Critical behavior of generic competing systems

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Generic higher character Lifshitz critical behaviors are described using field theory and ϵ_L expansion renormalization-group methods. These critical behaviors describe systems with arbitrary competing interactions. We derive the scaling relations and the critical exponents at the two-loop level for anisotropic and isotropic points of arbitrary higher character. The framework is illustrated for the N-vector ϕ^4 model describing a d-dimensional system. The anisotropic behaviors are derived in terms of many independent renormalization-group transformations, each one characterized by independent correlation lengths. The isotropic behaviors can be understood using only one renormalization-group transformation. Feynman diagrams are solved for the anisotropic behaviors using a different dimensional regularization associated to a generalized orthogonal approximation. The isotropic diagrams are treated using this approximation as well as with a different exact technique to compute the integrals. The entire procedure leads to the analytical solution of generic loop order integrals with arbitrary external momenta. The property of universality class reduction is also satisfied when the competing interactions are turned off. We show how the results presented here reduce to the usual m-fold Lifshitz critical behaviors for both isotropic and anisotropic criticalities.

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I. INTRODUCTION

Field-theoretic renormalization-group techniques are invaluable tools for studying usual critical phenomena as well as the critical behavior associated to the physics of systems presenting arbitrary short-range competing interactions. The universality classes of the ordinary critical behavior are characterized by the space dimension of the system d and the number of components of the (field) order parameter N.[1,2] Competing systems, on the other hand, possess different types of space directions known as competition axes.

The simplest type of competition directions can be most easily visualized using the terminology of magnetic systems via a generalized Ising model. One permits exchange ferromagnetic couplings between nearest neighbors ($J_1>0$) *and* antiferromagnetic interactions between second neighbors ($J_2<0$) occurring along m_2 dimensions. Whenever $m_2<d$ the system presents a (usual) second character anisotropic Lifshitz critical behavior whose universality classes are characterized by (N, d, m_2), whereas the isotropic behavior characterized by $d=m_2$ close to 8 was formerly described at the same time.[3] The phenomenological model corresponding to a uniaxial anisotropy ($m_2=1$) in a cubic lattice is known as axial next-nearest-neighbor Ising (ANNNI) model.[4] In the critical region, this sort of system is characterized by a disordered, a uniformly ordered, *and* a modulated ordered phase which meet in a uniaxial Lifshitz multicritical point, where the ratio J_2/J_1 is fixed at the corresponding Lifshitz temperature T_L. High-precision numerical Monte Carlo simulations were carried out for the critical exponents of this model[5] and checked using two different two-loop analytical calculations.[6–8] From the renormalization-group perspective there is an important difference between these Lifshitz critical behaviors. The anisotropic behaviors have two independent correlation lengths, ξ_{L2} perpendicular to the competing axes as well as ξ_{L4} parallel to the m_2 competing axes. The isotropic behavior has only one correlation length ξ_{L4}. (For an alternative field-theoretic approach for m-axial Lifshitz points, see Refs. 9 and 10.)

If we go on to include ferromagnetic couplings up to the third neighbors ($J_3>0$) along a single axis, the system will present a uniaxial third character Lifshitz point whenever J_2/J_1 and J_3/J_1 take certain fixed values at the corresponding Lifshitz temperature.[11] When this sort of competition takes place along m_3 spatial directions, the system presents a m_3-fold third character Lifshitz point. On the other hand, if simultaneous and independent competing interactions take place between second neighbors along m_2 space directions and third neighbors along m_3 space dimensions, the system presents a *generic* third character m_3-fold Lifshitz critical behavior. The generic third character universality classes are defined by (N, d, m_2, m_3), thus describing a wider sort of critical behavior when compared with the third character universality classes (N, d, m_3).

This idea can be extended in order to define the m_L-fold Lifshitz point of character L, when further alternate couplings are permitted up to the Lth neighbors along m_L directions, provided the ratios J_L/J_1, J_{L-1}/J_1,..., J_2/J_1 take special values at the associated Lifshitz temperature.[12–14] However, the most general anisotropic situation is to consider *several* types of competing axes occurring simultaneously in the system such that second neighbors interact along m_2 space directions, m_3 directions couple third neighbors, etc., up to the interactions of L neighbors along m_L dimensions, with all competing axes perpendicular among each other. In that case, the corresponding critical behavior is called a generic Lth character Lifshitz critical behavior.[15]

In this work we shall undertake an exploration of the field-theoretical renormalization-group structure of the most general competing system using ϵ_L-expansion techniques for

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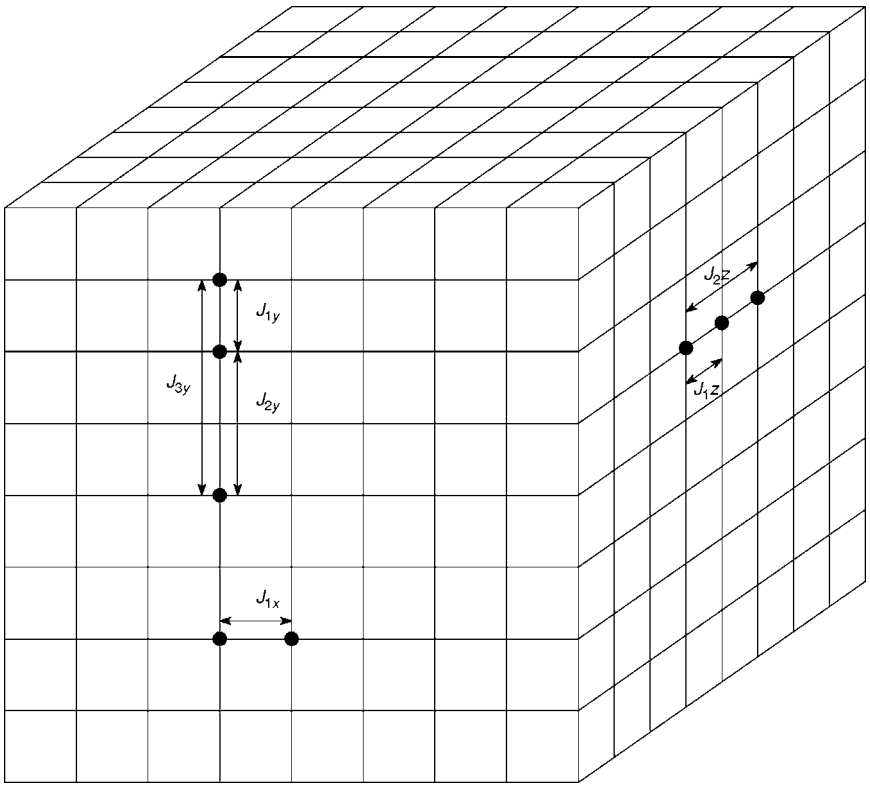

FIG. 1. The simplest example of the CECI model with uniaxial competing interactions between second neighbors as well as uniaxial couplings between third neighbors. This system has three independent correlation lengths and presents a generic third character anisotropic Lifshitz critical behavior. Note that turning off the interactions among second neighbors leads to the simpler third character critical behavior.

anisotropic and isotropic higher character Lifshitz critical behaviors. A rather brief description of this structure was set forth in a previous paper;[15] here we shall present the details and extend the formalism in order to incorporate the exact two-loop calculation for arbitrary isotropic higher character criticalities. Renormalization-group (RG) arguments are constructed in order to find out the scaling relations for the anisotropic as well as the isotropic critical behaviors. The discussion parallels that for the usual second character Lifshitz points.[8]

The arbitrary competing exchange coupling Ising model (CECI model) is the lattice model associated to this interesting critical behavior. It has a general renormalization-group structure which contains many independent length scales, and its construction can be utilized for both anisotropic and isotropic cases. In the anisotropic cases, the system has only nearest-neighbor interactions along $(d-m_2-\cdots-m_L)$, second-neighbor competing interactions along m_2 dimensions, and so on, up to Lth-neighbor competing interactions along m_L spatial directions. The distinct competing axes originate several types of independent correlation lengths, namely ξ_1 for directions parallel to the $(d-m_2-\cdots-m_L)$-dimensional noncompeting subspace, ξ_2 for directions parallel to the m_2-dimensional competing subspace, etc., and ξ_L characterizing the m_L-dimensional subspace. The simplest representative of the CECI model is better understood with the help of Fig. 1, which is the particular case $m_2=m_3=1$, $m_4=\cdots=m_L=0$. There are two competing subspaces and three types of correlation lengths which define three independent renormalization-group transformations. It defines a particular generic third character anisotropic Lifshitz critical behavior.

In the phase diagram of the ANNNI model, the parameters which are varied are the temperature T and $p=J_2/J_1$ which take a particular value at the uniaxial second character Lifshitz multicritical point as depicted in Fig. 2. It is a particular case of the CECI model whenever $m_2=1$, with $m_3=\cdots=m_L=0$. Although the ANNNI model has applications in several real physical systems (see, for example, Ref. 4), the prototype of second character Lifshitz points in magnetic materials is manganese phosphide (MnP). Experimental as well as theoretical investigations have determined that MnP presents a pure uniaxial Lifshitz point ($m_2=1$, $d=3$, $N=1$).[16,17]

When adding further competing interactions to the ANNNI model, the number of parameters in the phase diagram increases.[12] For instance, in the phase diagram of the model including uniaxial competing interactions up to third neighbors the parameters to be varied are the temperature T, $p_1=J_2/J_1$, and $p_2=J_3/J_1$. One can locate the third character Lifshitz point by looking at the projection of the phase diagram in the plane (p_1, p_2), as was demonstrated using numerical means.[12] In the example of the CECI model displayed in Fig. 1, let the competing exchange be completely independent along the different competing axes. In that case, the phase diagram can be described by T, $p_z=J_{2z}/J_{1z}$, $p_{1y}=J_{2y}/J_{1y}$, $p_{2y}=J_{3y}/J_{1y}$. A useful two-dimensional representation can be obtained by separating the phase diagrams in two parts. The diagram (T, p_z) characterizing the second charac-

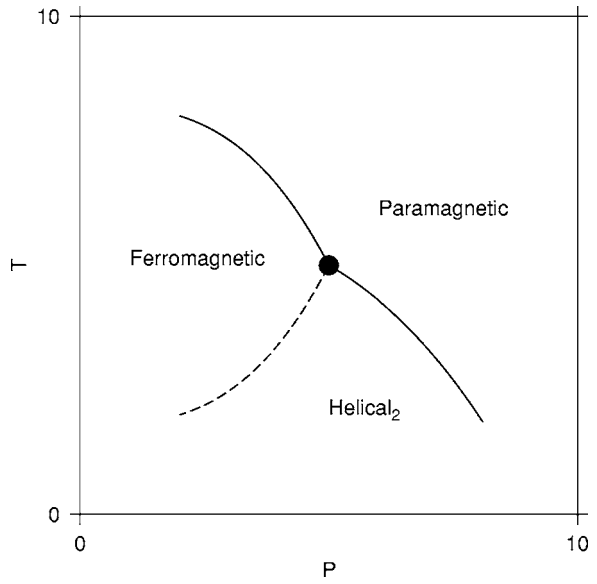

FIG. 2. The phase diagram of a typical uniaxial second character Lifshitz critical behavior. The dashed lines indicate a first-order transition between the uniformly ordered and modulated ordered phase which terminates at the Lifshitz point of second character. The parameter p is defined by $p=J_2/J_1$.

ter behavior (with p_{1y}, p_{2y} fixed) and the diagram (p_{1y}, p_{2y}) (with T, p_z fixed) corresponding to the third character behavior can be ploted independently. The superposition of the two diagrams at the generic third character Lifshitz point is indicated in Fig. 3. As a consequence, there is a uniformly ordered phase and two modulated phases called Helical_2 and Helical_3 in Fig. 3 which meet at the uniaxial generic third character anisotropic Lifshitz point. Now there are two first-

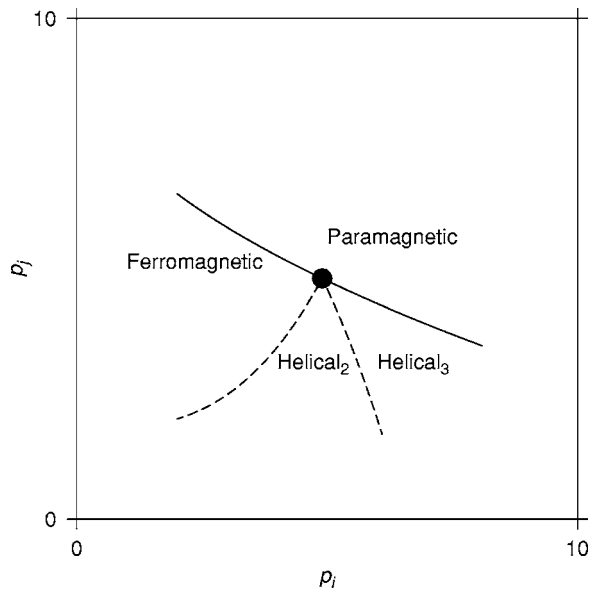

FIG. 3. The superposition of the two two-dimensional independent phase diagrams for the simplest CECI model. In this two-dimensional picture, the confluence of the two distinct modulated phases, the ferromagnetic and the paramagnetic phases occurs at the generic third character Lifshitz point.

order lines separating the ferromagnetic-Helical_2 and Helical_2-Helical_3 phases. Analogously, when there are arbitrary independent types of competing axes, we can consider several independent phase diagrams and each two-dimensional projection of them. The superposition of them in one two-dimensional diagram gives origin to a situation that resembles that illustrated in Fig. 3. Instead, there are *several* modulated phases and one uniformly ordered phase which meet at the generich Lth character anisotropic Lifshitz point. Each competing subspace has its own characteristic modulated phase along with its own independent correlation length. Although these objects go critical simultaneously at the Lifshitz critical temperature, they define independent renormalization-group transformations in each different subspace.

Therefore we find multiscale scaling laws as a consequence of this renormalization-group flow independence in parameter space. This implies that we find several independent coupling constants, each one depending on a definite momenta scale characterizing the particular competition axes under consideration. Nevertheless, all coupling constants flow to the same fixed point. The universality classes of this system are characterized by the parameters (N, d, $m_2,\ldots,m_L$), therefore generalizing the usual Lifshitz behavior. It is important to mention that when we turn off all the competing interactions between third and more distant neighbors, the universality classes of the generic higher character Lifshitz point turn out to reduce to that associated to the second character behavior (N, d, m_2). Notice that these anisotropic behaviors generalize previous lattice models with competing interactions[18] as it includes all types of competing axes. The isotropic critical behaviors $d=m_n$ have a distinct feature in which there is only one type of correlation length ξ_{4n}. Their universality classes are characterized by (N, d, n) where n is the number of neighbors coupled through competing interactions.

In addition, we compute the critical exponents at least at $O(\epsilon_L^2)$ using dimensional regularization to resolving the Feynman diagrams and normalization conditions (and) or minimal subtraction as the renormalization procedures. The computation is realized in momentum space. For the anisotropic cases, the Feynman diagrams are performed with an approximation which is the most general one consistent with the homogeneity of these integrals in the external momenta scales. The isotropic situations are treated using this approximation as well, but we also present the exact calculation at the same loop order and make a comparison with the above-mentioned approximation.

We present the functional integral representation of the model in terms of a $\lambda\phi^4$ setting and define the normalization conditions for this higher character Lifshitz critical behavior in Sec. II. We show that many sets of normalization conditions, each one corresponding to a specific type of competition axes, are convenient to have a satisfactory description of the problem in its maximal generality.

In Sec. III we present the renormalization-group analysis for the anisotropic critical behaviors. We construct the several renormalization functions appropriate to each competing subspace and study their flow with the various renormalization-group transformations. We find the proper

scaling relations to each competition subspace.

Section IV discusses the renormalization-group treatment for the various isotropic behaviors. We obtain the scaling relations and show that they reduce to the usual $\lambda\phi^4$ case when the interactions beyond the first neighbors are switched off.

