Tail states below the Thouless gap in superconductor–normal-metal–superconductor junctions: Classical fluctuations

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We study the tails of the density of states (DOS) in a diffusive superconductor–normal-metal–superconductor junction below the Thouless gap. We show that long-wave fluctuations of the concentration of impurities in the normal layer lead to the formation of subgap quasiparticle states, and calculate the associated subgap DOS in all effective dimensionalities. We compare the resulting tails with those arising from mesoscopic gap fluctuations, and determine the dimensionless parameters controlling which contribution dominates the subgap DOS. We observe that the two contributions are formally related to each other by a dimensional reduction.

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I. INTRODUCTION

The properties of hybrid superconductor–normal-metal (SN) structures continue to attract considerable attention both experimentally[1] and theoretically,[2–9] though the fundamental process governing the physics of such systems, Andreev reflection,[10] has been discovered long ago. In fact, while it is well known that generically the proximity to a superconductor leads to a modification of the density of states in the normal metal, the nature and extent of this effect depends on the details the hybrid structure. In particular, it was recently pointed out[2] that when a closed mesoscopic metallic region is contacted on one side to a superconductor, the resulting density of states (DOS) turns out to depend on its shape. If integrable, the DOS is finite everywhere but at the Fermi level, where it vanishes as a power law. On the contrary, in a generic chaotic metallic region one expects the opening of a gap around the Fermi level, the Thouless gap.[3] In analogy with the considerations above, a diffusive metallic region sandwiched between two bulk superconducting electrodes has been predicted to have a gapped density of states, the gap being at energies comparable to to the Thouless energy $E_{Th}=D/L_z^2$, where D is the diffusion constant and L_z the width of the normal layer[4–7] (see Fig. 1).

In a diffusive superconductor–normal-metal–superconductor (SNS) structure with transparent SN interfaces, the density of states in the normal part, averaged over its thickness, and at energies E right above the gap edge $E_g \simeq 3.12 E_{Th}$, is $\nu \propto 1/\pi V\sqrt{(E-E_g)/\Delta_0^3}$, where $\Delta_0=(E_g\delta^2)^{1/3}$, $\delta=1/(\nu_0 V)$, and $V=L_xL_yL_z$ is the volume of the normal region. This dependence is reminiscent of the density of states at the edge of a Wigner semicircle in random matrix theory (RMT), Δ_0 being the effective level spacing right above the gap edge. Using this analogy, Vavilov *et al.*[8] realized that the disorder-averaged DOS should not display a real gap, but have exponentially small tails below the gap edge, analogous to the Tracy-Widom tails[11] in RMT. A rigorous study in terms of a supersymmetric σ model description of the SNS structure has shown that this is indeed the case.[9] However, in analogy to the theory of Lifshits tails[12] in disordered conductors, the nature of the resulting subgap quasiparticle states depends additionally on the effective dimensionality d, determined by comparing the interface length scales L_x,L_y, with the typical length scale of a subgap quasiparticle state, $L_\perp$. In particular, if $L_x \gg L_\perp > L_y$ or $L_x, L_y \gg L_\perp$ the subgap quasiparticle states are localized either in the x direction or in the x-y plane along the interface, respectively. Correspondingly, the asymptotic tails of the DOS deviate from the universal RMT result, applicable only in the zero-dimensional case $(L_x, L_y < L_\perp)$.

