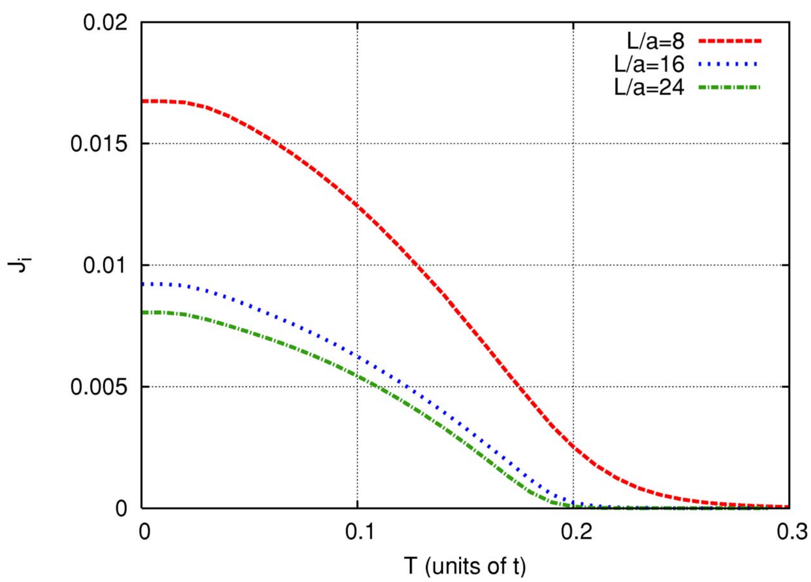


FIG. 13. (Color online) The $c$-axis dc Josephson current in the middle layer as a function of temperature for different lengths $L$ of the weak link for which $V=-4t$ and $V'=-2t$. The $c$-axis hopping amplitude is $t^{\perp}=0.5t^{\parallel}$.

from an “applied” pairing field produced by the outer layers. This accounts for the much higher leaking distance for a weak superconductor.

We also calculated the dc Josephson current, and extracted the maximum value. We observed that the current has a nonzero value for lengths of the weak link much larger than the “conventional″ leaking distance, and for temperatures well above $T_c'$. Although the divergence of the leaking distance at the critical temperature of the $N'$ layer enhances the Josephson current for temperatures above $T_c'$, it is not enough to explain the experimental measurement of the “giant proximity effect.”[1] As prompted by previous attempts to explain the giant proximity effect,[22,23] we considered areas of superconductivity with critical temperature $T_c > T_c'$, which are embedded in the $N'$ layer. Further enhancement of the Josephson current is observed. Depending on the volume of the superconducting pockets, nonzero values of the Josephson current are obtained even for temperatures $T > 2T_c'$. These results form the basis for a qualitative understanding of the giant proximity effect observed by Bozovic *et al.*[1]

## ACKNOWLEDGMENTS

We thank Wonkee Kim and Fatih Doğan for useful discussions. This work was supported in part by the Natural Sciences and Engineering Research Council of Canada (NSERC), by ICORE (Alberta), and by the Canadian Institute for Advanced Research (CIAR).

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