Section V is an exposition of the calculation of the critical exponents for the anisotropic cases using the scaling relations obtained in Sec. III. We describe two different ways of calculating the critical exponents either in normalization conditions or in minimal subtraction. We discuss the limit $L \to \infty$ and some of its implications. We point out the analogy of the Lifshitz critical region with an effective field theory which arises from a recent cosmological model including modifications of gravity in the long-distance limit.[19]

Sections VI and VII are an in-depth analysis of the critical exponents for the isotropic cases. The critical exponents for the isotropic cases are calculated using the orthogonal approximation and the scaling relations in Sec. VI. The exact calculation of the critical exponents and the comparison with those obtained from the orthogonal approximation are carried out in Sec. VII.

The particular case corresponding to the second character isotropic Lifshitz critical behavior is discussed explicitly in Sec. VIII. The resulting critical exponents are shown to generalize those obtained previously.[3] We discuss their relationship with those coming from the orthogonal approximation.

Section IX concludes this paper; a discussion of the ideas is summarized and some possible applications will be proposed. We calculate the Feynman integrals in the appendixes. We describe in detail the generalized orthogonal approximation for the calculation of higher loop integrals of the anisotropic behaviors in Appendix A. It will be shown there that one can obtain the answer in a simple analytical form for arbitrary external momenta scales. The property of homogeneity of these integrals along arbitrary external momenta scales is preserved. Then, we use the same approximation to compute diagrams for the isotropic behaviors in Appendix B. In addition, we perform the exact calculation for arbitrary isotropic cases in Appendix C. We also analyze the simple particular case associated to the usual second character isotropic behavior.

II. FIELD THEORY AND NORMALIZATION CONDITIONS FOR HIGHER CHARACTER LIFSHITZ POINTS

The field-theoretical representation can be expressed in terms of a modified $\lambda\phi^4$ field theory presenting arbitrary higher derivative terms due to the effect of competition along the different kinds of competing axes. The types of competing axes are defined by the number of neighbors that interact among each other via exchange competing couplings. Let m_n be the number of space directions whose competing interactions extend to the nth neighbor. Thus the m_n-dimensional competition subspace will be represented in the Lagrangian with even powers (up to the $2n$th) of the gradient acting on the order parameter scalar field. Thus the effect of the competition resides in the higher derivatives of the field.

The corresponding bare Lagrangian density can be written as[15]

$$L = \frac{1}{2}\left|\nabla_{\left(d-\sum_{n=2}^{L} m_n\right)}\phi_0\right|^2 + \sum_{n=2}^{L}\frac{\sigma_n}{2}|\nabla^n_{m_n}\phi_0|^2 + \sum_{n=2}^{L}\delta_{0n}\frac{1}{2}|\nabla_{m_n}\phi_0|^2 + \sum_{n=3}^{L-1}\sum_{n'=2}^{n-1}\frac{1}{2}\tau_{nn'}|\nabla^{n'}_{m_n}\phi_0|^2 + \frac{1}{2}t_0\phi_0^2 + \frac{1}{4!}\lambda_0\phi_0^4. \quad (1)$$

At the Lifshitz point, the fixed ratios among the exchange couplings explained above translate into this field-theoretic version though the conditions $\delta_{0n}=\tau_{nn'}=0$. All even momentum powers up to $2L$ become relevant in the free propagator.[20] This condition simplifies the treatment of the system since it allows the decoupling of the several competing subspaces of Feynman integrals in momentum space. It indeed makes possible to solve these diagrams to any desired order in a perturbative approach. Within the loop order chosen, we can set a perturbative regime with maximal generality as far as critical behavior of competing systems are concerned. So we need the small loop parameter, which is intimately connected to the critical dimension of the theory.

It is instructive to find the critical dimension of this field theory through the use of the Ginzburg criterion.[21–23] From the perspective of magnetic systems in the above Lagrangian density, $t_0=t_{0L}+(T-T_L)$ measures the temperature difference from the critical temperature T_L. In the mean-field approximation the inverse susceptibility is proportional to $(T-T_L)$, but has no longer this behavior when fluctuations get bigger due to the closenesss of the critical point and the mean-field argument breaks down. This immediately leads to the critical dimension $d_c=4+\Sigma_{n=2}^{L}[(n-1)/n]m_n$. Above this critical dimension the mean-field behavior dominates the system. Consequently, the small parameter for a consistent perturbative expansion is $\epsilon_L=4+\Sigma_{n=2}^{L}[(n-1)/n]m_n-d$. The same type of argument can be constructed for the isotropic behaviors. When $d=m_n$, the critical dimension is $d_c=4n$.

The renormalized theory can be defined starting from the bare Lagrangian (1). The approach we shall follow here is completely analogous to that exposed in Ref. 8 (see also Ref. 1) and the reader is invited to consult that reference in order to be familiarized with the notation employed. The renormalization functions are determined in terms of the renormalized reduced temperature and order parameter (magnetization in the context of magnetic systems) as $t=Z_{\phi^2}^{-1}t_0$, $M=Z_{\phi}^{-1/2}\phi_0$ and depend on Feynman graphs. When the theory is renormalized at the critical temperature ($t=0$), a nonvanishing external momenta must be used to define the renormalized theory. Consequently, the renormalization constants at the critical temperatute T_L depend on the external momenta scales involved in the renormalization algorithm.

Let us analyze the anisotropic behaviors. The Feynman integrals depend on various external momenta scales, namely that characterizing the $(d-m_2-\cdots-m_L)$-dimensional noncompeting subspace, a momentum scale associated to the m_2 space directions, etc., up to the momentum scale corresponding to the m_L competing axes. Thus it is appropriate to define L sets of normalization conditions in order to compute the critical exponents associated to correlations either perpendicular to or along the several types of competition axes. We define κ_1 to be the external momenta scale asssociated to the

$(d-m_2-\cdots-m_L)$-dimensional noncompeting directions. If we define the noncompeting directions to be along the m_1-dimensional subspace, where $m_1=d-m_2-\cdots-m_L$, we can unify the language by stating that κ_n is the typical external momenta scale characterizing the m_n competing axes ($n=1,\ldots, L$).

We now turn our attention to the definition of the symmetry points (SPs). In case we wish to evaluate the critical exponents along the jth type of competing axes, we set $\kappa_n=0$ for $n\neq j$ keeping, however, $\kappa_j\neq 0$. The proper normalization conditions to evaluating exponents along the various competition axes can be defined as follows. If $k'_{i(n)}$ is the external momenta along the competition axes associated to a generic one-particle irreducible (1PI) vertex part, the external momenta along the nth type of competing directions are chosen as $k'_{i(n)}k'_{j(n)}=(\kappa_n^2/4)(4\delta_{ij}-1)$. This implies that $(k'_{i(n)}+k'_{j(n)})^2=\kappa_n^2$ for $i\neq j$. The momentum scale of the two-point function is defined by $k'^2_{(n)}=\kappa_n^2=1$. The set of renormalized 1PI vertex parts is given by

$$\Gamma^{(2)}_{R(n)}(0,g_n)=0, \tag{2a}$$

$$\left.\frac{\partial\Gamma^{(2)}_{R(n)}(k'_{(n)},g_n)}{\partial k'^{2n}_{(n)}}\right|_{k'^{2n}_{(n)}=\kappa_n^{2n}}=1, \tag{2b}$$

$$\Gamma^{(4)}_{R(n)}(k'_{i(n)},g_n)\big|_{SP_n}=g_n, \tag{2c}$$

$$\Gamma^{(2,1)}_{R(n)}(k'_{1(n)},k'_{2(n)},k',g_n)\big|_{S\bar{P}_n}=1, \tag{2d}$$

$$\Gamma^{(0,2)}_{R(n)}(k'_{(n)},g_n)\big|_{k'^{2n}_{(n)}=\kappa_n^{2n}}=0. \tag{2e}$$

These L systems of normalization conditions seem to provide L renormalized coupling constants. The origin of this overcounting is a consequence of the L independent flow in the renormalization momenta scales $\kappa_1,\ldots,\kappa_L$. The analysis works with L coupling constants, namely $g_n=u_n(\kappa_n^{2n})^{\epsilon_L/2}$ [and $\lambda_n=u_{0n}(\kappa_n^{2n})^{\epsilon_L/2}$] characterizing the flow along the momenta components parallel to each m_n-dimensional competing subspace. This is really a disguise since the situation becomes simpler at the fixed point: the many couplings will flow to the same fixed point, at the two-loop level, giving a clear indication that this property is kept in higher-loop calculations. The explicit demonstration of this fact will be tackled in Sec. V. (Let us mention briefly another type of anisotropic critical behavior whose theoretical existence is granted from the CECI model structure. In magnetic systems, the absence of the ferromagnetic phase leads to a structure where several modulated phases are mixed among each other. This situation is transliterated in the condition $d=m_2+\cdots+m_L$. In that case, the normalization conditions can be defined without making reference to the noncompeting subspace. In general the critical dimension will increase and the lowest character modulated phase will play the role of the former ferromagnetic ordered phase. We shall not delve any further into this situation in this paper, but leave the analysis for future work.)

The normalization conditions for the isotropic case ($m_n=d$ near $4n$) can be defined in a close analogy to its second character isotropic particular case.[8] If k'_i is the external momenta along the m_n competition axes, the external momenta along the $4n$ directions are chosen as $k'_ik'_j=(\kappa_n^{2n}/4)(4\delta_{ij}-1)$. This implies that $(k'_i+k'_j)^2=\kappa_n^2$ for $i\neq j$. The momentum scale of the two-point function is fixed through $k'^{2n}=\kappa_n^{2n}=1$. Then we have the same normalization conditions, Eqs. (2), but now there is solely one type of external momenta scale. The others are absent in this situation as an effect of the Lifshitz condition $\delta_{0n}=\tau_{nn'}=0$.

We can express all the renormalization functions and bare coupling constants in terms of the dimensionless couplings in a unified perspective for both anisotropic and isotropic behaviors. The subscript $n=1, 2, 3,\ldots, L$ labels the different external momenta scales belonging to the general Lifshitz critical behavior, as defined above for the anisotropic and isotropic cases. Expansion of the dimensionless bare coupling constants u_{on} and the normalization constants $Z_{\phi(n)}$, $\bar{Z}_{\phi^2(n)}=Z_{\phi(n)}Z_{\phi^2(n)}$ as functions of the dimensionless renormalized couplings u_n up to two-loop order as

$$u_{on}=u_n(1+a_{1n}u_n+a_{2n}u_n^2), \tag{3a}$$

$$Z_{\phi(n)}=1+b_{2n}u_n^2+b_{3n}u_n^3, \tag{3b}$$

$$\bar{Z}_{\phi^2(n)}=1+c_{1n}u_n+c_{2n}u_n^2, \tag{3c}$$

along with dimensional regularization will be sufficient to find out all critical exponents.

III. SCALING THEORY FOR THE ANISOTROPIC CASES

The anisotropic behaviors are characterized by correlation lengths $\xi_1,\ldots,\xi_L$. When considered independently they define independent renormalization-group transformations along the several competing directions. In momentum space, they induce independent flows in each external momenta scale $\kappa_1,\ldots,\kappa_L$.

In order to define the renormalized vertex parts we consider a set of cutoffs Λ_j ($j=1,\ldots, L$), each of them characterizing a different competing subspace. As functions of the bare vertices and normalization constants they read

$$\begin{aligned}\Gamma^{(N,M)}_{R(n)}&(p_{i(n)},Q_{i(n)},g_n,\kappa_n)\\ &=Z^{N/2}_{\phi(n)}Z^M_{\phi^2(n)}(\Gamma^{(N,L)}(p_{i(n)},Q_{i(n)},\lambda_n,\Lambda_n)\\ &\quad-\delta_{N,0}\delta_{L,2}\Gamma^{(0,2)}_{(n)}(Q_{(n)},Q_{(n)},\lambda_n,\Lambda_n)\big|_{Q^2_{(n)}=\kappa_n^2}),\end{aligned} \tag{4}$$

where $p_{i(n)}$ ($i=1,\ldots, N$) are the external momenta associated to the vertex functions $\Gamma^{(N,L)}_{R(n)}$ with N external legs and $Q_{i(n)}$ ($i=1,\ldots, M$) are the external momenta associated to the M insertions of ϕ^2 operators. From the last section, u_{0n}, $Z_{\phi(n)}$, and $Z_{\phi^2(n)}$ are represented as power series in u_n. In order to write the renormalization-group equations in terms

of dimensionless bare and renormalized coupling constants, we shall discuss the central idea which underlies the subsequent scaling theory.

Consider the volume element in momentum space for calculating an arbitrary Feynman integral. It is given by $d^{d-\Sigma_{i=2}^{L}m_i}q\Pi_{i=2}^{L}d^{m_i}k_{(i)}$. Recall that $\vec{q}$ represents a $(d-\Sigma_{i=2}^{L}m_i)$-dimensional vector perpendicular to the competing axes and $\vec{k}_{(i)}$ denotes an m_i-dimensional vector along the ith competing subspace, respectively. The Lifshitz condition $\delta_{0n}=\tau_{nn'}=0$ suppresses the quadratic part of the momentum along the m_2 competition axes, the quadratic and quartic part of the momentum along the m_3 competing directions, and so on, such that the m_L competing subspace is represented by a $2L$th power of momentum in the inverse free critical $(t=0)$ propagator, i.e., $G_0^{(2)-1}(q,k)=q^2+\Sigma_{n=2}^{L}\sigma_n(k_{(n)}^2)^n$. In order to be dimensionally consistent, the canonical dimension in mass units of the various terms in the propagator should be equal.

Our normalization conditions give us a hint that we can get rid of the σ_n parameters provided we make simultaneously dimensional redefinitions of the momenta components along each type of competition subspace in a complete analogy to the second character case. Let $[\vec{q}]=M$ be the mass dimension of the quadratic momenta. Since all momentum terms in the propagator should have the same canonical dimension, this requires that $[\vec{k}_{(i)}]=M^{1/i}$. These simultaneous dimensional redefinitions of the momenta along the competing axes is only possible due to the Lifshitz condition. The volume element in momentum space $d^{d-\Sigma_{i=2}^{L}m_i}q\Pi_{i=2}^{L}d^{m_i}k_{(i)}$ has mass dimension $M^{d-\Sigma_{i=2}^{L}[(i-1)m_i/i]}$. The dimension of the field can be found from the requirement that the volume integral of the Lagrangian density (1) is dimensionless in mass units. In other words, one obtains $[\phi]=M^{(1/2)\{d-\Sigma_{i=2}^{L}[(i-1)m_i/i]\}-1}$. In momentum space the one-particle irreducible (1PI) vertex functions have canonical dimension $[\Gamma^{(N)}(k_i)]=M^{N+\{d-\Sigma_{i=2}^{L}[(i-1)m_i/i]\}-N\{d-\Sigma_{i=2}^{L}[(i-1)m_i/i]\}/2}$.

Let us describe the theory in terms of dimensionless parameters. As the coupling constants are associated to $\Gamma^{(4)}$, we can write $g_n=u_n(\kappa_n^{2n})^{\epsilon_L/2}$, and $\lambda_n=u_{0n}(\kappa_n^{2n})^{\epsilon_L/2}$, where $\epsilon_L=4+\Sigma_{n=2}^{L}[(n-1)/n]m_n-d$. Expressed in terms of these dimensionless coupling constants, the renormalization-group equation can be cast in the form

$$\left(\kappa_n\frac{\partial}{\partial\kappa_n}+\beta_n\frac{\partial}{\partial u_n}-\frac{1}{2}N\gamma_{\phi(n)}(u_n)+L\gamma_{\phi^2(n)}(u_n)\right)\Gamma_{R(n)}^{(N,L)}=\delta_{N,0}\delta_{L,2}(\kappa_n^{-2n})^{\epsilon_L/2}B_n(u_n). \tag{5}$$

The functions

$$\beta_n=\left(\kappa_n\frac{\partial u_n}{\partial\kappa_n}\right), \tag{6a}$$

$$\gamma_{\phi(n)}(u_n)=\beta_n\frac{\partial lnZ_{\phi(n)}}{\partial u_n}, \tag{6b}$$

$$\gamma_{\phi^2(n)}(u_n)=-\beta_n\frac{\partial lnZ_{\phi^2(n)}}{\partial u_n} \tag{6c}$$

are calculated at fixed bare coupling λ_n. The β_n functions can be rewitten in terms of dimensionless quantities as

$$\beta_n=-n\epsilon_L\left(\frac{\partial lnu_{0n}}{\partial u_n}\right)^{-1}. \tag{7}$$

Note that the beta function corresponding to the flow in κ_n has a factor of n compared to that associated to the flow in κ_1.