The analogy with RMT applies, within the appropriate symmetry class, to other physical situations, such as diffusive superconductors containing magnetic impurities,[8,13,14] and superconductors with inhomogeneous coupling constants.[15] In both cases, at mean field level the density of states has a square-root singularity close to the gap edge.[16,17] Correspondingly, accounting for mesoscopic RM-like fluctuation, the disorder-averaged density of states has tails below the gap edge, with an asymptotics similar to the one calculated in Ref. 9 for SNS structures. On the other hand, in the case of diffusive superconductors containing magnetic impurities, it was shown[18,19] that, in addition to *mesoscopic fluctuations*, subgap quasiparticle states can form as a result of *classical fluctuations*, i.e., long-wave fluctuations of the concentration of magnetic impurities associated with their Poissonian statistics. Similarly, also in superconductors with inhomogeneous coupling constant long-wave fluctuations of the coarse-grained gap lead to the appearance of subgap quasiparticle states, and consequently to tails of the DOS.[17] Interestingly, in both cases the tails originating from mesoscopic fluctuations and from classical ones are formally related by a dimensional reduction.[18]

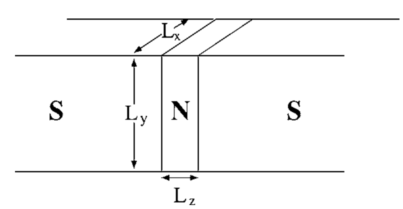

FIG. 1. A schematic plot of a SNS junction: two bulk superconducting electrodes (S) connected to a diffusive metal (N) of thickness L_z. The interfaces have linear size L_x,L_y.

In this paper, we close this set of analogies, studying the contribution to the subgap tails of the DOS in a diffusive SNS junction arising from long-wave fluctuations of the concentration of impurities in the normal layer. Combining the results of this analysis with those obtained by Ostrovsky, Skvortsov, and Feigel'man,[9] who considered the subgap tails originating from mesoscopic fluctuations, we provide a consistent picture of the physics of the subgap states. In particular, a quantitative comparison of the two contribution shows that mesoscopic fluctuations dominate in long and dirty junctions, while classical fluctuations dominate in wider and/or cleaner ones. In analogy with diffusive superconductors with magnetic impurities, and superconductors with inhomogeneous coupling constants, also in the present case the two contributions to the subgap tails, arising from mesoscopic and classical fluctuations, are related by a dimensional reduction.

The rest of the paper is organized as follows. In Sec. II we present the details of the analysis of the subgap DOS arising from fluctuations of the concentration of impurities n_{imp} in an SNS junction. In Sec. III, we compare the two contributions to the subgap DOS associated to mesoscopic and classical fluctuations. In Sec. IV, we present our conclusions.

II. SUBGAP DOS ASSOCIATED WITH FLUCTUATIONS OF n_{imp}

Let us start by considering a diffusive metallic layer between two superconducting bulk electrodes, a geometry represented schematically in Fig. 1. Assuming $k_F l \gg 1$, where l is the mean free path, this system can be described in terms of the quasiclassical approximation. In particular, at mean-field level (i.e., neglecting both mesoscopic and classical fluctuations), neglecting electron-electron interaction, and assuming the thickness of the metallic layer $L_z \gg l$, one can describe the SNS structure by the Usadel equation[20,21]

$$\frac{D}{2}\nabla^2\theta + iE\,\sin(\theta) = 0, \tag{1}$$

where $D = v_F^2\tau/3$ is the diffusion constant, and E is the energy measured from the Fermi level, assumed to be $|E| \ll \Delta$, where Δ is the gap in the bulk electrodes. The field θ is related to the quasiclassical Green's functions and the anomalous Green's function by the relations $g(\mathbf{r},E) = \cos[\theta(\mathbf{r},E)]$, $f(\mathbf{r},E) = i\sin[\theta(\mathbf{r},E)]$. In addition, assuming the interfaces to be perfectly transparent, the proximity to the two superconducting regions can be described by the boundary conditions $\theta(z = \pm L_z/2) = \pi/2$.

It is convenient to measure all lengths in units of L_z, and set $\theta = \pi/2 + i\Psi$. Therefore, Eq. (1) becomes

$$\nabla^2\Psi + 2\frac{E}{E_{Th}}\cosh(\Psi) = 0, \tag{2}$$

where $E_{Th} = D/L_z^2$ is the Thouless energy. The boundary conditions for the field Ψ are simply $\Psi(z = \pm 1/2) = 0$.

