

# Conformal properties of one-dimensional quantum systems with long-range interactions

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(Received 18 October 2005; published 13 December 2005)

We investigate conformal properties of a one-dimensional quantum system with a long-range, Coulomb-like potential of the form  $1/|x|^\sigma$ , with  $\sigma > 0$ . We compute the conformal anomaly  $c$  as function of  $\sigma$ . We obtain  $c=0$  for  $\sigma < 1$  (Wigner crystal) and  $c=1$  for  $\sigma > 1$  (Luttinger liquid). By studying the scaling regime of the system when it is deformed by the inclusion of a term  $t\rho(x)$  [with  $\rho(x)$  a scaling operator], we show that, for  $\sigma > 1$ , the correlation length  $\xi$  gets an additional interaction dependent factor. This leads to a necessary redefinition of  $\xi$  in order to avoid an unphysical dependence of the central charge on the coupling constant.

DOI: [10.1103/PhysRevB.72.245110](https://doi.org/10.1103/PhysRevB.72.245110)

PACS number(s): 71.10.Pm, 05.30.Fk, 05.30.Jp, 64.60.Fr

## I. INTRODUCTION

The principle of conformal invariance (CI) at a critical point is one of the most powerful tools for our understanding of one-dimensional (1D) quantum systems.<sup>1</sup> This invariance constrains the possible behavior of a given critical theory, allowing a characterization of universality classes in terms of a dimensionless parameter  $c$  (the conformal anomaly) which is the central charge of the quantum extension of the (infinite-dimensional) conformal algebra. The central charge gives a measure of the sensitivity of a critical theory when one changes the space geometry. It has also been related to the low-temperature specific heat in quantum chains.<sup>2</sup> As a striking achievement, in the context of classical two-dimensional statistical mechanics models, conformal invariance led to the derivation of all critical exponents corresponding to theories with  $c < 1$ .<sup>3</sup> When the model possesses an additional continuous symmetry a classification of critical models is also possible for  $c \geq 1$ .<sup>4</sup>

CI is obtained by assuming translational, rotational and scale invariance, together with short-range interactions. If any of these conditions are not verified its validity can not be assured. However, the concepts of CI have been extended in several directions. Cardy, has derived a relationship between the second moment of the correlation function of a scaling operator away from criticality (in the scaling region) and the central charge and the scaling dimension corresponding to the critical theory.<sup>5</sup> CI has been also applied, as a principle, to 1D quantum systems with inverse-square long-range interactions.<sup>6,7</sup> In particular, the continuum limit of the  $S=1/2$  Heisenberg chain with inverse-square exchange was associated to a  $c=1SU(2)$  Kac-Moody theory<sup>8</sup> and Sutherland's model with interaction  $g/r^2$  (Ref. 6) was analyzed through finite size scaling.<sup>9,10</sup> Finite size corrections to the ground state led to a  $g$ -dependent central charge, whereas the low temperature expansion of the free energy yielded  $c=1$ , which is consistent with the scaling dimensions obtained for density operators and momentum distributions. The reason for this discrepancy has not been fully explained, although it was suggested that it might be related to the finite-size approximation to the long-range potential<sup>9</sup> and to the possible differences between time and space correlations in long-range models.<sup>10</sup>

On the other hand, in the context of low-dimensional condensed matter systems, the interest in long-range interactions is not a purely academic one. For instance, in quasi-one-dimensional systems one usually considers contact electron-electron interactions assuming that the presence of adjacent chains effectively screens the basic Coulomb interaction. However, as the dimensionality of the system decreases charge screening effects become less important and consequently the long-range interaction plays an observable role.<sup>11</sup> The effect of power-law long-range interactions is also of interest for the study of antiferromagnetic spin chains,<sup>12</sup> in which possible experimental realizations might be found, for instance, in systems with Ruderman-Kittel-Kasuya-Yosida interactions.<sup>13</sup>

The purpose of this paper is to explore the interplay between the principle of conformal invariance and the effect of long-range, distance-dependent interactions. By performing analytical computations of the central charge  $c$  in a continuous theory, we shed light on the effect of anisotropy on the conformal properties of a 1D quantum system. In particular, in connection to the above mentioned discrepancy, we will explicitly show how the asymmetry between long-time and long-space behaviors of the correlations must be treated in order to obtain sensible results for the conformal anomaly.

Most of the previous calculations of  $c$  for long-ranged systems have been done on the basis of a discrete model and for specific values of  $\sigma$ . The exception is the important paper by Mironov and Zabrodin,<sup>14</sup> who were able to determine scaling dimensions and correlation functions for a general class of potentials. However, in their formulation it was not possible to specify the conditions to be satisfied by the potentials in order to assure the absence of a gap. Moreover, the value of  $c$  was not computed but inferred from the form of some scaling dimensions.