For the anisotropic case, the multiparameters' group of invariance is manifest in the solution of the renormalization-group equation, which is given by

$$\Gamma_{R(n)}^{(N)}(k_{i(n)},u_n,\kappa_n)=\exp\left[-\frac{N}{2}\int_1^{\rho_n}\gamma_{\phi(n)}(u_n(\rho_n))\frac{dx_n}{x_n}\right]\times\Gamma_{R(n)}^{(N)}(k_{i(n)},u_n(\rho_n),\kappa_n\rho_n). \tag{8}$$

From the above analysis, the dimensional redefinitions of the momenta along the distinct competing axes result in an effective space dimension for the anisotropic case, namely $\{d-\Sigma_{n=2}^{L}[(n-1)/n]m_n\}$. We discover the following behavior for the 1PI vertex parts $\Gamma_{R(n)}^{(N)}$ under flows in the external momenta:

$$\Gamma_{R(n)}^{(N)}(\rho_n k_{i(n)},u_n,\kappa_n)=\rho_n^{n(N+\{d-\Sigma_{n=2}^{L}[(n-1)/n]m_n\}-N\{d-\Sigma_{n=2}^{L}[(n-1)/n]m_n\}/2)}\times\exp\left(-\frac{N}{2}\int_1^{\rho_n}\gamma_{\phi(n)}(u_n(x_n))\frac{dx_n}{x_n}\right)\times\Gamma_{R(n)}^{(N)}(k_{i(n)},u_n(\rho_n),\kappa_n). \tag{9}$$

The behavior of the vertex functions at the infrared regime is worthwhile, since their fixed point structure will determine the scaling laws and the critical exponents for arbitrary m_n-dimensional competing subspace. These L independent fixed points are defined by $\beta_n(u_n^*)=0$. At the fixed points the simple scaling property holds:

$$\Gamma_{R(n)}^{(N)}(\rho_n k_{i(n)},u_n^*,\kappa_n)=\rho_n^{n(N+\{d-\Sigma_{n=2}^{L}[(n-1)/n]m_n\}-N\{d-\Sigma_{n=2}^{L}[(n-1)/n]m_n\}/2)-(N/2)\gamma_{\phi(n)}(u_n^*)}\times\Gamma_{R(n)}^{(N)}(k_{i(n)},u_n^*,\kappa_n). \tag{10}$$

For $N=2$, we have

$$\Gamma_{R(n)}^{(2)}(\rho_n k_{(n)},u_n^*,\kappa_n)=\rho_n^{2n-\gamma_{\phi(n)}(u_n^*)}\Gamma_{R(n)}^{(2)}(k_{(n)},u_n^*,\kappa_n). \tag{11}$$

The quantity $\gamma_{\phi(n)}(u_n^*)$ can be identified as the anomalous dimension of the competing subspace under consideration.

This can be readily generalized to include M insertions of ϕ^2 operators such that the RG equations at the fixed point lead to the solution $[(N,M)\neq(0,2)]$

$$\Gamma^{(N,M)}_{R(n)}(\rho k_{i(n)},\rho p_{i(n)},u^*_n,\kappa_n)=\rho_n^{n(N+\{d-\Sigma_{n=2}^{L}[(n-1)/n]m_n\}-N\{d-\Sigma_{n=2}^{L}[(n-1)/n]m_n\}/2-2M)-N\gamma^*_{\phi(n)}/2+M\gamma^*_{\phi^2(n)}}\times\Gamma^{(N,M)}_{R(n)}(k_{i(n)},p_{i(n)},u^*_n,\kappa_n). \quad (12)$$

Writing this at the fixed point as

$$\Gamma^{(N,M)}_{R(n)}(\rho k_{i(n)},\rho p_{i(n)},u^*_n,\kappa_n)=\rho_n^{n(\{d-\Sigma_{n=2}^{L}[(n-1)/n]m_n\}-Nd_\phi)+Md_{\phi^2}}\Gamma^{(N,M)}_{R(n)}(k_{i(n)},p_{i(n)},u^*_n,\kappa_n), \quad (13)$$

the anomalous dimensions of the insertions of ϕ^2 operators are $d_{\phi^2}=-2n+\gamma_{\phi^2(n)}(u^*_n)$.

The scaling relations can be found by going away from the Lifshitz critical temperature ($t\neq 0$) staying, however, at the critical region $\delta_{0n}=\tau_{nn'}=0$, which is the generalization of that from the ordinary second character Lifshitz behavior. Above the Lifshitz critical temperature, the renormalized vertices for $t\neq 0$ can be expressed as a power series in t around the renormalized vertex parts at $t=0$, as long as $N\neq 0$. Then one can show that the RG equations for the vertex parts when $t\neq 0$ are given by

$$\left(\kappa_n\frac{\partial}{\partial\kappa_n}+\beta_n\frac{\partial}{\partial u_n}-\frac{1}{2}N\gamma_{\phi(n)}(u_n)+\gamma_{\phi^2(n)}(u_n)t\frac{\partial}{\partial t}\right)\times\Gamma^{(N)}_{R(n)}(k_{i(n)},t,u_n,\kappa_n)=0. \quad (14)$$

The key property of the solution is that it is a homogeneous function of the product of $k_{i(n)}$ (to some power) and t only at the fixed point u^*_n. As the value of u_n is fixed at u^*_n, we shall omit it from the notation of this section henceforth. Thus, at the fixed point the solution of the RG equation reads

$$\Gamma^{(N)}_{R(n)}(k_{i(n)},t,\kappa_n)=\kappa_n^{N\gamma^*_{\phi(n)}/2}F^{(N)}_{(n)}(k_{i(n)},\kappa_n t^{-1/\gamma^*_{\phi^2(n)}}). \quad (15)$$

Defining $\theta_n=-\gamma^*_{\phi^2(n)}$, and using dimensional analysis, it is easy to show that

$$\Gamma^{(N)}_{R(n)}(k_{i(n)},t,\kappa_n)=\rho_n^{n(N+\{d-\Sigma_{n=2}^{L}[(n-1)/n]m_n\}-(N/2)\{d-\Sigma_{n=2}^{L}[(n-1)/n]m_n\})-(N/2)\eta_n}\kappa_n^{(N/2)\eta_n}F^{(N)}_{(n)}(\rho_n^{-1}k_{i(n)},(\rho_n^{-1}\kappa_n)(\rho_n^{-2n}t)^{-1/\theta_n}). \quad (16)$$

The choice $\rho_n=\kappa_n(t/\kappa_n^{2n})^{[1/(\theta_n+2n)]}$, can be substituted back in Eq. (16), implying that the vertex function depends only on the combination $k_{i(n)}\xi_n$ apart from a power of t. Since the correlation lengths ξ_n are proportional to $t^{-\nu_n}$, it implies that the critical exponents ν_n satisfy the identity

$$\nu_n^{-1}=2n+\theta^*_n=2n-\gamma^*_{\phi^2(n)}. \quad (17)$$

For convenience we could have defined the function

$$\bar{\gamma}_{\phi^2(n)}(u_n)=-\beta_n\frac{\partial\ln(Z_{\phi^2(n)}Z_{\phi(n)})}{\partial u_n}. \quad (18)$$

In that case we would have discovered the equivalent relations

$$\nu_n^{-1}=2n-\eta_n-\bar{\gamma}_{\phi^2(n)}(u^*_n). \quad (19)$$

For $N=2$ we choose $\rho_n=k_{(n)}$, the external momenta. Then $\Gamma^{(2)}_{R(n)}(k_{(n)},t,\kappa_n)=k^{2n-\eta_n}\kappa_n^{\eta_n}f(k_{(n)}\xi_n)$. The infrared regime corresponds to $\xi_n\to\infty$ and $k_{(n)}\to 0$ such that $f(k_{(n)}\xi_n)\to$const. The definition $f_n=(k_{(n)}\xi_n)^{2n-\eta_n}f(k_{(n)}\xi_n)$ leads to

$$\Gamma^{(2)}_{R(n)}(k_{(n)},t,\kappa_n)=(k_{(n)}\xi_n)^{2n-\eta_n}\kappa_n^{\eta_n}f_n(k_{(n)}\xi_n). \quad (20)$$

Since the susceptibility is proportional to $t^{-\gamma_n}$ as $k_{(n)}\to 0$, and $\Gamma^{(2)}_{R(n)}=\chi^{-1}_{(n)}$, the susceptibility critical exponents are given by

$$\gamma_n=\nu_n(2n-\eta_n). \quad (21)$$

We now discuss the scaling law appropriate to relate the specific heat critical exponent to the others critical indices. The analysis of the RG equation for $\Gamma^{(0,2)}_{R(n)}$ above T_L at the fixed point yields information about the specific heat exponents. In that case it reads

$$\left[\kappa_n\frac{\partial}{\partial\kappa_n}+\gamma^*_{\phi^2(n)}\left(2+t\frac{\partial}{\partial t}\right)\right]\Gamma^{(0,2)}_{R(n)}=\left(\kappa_n^{-2n}\right)^{\epsilon_L/2}B_n(u^*_n), \quad (22)$$

where $B_n(u^*_n)$ is given by

$$(\kappa_n^{-2n})^{\epsilon_L/2}B_n(u^*_n)=-Z^2_{\phi^2(n)}\kappa_n\frac{\partial}{\partial\kappa_n}\Gamma^{(0,2)}_{(n)}(Q_{(n)};-Q_{(n)},\lambda_n)_{Q^2_{(n)}=\kappa_n^2}. \quad (23)$$

The general discussion given up to now for the vertex part $\Gamma^{(N,M)}_{R(n)}$ will be useful to uncover the homogeneous part of the solution. In fact, at the fixed point the generalization of the solution for $\Gamma^{(N,M)}_{R(n)}$ is written as

$$\Gamma^{(N,M)}_{R(n)}(p_{i(n)},Q_{i(n)},t,\kappa_n)=\kappa_n^{(1/2)N\gamma^*_{\phi(n)}-M\gamma^*_{\phi^2(n)}}F^{(N,M)}_n(p_{i(n)},Q_{i(n)},\kappa_n t^{-1/\gamma^*_{\phi^2(n)}}). \quad (24)$$

At the fixed point, the temperature-dependent homogeneous part for $\Gamma^{(0,2)}_{R(n),h}$ has the following property:

$$\Gamma^{(0,2)}_{R(n),h}(Q_{(n)},-Q_{(n)},t,\kappa_n) = \kappa_n^{-2\gamma^*_{\phi^2(n)}} F^{(0,2)}_n(Q_{(n)},-Q_{(n)},\kappa_n t^{-1/\gamma^*_{\phi^2(n)}}). \quad (25)$$

This is going to be identified with the specific heat at zero external momentum insertion $Q_{(n)}=0$. Using the dimensional analysis results, one can show that

$$\Gamma^{(0,2)}_{R(n),h}(Q_{(n)},-Q_{(n)},t,\kappa_n) = \rho_n^{n(\{d-\Sigma_{n=2}^{L}[(n-1)/n]m_n\}-4)+2\gamma^*_{\phi^2(n)}} \times \Gamma^{(0,2)}_{R(n),h}(\rho_n^{-1}Q_{(n)}, -\rho_n^{-1}Q_{(n)},\rho_n^{-2n}t,\rho_n^{-1}\kappa_n), \quad (26)$$

and substituting this into the solution at the fixed point, it yields

$$\Gamma^{(0,2)}_{R(n),h}(Q_{(n)},-Q_{(n)},t,\kappa_n) = \rho_n^{n(\{d-\Sigma_{n=2}^{L}[(n-1)/n]m_n\}-4)+2\gamma^*_{\phi^2(n)}} \kappa_n^{-2\gamma^*_{\phi^2(n)}} \times F^{(0,2)}_n(\rho_n^{-1}Q_{(n)}, -\rho_n^{-1}Q_{(n)},\rho_n^{-1}\kappa_n(\rho_n^{-2n}t)^{-1/\gamma^*_{\phi^2(n)}}). \quad (27)$$

Once more, choose $\rho_n=\kappa_n(t/\kappa_n^{2n})1/(\theta_n+2n)$. Replace this in the last equation, take the limit $Q_{(n)}\to 0$, and identify the power of t with the specific-heat exponent α_n. The result is

$$\alpha_n = 2 - n\left(d - \sum_{n=2}^{L}\frac{(n-1)}{n}m_n\right)\nu_n. \quad (28)$$

The inhomogeneous part can now be discussed. Take $Q_{(n)}=0$ and choose a particular solution of the form:

$$C_p(u_n) = (\kappa_n^{2n})^{-\epsilon_L/2}\widetilde{C}_p(u_n). \quad (29)$$

When this is replaced into the RG equation for $\Gamma^{(0,2)}_{R(n)}$ at the fixed point, we learn that

$$C_p(u_n^*) = (\kappa_n^{2n})^{-\epsilon_L/2}\frac{\nu_n}{\nu_n n\left(d-\sum_{n=2}^{L}\frac{(n-1)}{n}m_n\right)-2}B_n(u_n^*). \quad (30)$$

Summing up both terms gives the following general solution at the fixed point:

$$\Gamma^{(0,2)}_{R(n)} = (\kappa_n^{-2n})^{\epsilon_L/2}\left[C_n\left(\frac{t}{\kappa_n^{2n}}\right)^{-\alpha_n} + \frac{\nu_n}{\nu_n n\left(d-\sum_{n=2}^{L}\frac{(n-1)}{n}m_n\right)-2}B_n(u_n^*)\right]. \quad (31)$$

The situation for $T<T_L$ is as follows. For simplicity consider the case of magnetic systems. The renormalized equation of state furnishes a relation between the renormalized magnetic field and the renormalized vertex parts for $t<0$ via a power series in the magnetization M, i.e.,

$$H_{(n)}(t,M,u_n,\kappa_n) = \sum_{N=1}^{\infty}\frac{1}{N!}M^N\Gamma^{(1+N)}_{R(n)}(k_{i(n)}=0;t,u_n,\kappa_n), \quad (32)$$

where the zero momentum limit must be taken after performing the summation. The magnetic field satisfies the following RG equation:

$$\left[\kappa_n\frac{\partial}{\partial\kappa_n}+\beta_n\frac{\partial}{\partial u_n}-\frac{1}{2}N\gamma_{\phi(n)}\left(N+M\frac{\partial}{\partial M}\right)+\gamma_{\phi^2(n)}t\frac{\partial}{\partial t}\right] \times H_{(n)}(t,M,u_n,\kappa_n)=0. \quad (33)$$

The equation of state has the following form at the fixed point:

$$H_{(n)}(t,M,\kappa_n) = \kappa_n^{\eta_n/2}h_{1n}(\kappa_n M^{2/\eta_n},\kappa_n t^{-1/\gamma_{\phi^2(n)}}). \quad (34)$$

Dimensional analysis arguments can be used to determine how a flow in the external momenta affects the renormalized magnetic field. The flow produces the following expression:

$$H_{(n)}(t,M,\kappa_n) = \rho_n^{n(\{d-\Sigma_{n=2}^{L}[(n-1)/n]m_n\}/2+1)}H_{(n)} \times\left(\frac{t}{\rho_n^{2n}},\frac{M}{\rho_n^{n(\{d-\Sigma_{n=2}^{L}[(n-1)/n]m_n\}/2-1)}},\frac{\kappa_n}{\rho_n}\right). \quad (35)$$

The standard choice corresponds to ρ_n being a power of M,

$$\rho_n = \kappa_n\left(\frac{M}{\kappa_n^{(n/2)(\{d-\sum_{n=2}^{L}[(n-1)/n]m_n\}-2)}}\right)^{2/[n(\{d-\Sigma_{n=2}^{L}[(n-1)/n]m_n\}-2)+\eta_n]}. \quad (36)$$

Replacing this into Eq. (35) and from the scaling form of the equation of state $H_{(n)}(t,M)=M^{\delta_n}f(t/M^{1/\beta_n})$, we obtain the remaining scaling laws,

$$\beta_n = \frac{1}{2}\nu_n\left[n\left(d-\sum_{i=2}^{L}\frac{(i-1)}{i}m_i\right)-2n+\eta_n\right], \quad (37a)$$

$$\delta_n = \frac{n\left(d-\sum_{i=2}^{L}\frac{(i-1)}{i}m_i\right)+2n-\eta_n}{n\left(d-\sum_{i=2}^{L}\frac{(i-1)}{i}m_i\right)-2n+\eta_n}, \quad (37b)$$

which imply the Widom $\gamma_n=\beta_n(\delta_n-1)$ and Rushbrooke $\alpha_n+2\beta_n+\gamma_n=2$ relations for *arbitrary* competing or noncompeting subspaces, since $n=1,\ldots,L$. We note that there is one set of scaling relations for each competing subspace. This suggests that all the critical exponents take different values in distinct subspaces. We are going to see that this is not necessarily true, since the fixed-point structure restricts the values of most critical exponents to be the same in different competing subspaces. It is a direct consequence that there is

only one fixed point independent of the space directions under consideration.