In terms of Ψ the DOS is $\nu = 2\nu_0 \mathrm{Im}[\sinh(\Psi)]$, where ν_0 is the density of states of the normal metal at the Fermi level. The DOS can be calculated by looking for solutions of Eq. (2) uniform in the x-y plane.[4–6,9] In particular, for $E < E_g \equiv C_2 E_{Th} (C_2 \simeq 3.122)$ all solutions of Eq. (2) are real, implying $\nu = 0$. Therefore, one identifies E_g with the proximity-induced gap within the normal metal layer. The mean-field DOS right above E_g averaged over the z direction is found to be

$$\nu \simeq 3.72\nu_0\sqrt{\frac{E - E_g}{E_g}}. \tag{3}$$

Let us proceed by analyzing the tails of the DOS at energies $E < E_g$ arising from fluctuations of the concentration of impurities, i.e., long-wave inhomogeneities in the x-y plane of $1/\tau$. We first consider a SNS structure such that the linear size of the SN interfaces is much larger than the thickness of the metallic layer ($L_x, L_y \gg L_z$). In the framework of the Usadel description of the metallic layer [Eq. (2)] one can account for long-wave transversal fluctuations of the concentration of impurities by promoting E_{Th}, or equivalently $E_g = C_2 E_{Th}$, to be a position-dependent random variable, characterized by the statistics

$$E_g(\mathbf{x}) = E_g + \delta E_g(\mathbf{x}), \tag{4}$$

$$\langle \delta E_g(\mathbf{x}) \rangle = 0, \tag{5}$$

$$\langle \delta E_g(\mathbf{x})\delta E_g(\mathbf{x}') \rangle = \frac{E_g^2}{n_d L_z^d}\delta(\mathbf{x} - \mathbf{x}'), \tag{6}$$

where d is the effective dimensionality of the system and n_d the effective concentration of impurities. As shown below, d is determined by comparing the linear sizes of the interface L_x, L_y to the linear scale of the subgap states $L_\perp \simeq L_z/[(E_g - E)/E_g]^{1/4}$. If $L_x, L_y \gg L_\perp$ the system is effectively two dimensional, and $n_2 = n_{imp} L_z$. On the other hand, if $L_x < L_\perp \ll L_y$ (or $L_y < L_\perp \ll L_x$), the system is effectively one dimensional, and $n_1 = n_{imp} L_z L_x$.

Accounting for these fluctuations, the Usadel equation Eq. (2) becomes

$$\partial_z^2\Psi + \nabla_{\mathbf{x}}^2\Psi + 2C_2\frac{E}{E_g}[1 - \delta\epsilon_g(\mathbf{x})]\cosh(\Psi) = 0, \tag{7}$$

where $\delta\epsilon_g = \delta E_g / E_g$.

Our purpose is to calculate the DOS averaged over fluctuations of δE_g at energies $E < E_g$. For this sake, let us introduce $\delta E = E_g - E$, and $\delta\Psi(z,\mathbf{x}) = \Psi(z,\mathbf{x}) - \Psi_0(z)$, where Ψ_0 is the solution of Eq. (2) at $E = E_g$. Expanding Eq. (7) and keeping the lowest-order nonlinearity in $\delta\Psi$ one obtains

$$[\partial_z^2 + f_0(z)]\delta\Psi + \nabla_{\mathbf{x}}^2\delta\Psi + \frac{g_0(z)}{2}\delta\Psi^2 = g_0(z)(\delta\epsilon - \delta\epsilon_g), \tag{8}$$

where $\delta\epsilon = \delta E / E_g$, $g_0(z) = 2C_2\cosh[\Psi_0(z)]$, and $f_0(z) = 2C_2\sinh[\Psi_0(z)]$.