## II. THE MODEL AND THE LONG-RANGE POTENTIAL

We will study a bosonic non local (1+1)-dimensional quantum field theory with Lagrangian density

$$\mathcal{L}_0 = \frac{1}{2} [\partial_\mu \phi(x)]^2 + \frac{1}{2} \int d^2y \partial_1 \phi(x) V(x-y) \partial_1 \phi(y). \quad (1)$$

This model is the bosonized version of a Thirring-Tomonaga-Luttinger fermionic model containing only forward-scattering interactions.<sup>15</sup> It describes charge-density oscillations with sound velocity  $v = \sqrt{1 + V(p_1)}$ , where  $V(p_1)$  is the Fourier transform of the distance-dependent potential. Although the model under consideration is not Lorentz covariant, we shall use, for convenience, a notation which is reminiscent of relativistic problems ( $d^2x = dx_0 dx_1$ ,  $d^2p = dp_0 dp_1$ ).

We will consider an instantaneous, power-law decaying potential of the form

$$V(x) = g \delta(x_0) (x_1^2 + \alpha^2)^{-\sigma/2}, \quad (2)$$

where  $\sigma$  is an arbitrary positive real number and  $\alpha$  is a regulator parameter that can be associated to the lattice spacing.

Since we shall deal with all positive values of  $\sigma$  we will need the Fourier transformed potential which depends on  $\sigma$  as follows:

$$V(p_1) = \frac{2g\sqrt{\pi}(2\alpha)^{(1-\sigma)/2}}{\Gamma(\sigma/2)} |p_1|^{(1-\sigma)/2} K_{(1-\sigma)/2}(\alpha|p_1|), \quad (3)$$

for  $\sigma > 1$ . Here  $K_{(1-\sigma)/2}(\alpha|p_1|)$  is a modified Bessel function. Note that, in this case, the potential is renormalized. This renormalization is associated to the divergence of the integrand in the Fourier integral, for  $x_1=0$ . This divergence is not present when  $\sigma < 1$ . Defining the renormalized coupling constant  $g_R = (\alpha)^{1-\sigma} g$  and taking into account the behavior of the Bessel function for small arguments we get

$$V(p_1) = f_>(g_R, \sigma), \quad (4)$$

with  $f_>(g_R, \sigma) = g_R \sqrt{\pi} \Gamma[(\sigma-1)/2] / \Gamma(\sigma/2)$ .

For  $\sigma < 1$  the Fourier-transformed potential reads

$$V(p_1) = f_<(g, \sigma) |p_1|^{(\sigma-1)}, \quad (5)$$

with  $f_<(g, \sigma) = g \sqrt{\pi} (2)^{1-\sigma} \Gamma[(1-\sigma)/2] / \Gamma(\sigma/2)$ .

As we shall see, the Coulomb potential corresponding to  $\sigma=1$  is a special case that can be considered as the frontier between two regions of qualitatively different conformal behaviors. This case was investigated, from another point of view, by Schulz, in his work on Wigner crystal-like formation in a Luttinger system.<sup>16</sup> (See also Ref. 17 for an explicit computation of the electronic single-particle function.) For  $\sigma=1$  one has

$$V(p_1) = 2gK_0(\alpha|p_1|) \approx -2g \ln(\alpha|p_1|). \quad (6)$$

### III. COMPUTATION OF THE CONFORMAL ANOMALY

Exactly as it was done in Refs. 9 and 10 with the Sutherland's model, for the special case  $\sigma=2$ , in this paper we shall employ the model given by Eq. (1) (for arbitrary real and positive  $\sigma$ ) as a testing ground to examine the consequences of imposing CI in the presence of anisotropic interactions. Our idea is to compute the central charge following two different routes. First, we will compute the low-temperature behaviour of the specific heat, from which we can read the

value of  $c$ , according to the results of Refs. 2. Secondly we will perturb the critical Lagrangian (1) by adding a simple scaling operator  $\rho(x)$  that takes the system away from criticality. Then, by considering the second moment of the two-point correlations of  $\rho(x)$  in the scaling regime, we can obtain the value of the conformal anomaly by using the result derived by Cardy<sup>5</sup>

$$\int d^2x |x|^2 \langle \rho(x) \rho(0) \rangle_t = \frac{c}{3\pi t^2 (2 - \Delta_\rho)^2}, \quad (7)$$

where  $\Delta_\rho$  is the scaling dimension of  $\rho$  and  $t$  is the coupling constant of the added term that spoils CI [ $t \propto \xi^{-1} \propto (T - T_c)$ ]. Although somewhat indirect, this method is interesting because it involves both large spatial and temporal dependence of the correlations. This fact will enable us to unravel some subtleties involved in the application of CI concepts to anisotropic models. Let us point out that, eventhough the above equation is valid for any scaling operator,  $\rho(x)$  is the density operator throughout this work.