The perturbative calculation of the critical exponents and other universal quantities follows from a diagrammatic expansion whose basic objects are Feynman diagrams. We shall use the loop expansion for the anisotropic integrals with the perturbation parameter $\epsilon_L=4+\Sigma_{n=2}^{L}[(n-1)/n]m_n-d$. The solution of the Feynman diagrams in terms of ϵ_L results in the ϵ_L expansion for the universal critical ammounts of the anisotropic criticalities. The anisotropic integrals are described using the generalized orthogonal approximation in Appendix A. This approximation yields a solution which is the most general one compatible with the homogeneity of the Feynman integrals for *arbitrary* external momenta scales. With this technique all the critical exponents in the anisotropic cases can be obtained as will be shown in Sec. V.

IV. SCALING THEORY FOR THE ISOTROPIC BEHAVIORS

To begin with let us promote a slight change of notation with respect to the conventions presented in our previous paper.[15] There, the subscript associated to each type of m_n competing axes was chosen as $4n$. Here, we choose the subscript n to express the same thing. This will cause no confusion to the reader since the anisotropic and isotropic cases are considered separately in this work. Then an arbitrary ammount A_{4n}, should be changed to A_n. In particular, the perturbative parameter discussed in Sec. II is now represented as ϵ_n. Obviously, whenever $d=m_n$, the volume element in momentum space is given by $d^{m_n}k$. Setting $\sigma_n=1$ we perform a dimensional redefinition of the momenta such that $[k]=M^{1/n}$. Accordingly, the volume element has dimension $[d^{m_n}k]=M^{m_n/n}$. The dimension of the field in mass units is $[\phi]=M^{m_n/2n-1}$. The 1PI vertex parts have dimensions $[\Gamma^{(N)}(k_n)]=M^{N+m_n/n-N(m_n/2n)}$. Then, make the continuation $m_n=4n-\epsilon_n$. The coupling constant has dimension $\lambda_{4n}=M^{(4n-m_n)/n}=M^{\epsilon_n/n}$. In terms of dimensionless quantities, one has the renormalized $g_n=u_n(\kappa_n^{2n})^{\epsilon_n/2n}$ and bare $\lambda_n=u_{0n}(\kappa_n^{2n})^{\epsilon_n/2n}$ coupling constants, respectively. Again, the functions

$$\beta_n=\left(\kappa_n\frac{\partial u_n}{\partial \kappa_n}\right), \tag{38a}$$

$$\gamma_{\phi(n)}(u_n)=\beta_n\frac{\partial \ln Z_{\phi(n)}}{\partial u_n}, \tag{38b}$$

$$\gamma_{\phi^2(n)}(u_n)=-\beta_n\frac{\partial \ln Z_{\phi^2(n)}}{\partial u_n} \tag{38c}$$

are computed at fixed bare coupling constant λ_n. The beta functions in terms of dimensionless quantities are given by $\beta_n=-\epsilon_n(\partial \ln u_{0n}/\partial u_n)^{-1}$. Notice that the beta function for the isotropic case does not possess the overall factor of n present in the anisotropic beta function β_n obtained from renormalization-group transformations over the m_n-dimensional competing subspace. This is a very close analogy to the second character behaviors and a general property of Lifshitz critical behaviors.

The dimensional redefinition of the momenta along the m_n competing axes leads to an effective space dimension for the isotropic case, i.e., (m_n/n). Under a flow in the external momenta we find the following behavior for the 1PI vertex parts $\Gamma_{R(n)}^{(N)}$:

$$\Gamma_{R(n)}^{(N)}(\rho_n k_i,u_n,\kappa_n)=\rho_n^{n[N+(m_n/n)-N(m_n/2n)]}\exp\left[-N/2\int_1^{\rho_n}\gamma_{\phi(n)}(u_n(x_n))dx_n/x_n\right]\times\Gamma_{R(n)}^{(N)}(k_i,u_n(\rho_n),\kappa_n). \tag{39}$$

Note that since there is only one type of space directions in the isotropic behaviors, we do not need to use a label in the external momenta specifying the type of competing axes considered as we did in the anisotropic cases. At the fixed point, the simple scaling property for the vertex parts $\Gamma_{R(n)}^{(N)}$ follows:

$$\Gamma_{R(n)}^{(N)}(\rho_n k_i,u_n^*,\kappa_n)=\rho_n^{n[N+(m_n/n)-N(m_n/2n)]-(N/2)\gamma_{\phi(n)}(u_n^*)}\times\Gamma_{R(n)}^{(N)}(k_i,u_n^*,\kappa_n). \tag{40}$$

For $N=2$, we have

$$\Gamma_{R(n)}^{(2)}(\rho_n k,u_n^*,\kappa_n)=\rho_n^{2n-\gamma_{\phi(n)}(u_n^*)}\Gamma_{R(n)}^{(2)}(k,u_n^*,\kappa_n). \tag{41}$$

In the noninteracting theory $d_\phi^0=(m_n/n)/2-1$ is the naive dimension of the field. At the isotropic fixed point, the presence of interactions modify it such that $d_\phi=(m/n)/2-1+\eta_n/2n$. The generalization to include L insertions of ϕ^2 operators can be written at the fixed point as $[(N,L)\neq(0,2)]$

$$\Gamma_{R(n)}^{(N,L)}(\rho_n k_i,\rho_n p_i,u_n^*,\kappa_n)=\rho_n^{n[N+m_n/n-N(m_n/n)/2-2L]-N\gamma_{\phi(n)}^*/2+L\gamma_{\phi^2(n)}^*}\times\Gamma_{R(n)}^{(N,L)}(k_i,p_i,u_n^*,\kappa_n). \tag{42}$$

Writing at the fixed point

$$\Gamma_{R(n)}^{(N,L)}(\rho_n k_i,\rho_n p_i,u_n^*,\kappa_n)=\rho_n^{m_n-Nd_\phi+Ld_{\phi^2}}\Gamma_{R(n)}^{(N,L)}(k_i,p_i,u_n^*,\kappa_n), \tag{43}$$

the anomalous dimension of the insertions of ϕ^2 operators is given by $d_{\phi^2}=-2n+\gamma_{\phi^2(n)}(u_n^*)$.

Above the Lifshitz critical temperature we find the following RG equation:

$$\left(\kappa_n\frac{\partial}{\partial\kappa_n}+\beta_n\frac{\partial}{\partial u_n}-\frac{1}{2}N\gamma_{\phi(n)}(u_n)+\gamma_{\phi^2(n)}(u_n)t\frac{\partial}{\partial t}\right)\times\Gamma_{R(n)}^{(N)}(k_i,t,u_n,\kappa_n)=0. \tag{44}$$

The solution at the fixed point is given by

$$\Gamma_{R(n)}^{(N)}(k_{i(n)},t,u_n^*,\kappa_n)=\kappa_n^{N\gamma_{\phi(n)}^*/2}F_{(n)}^{(N)}(k_i,\kappa_n t^{-1/\gamma_{\phi^2(n)}^*}). \tag{45}$$

If we define $\theta_n=-\gamma_{\phi^2(n)}^*$, we can use dimensional analysis to obtain

$$\Gamma^{(N)}_{R(n)}(k_i,t,\kappa_n)=\rho_n^{n[N+m_n/n-(N/2)(m_n/n)]-(N/2)\eta_n}\kappa_n^{(N/2)\eta_n}\times F^{(N)}_{(n)}(\rho_n^{-1}k_i,(\rho_n^{-1}\kappa_n)(\rho_n^{-2n}t)^{1/\theta_n}). \quad (46)$$

We can choose $\rho_n=\kappa_n(t/\kappa_n^{2n})^{1/(\theta_n+2n)}$, and replacing it in the last two equations, the vertex parts depend only on the combination $k_i\xi_n$ apart from a power of t. As ξ_n is proportional to $t^{-\nu_n}$ we can identify the critical exponent ν_n as

$$\nu_n^{-1}=2n+\theta_n^*=2n-\gamma^*_{\phi^2(n)}. \quad (47)$$

For the sake of convenience we define the function

$$\bar{\gamma}_{\phi^2(n)}(u_n)=-\beta_n\frac{\partial\ln(Z_{\phi^2(n)}Z_{\phi(n)})}{\partial u_n}. \quad (48)$$

In terms of this function the last equation turns into the following relation:

$$\nu_n^{-1}=2n-\eta_n-\bar{\gamma}_{\phi^2(n)}(u_n^*). \quad (49)$$

For $N=2$ we choose $\rho_n=k$, the external momenta. When $\xi_n\to\infty$ and $k\to 0$, simultaneously, then $f(k\xi_n)\to$ const. The susceptibility is proportional to $t^{-\gamma_n}$ as $k_i\to 0$. As $\Gamma^{(2)}_R=\chi^{-1}$, the susceptibility critical exponent follows

$$\gamma_n=\nu_n(2n-\eta_n). \quad (50)$$

From the RG equation for $\Gamma^{(0,2)}_{R(n)}$ above T_L at the fixed point, the scaling relation for the specific-heat exponent can be found. The RG equation is

$$\left[\kappa_n\frac{\partial}{\partial\kappa_n}+\gamma^*_{\phi^2(n)}\left(2+t\frac{\partial}{\partial t}\right)\right]\Gamma^{(0,2)}_{R(n)}=(\kappa_n^{-2})^{\epsilon_n/2}B_n(u_n^*), \quad (51)$$

where

$$(\kappa_n^{-2n})^{\epsilon_n/2n}B_n(u_n^*)=-Z^2_{\phi^2(n)}\kappa_n\frac{\partial}{\partial\kappa_n}\left[\Gamma^{(0,2)}_{(n)}(Q;-Q,\lambda_n)\Big|_{Q^2=\kappa_n^2}\right]. \quad (52)$$

The $\Gamma^{(N,L)}_{R(n)}$ can be generalized to

$$\Gamma^{(N,L)}_{R(n)}(p_i,Q_i,t,\kappa_n)=\kappa_n^{(1/2)N\gamma^*_{\phi(n)}-L\gamma^*_{\phi^2(n)}}F^{(N,L)}_n(p_i,Q_i,\kappa_n t^{-1/\gamma^*_{\phi^2(n)}}). \quad (53)$$

The homogeneous part of the solution for $\Gamma^{(0,2)}_{R(n),h}$ is temperature dependent and scales at the fixed point as

$$\Gamma^{(0,2)}_{R(n),h}(Q,-Q,t,\kappa_n)=\kappa_n^{-2\gamma^*_{\phi^2(n)}}F^{(0,2)}_n(Q,-Q,\kappa_n t^{-1/\gamma^*_{\phi^2(n)}}). \quad (54)$$

This vertex function is going to be identified with the specific heat at zero external momentum insertion $Q=0$. Use of dimensional analysis yields the result

$$\Gamma^{(0,2)}_{R(n),h}(Q,-Q,t,\kappa_n)=\rho_n^{n[m_n/n-4]+2\gamma^*_{\phi^2(n)}}\Gamma^{(0,2)}_{R(n),h}(\rho_n^{-1}Q,-\rho_n^{-1}Q,\rho_n^{-2n}t,\rho_n^{-1}\kappa_n). \quad (55)$$

Substituting this equation in the solution at the fixed point leads to

$$\Gamma^{(0,2)}_{R(n),h}(Q,-Q,t,\kappa_n)=\rho_n^{n[m_n/n-4]+2\gamma^*_{\phi^2(n)}}\kappa_n^{-2\gamma^*_{\phi^2(n)}}F^{(0,2)}_n(\rho_n^{-1}Q,-\rho_n^{-1}Q,\rho_n^{-1}\kappa_n(\rho_n^{-2n}t)^{-1/\gamma^*_{\phi^2(n)}}). \quad (56)$$

The choice $\rho_n=\kappa_n(t/\kappa_n^{2n})1/(\theta_n+2n)$ can be made. Substitution of this choice into the last equation in the limit $Q\to 0$ and identifying the power of t with the specific-heat exponent α_n, we obtain

$$\alpha_n=2-m_n\nu_n. \quad (57)$$

The inhomogeneous part can be found by taking $Q=0$ and choosing a particular solution in the standard way. Therefore the general solution at the fixed point is given by

$$\Gamma^{(0,2)}_{R(n)}=(\kappa_n^{-2n})^{\epsilon_n/2n}\left[C_n\left(\frac{t}{\kappa_n^{2n}}\right)^{-\alpha_n}+\frac{\nu_n}{\nu_n m_n-2}B_n(u_n^*)\right]. \quad (58)$$

Next, let us concentrate ourselves in the scaling relations when the system is below the Lifshitz critical temperature $T<T_L$. The relation among the renormalized magnetic field, the renormalized vertex parts for $t<0$ and the magnetization M is given by

$$H_{(n)}(t,M,u_n,\kappa_n)=\sum_{N=1}^{\infty}\frac{1}{N!}M^N\Gamma^{(1+N)}_{R(n)}(k_i=0;t,u_n,\kappa_n). \quad (59)$$

The RG equation satisfied by the magnetic field is

$$\left[\kappa_n\frac{\partial}{\partial\kappa_n}+\beta_n\frac{\partial}{\partial u_n}-\frac{1}{2}N\gamma_{\phi(n)}(u_n)\left(N+M\frac{\partial}{\partial M}\right)+\gamma_{\phi^2(n)}t\frac{\partial}{\partial t}\right]H_{(n)}(t,M,u_n,\kappa_n)=0. \quad (60)$$

The solution of the equation of state at the fixed point has the following property:

$$H_{(n)}(t,M,\kappa_n)=\kappa_n^{\eta_n/2}h_n(\kappa_n M^{2/\eta_n},\kappa_n t^{-1/\gamma_{\phi^2(n)}}). \quad (61)$$

The scale change in the magnetic field followed by a flow in the external momenta can be written in the form

$$H_{(n)}(t,M,\kappa_n)=\rho_n^{n[m_n/2n+1]}H_n\left(\frac{t}{\rho_n^{2n}},\frac{M}{\rho_n^{n[m_n/2n-1]}},\frac{\kappa_n}{\rho_n}\right). \quad (62)$$

The flow parameter ρ_n is chosen as a power of M such that

$$\rho_n=\kappa_n\left(\frac{M}{\kappa_n^{[m_n/2-n]}}\right)^{\frac{2}{m_n-2n+\eta_n}}, \quad (63)$$

and from the scaling form of the equation of state $H_{(n)}(t,M)=M^{\delta_n}f_{(n)}[(t/M^{(1/\beta_n)})]$, we obtain the following scaling relations:

$$\delta_n=\frac{m_n+2n-\eta_n}{m_n-2n+\eta_n}, \quad (64a)$$

$$\beta_n = \frac{1}{2}\nu_n(m_n - 2n + \eta_n), \tag{64b}$$

which imply the Widom $\gamma_n = \beta_n(\delta_n - 1)$ and Rushbrooke $\alpha_n + 2\beta_n + \gamma_n = 2$ relations.