In order to simplify further Eq. (8), it is useful to notice that the operator $\mathcal{H} = -\partial_z^2 - f_0(z)$, diagonalized with zero boundary conditions at $\pm 1/2$, admits an eigenstate Φ_0 with zero eigenvalue. Physically, Φ_0 determines the shape of the mean-field z-dependent DOS obtained from Eq. (2). Therefore, it is natural to set

$$\delta\Psi(z,\mathbf{x}) \simeq \sqrt{A_1/A_2}\chi(\mathbf{x})\Phi_0(z), \tag{9}$$

with $A_1=\int dz\, g_0\Phi_0 \simeq 7.18$, and $A_2=\int dz(g_0/2)\Phi_0^3 \simeq 2.74$.

Substituting Eq. (9) in Eq. (8), and projecting the resulting equation on Φ_0, one obtains

$$\nabla^2\chi + \chi^2 = \delta\epsilon - \delta\epsilon_g(\mathbf{x}), \tag{10}$$

where we rescaled the length by $(A_1A_2)^{-1/4}$, and

$$\langle \delta\epsilon_g(\mathbf{x})\delta\epsilon_g(\mathbf{x}')\rangle = \eta\ \delta(\mathbf{x}-\mathbf{x}'), \tag{11}$$

with $\eta \equiv (A_1A_2)^{1/4}/(n_d L_z^d)$.

Let us now split $\chi=-u+iv$, and obtain the system

$$-\nabla^2 u + u^2 - v^2 = \delta\epsilon - \delta\epsilon_g, \tag{12}$$

$$-\frac{1}{2}\nabla^2 v + u\ v = 0. \tag{13}$$

Interestingly, this set of equations is analogous to the equations obtained by Larkin and Ovchinikov in the context of the study of gap smearing in inhomogeneous superconductors,[17] and to the equations obtained by Silva, and Ioffe in the context of the study of subgap tails in diffusive superconductors containing magnetic impurities.[18]

Let us now proceed with the calculation of the DOS. In the present notation, the DOS averaged over the thickness of the normal layer is given by

$$\frac{\nu[\mathbf{x},\delta\epsilon|\delta\epsilon_g(\mathbf{x})]}{\nu_0} \simeq 3.72\ v[\mathbf{x},\delta\epsilon|\delta\epsilon_g(\mathbf{x})]. \tag{14}$$

We are interested in calculating the average density of states $\langle\nu\rangle/\nu_0 \simeq 3.72\langle v\rangle$ at energies below the Thouless gap ($\delta\epsilon > 0$). In this parameter range, the corresponding functional integral

$$\langle v\rangle \simeq \frac{\int D(\delta\epsilon_g) v[\mathbf{x},\delta\epsilon|\delta\epsilon_g(\mathbf{x})]\exp(-1/(2\eta)\int d\mathbf{x}[\delta\epsilon_g(\mathbf{x})]^2)}{\int D(\delta\epsilon_g)\exp(-1/(2\eta)\int d\mathbf{x}[\delta\epsilon_g(\mathbf{x})]^2)}, \tag{15}$$

receives its most important contributions by exponentially rare instanton configurations of $\delta\epsilon_g$ such that, at specific locations along the interfaces of the junction, $\delta\epsilon_g(\mathbf{x}) \geqslant \delta\epsilon$. The remaining task is to select among all these fluctuations the one that dominates the functional integral Eq. (15), i.e., the *optimal fluctuation*.

The action associated with a configuration of $\delta\epsilon_g$ is

$$S = \frac{1}{2\eta}\int d\mathbf{x}(\delta\epsilon_g)^2 \simeq \int d\mathbf{x}(\nabla^2 u - u^2 + \delta\epsilon)^2, \tag{16}$$

where we used Eq. (12) to express $\delta\epsilon_g$ in terms of u, v and, with exponential accuracy, neglected the term v^2 in the action. In order to find the optimal fluctuation one has to find a nontrivial saddle point u_0 of S, tending asymptotically to the solution of the homogeneous problem ($u_0 \to \sqrt{\delta\epsilon}$), and subject to the constraint of having nontrivial solutions for v of Eq. (13).