#### A. First method: free-energy calculation

In order to compute the low temperature behavior of the specific heat we study Eq. (1) in the imaginary-time formalism.<sup>18,19</sup> As usual, we build the partition function  $\mathcal{Z}$  as a functional integral extended over the paths with periodicity conditions in the Euclidean time variable  $x_0$ :  $\phi(x_0 + \beta, x_1) = \phi(x_0, x_1)$ , where  $\beta = 1/T$ ,  $k_B = 1$ . After some standard algebraic steps (see Ref. 19 for a detailed similar computation in a fermionic model) the Helmholtz free energy  $\mathcal{F} = -(1/\beta) \ln \mathcal{Z}$  can be expressed as

$$\mathcal{F} = \mathcal{F}_0 + \frac{1}{\beta} \int_{-\infty}^{\infty} \frac{dp_1}{2\pi} \ln(1 - e^{-\beta|p_1|v(p_1)}), \quad (8)$$

where  $\mathcal{F}_0$  is the zero-point, vacuum free energy. Now we have to evaluate the integral and read the value of  $c$ . This is easily done for  $\sigma > 1$ , since  $v = \sqrt{1 + f_>(g_R, \sigma)} = v(g_R, \sigma)$  is independent of the momentum. The result is

$$\mathcal{F} = \mathcal{F}_0 - \frac{\pi}{6\beta^2 v(g_R, \sigma)}, \quad (9)$$

which means that  $c=1$ .<sup>2</sup> For  $\sigma < 1$  the computation is a little bit more subtle because the sound velocity depends on  $p_1$ . However, taking into account that one is interested in the large- $\beta$  behavior of  $\mathcal{F}$ , the integral can be approximated, yielding a result proportional to  $\beta^{-(\sigma+3)/(\sigma+1)}$ . For  $\sigma < 1$  the exponent  $(\sigma+3)/(\sigma+1) > 2$ , from which one concludes that  $c=0$ . Conformal field theories with  $c=0$  have been previously found in the study of second order phase transitions in the presence of quenched disorder.<sup>20</sup> Some of their interesting properties have been analyzed quite recently.<sup>21</sup>

#### B. Second method: deformation of a conformal field theory

Let us now undertake the computation of  $c$  following the second method described above. We choose as perturbing operator  $\rho(x) = \partial_1 \phi(x)$ . In order to determine the scaling di-

mension we compute the critical two-point correlation. A straightforward manipulation yields

$$\langle \rho(x)\rho(0) \rangle_0 = \frac{1}{(2\pi)^2} \int d^2p \frac{p_1^2 e^{ip \cdot x}}{p_0^2 + p_1^2 v^2 (p_1)}. \quad (10)$$

As a first step we evaluate the scaling dimensions by considering equal-time correlations. Performing a long-distance approximation in the above integral we get

$$\langle \rho(x_1)\rho(0) \rangle_0 = \frac{1}{4\pi v} \frac{1}{|x_1|^2}, \quad (11)$$

for  $\sigma > 1$ , from which we obtain  $\Delta_\rho = 1$ . Note that in this case  $v = \sqrt{1 + f_{>}(g_R, \sigma)} = v(g_R, \sigma)$ .

The analogous calculation for  $\sigma < 1$  gives a lengthy expression in terms of a confluent hypergeometric function. Analyzing the asymptotic long distance limit we obtain

$$\langle \rho(x_1)\rho(0) \rangle_0 \sim \frac{1}{\pi(5 - \sigma) \sqrt{f_{<}(g, \sigma)} |x_1|^{2[1+(1-\sigma)/4]}}, \quad (12)$$

from which we get  $\Delta_\rho = 1 + (1 - \sigma)/4$ . Note that both results, for  $\sigma > 1$  and  $\sigma < 1$  coincide for  $\sigma = 1$ , which suggests that  $\Delta_\rho$  is continuous for all  $\sigma > 0$ . We have explicitly verified the correctness of this assertion

$$\langle \rho(x)\rho(0) \rangle_0 = \frac{e^{1/g} \sqrt{\pi/2g}}{\alpha^2} \text{Erfc} \left[ \sqrt{\frac{1}{g} - 2 \ln \frac{\alpha}{|x_1|}} \right], \quad (13)$$

where Erfc stands for the complementary error function. In the long-distance limit this expression behaves as  $|x_1|^{-2}$ , i.e.,  $\Delta_\rho = 1$  for  $\sigma = 1$ . We will use this result later.