Except for some minor modifications, the renormalization-group treatment of each isotropic behavior is equivalent to treat separately each competing subspace appearing in the most general anisotropic behavior. It is easy to see that all scaling laws reduce to those from the usual critical behavior described by a ϕ^4 field theory for $n=1$. Then, the usual critical behavior is actually a first character isotropic Lifshitz critical behavior. For $n=2$, they easily reproduce those associated to the second character Lifshitz point.[8] Therefore a different nomenclature emerges from the study of these higher character Lifshitz critical behaviors: the number of neighbors coupled through competing interactions is a fundamental parameter, generalizing the concept of universality class. Thus the universality classes of isotropic behaviors ($d=m_n$) are characterized by (N, d, n). These statements will be put on a firmer ground when we calculate the critical exponents, as we shall see in the next sections.

The isotropic behaviors are calculated using the generalized orthogonal approximation as well as exactly, without the resource of any approximation. The approximation is useful to complete the unified analytical description of the higher character Lifshitz critical behavior in its full generality, at least at the loop order considered here. On the other hand, the analytic exact (perturbative) solution is a conceptual step forward towards a better comprehension of this sort of system. At this point, the reader should consult the Appendix B in order to access the computation of the Feynman integrals using the generalized orthogonal approximation and Appendix C to see the exact computation required to finding the critical exponents.

V. CRITICAL EXPONENTS FOR THE ANISOTROPIC BEHAVIORS

In this section we compute the critical exponents using the generalized orthogonal approximation with the results derived in Appendix A. First, we attack the problem using normalization conditions. Second, the results are checked using a variant of the minimal subtraction scheme developed in Ref. 8 for the anisotropic second character m_2-fold Lifshitz point.

A. Normalization conditions and critical exponents

The bare coupling constants and renormalization functions were defined in Sec. II. They are given by

$$u_{on} = u_n(1 + a_{1n}u_n + a_{2n}u_n^2), \tag{65a}$$

$$Z_{\phi(n)} = 1 + b_{2n}u_n^2 + b_{3n}u_n^3, \tag{65b}$$

$$\bar{Z}_{\phi^2(n)} = 1 + c_{1n}u_n + c_{2n}u_n^2, \tag{65c}$$

where the constants a_{in}, b_{in}, c_{in} depend on Feynman diagrams computed at suitable symmetry points. The critical exponents associated to correlations perpendicular or parallel to the arbitrary competing m_n-dimensional subspace can be calculated through the specification of the corresponding symmetry point.

In terms of the constants defined above, the beta functions and renormalization constants can be cast in the form

$$\beta_n = -n\epsilon_L u_n[1 - a_{1n}u_n + 2(a_{1n}^2 - a_{2n})u_n^2], \tag{66a}$$

$$\gamma_{\phi(n)} = -n\epsilon_L u_n[2b_{2n}u_n + (3b_{3n} - 2b_{2n}a_{1n})u_n^2], \tag{66b}$$

$$\bar{\gamma}_{\phi^2(n)} = n\epsilon_L u_n[c_{1n} + (2c_{2n} - c_{1n}^2 - a_{1n}c_{1n})u_n]. \tag{66c}$$

The above coefficients can be figured out as functions of the integrals calculated at the symmetry points. We find

$$a_{1n} = \frac{N+8}{6\epsilon_L}[1 + h_{m_L}\epsilon_L], \tag{67a}$$

$$a_{2n} = \left(\frac{N+8}{6\epsilon_L}\right)^2 + \left(\frac{(N+8)^2}{18}h_{m_L} - \frac{(3N+14)}{24}\right)\frac{1}{\epsilon_L}, \tag{67b}$$

$$b_{2n} = -\frac{(N+2)}{144\epsilon_L}\left[1 + \left(2h_{m_L} + \frac{1}{4}\right)\epsilon_L\right], \tag{67c}$$

$$b_{3n} = -\frac{(N+2)(N+8)}{1296\epsilon_L^2} + \frac{(N+2)(N+8)}{108\epsilon_L}\left(-\frac{1}{4}h_{m_L} + \frac{1}{48}\right), \tag{67d}$$

$$c_{1n} = \frac{(N+2)}{6\epsilon_L}[1 + h_{m_L}\epsilon_L], \tag{67e}$$

$$c_{2n} = \frac{(N+2)(N+5)}{36\epsilon_L^2} + \frac{(N+2)}{3\epsilon_L}\left(\frac{(N+5)}{3}h_{m_L} - \frac{1}{4}\right). \tag{67f}$$

These expressions are sufficient to find out the fixed points at $O(\epsilon_L^2)$, which are defined by $\beta_n(u_n^*)=0$. As was seen in Appendix A, every integral computed at arbitrary symmetry points $SP_1, \ldots, SP_L$ gives the same result, irrespective of the considered subspace. The overall factor of $n=1, \ldots, L$ in the β_n functions drops out at the fixed points, such that the renormalization-group transformations realized over $\kappa_1, \ldots,$ and κ_L will flow to the same fixed point given by $(u_1^* = \cdots = u_L^* \equiv u^*)$

$$u^* = \frac{6}{8+N}\epsilon_L\left[1 + \epsilon_L\left(-h_{m_L} + \frac{(9N+42)}{(8+N)^2}\right)\right]. \tag{68}$$

It is instructive to separate explicitly the noncompeting and competing subspaces. The functions $\gamma_{\phi(1)}$ and $\bar{\gamma}_{\phi^2(1)}$ read

$$\gamma_{\phi(1)} = \frac{(N+2)}{72}\left[1 + \left(2h_{m_L} + \frac{1}{4}\right)\epsilon_L\right]u_1^2 - \frac{(N+2)(N+8)}{864}u_1^3, \tag{69}$$

$$\bar{\gamma}_{\phi^2(1)} = \frac{(N+2)}{6} u_1 \left(1 + h_{m_L} \epsilon_L - \frac{1}{2} u_1 \right). \tag{70}$$

When the value of the fixed point is substituted into these equations, using the relation among these functions and the critical exponents $\eta_1 (\equiv \eta_{L2})$ and $\nu_1 (\equiv \nu_{L2})$, we find

$$\eta_1 = \frac{1}{2} \epsilon_L^2 \frac{N+2}{(N+8)^2} \left[1 + \epsilon_L \left(\frac{6(3N+14)}{(N+8)^2} - \frac{1}{4} \right) \right], \tag{71}$$

$$\nu_1 = \frac{1}{2} + \frac{(N+2)}{4(N+8)} \epsilon_L + \frac{1}{8} \frac{(N+2)(N^2+23N+60)}{(N+8)^3} \epsilon_L^2. \tag{72}$$

The coefficient of each power of ϵ_L is the same as that coming from the second character Lifshitz behavior $m_3 = \cdots = m_L = 0$. Consequently, the reduction to the Ising-like universality class $m_2 = 0$ case is warranted. The several beta functions corresponding to distinct competing axes satisfy the property $\beta_n = n\beta_1$. This implies that $\gamma_{\phi(n)} = n\gamma_{\phi(1)}$ and $\bar{\gamma}_{\phi^2(n)} = n\bar{\gamma}_{\phi^2(1)}$. Then, we have

$$\eta_n = \frac{n}{2} \left\{ \epsilon_L^2 \frac{(N+2)}{(N+8)^2} \left[1 + \epsilon_L \left(\frac{6(3N+14)}{(N+8)^2} - \frac{1}{4} \right) \right] \right\}, \tag{73}$$

$$\nu_n = \frac{1}{n} \left(\frac{1}{2} + \frac{(N+2)}{4(N+8)} \epsilon_L + \frac{1}{8} \frac{(N+2)(N^2+23N+60)}{(N+8)^3} \epsilon_L^2 \right). \tag{74}$$

At $O(\epsilon_L^3)$, the relation $\eta_n = n\eta_1$ is satisfied, whereas at $O(\epsilon_L^2)$, the relation $\nu_n = (1/n)\nu_1$ holds. Strong anisotropic scale invariance[24] is *exact* to the perturbative order considered here, and within the generalized orthogonal approximation it is expected to hold at arbitrary higher loop order. Differently from the critical indices η_n and ν_n which depend explicitly on the m_n-dimensional subspace under consideration, the other exponents take the same value in each subspace even though they are obtained through independent scaling relations along the distinct competing axes. They are given by

$$\gamma_L = 1 + \frac{(N+2)}{2(N+8)} \epsilon_L + \frac{(N+2)(N^2+22N+52)}{4(N+8)^3} \epsilon_L^2, \tag{75}$$

$$\alpha_L = \frac{(4-N)}{2(N+8)} \epsilon_L - \frac{(N+2)(N^2+30N+56)}{4(N+8)^3} \epsilon_L^2, \tag{76}$$

$$\beta_L = \frac{1}{2} - \frac{3}{2(N+8)} \epsilon_L + \frac{(N+2)(2N+1)}{2(N+8)^3} \epsilon_L^2, \tag{77}$$

$$\delta_L = 3 + \epsilon_L + \frac{(N^2+14N+60)}{2(N+8)^2} \epsilon_L^2. \tag{78}$$

The exponents correctly reduce to those from the second character behavior,[8,15] with a further reduction to the Ising-like case when $m_2 = 0$. In fact the universality classes reduction of generic higher character anisotropic Lifshitz points to that from Ising-like critical points is manifest in *all* critical exponents. Hence this universality class reduction is a generic property of arbitrary competing systems.

To check the correctness of these exponents, it is convenient to calculate them in another renormalization procedure. Let us check these results using minimal subtraction of dimensional poles, as we are going to show next.

B. Minimal subtraction and critical exponents

In the minimal subtraction renormalization scheme, the common situation is to have just one momenta scale μ,[25] which is called κ in the present paper. Nevertheless, the dimensional redefinitions performed over the external momenta characterizing arbitrary types of competing axes permit a picture of the anisotropic cases with L independent momenta scales.

The calculation of the critical exponents along an arbitrary kind of competition subspace can be done, provided all external momenta not belonging to that subspace are set to zero. Then, we define κ_j to be the typical scale parameter of the jth subspace, calculate the renormalization functions for arbitrary external momenta along the m_jth space directions and require minimal subtraction of dimensional poles. This procedure is inspired in the method formerly discussed in the second character anisotropic Lifshitz behaviors.[8] Thus although the external momentum associated to the competing subspace under consideration is kept arbitrary in all stages of the computation, the same is not true for all other external momenta corresponding to distinct competing subspaces. This restriction on the values of all the external momenta is the price to be paid in order to describe independently the scale transformations of each inequivalent subspace. This is the main difference of this minimal subtraction scheme with several independent momentum scales from the conventional method with just one momentum scale.

Here we will content ourselves in showing that the diagrammatic procedure to calculate the fixed point using minimal subtraction results in the same functions $\gamma_{\phi(n)}$ and $\bar{\gamma}_{\phi^2(n)}$ at the fixed point as those coming from normalization conditions. This is equivalent to prove the renormalization scheme independence of all critical indices.

In minimal subtraction, the dimensionless bare coupling constants and the renormalization functions are defined by

$$u_{0n} = u_n \left(1 + \sum_{i=1}^{\infty} a_{in}(\epsilon_L) u_n^i \right), \tag{79a}$$

$$Z_{\phi(n)} = 1 + \sum_{i=1}^{\infty} b_{in}(\epsilon_L) u_n^i, \tag{79b}$$

$$\bar{Z}_{\phi^2(n)} = 1 + \sum_{i=1}^{\infty} c_{in}(\epsilon_L) u_n^i. \tag{79c}$$

The renormalized vertex parts

$$\Gamma^{(2)}_{R(n)}(k_{(n)}, u_n, \kappa_n) = Z_{\phi(n)} \Gamma^{(2)}_{(n)}(k_n, u_{0n}, \kappa_n), \tag{80a}$$

$$\Gamma^{(4)}_{R(n)}(k_{i(n)}, u_n, \kappa_n) = Z^2_{\phi(n)} \Gamma^{(4)}_{(n)}(k_{i(n)}, u_{0n}, \kappa_n), \tag{80b}$$

$$\Gamma^{(2,1)}_{R(n)}(k_{1(n)},k_{2(n)},p_{(n)};u_n,\kappa_n) = \bar{Z}_{\phi^2(n)}\Gamma^{(2,1)}_{(n)}(k_{1(n)},k_{2(n)},p_{(n)},u_{0n},\kappa_n) \tag{80c}$$

are finite by construction when $\epsilon_L \to 0$, order by order in u_n. One should bear in mind that the external momenta in the bare vertices are mutiplied by κ_n^{-1}. Since $k_{i(1)}=p_i$ are the external momenta perpendicular to the competing axes, whereas $k_{i(n)}=k'_{i(n)}$ are the external momenta parallel to the m_n-dimensional type of competing subspace, the coefficients $a_{in}(\epsilon_L)$, $b_{in}(\epsilon_L)$, and $c_{in}(\epsilon_L)$ are obtained by requiring that the poles in ϵ_L be minimally subtracted. The bare vertices are written in the form

$$\Gamma^{(2)}_{(n)}(k_{(n)},u_{0n},\kappa_n) = k^{2n}_{(n)}(1 - B_{2n}u^2_{0n} + B_{3n}u^3_{0n}), \tag{81a}$$

$$\Gamma^{(4)}_{(n)}(k_{i(n)},u_{0n},\kappa_{(n)}) = \kappa_n^{n\epsilon}u_{0n}[1 - A_{1n}u_{0n} + (A^{(1)}_{2n} + A^{(2)}_{2n})u^2_{0n}], \tag{81b}$$

$$\Gamma^{(2,1)}_{(n)}(k_{1(n)},k_{2(n)},p_{(n)};u_{0n},\kappa_n) = 1 - C_{1n}u_{0n} + (C^{(1)}_{2n} + C^{(2)}_{2n})u^2_{0n}. \tag{81c}$$

Remember that B_{2n} is proportional to the integral I_3 and B_{3n} is proportional to I_5 which are calculated with all external momenta not belonging to the m_n-dimensional subspace set to zero.