Since the normal metal layer is diffusive, and momentum scattering isotropic, it is natural to assume the optimal fluctuation to be spherically symmetric. The Euler-Lagrange equation associated with S is

$$\left(-\frac{1}{2}\Delta^{(d)} + u\right)(\Delta^{(d)}u - u^2 + \delta\epsilon) = 0 \tag{17}$$

where

$$\Delta^{(d)} \equiv \partial_r^2 + \frac{d-1}{r}\partial_r \tag{18}$$

is the radial part of the Laplacian in spherical coordinates. An obvious solution to Eq. (17) is obtained by setting

$$\Delta^{(d)}u - u^2 + \delta\epsilon = 0. \tag{19}$$

This equation is equivalent to the homogeneous Usadel equation with uniform E_g, i.e., Eq. (10) with $\delta\epsilon_g=0$. Although this equation has definitely nontrivial instanton solutions for u with the appropriate asymptotics, it is possible to show that the constraint of Eq. (13) is satisfied only by $v=0$. This is physically obvious since Eq. (19) describes a uniform system where all long-wave fluctuations of $1/\tau$ have been suppressed, and thus, within the present approximation scheme, the subgap DOS must vanish. However, it should be pointed out that, accounting for mesoscopic fluctuations, the instanton solutions of Eq. (19) describe the optimal fluctuation associated with mesoscopic gap fluctuations, as shown in Ref. 9.

Let us now look for the nontrivial saddle point. Equation (17) is equivalent to the system

$$\left(-\frac{1}{2}\Delta^{(d)} + u\right)h = 0, \tag{20}$$

$$\Delta^{(d)}u - u^2 + \delta\epsilon = h, \tag{21}$$

which can be reduced to a single second-order instanton equation setting $h=(2\partial_r u)/r$. With this substitution, Eq. (20) becomes the derivative of Eq. (21), which now reads

$$\Delta^{(d-2)}u - u^2 + \delta\epsilon = 0. \tag{22}$$

Notice that this equation is, upon reduction of the dimensionality by 2, identical in form to the one associated with mesoscopic fluctuations, Eq. (19). As we will see later, this reduction of dimensionality relates in a similar way the dependence of the action associated with classical and mesoscopic fluctuations on $\delta\epsilon$.

It is now straightforward to see that the instanton solution u_0 of this equation with the appropriate asymptotics describes indeed the optimal fluctuation, the constraint of Eq. (13) being automatically satisfied by virtue of Eq. (20), with $v_0 \propto (2\partial_r u_0)/r$. Moreover, the corresponding optimal fluctuation of $\delta\epsilon_g$ is $\delta\epsilon_g = 2\partial_r u_0/r$.

It is clear that the instanton solutions of Eq. (22) must have the form $u_0=\sqrt{\delta\epsilon}\,\upsilon(r/\lambda)$, with $\lambda=1/(\delta\epsilon)^{1/4}$. The corresponding equation for $\upsilon(r)$ is $\partial_r^2\upsilon+(d-3)/r\partial_r\upsilon-\upsilon^2+1=0$. The instanton solution of this equation can be easily found numerically, and the corresponding action S calculated. The result is

$$S_d = a_d n_d L_z^d \delta\epsilon^{(8-d)/4} \tag{23}$$

where the constants a_d are $a_1 \simeq 0.88$ and $a_2 \simeq 7.74$.

Within our approximation scheme, the density of states is $\langle \nu \rangle \propto W \exp(-S)$, where W is a prefactor due to Gaussian fluctuations around the instanton saddle point. The calculation of W can be performed using the standard technique due to Zittarz and Langer,[22] and is similar to those reported in Refs. 18,17. To leading order in the saddle point approximation, the final result is

$$\frac{\langle \nu \rangle}{\nu_0} \simeq \beta_d \sqrt{n_d L_z^d} \delta\epsilon^{[d(10-d)-12]/8} e^{-S_d}, \tag{24}$$

where $\beta_1 \approx 0.1$ and $\beta_2 \approx 0.5$.