At this point we start considering the modified, non critical theory described by

$$\mathcal{L} = \mathcal{L}_0 + t\rho(x), \quad (14)$$

and try to compute the left hand side of Eq. (7). This computation involves correlations slightly away from criticality, in the scaling regime in which both  $|x_1|$  and the correlation length  $\xi$  are very large, but with  $|x_1| \ll \xi$ . Besides, both spatial and temporal dependence of the correlations are needed. In fact, regarding the possible application of CI to an anisotropic model such as the one considered here, this is a crucial point. Indeed, it is very instructive to compute again  $\langle \rho(x)\rho(0) \rangle_0$ , not in the equal-time case but for  $x_0, x_1 \neq 0$ . This is quite easily done for  $\sigma > 1$ . The result is

$$\langle \rho(x)\rho(0) \rangle_0 = \frac{e^{-|x_0|v/|x_1|} \left( e^{|x_0|v/|x_1|} - 1 - \frac{|x_0|v}{|x_1|} \right)}{2\pi v^3 |x_0|^2}. \quad (15)$$

Performing an expansion of the numerator around  $x_0 = 0$  we reobtain our previous result for the equal-time correlator (11). The important observation here is that although one can define a scaling dimension which is uniquely determined for large distances and long times, the precise functional form of the correlation depends on the direction in which large scales are observed. The interesting point is that in contrast to previous calculations, our procedure is able to take into account this expected anisotropy. Since the integrand in the left hand

side of Eq. (7) involves the general (not equal-time) behavior of the correlator, our result for  $c$  will reflect the contributions coming from different directions of space time. In view of these remarks we now compute the left hand side of Eq. (7). In our calculation the correlation length is associated to spatial fluctuations only. This parameter is used as an infrared cutoff for the spatial integral. For  $\sigma < 1$  we get

$$\int d^2x |x|^2 \langle \rho(x)\rho(0) \rangle_t = C_1 \xi^\sigma + C_2 \xi^{\sigma+1} + C_3 \xi^{2\sigma}, \quad (16)$$

where  $C_1$ ,  $C_2$ , and  $C_3$  are constants depending on  $g$  and  $\sigma$ . Since  $t \propto \xi^{-1}$  one sees that  $c \rightarrow 0$ , in agreement with the result obtained by computing the free energy (8). It is more illuminating the case  $\sigma > 1$  which gives

$$c = \left( \frac{1}{v^2} + \frac{6}{v^4} \right) \xi^2 t^2. \quad (17)$$

Since we know from Eq. (9) that  $c = 1$ , we obtain an expression for the correlation length in terms of  $t$  and  $v$ . The extra dependence of  $\xi$  on  $v$  is due to the anisotropy of the system, although the precise form of this dependence is a consequence of our regularization prescription. Please note that if we assume that  $\xi$  is a function of  $t$  only, the above equation predicts a conformal anomaly that depends on the long-range potential through the sound velocity, in analogy with the results previously obtained in Refs. 9 and 10 (for a different model and for  $\sigma = 2$ ). This situation reflects the fact that the correlation length is defined up to a multiplicative nonuniversal constant.

For the particular value  $\sigma = 1$  the above procedures are not easy to implement due to the peculiar form of the potential (6). However, based on the results (11) and (13) it is straightforward to conclude that  $c = 1$  for  $\sigma = 1$  also.

#### IV. SUMMARY AND CONCLUSIONS

In summary, we have presented an explicit and analytical computation of the conformal anomaly in a system with long-range Coulomb-like interactions. For illustrative purposes we have considered the simple bosonic QFT defined in Eq. (1). However, this model is interesting by itself since it is a bosonized version of the Tomonaga-Luttinger model with a distance-dependent forward-scattering potential. We obtained the central charge  $c$  as a function of  $\sigma$ . We predict  $c = 0$  for a very weakly decaying potential corresponding to  $\sigma < 1$ . According to the results of Schulz<sup>16</sup> this corresponds to the regime in which the system behaves similar to a 1D version of a Wigner crystal. For  $\sigma \geq 1$  we obtain  $c = 1$ , in agreement with ‘‘common knowledge’’ on Luttinger liquids.<sup>14</sup> However, in previous computations the value of  $c$  was inferred from the scaling dimensions, whereas we are now providing a direct determination for a whole family of well specified Coulombian potentials. In contrast to all previous studies of CI in long-ranged models we have explicitly computed correlators for different spatial and temporal points, which allowed us to capture the effect of anisotropy on  $c$  in a more complete fashion. Moreover, by considering as a mathematical tool a

deformation of the critical theory given by Eq. (14) and using a quantitative prediction relating  $c$  with properties of the deformed model in the scaling regime, we showed how, for  $\sigma > 1$ , the correlation length acquires a finite, interaction dependent renormalization factor. If one naively ignores this factor, an unphysical, interaction dependent conformal anomaly is obtained.

## ACKNOWLEDGMENTS

The authors are grateful to Daniel Cabra, Eduardo Fradkin and Gerardo Rossini for valuable discussions. This work was partially supported by Universidad Nacional de La Plata (Argentina) and Consejo Nacional de Investigaciones Científicas y Técnicas, CONICET (Argentina).

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