The coefficients are expressed explicitly by the following integrals:

$$A_{1n} = \frac{(N+8)}{18}\left[I_2\left(\frac{k_{1(n)}+k_{2(n)}}{\kappa_n}\right) + I_2\left(\frac{k_{1(n)}+k_{3(n)}}{\kappa_n}\right) + I_2\left(\frac{k_{2(n)}+k_{3(n)}}{\kappa_n}\right)\right], \tag{82a}$$

$$A^{(1)}_{2n} = \frac{(N^2+6N+20)}{108}\left[I^2_2\left(\frac{k_{1(n)}+k_{2(n)}}{\kappa_n}\right) + I^2_2\left(\frac{k_{1(n)}+k_{3(n)}}{\kappa_n}\right) + I^2_2\left(\frac{k_{2(n)}+k_{3(n)}}{\kappa_n}\right)\right], \tag{82b}$$

$$A^{(2)}_{2n} = \frac{(5N+22)}{54}\left[I_4\left(\frac{k_{i(n)}}{\kappa_n}\right) + 5 \text{ permutations}\right], \tag{82c}$$

$$B_{2n} = \frac{(N+2)}{18}I_3\left(\frac{k_{(n)}}{\kappa_n}\right), \tag{82d}$$

$$B_{3n} = \frac{(N+2)(N+8)}{108}I_5\left(\frac{k_n}{\kappa_n}\right), \tag{82e}$$

$$C_{1n} = \frac{N+2}{18}\left[I_2\left(\frac{k_{1(n)}+k_{2(n)}}{\kappa_n}\right) + I_2\left(\frac{k_{1(n)}+k_{3(n)}}{\kappa_n}\right) + I_2\left(\frac{k_{2(n)}+k_{3(n)}}{\kappa_n}\right)\right], \tag{82f}$$

$$C^{(1)}_{2n} = \frac{(N+2)^2}{108}\left[I^2_2\left(\frac{k_{1(n)}+k_{2(n)}}{\kappa_n}\right) + I^2_2\left(\frac{k_{1(n)}+k_{3(n)}}{\kappa_n}\right) + I^2_2\left(\frac{k_{2(n)}+k_{3(n)}}{\kappa_n}\right)\right], \tag{82g}$$

$$C^{(2)}_{2n} = \frac{N+2}{36}\left[I_4\left(\frac{k_{i(n)}}{\kappa_n}\right) + 5 \text{ permutations}\right]. \tag{82h}$$

We have at hand all we need to determine the normalization constants at least at two-loop order. Requiring minimal subtraction for the renormalized vertex parts above listed, it can be verified that all the logarithmic integrals depending upon each arbitrary (nonvanishing) external momenta subspace appearing in I_2, I_3, I_4, and I_5 cancel out. This leads to the following expressions for the normalization functions and coupling constants:

$$u_{0n} = u_n\left[1 + \frac{(N+8)}{6\epsilon_L}u_n + \left(\frac{(N+8)^2}{36\epsilon^2_L} - \frac{(3N+14)}{24\epsilon_L}\right)u^2_n\right], \tag{83a}$$

$$Z_{\phi(n)} = 1 - \frac{N+2}{144\epsilon_L}u^2_n + \left(-\frac{(N+2)(N+8)}{1296\epsilon^2_L} + \frac{(N+2)(N+8)}{5184\epsilon_L}\right)u^3_n, \tag{83b}$$

$$\bar{Z}_{\phi^2(n)} = 1 + \frac{N+2}{6\epsilon_L}u_n + \left(\frac{(N+2)(N+5)}{36\epsilon^2_L} - \frac{(N+2)}{24\epsilon_L}\right)u^2_n. \tag{83c}$$

Using the renormalization constants we obtain

$$\gamma_{\phi(n)} = n\left(\frac{(N+2)}{72}u^2_n - \frac{(N+2)(N+8)}{1728}u^3_n\right), \tag{84}$$

$$\bar{\gamma}_{\phi^2(n)} = n\frac{(N+2)}{6}u_n\left(1 - \frac{1}{2}u_n\right). \tag{85}$$

The fixed points are defined by $\beta_n(u^*_n)=0$. Then, we learn that the fixed points generated by renormalization-group transformations over $\kappa_1,\ldots,$ and κ_L are the same, namely

$$u^*_n = \frac{6}{8+N}\epsilon_L\left[1 + \epsilon_L\left(\frac{(9N+42)}{(8+N)^2}\right)\right]. \tag{86}$$

When this result is replaced in the renormalization constants at the fixed point it yields $\gamma^*_{\phi(n)} = \eta_n$, where η_n are given by Eqs. (71) and (73). In addition, we have

$$\bar{\gamma}^*_{\phi^2(n)} = n\frac{(N+2)}{(N+8)}\epsilon_L\left(1 + \frac{6(N+3)}{(N+8)^2}\epsilon_L\right). \tag{87}$$

The reader can verify that the exponents ν_n encountered by using the last equation are the same as those obtained via normalization conditions Eqs. (72) and (74). This proves the consistency of either renormalization scheme for the anisotropic Lifshitz critical behaviors.

C. Discussion

The generic Lth character Lifshitz critical behavior naturally extends the comprehension of the usual second character Lifshitz criticality. The latter is characterized by one noncompeting subspace and only one competing subspace, whereas the former is characterized by several types of competing subspaces. The expressions for the critical exponents can be analyzed to extract further information concerning those systems. For instance, when $m_3 \neq 0$ and $m_2 = m_4 = \cdots = m_L = 0$, the third character behavior is recovered and correctly reduces to the Ising-like behavior for $m_3 = 0$. The main characteristic of generic third character behavior is that there are m_2, $m_3 \neq 0$ competing axes with $m_4 = \cdots = m_L = 0$, and so on. From a phenomenological perspective in magnetic systems, may be it is worthy to assemble all magnetic materials presenting Lishitz critical behavior and analyze their critical exponents. Choosing those alloys in the same conjectured universality class, the greater the difference in their critical exponents the more likely they are in an alternative universality class contained in the CECI model analyzed here.

It is interesting to note that exact strong anisotropic scale invariance[24] is valid in the CECI model as a result of the generalized orthogonal approximation. Since the model describes the physics of short-ranged competing systems, the limit $L \to \infty$ in the Lth character Lifshitz point should not be taken, since it would describe *long-range* competing systems. Nevertheless, since no restriction on L was made in the beginning of the discussion, the CECI model can be viewed as describing a particular type of long-range competing interactions. If we go on and take this limit in the expression of the critical exponents we find that η_L tends to infinity, whereas $\nu_L \to 0$ as a consequence of the strong anisotropic scale invariance. This implies that close to the Lifshitz critical temperature the correlation length ξ_L does not diverge in that limit, instead of having a usual power-law divergence. This fact is a nonperturbative result valid to all orders in perturbation theory within the context of the ϵ_L expansion.

Another feature emerges from this limit by looking at the critical dimension $d_c = 4 + m_\infty$. (Recall that in the Lth anisotropic character critical behavior the system has m_L competing axes and $d - m_L$ noncompeting space directions, i.e., $m_2 = \cdots = m_{L-1} = 0$.) This limit yields a mechanism which is a natural way to study extra dimensions without destroying the renormalizability of the corresponding field theory, while retaining the nontrivial aspect of the fixed point. In spite of the divergence of the anomalous dimension and the vanishing of the correlation length exponent along the m_L competing directions in this limit, all other exponents are well behaved and have the same value of those corresponding to space directions without competition, as can be seen explicitly by the expressions for the exponents. This is so since the anisotropic scaling laws only contain the safe combination $L\nu_L$ which is always finite in the limit $L \to \infty$. Once more, note that $m_\infty \to 0$ reduces to the previous ϕ^4 universality classes. Recall that the isotropic case is characterized by $d = m_L = 4L$. Some care must be taken in the interpretation of this case in the limit $L \to \infty$, for the approach from the anisotropic ($0 \leqslant m_L < 4L$) to the isotropic case takes infinite steps, which is not as simple as in the case when L is finite. This point deserves its own analysis for future work.

On the other hand, the last few years have witnessed some ideas in quantum field theories that resemble very much issues contained in the analysis of Lifshitz critical phenomena. The closest analogy with the Lifshitz field-theoretic tools presented here is the very recent idea of ghost condensation producing a consistent modification of gravity in the infrared (long-distance limit).[19,26,27] In this framework, gravity is modified to have attractive *as well as* repulsive components. The latter mimics a kind of dark energy,[28] whose ghost condensate is a physical fluid arising from a theory where a real scalar field changes with a constant velocity. The ghost condensate appears as the physical scalar excitation around the background generated by the scalar field whose vacuum expectation value is defined by a constant value of its time derivative. Consequently, the effective action for the field representing the ghost condensate has kinetic terms with quadratic time derivatives and quartic space derivatives of the field, therefore breaking Lorentz invariance.[19] The instability in momentum space is manifest in the absence of kinetic terms quadratic in the space derivatives of the ghost condensate. This characteristic is a result of the competing nature between the attractive and repulsive components of the gravitational force. This is a precise analogy with the Lifshitz critical region in the case of the usual second character behavior included in the discussion given in the present paper. The utilization of an analogous reasoning leads us to conclude that the CECI model can be used to extract further insights from these effective quantum field theories where Lorentz invariance is broken. It permits generalizations for the ghost condensate when higher powers in space derivatives of this field are present in its corresponding effective action. For example, at large distances the case of quadratic time derivatives and sixth space derivatives in the kinetic term would correspond to a gravitational interaction with attractive-repulsive-attractive competing components and so on. Further analysis of the model might be helpful to address the perturbative calculations regarding the ghost condensate in that scalar field background.

Numerical methods to probe the results obtained here are in their infancy for the CECI model. Earlier Monte Carlo simulations for a uniaxial third character behavior were performed[11] and the existence of the corresponding Lifshitz point at nonzero temperature was established. Unfortunately, no critical exponent was determined for the third character Lifshitz point. Since then, perhaps due to the fact that these higher character criticalities were not well understood theoretically, these methods are still waiting for more investigations. Recently, a model with antiferromagnetic couplings between nearest neighbors as well as antiferromagnetic exchange interactions between third neighbors was studied using Monte Carlo simulations and some quantum properties were investigated.[29] No ferromagnetic couplings appear among arbitrary neighbors. Even though there is a quantum Lifshitz point there, it pertains to a universality class which might be different from that representing third character critical points discussed in the present paper. Further numerical studies motivated by the CECI model are necessary to understand completely the classical and quantum properties of its critical regions.

VI. ISOTROPIC CRITICAL EXPONENTS IN THE ORTHOGONAL APPROXIMATION

We now address the calculation of critical exponents using the generalized orthogonal approximation from the results obtained in Appendix B. As we have seen in the calculation of Feynman integrals, this problem can also be tackled without performing any approximation during all steps of the calculation. This technique shall be postponed until next section. In one way or another, the approach is equivalent to treat each competing subspace separately. Though very similar, the framework of this section turns out to be more economical than that reported in the anisotropic cases.

A. Critical exponents in normalization conditions

The basic definitions of the bare coupling constants and renormalization functions were encountered before and are given by

$$u_{0n} = u_n(1 + a_{1n}u_n + a_{2n}u_n^2), \tag{88a}$$

$$Z_{\phi(n)} = 1 + b_{2n}u_n^2 + b_{3n}u_n^3, \tag{88b}$$

$$\bar{Z}_{\phi^2(n)} = 1 + c_{1n}u_n + c_{2n}u_n^2, \tag{88c}$$

where the constants a_{in}, b_{in}, c_{in} depend on Feynman integrals at the symmetry point called SP_n. Let κ_n denote the competing m_n-dimensional subspace in the isotropic cases.

The beta function and renormalization constants can be expressed in the form

$$\beta_n = -\epsilon_n u_n[1 - a_{1n}u_n + 2(a_{1n}^2 - a_{2n})u_n^2], \tag{89a}$$

$$\gamma_{\phi(n)} = -\epsilon_n u_n[2b_{2n}u_n + (3b_{3n} - 2b_{2n}a_{1n})u_n^2], \tag{89b}$$

$$\bar{\gamma}_{\phi^2(n)} = \epsilon_n u_n[c_{1n} + (2c_{2n} - c_{1n}^2 - a_{1n}c_{1n})u_n]. \tag{89c}$$

The coefficients above obtained as functions of the integrals calculated at the symmetry point read

$$a_{1n} = \frac{N+8}{6\epsilon_n}\left(1 + \frac{1}{2n}\epsilon_n\right), \tag{90a}$$

$$a_{2n} = \left(\frac{N+8}{6\epsilon_n}\right)^2 + \left(\frac{2N^2+23N+86}{72n\epsilon_n}\right), \tag{90b}$$

$$b_{2n} = -\frac{(N+2)}{144n\epsilon_n}\left(1 + \frac{5}{4n}\epsilon_n\right), \tag{90c}$$

$$b_{3n} = -\frac{(N+2)(N+8)}{1296n\epsilon_n^2} - 5\frac{(N+2)(N+8)}{5184n^2\epsilon_n}, \tag{90d}$$

$$c_{1n} = \frac{(N+2)}{6\epsilon_n}\left(1 + \frac{1}{2n}\epsilon_n\right), \tag{90e}$$

$$c_{2n} = \frac{(N+2)(N+5)}{36\epsilon_n^2} + \frac{(N+2)(2N+7)}{72n\epsilon_n}. \tag{90f}$$

The equation $\beta_n(u_n^*)=0$ defines the fixed point. Thus we find

$$u_n^* = \frac{6}{8+N}\epsilon_n\left[1 + \epsilon_n\frac{1}{n}\left(-\frac{1}{2} + \frac{(9N+42)}{(8+N)^2}\right)\right]. \tag{91}$$

We stress that this fixed point is different from that arising in the anisotropic behavior and cannot be obtained from it within the ϵ_L expansion described above. The functions $\gamma_{\phi(n)}$ and $\bar{\gamma}_{\phi^2(n)}$ are found to be

$$\gamma_{\phi(n)} = \frac{(N+2)}{72n}\left(1 + \frac{5}{4n}\epsilon_n\right)u_n^2 - \frac{(N+2)(N+8)}{864n^2}u_n^3, \tag{92}$$

$$\bar{\gamma}_{\phi^2(n)} = \frac{(N+2)}{6}u_n\left(1 + \frac{1}{2n}\epsilon_n - \frac{1}{2n}u_n\right). \tag{93}$$

When the fixed point is replaced inside these equations, using the relation among these functions and the critical exponents η_n and ν_n, we find

$$\eta_n = \frac{1}{2n}\epsilon_n^2\frac{N+2}{(N+8)^2}\left[1 + \epsilon_n\frac{1}{n}\left(\frac{6(3N+14)}{(N+8)^2} - \frac{1}{4}\right)\right], \tag{94}$$

$$\nu_n = \frac{1}{2n} + \frac{(N+2)}{4n^2(N+8)}\epsilon_n + \frac{1}{8n^3}\frac{(N+2)(N^2+23N+60)}{(N+8)^3}\epsilon_n^2. \tag{95}$$

The coefficient of the ϵ_n^2 term in the exponent η_n is positive, consistent with its counterpart in the anisotropic cases as well as in the Ising-like case. In the generalized orthogonal approximation the competing momenta are not sufficient to induce its change of sign.

Now using the scaling relations derived for the isotropic case we obtain immediately

$$\gamma_n = 1 + \frac{(N+2)}{2n(N+8)}\epsilon_n + \frac{(N+2)(N^2+22N+52)}{4n^2(N+8)^3}\epsilon_n^2, \tag{96}$$

$$\alpha_n = \frac{(4-N)}{2n(N+8)}\epsilon_n - \frac{(N+2)(N^2+30N+56)}{4n^2(N+8)^3}\epsilon_n^2, \tag{97}$$

$$\beta_n = \frac{1}{2} - \frac{3}{2n(N+8)}\epsilon_n + \frac{(N+2)(2N+1)}{2n^2(N+8)^3}\epsilon_n^2, \tag{98}$$

$$\delta_n = 3 + \frac{1}{n}\epsilon_n + \frac{(N^2+14N+60)}{2n^2(N+8)^2}\epsilon_n^2. \tag{99}$$

The explicit dependence on the number of neighbors coupled through competing interactions is manifest in the above exponents. Hence the universality classes for an arbitrary isotropic ($d=m_n$) competing system are determined by (N,d,n). The interesting fact is that the results for the ordinary critical behavior are correctly recovered in the limit $n \to 1$.

The orthogonal approximation is a good approximation even for isotropic higher character Lifshitz critical behaviors for at least three main reasons. It preserves the homogenity of the Feynman integrals in the external momenta scales. Second, it indicates which parameters are important to describe the universality classes of isotropic systems. And last, but not least, it manifests one of the most important proper-

ties of competing systems, namely the reduction to the Ising-like universality classes in the limit when all interactions beyond first neighbors are turned off.