The result in Eq. (24) relies on a saddle point approximation, which is justified provided $S_d \gg 1$. This translates into the condition

$$\delta\epsilon \gg \left(\frac{1}{a_d n_d L_z^d} \right)^{4/(8-d)}. \tag{25}$$

As mentioned before, the effective dimensionality, and therefore the asymptotic density of states, is determined by comparing the linear size of the optimal fluctuation, in dimensionfull units $L_\perp \simeq L_z \lambda = L_z / \delta\epsilon^{1/4}$, to the linear dimensions of the interfaces L_x, L_y. If $L_x, L_y \gg L_\perp$ the asymptotics is effectively two dimensional ($d=2$), while for $L_y \gg L_\perp$, $L_x \ll L_\perp$ the asymptotic DOS is effectively one dimensional (1D) ($d=1$). Since $L_\perp$ increases as the energy gets closer to the average gap edge, it is clear that in any finite size system the applicable asymptotics might exhibit various crossovers, $2D \to 1D \to 0D$, as $\delta\epsilon \to 0$. In particular, the tails are zero dimensional when $L_x, L_y < L_\perp$, in which case the asymptotic form of the DOS is obtained by calculating the integral

$$\frac{\langle \nu \rangle}{\nu_0} \simeq 3.72 \int \frac{d(\delta\epsilon_g)}{\sqrt{2\pi\eta_0}} \sqrt{\delta\epsilon_g - \delta\epsilon}\, e^{-\delta\epsilon_g^2/2\eta_0} \approx \frac{1}{\delta\epsilon^{3/2}} e^{-S_0}, \tag{26}$$

where $\eta_0 = 1/(n_{imp} V)$ ($V = L_x L_y L_z$) and $S_0 = 1/(2\eta_0)\delta\epsilon^2$.

III. MESOSCOPIC VS CLASSICAL FLUCTUATIONS

In the previous section we have discussed the asymptotic density of states below the Thouless gap originating from classical fluctuations, i.e., inhomogeneities in the concentration of impurities or equivalently in $1/\tau$. As discussed in the Introduction, this mechanism to generate subgap states is complementary to mesoscopic fluctuations of the gap edge.

The tails associated with mesoscopic gap fluctuations have been calculated by Ostrovsky, Feigel'man, and Skvortsov in Ref. 9. To exponential accuracy, the subgap DOS associated with mesoscopic fluctuations is $\langle \nu \rangle / \nu_0 \propto \exp(-\tilde{S}_d)$, where

$$\tilde{S}_d \simeq \tilde{a}_d G_d (\delta\epsilon)^{(6-d)/2}, \tag{27}$$

where $\tilde{a}_d$ is a constant ($\tilde{a}_0 \simeq 1.9$, $\tilde{a}_1 \simeq 4.7$, and $\tilde{a}_2 \simeq 10$), and G_d is the effective dimensionless conductance

$$G_0 = 4\pi\nu_0 D \frac{L_x L_y}{L_z}, \tag{28}$$

$$G_1 = 4\pi\nu_0 D L_x, \tag{29}$$

$$G_2 = 4\pi\nu_0 D L_z. \tag{30}$$

The scale of the optimal fluctuation associated with mesoscopic fluctuations is also $L_\perp \simeq L_z / (\delta\epsilon)^{1/4}$. Therefore, the effective dimensionality d is to be determined according to the criteria presented in the previous section.