It is important to emphasize a technical detail concerning the approximation just employed. The leading singularities from the one- and two-loop diagrams contributing to the four-point 1PI vertex part do not get modified from the usual ϕ^4. On the other hand, the leading singularities of the two- and three-loop for the two-point 1PI vertex function do get a factor of $1/n$ with respect to those from the usual critical behavior. These integrals also do not change signs under the generalized orthogonal approximations. The calculation performed without approximations shows that the leading singularities of diagrams contributing to the 1PI two-point function change in a more complicated way, also changing sign depending on the value of n. We shall compare the differences in the values of the exponents utilizing the orthogonal approximation and the exact treatment later on. In particular, we shall perform a numerical analysis for the isotropic second character behavior to understanding the deviations in both approaches. In order to check these results, let us analyze the situation using the minimal subtraction scheme.

B. Critical exponents in minimal subtraction

Minimal subtraction of dimensional poles in the renormalized vertex $\Gamma^{(4)}_{R(n)}$ can be used to show that all momentum-dependent logarithimic integrals are eliminated in the renormalization process leading the bare dimensionless coupling constant to be written as

$$u_{0n} = u_n\left[1 + \frac{(N+8)}{6\epsilon_n}u_n + \left(\frac{(N+8)^2}{36\epsilon_n^2} - \frac{(3N+14)}{24n\epsilon_n}\right)u_n^2\right]. \tag{100}$$

The fixed point can be found out:

$$u_n^* = \frac{6}{(N+8)}\epsilon_n + \frac{18(3N+14)}{n(N+8)^3}\epsilon_n^2. \tag{101}$$

In addition, the normalization constants are given by

$$Z_{\phi(n)} = 1 - \frac{(N+2)}{144n\epsilon_n}u_n^2 + \left(-\frac{(N+2)(N+8)}{1296n\epsilon_n^2} + \frac{(N+2)(N+8)}{5184n^2\epsilon_n}\right)u_n^3, \tag{102}$$

$$\bar{Z}_{\phi^2(n)} = 1 + \frac{(N+2)}{6\epsilon_n}u_n + \left(\frac{(N+2)(N+5)}{36\epsilon_n^2} - \frac{(N+2)}{24n\epsilon_n}\right)u_n^2. \tag{103}$$

Consequently, the functions $\gamma_{\phi(n)}$ and $\bar{\gamma}_{\phi^2(n)}$ can be obtained directly,

$$\gamma_{\phi(n)} = \frac{(N+2)}{72n}u_n^2 - \frac{(N+2)(N+8)}{1728n^2}u_n^3, \tag{104a}$$

$$\gamma_{\phi^2(n)} = \frac{(N+2)}{6}\left(u_n - \frac{1}{2n}u_n^2\right). \tag{104b}$$

Substitution of these expressions in the function $\gamma^*_{\phi(n)}$ at the fixed point, one obtains the value of η_n as obtained in Eq. (94). The function $\bar{\gamma}^*_{\phi^2(n)}$ at the fixed point reads

$$\bar{\gamma}^*_{\phi^2(n)} = \frac{(N+2)}{(N+8)}\epsilon_n\left(1 + \frac{6(N+3)}{n(N+8)^2}\epsilon_n\right), \tag{105}$$

that is, the same as the one obtained in the fixed point using normalization conditions. Therefore it yields the same critical exponent ν_n from Eq. (95) as can be easily cheched. This constitutes the equivalence between the two renormalization schemes.

VII. ISOTROPIC CRITICAL EXPONENTS IN THE EXACT CALCULATION

We now turn our attention to the calculation of the isotropic critical exponents exactly, i.e., not using the orthogonal approximation. The algorithm we need to employ to obtain the critical indices is pretty much the same as that used in the orthogonal approximation, since the renormalization program and the scaling laws are approximation independent. The difference is that one should replace the Feynman integrals by their exact values already calculated in Appendix C. As we are going to discuss the case $n=2$ in the next section using normalization conditions and minimal subtraction, we shall take normalization conditions in this section. Nevertheless, with the resources furnished in the text, the reader should be able to check the exponents using minimal subtraction as well.

Using the definitions given for the power series of the bare dimensionless coupling constant and renormalization functions in terms of the dimensionless renormalized coupling constant, we find the following values for the coefficients of each power of u_n:

$$a_{1n} = \frac{N+8}{6\epsilon_n}[1 + D(n)\epsilon_n], \tag{106a}$$

$$a_{2n} = \left(\frac{N+8}{6\epsilon_n}\right)^2 + \frac{1}{\epsilon_n}\left[\frac{(N^2+21N+86)D(n)}{18} - \frac{5N+22}{18} - \frac{(N+2)(-1)^n\Gamma(2n)^2}{36\Gamma(n+1)\Gamma(3n)} - \frac{5N+22}{18} \times\left(\sum_{p=1}^{2n-2}\frac{1}{2n-p} - 2\sum_{p=1}^{n-1}\frac{1}{n-p}\right)\right], \tag{106b}$$

$$b_{2n} = (-1)^n\frac{(N+2)\Gamma(2n)^2}{72\Gamma(n+1)\Gamma(3n)\epsilon_n}\left[1 + \left(D(n) + \frac{3}{4} - \frac{3}{2}\sum_{p=2}^{2n-1}\frac{1}{p} + \frac{1}{2}\sum_{p=1}^{n-1}\frac{1}{n-p} + \frac{3}{2}\sum_{p=3}^{3n-1}\frac{1}{p}\right)\epsilon_n\right], \tag{106c}$$

$$b_{3n} = (-1)^n \frac{(N+2)(N+8)\Gamma(2n)^2}{108\Gamma(n+1)\Gamma(3n)} \left[\frac{1}{6\epsilon_n^2} + \frac{1}{\epsilon_n} \left(\frac{D(n)}{3} + \frac{1}{24} - \frac{1}{12} \sum_{p=2}^{2n-1} \frac{1}{p} - \frac{1}{12} \sum_{p=1}^{n-1} \frac{1}{n-p} + \frac{1}{12} \sum_{p=3}^{3n-1} \frac{1}{p} \right) \right], \tag{106d}$$

$$c_{1n} = \frac{(N+2)}{6\epsilon_n} [1 + D(n)\epsilon_n], \tag{106e}$$

$$c_{2n} = \frac{(N+2)(N+5)}{36\epsilon_n^2} + \frac{(N+2)}{6\epsilon_n} \left[\frac{(N+8)}{3} D(n) - \frac{1}{2} \left(1 + D(n) + \sum_{p=1}^{2n-2} \frac{1}{2n-p} - 2 \sum_{p=1}^{n-1} \frac{1}{n-p} \right) \right]. \tag{106f}$$

Here $D(n) = 1/2\psi(2n) - \psi(n) + 1/2\psi(1)$. The fixed point at two-loop level is defined as the zero of the β function and it is given by the following expression:

$$u_n^* = \frac{6\epsilon_n}{(N+8)} \left\{ 1 + \epsilon_n \left[\frac{1}{(N+8)^2} \frac{(-1)^n \Gamma(2n)^2 (2N+4)}{\Gamma(n+1)\Gamma(3n)} + (20N+88)[1 - D(n)] - D(n) + \frac{(20N+88)}{(N+8)^2} \times \left(\sum_{p=1}^{2n-2} \frac{1}{2n-p} - 2 \sum_{p=1}^{n-1} \frac{1}{2n-p} \right) \right] \right\}. \tag{107}$$

The renormalization functions $\gamma_{\phi(n)}(u_n)$ and $\overline{\gamma}_{\phi^2(n)}(u_n)$ can be expressed in the following simple manner in terms of u_n:

$$\gamma_{\phi(n)}(u_n) = (-1)^{n+1} \frac{(N+2)\Gamma(2n)^2}{36\Gamma(n+1)\Gamma(3n)} \left[1 + \left(D(n) + \frac{3}{4} - \frac{3}{2} \sum_{p=2}^{2n-1} \frac{1}{p} + \frac{1}{2} \sum_{p=1}^{n-1} \frac{1}{n-p} + \frac{3}{2} \sum_{p=3}^{3n-1} \frac{1}{p} \right) \epsilon_n \right] u_n^2 + (-1)^{n+1} \frac{(N+2)(N+8)\Gamma(2n)^2}{216\Gamma(n+1)\Gamma(3n)} \times \left(-\frac{1}{2} + \sum_{p=2}^{2n-1} \frac{1}{p} - \sum_{p=1}^{n-1} \frac{1}{n-p} - \sum_{p=3}^{3n-1} \frac{1}{p} \right) u_n^3; \tag{108}$$

$$\overline{\gamma}_{\phi^2(n)}(u_n) = \frac{(N+2)}{6} u_n [1 + D(n)\epsilon_n - D(n)u_n]. \tag{109}$$

Substitution of the fixed point in the first equation gives directly the anomalous dimensions η_n. Using the scaling law relating the second expression with the exponents η_n and ν_n, one obtains the exponent ν_n. Thus we get

$$\eta_n = (-1)^{n+1} \frac{(N+2)\Gamma(2n)^2}{(N+8)^2\Gamma(n+1)\Gamma(3n)} \epsilon_n^2 + (-1)^{n+1} \frac{(N+2)\Gamma(2n)^2 F(N,n)}{(N+8)^2\Gamma(n+1)\Gamma(3n)} \epsilon_n^3; \tag{110}$$

where

$$F(N,n) = \left[\left((-1)^n \frac{\Gamma(2n)^2(4N+8)}{\Gamma(n+1)\Gamma(3n)} + (40N+176)D(n) \right) \frac{1}{(N+8)^2} - \frac{3}{4} - \sum_{p=1}^{2n-1} \frac{1}{p} + \frac{1}{2} \sum_{p=1}^{n-1} \frac{1}{p} + \frac{1}{2} \sum_{p=1}^{3n-1} \frac{1}{p} \right]. \tag{111}$$

Analogously,

$$\nu_n = \frac{1}{2n} + \frac{(N+2)}{4n^2(N+8)} \epsilon_n + \frac{(N+2)}{4n^2(N+8)^3} \times \epsilon_n^2 \left((-1)^n (N-4) \frac{\Gamma(2n)^2}{\Gamma(n+1)\Gamma(3n)} + \frac{(N+2)(N+8)}{2n} + (14N+40)D(n) \right). \tag{112}$$

Now, we use the scaling relations to obtain the remaining critical exponents. We find

$$\gamma_n = 1 + \frac{(N+2)}{2n(N+8)} \epsilon_n + \frac{(N+2)}{4n^2(N+8)^3} \times \epsilon_n^2 \left((-1)^n \frac{2n(2N+4)\Gamma(2n)^2}{\Gamma(n+1)\Gamma(3n)} + (N+2)(N+8) + 2n(14N+40)D(n) \right), \tag{113}$$

$$\alpha_n = \frac{(4-N)}{2n(N+8)} \epsilon_n - \frac{(N+2)}{4n^2(N+8)^3} \epsilon_n^2 \left((-1)^n \frac{4n(N-4)\Gamma(2n)^2}{\Gamma(n+1)\Gamma(3n)} + (N-4)(N+8) + 4n(14N+40)D(n) \right), \tag{114}$$

$$\beta_n = \frac{1}{2} - \frac{3}{2n(N+8)} \epsilon_n + \frac{(N+2)}{4n^2(N+8)^3} \times \epsilon_n^2 \left((-1)^{n+1} \frac{12n\Gamma(2n)^2}{\Gamma(n+1)\Gamma(3n)} - 3(N+8) + n(14N+40)D(n) \right), \tag{115}$$

$$\delta_n = 3 + \frac{\epsilon_n}{n} + \frac{\epsilon_n^2}{2n^2(N+8)^2} \times \left((N+8)^2 + (-1)^n \frac{4n(N+2)\Gamma(2n)^2}{\Gamma(n+1)\Gamma(3n)} \right). \tag{116}$$

Two important features emerge from this exact picture of the critical exponents shown above. It is instructive to discuss the exponent η_n. First, the sign of the lowest $[O(\epsilon_n^2)]$ contribution to the exponent η_n is determined by the value of n. Remember that this fact merely reflects the change of sign of the two-point diagrams depending on the value of n. Second, instead of a global factor proportional to n the exact solution has a dependence on n coming from a product of Γ functions having n on their arguments as well as a finite sum with terms which depend on n. This property is valid for all critical exponents and appears explicitly at two- and higher-loop corrections. This is quite a remarkable feature of arbitrary isotropic competing systems, going beyond the simple polynomial dependence on n found using the orthogonal approximation.

The universality class reduction to that from the Ising-like one in the limit $n \to 1$ is obvious. Moreover, the case $n=2$ correctly reproduces the solution of an earlier two-loop calculation. Actually, the results given above extend the calculation of η_n for arbitrary n to include $O(\epsilon_n^3)$ corrections. In addition, all critical exponents presented here at least up to two-loop order generalize all previous results for arbitrary higher character isotropic Lifshitz points. Therefore at the loop order considered the results above represent the complete solution to the critical exponents of the CECI model for arbitrary types of isotropic competing interactions.

A numerical analysis for comparison between the results obtained either using the orthogonal approximation or in the exact calculation are worthwhile. The case $n=2$ will be analyzed later. Here we display the numerical results for the cases $n=3$, 4, 5, 6.

It would be appropriate to calculate the exponents in either approach in a particular case in order to see if the difference is meaningful. The usual ϵ expansion is good enough for the numerical estimation of critical exponents associated to three-dimensional critical systems even though the expansion parameter is not small. We can then ask ouselves if the same analogy is valid in order to extract concrete results from a fixed value of the space dimension and number of components of the order-parameter field for arbitrary isotropic higher character Lifshitz points. We shall look at space dimensions which yields $\epsilon_n=1$ in analogy to the method used to extract numerical results from the ordinary ϵ expansion for three-dimensional systems.

For an 11-dimensional lattice, take $N=1$ and $n=3$ for the isotropic third character behavior. The orthogonal approximation yields $\eta_3=0.006$, $\nu_3=0.177$, $\alpha_3=0.046$, $\beta_3=0.446$, $\gamma_3=1.064$, and $\delta_3=3.385$. The exact calculation produces the exponents $\eta_3=0.002$, $\nu_3=0.174$, $\alpha_3=0.085$, $\beta_3=0.435$, $\gamma_3=1.046$, and $\delta_3=3.390$. The maximal deviation (4.1%) occurs for the exponent α_3 followed by the deviation in the γ_3 exponent (1.8%) and an error in the exponent β_3 around 1%. The other exponents have deviations smaller than 0.5%.

Consider the case $N=1$, $d=15$, $n=4$. The results from the orthogonal approximation are $\eta_4=0.005$, $\nu_4=0.131$, $\alpha_4=0.036$, $\beta_4=0.459$, $\gamma_4=1.046$, and $\delta_4=3.279$. The exact calculation yields $\eta_4=-0.001$, $\nu_4=0.129$, $\alpha_4=0.058$, $\beta_4=0.449$, $\gamma_4=1.029$, and $\delta_4=3.282$. The maximal deviations takes place in $\alpha_4(2.2\%)$, $\gamma_4(1.7\%)$, and $\beta_4(1\%)$, while the other exponents have errors smaller than 0.6%.

Let us examine the case $N=1$, $d=19$, and $n=5$. The isotropic exponents in the orthogonal approximation are $\eta_5=0.004$, $\nu_5=0.104$, $\alpha_5=0.030$, $\beta_5=0.467$, $\gamma_5=1.036$, and $\delta_5=3.219$. The exact exponents are $\eta_5=0.001$, $\nu_5=0.102$, $\alpha_5=0.064$, $\beta_5=0.475$, $\gamma_5=1.020$, and $\delta_5=3.220$. The maximal deviations are in $\alpha_5(3.4\%)$ and $\gamma_5(1.6\%)$, whereas the remaining exponents have errors smaller than 0.8%. (The maximal deviations also occur for α_n and γ_n for $n=6$, 7, 8 and decrease for increasing n. For $n=8$, the maximal error for α_8 is about 2.7%, whereas the maximal deviation for γ_8 is 1.3%.)

This analysis leads us to conclude that the orthogonal approximation is very precise to predict numerical values in specific situations, since the deviations are negligible when compared with the exact calculation. Moreover, the above data indicate that the deviations are under control no matter how the number of neighbors increases.