Before discussing the comparison of mesoscopic and classical fluctuations, let us first explain the rationale behind the separation these two contributions. Although it is clear that the only physical fluctuations in a real sample are associated with fluctuations in the positions of impurities, these fluctuations can affect the DOS in two ways: (i) depress the Thouless gap edge by increasing locally the scattering rate (classical fluctuations), or (ii) take advantage of interference effects in the quasiparticle wave functions to generate quasiparticle states that couple inefficiently to the superconducting banks (mesoscopic fluctuations). It makes sense to think of two types of effects separately if the actions associated with them are very different in magnitude ($\tilde{S} \gg S$ or vice versa). Obviously, in the crossover region, where $S \approx \tilde{S}$ the separation of these two mechanisms is meaningless, because the system can take advantage of both at the same time.

With this caveat, let us proceed in the comparison of these two contributions, starting with the zero-dimensional case. Since the dimensionless conductance is $G_0 \approx E_g / \delta$, where $\delta \approx 1/(\nu_0 V)$ is the level spacing, then the $d=0$ action associated with mesoscopic fluctuations can be written as

$$\tilde{S}_0 \approx \left(\frac{\delta E}{\Delta_0} \right)^{3/2}, \tag{31}$$

where $\Delta_0 = (E_g \delta^2)^{1/3}$, where $\delta = 1/(\nu_0 V)$ is the level spacing in the metallic layer. Physically, Δ_0 can be interpreted as being the *effective* level spacing right above the gap edge. Indeed, from Eq. (3) one sees that

$$\nu \approx \frac{1}{\pi V} \sqrt{\frac{\delta E}{\Delta_0^3}}. \tag{32}$$

Therefore, the result of Eq. (31) indicates that tails originating from mesoscopic fluctuations of the gap edge are universal (in $d=0$), in accordance with the conjecture formulated in Ref. 8 on the basis of random matrix theory. In turn, in the zero-dimensional case the action associated with classical fluctuations is

$$S_0 \approx \left(\frac{\delta E}{\delta E_0} \right)^2, \tag{33}$$

where $\delta E_0 = E_g/\sqrt{n_{imp}V}$ is the scale of typical fluctuations of the gap edge associated with fluctuations of the concentration of impurities. The dimensionless parameter controlling which mechanism dominates is therefore

$$\gamma_0 = \frac{\Delta_0}{\delta E_0}. \tag{34}$$

Clearly, for $\gamma_0 \gg 1$ mesoscopic fluctuations dominate the subgap tails, while for $\gamma_0 \ll 1$ classical fluctuations give the largest contribution to the subgap DOS.[23]

Let us now write γ_0 in terms of elementary length scales. One can estimate

$$\gamma_0 \approx \frac{1}{k_F l}\frac{1}{\sqrt{k_F^2\sigma}}\frac{(L_z/l)^{7/6}}{(L_xL_y/l^2)^{1/6}} \approx \frac{1}{k_F l}\frac{(L_z/l)^{7/6}}{(L_xL_y/l^2)^{1/6}}, \tag{35}$$

where we used the fact that the scattering cross section of a single impurity σ is typically of the same order as λ_F^2. Within the assumptions of the theory, γ_0 is the ratio of two large numbers, and therefore its precise value depends on the system parameters. However, from Eq. (35) we see that making the junction longer and longer, i.e., increasing L_z, tends to favor mesoscopic fluctuations. Intuitively, this is due to the fact that as L_z increases, the dimensionless conductance of the junction diminishes while the average number of impurities increases, therefore suppressing the associated fluctuations of the gap edge. At the same time, increasing the area of the junction, or making them cleaner, reverses the situation. In summary, mesoscopic fluctuations are favored in *long and dirty* junctions, while classical fluctuations are favored in *wider and/or cleaner* ones.

Since in higher dimensionalities the linear scale of the optimal fluctuation associated with the two mechanisms is identical $[L_\perp = L_z/(\delta\epsilon)^{1/4}]$, it is possible, and physically suggestive, to reduce the form of the actions in $d=1$, 2 to a zero-dimensional action calculated within the typical volume of the optimal fluctuation. The latter is $V_\perp = L_xL_\perp L_z$ for $d=1$, and $V_\perp = L_\perp^2 L_z$ in $d=2$. For example, for $d=1$ one can write

$$S_1 \approx n_{imp}L_xL_\perp L_z(\delta\epsilon)^2 \approx \left(\frac{\delta E}{\delta E_{eff}}\right)^2, \tag{36}$$

where $\delta E_{eff} = E_g/\sqrt{n_{imp}V_\perp}$. Similarly,

$$\tilde{S}_1 \approx \left(\frac{\delta E}{\Delta_{eff}}\right)^2, \tag{37}$$

where $\Delta_{eff} = (E_g\delta_{eff}^2)^{1/3}$, $\delta_{eff} = 1/(\nu_0 V_\perp)$ being the level spacing in the volume of the optimal fluctuation. In analogy to the zero-dimensional case, one is therefore led to conclude that also for one-dimensional tails long and dirty junctions are dominated by mesoscopic fluctuations, while wider and/or cleaner junctions favor classical ones. This qualitative statement is indeed correct, but the proof is complicated by the energy dependence $L_\perp$.

The appropriate way to proceed for $d=1,2$ is to write the actions associated with classical and mesoscopic fluctuations in compact form as

$$S = \left(\frac{E_g - E}{\delta E_d}\right)^{(8-d)/4}, \tag{38}$$

$$\tilde{S} = \left(\frac{E_g - E}{\Delta_d}\right)^{(6-d)/4}, \tag{39}$$

where $\delta E_d = E_g/(a_d n_d L_z^d)^{4/(8-d)}$ and $\Delta_d = E_g/(\tilde{a}_d G_d)^{4/(6-d)}$. Therefore, the dimensionless parameter that determines which contributions dominates the subgap DOS is

$$\gamma_d \equiv \frac{\Delta_d}{\delta E_d}. \tag{40}$$

If $\gamma_d \gg 1$, the subgap DOS is dominated by mesoscopic gap fluctuations, and the applicable result is Eq. (27). On the other hand, for $\gamma_d \ll 1$ the DOS below the gap is determined by long-wave fluctuations of $1/\tau$ [Eq. (24)]. Finally, estimating γ_d in terms of elementary length scales, one obtains

$$\gamma_1 \approx \frac{1}{(k_F l)^{16/35}}\frac{(L_z/l)^{8/7}}{(L_x/l)^{8/35}}, \tag{41}$$

$$\gamma_2 \approx \frac{1}{(k_F l)^{2/3}}(L_z/l). \tag{42}$$

In analogy to Eq. (35), the fact that γ_d is proportional to a power of L_z/l implies that mesoscopic fluctuations are dominant in long junctions, while the inverse proportionality of γ_d on a power of $k_F l$ and of the linear size of the interface (in $d=0,1$) implies that wide interfaces and/or cleaner samples may favor the contribution arising from classical fluctuations.

IV. CONCLUSIONS

In this paper, we discussed the effect of inhomogeneous fluctuations of the concentration of impurities, or equivalently of $1/\tau$, on the tails of the DOS below the Thouless gap in diffusive SNS junctions. We have shown that these classical fluctuations lead to the formation of subgap quasiparticle states and are complementary to mesoscopic fluctuations in determining the asymptotic DOS. Finding the dimensionless parameter that controls which mechanism gives the dominant contribution to the subgap tails, one finds that, qualitatively, mesoscopic fluctuations are favored in long and dirty junctions, while classical ones dominate in wider and/or cleaner ones.

We have observed that, as for diffusive superconductors containing magnetic impurities, and for diffusive superconductors with an inhomogeneous coupling constant, the two contributions are formally related by a dimensional reduction by 2, both at the level of instanton equations determining the optimal fluctuation, and in the dependence of the DOS on the distance from the gap edge $\delta\epsilon$. As in other physical systems,[24] it is natural to expect that supersymmetry is at the root of dimensional reduction also in this context. This fact could in principle be elucidated generalizing the σ model describing mesoscopic fluctuations to include the physics associated with classical fluctuations.

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