The extra insight from the field-theoretic viewpoint is that the more neighbors are introduced and coupled through isotropic competing interactions the more the space dimensions seem to split. Then, one line with competing interactions up to second neighbors behaves for all practical purposes as having two dimensions. Pushing the argument further, a line with n neighbors interacting through competing interactions seem to have n dimensions. This is a striking general property of the field theory under consideration: in the massless limit presented here, when the free critical propagator is proportional to a $2n$th power of momenta each space direction "splits" in n dimensions. This is a kind of degeneracy in the dimension, which can only be unveiled when more participants (neighbors) are allowed to take place in the isotropic competing system. This is a different aspect of systems whose Lagrangians have kinetic terms described by higher derivatives of the field. Further implications of this phenomenon remain to be investigated.

Now, let us show that these findings easily reproduce and extend the original calculation done by Hornreich, Luban, and Shtrikman[3] for the $n=2$ case.

VIII. EXACT ISOTROPIC EXPONENTS FOR THE SECOND CHARACTER CASE

In the last section we found the critical exponents for the isotropic case when competing interactions among arbitrary neighbors are allowed. We now discuss its reduction for the usual second character Lifshitz critical behavior. Since we want to compare our results with others which already appeared in the literature we shall derive the critical indices using normalization conditions and minimal subtraction of poles.

A. Critical exponents in normalization conditions

The fixed point can be calculated by replacing $n=2$ in Eq. (107) in order to obtain

$$u_2^* = \frac{6\epsilon_2}{(N+8)}\left[1 - \frac{1}{3}\epsilon_2\left(\frac{(41N+202)}{(N+8)^2} - \frac{1}{4}\right)\right]. \quad (117)$$

The renormalization functions $\gamma_{\phi(2)}(u_2)$ and $\bar{\gamma}_{\phi^2(2)}(u_2)$ can be expressed in the following simple manner in terms of u_2:

$$\gamma_{\phi(2)}(u_2) = -\frac{(N+2)}{240}\left(1+\frac{131}{120}\epsilon_2\right)u_2^2 + \frac{29(N+2)(N+8)}{28800}u_2^3; \tag{118}$$

$$\bar{\gamma}_{\phi^2(2)}(u_2) = \frac{(N+2)}{6}u_2\left(1-\frac{1}{12}\epsilon_2+\frac{1}{12}u_2\right). \tag{119}$$

Substitution of the fixed point in the first equation gives directly the anomalous dimensions η_2. Thus we get

$$\eta_2 = -\frac{3(N+2)}{20(N+8)^2}\epsilon_2^2 + \frac{(N+2)}{10(N+8)^2}\left(\frac{(41N+202)}{10(N+8)^2}+\frac{23}{80}\right)\epsilon_2^3. \tag{120}$$

From the scaling law relating η_2 with ν_2 we obtain

$$\nu_2 = \frac{1}{4} + \frac{(N+2)}{16(N+8)}\epsilon_2 + \frac{(N+2)(15N^2+89N+4)}{960(N+8)^3}\epsilon_2^2. \tag{121}$$

Using the scaling relations to obtain the remaining critical exponents, we find

$$\gamma_2 = 1 + \frac{(N+2)}{4(N+8)}\epsilon_2 + \frac{(N+2)(15N^2+98N+76)}{240(N+8)^3}\epsilon_2^2; \tag{122}$$

$$\alpha_2 = \frac{(4-N)}{4(N+8)}\epsilon_2 + \frac{(N+2)(-15N^2+62N+952)}{240(N+8)^3}\epsilon_2^2; \tag{123}$$

$$\beta_2 = \frac{1}{2} - \frac{3}{4(N+8)}\epsilon_2 - \frac{(N+2)(80N+514)}{240(N+8)^3}\epsilon_2^2; \tag{124}$$

$$\delta_2 = 3 + \frac{1}{2}\epsilon_2 + \frac{(5N^2+86N+332)}{40(N+8)^2}\epsilon_2^2. \tag{125}$$

Since the earlier calculations in the literature were performed using minimal subtraction, we shall discuss it next.

B. Critical exponents in minimal subtraction

Now, let us obtain some renormalization functions at the fixed point using minimal subtraction. Since these objects are universal, finding them is equivalent to finding the fixed point. Requiring minimal subtraction of the renormalized vertex $\Gamma^{(4)}_{R(2)}$, the cancellations among logarithmic integrals for arbitrary external momenta indeed take place as expected. It leads to the expression of the bare dimensionless coupling constant u_{02} written in terms of the renormalized dimensionless coupling constant u_2, namely

$$u_{02} = u_2\left[1+\frac{(N+8)}{6\epsilon_2}u_2 + \left(\frac{(N+8)^2}{36\epsilon_2^2}+\frac{(41N+202)}{2160\epsilon_2}\right)u_2^2\right]. \tag{126}$$

From the definitions of the normalization constants, we can write the following expressions:

$$Z_{\phi(2)} = 1 + \frac{(N+2)}{480\epsilon_2}u_2^2 + \frac{(N+2)(N+8)}{4320}\left(\frac{1}{\epsilon_2^2}-\frac{23}{518400\epsilon_2}\right)u_2^3, \tag{127}$$

$$\bar{Z}_{\phi^2(2)} = 1 + \frac{(N+2)}{6\epsilon_2}u_2 + \left(\frac{(N+2)(N+5)}{36\epsilon_2^2}+\frac{N+2}{144\epsilon_2}\right)u_2^2. \tag{128}$$

Consequently, the functions $\gamma_{\phi(2)}(u_2)$ and $\bar{\gamma}_{\phi^2(2)}(u_2)$ read

$$\gamma_{\phi(2)}(u_2) = -\frac{(N+2)}{240}u_2^2 + \frac{23(N+2)(N+8)}{172800}u_2^3; \tag{129}$$

$$\bar{\gamma}_{\phi^2(2)}(u_2) = \frac{(N+2)}{6}u_2\left(1+\frac{1}{12}u_2\right). \tag{130}$$

The fixed point can be calculated and shown to be

$$u_2^* = \frac{6}{(N+8)}\epsilon_2 - \frac{(41N+202)}{5(N+8)^3}\epsilon_2^2. \tag{131}$$

Replacement of this expression for the function $\gamma_{\phi(2)}(u_2^*)$ gives precisely the exponent η_2 of the last section up to $O(\epsilon_2^3)$. The corresponding expression for $\bar{\gamma}_{\phi^2(2)}(u_2^*)$ is given by

$$\bar{\gamma}^*_{\phi^2(2)} = \frac{(N+2)}{(N+8)}\epsilon_2\left(1-\frac{(N+2)(13N+41)}{15(N+8)^2}\epsilon_2\right). \tag{132}$$

The last expression is actually the same as that coming from normalization conditions, therefore leading to the same exponent ν_2 whereas the remaining exponents are obtained using the scaling laws. Thus the equivalence between the two renormalization schemes is complete.

C. Discussion

First of all, our results for the isotropic $n=2$ case generalizes those in the seminal work by Hornreich, Luban, and Shtrikman.[3] To see this, we make the identifications $\epsilon_\alpha \equiv \epsilon_2$, $\eta_{l4} \equiv \eta_2$, and $\nu_{l4} \equiv \nu_2$. Equations (10a) and (10b) in Ref. 3 are identical to our results for η_2 and ν_2 Eqs. (120) and (121). The step forward within our method is the exponent η_2 which is obtained up to $O(\epsilon_2^3)$. Furthermore, using the scaling relations reported in our previous paper,[15] we found all critical exponents [Eqs. (122)–(125)] exactly, at least up to two-loop order which constitutes another interesting result for the usual isotropic case.

Next, let us confront the results coming from the generalized orthogonal approximation with those from the exact solution at the same loop order. Both the exact and the approximated one-loop exponents are the same. This can be seen

from the $n=2$ particular case or from the generic n isotropic criticality by the direct examination of the explicit results shown in the present paper. Two-loop deviations start at $n=2$ and for higher n.

Consider the case $n=2$. Take a magnetic system which has $N=1$ in a seven-dimensional lattice and analyze the exponents in each case. We shall restrict ourselves to three significant algorithms in our discussion. First use the orthogonal approximation. At two-loop order, the correlation length exponent yields the result $\nu_2=0.276$, and the anomalous dimension is given by $\eta_2=0.009$. The susceptibility, specific heat, magnetization, and magnetic field exponents are given by $\gamma_2=1.103$, $\alpha_2=0.061$, $\beta_2=0.418$, $\delta_2=3.616$. Now use the exact two-loop calculation. We obtain the following numerical values for the critical indices: $\nu_2=0.271$, $\eta_2=-0.006$, $\gamma_2=1.087$, $\alpha_2=0.100$, $\beta_2=0.406$, $\delta_2=3.631$.

Therefore for all the exponents the difference using either approach starts in the second significative algorithm. Specifically, the maximal error made by using the orthogonal approximation takes place for the specific-heat exponent with a deviation of 3.9%, whereas the minimal error occurs in the correlation length exponent whose difference is approximately 0.5%. This is a strong evidence that the orthogonal approximation is very good to give reliable information for the isotropic case in this specific situation. This feature was already encountered for uniaxial anisotropic cases, where the approximation showed its usefulness for three-dimensional uniaxial systems. We hope that these results may stimulate the search for these exponents using Monte Carlo numerical simulations in the particular case of second character isotropic Lifshitz critical points.

The remarkable agreement between the numerical results for the above-mentioned critical indices either using the orthogonal approximation or the exact two-loop calculations corroborates previous conjectures that the orthogonal approximation is not only good to describe uniaxial systems pertaining to the anisotropic second character Lifshitz critical behaviors but also arbitrary isotropic higher character Lifshitz points.

In fact, the numerical analysis can be carried out for higher-dimensional lattices for higher values of n. Extending this argument, the case $L=4n$, $N=1$, and $d=L-1$ yields the same value for the expansion parameter and should not deviate very much when both calculations are compared. Perhaps the study of the most general arbitrary isotropic points via numerical tools might be worthwhile as well, since now we have satisfactory numerical results coming from a purely analytical field-theoretical investigation.

IX. CONCLUSIONS AND PERSPECTIVES

In this paper we have discussed the field-theoretic description of the most general competing system, which has a simple lattice model representation named CECI model. It consists of a modified Ising model presenting the most general type of competing exchange couplings among arbitrary neighbors and includes other models previously reported. We have derived explicitly the critical exponents in the anisotropic as well as in the isotropic situations at least up to two-loop level. The CECI model and its field-theory representation generalize the description of the second character Lifshitz universality classes in a nontrivial way. In particular, strong anisotropic scale invariance is exact up to the loop order considered here. The universality class reduction is a general property of both anisotropic and isotropic critical behaviors. It implies that when the interactions beyond first neighbors are turned off, the Ising-like universality classes are recovered. This feature is manifest in all exponents.

The anisotropic exponents were calculated by using the generalized orthogonal approximation. The calculation of loop integrals is consistent, since it is rooted in the physical property of homogeneity. It is required for a satisfactory renormalization-group treatment with several independent relevant length scales represented by each correlation length associated to the several competing axes. The fixed point is the same irrespective of the competing axes under consideration. In close analogy to the second character case, this result is expected to hold in arbitrary perturbative orders. The second character Lifshitz exponents are easily recovered as a particular case of the generic anisotropic situation described in the paper. Although it is desirable to have an approach that does not require the use of approximations for the anisotropic behaviors, in the present moment it is not obvious. It is a consequence of the appearance of many competing subspaces simultaneously which makes the exact calculation (if not impossible) very difficult. We hope the techniques developed in the present work shed light on a quest for a solution without the necessity of approximations for the anisotropic cases.

The isotropic behaviors were calculated using two different trends to evaluate the Feynman diagrams. The first of them makes use of the orthogonal approximation. It can be noticed that the isotropic behavior cannot be obtained from any type of anisotropic behavior within the framework of the ϵ_L expansions developed in the present work even though the same kind of approximations are employed in both cases. This generalizes the previous situation taking place for second character Lifshitz points. Next, we attacked the diagrams without making any approximation. The result is that deviations in the calculation of the critical exponents start at the two-loop level in comparison to the outcome provenient of the orthogonal approximation. We obtain as a particular case of the generic isotropic behavior the second character isotropic behavior, which extends the results derived by Hornreich, Luban, and Shtrikman.[3] In this way, we obtain η_2 up to three-loop order and ν_2 at two loops as well as the remaining exponents via the scaling relations derived previously in Ref. 15, a result which was lacking since the discovery of the second character isotropic Lifshitz point.

The most immediate application of the formalism just presented is the calculation of all universal amplitude ratios of certain quantities close to generic higher character Lifshitz points. It is amazing that a detailed scale theory for these quantities is lacking in the literature, even for the simple second character Lifshitz points. In fact, some results were presented for the susceptibility[30] and specific heat[31] ampli-

tude ratios at one-loop order. The latter amplitude ratio proved to be reliable to explain the experimental result associated to the magnetic material MnP.[32] Nevertheless, a thorough renormalization-group analysis is necessary in order to have a better comprehension of the several scale transformations in each competing subspace and the role played by them in the treatment of generic amplitude ratios. In principle we can extend the formalism presented here for the calculations of amplitude ratios including the most general competing system.

The treatment of finite-size effects for the most general Lifshitz critical behavior can be developed as a direct extension of the approach to the Ising-like critical behavior.[33,34] The applications might include systems which are finite (or semi-infinite) along one (or several) of their dimensions, but of infinite extent in the remaining directions. It would be interesting to see how different competing axes alter the approach to the bulk criticalities, for example in parallel plate geometries. The utilization of different boundary conditions in a layered geometry would be particularly simple and instructive to see how the generalization works for arbitrary competing systems. It could be used to investigate how amplitude ratios change with different boundary conditions with respect to the situation occurring in bulk systems.[35]

Typical examples are systems which are finite in all directions, such as a (hyper)cube of size L, and systems which are of infinite size in $d'=d-1$ dimensions but are either of finite thickness L along the remaining direction (d-dimensional layered geometry) or of a semi-infinite extension. The presence of geometrical restrictions on the domain of systems also requires the introduction of boundary conditions (periodic, antiperiodic, Dirichlet, and Neumann) satisfied by the order parameter on the surfaces. In particular, the validity limits of the ϵ_L expansion for these systems and the approach to bulk criticality in a layered geometry can be studied.[35]

One interesting aspect of the generalized orthogonal approximation is that it can actually address the problem of calculating Feynman integrals originating in field theories in the massless limit with odd (greater than 2) powers of momentum in the propagator as well. This subject goes beyond the realm of critical phenomena in competing systems. It might be useful for treating perturbatively a recent proposal of an effective quantum field theory with cubic kinetic terms.[36] In addition, the general framework can be used to treat perturbatively other effective quantum field theories with higher derivative kinetic terms which break Lorentz invariance in the infrared regime as the effect of combined gravitational attraction and repulsion.[19] This type of theory resembles very much the second character Lifshitz critical behavior discussed above. One can expect that introducing more and more competing gravitational interactions in the infrared (long distance) limit higher powers will appear in the kinetic terms of the corresponding effective field theory. The perturbative analysis of this system would be quite analogous to the generic higher character discussed in the present work.

To summarize, we have described the generic higher character Lifshitz critical behaviors. Different field-theoretical tools were exposed resulting in interesting analytical expressions for all the critical indices in the isotropic as well as in the anisotropic cases at least at $O(\epsilon_L^2)$. Other aspects like crossover phenomena and tricritical behavior for this model remain to be studied. Interesting anisotropic behaviors in the absence of a uniform ordered phase for the CECI model are under current investigation.

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APPENDIX A: FEYNMAN INTEGRALS FOR ANISOTROPIC BEHAVIORS

In general the critical exponents and other universal ammounts are independent of the renormalization-group scheme employed. The explicit calculations in this section are presented in such a way that the results can be checked using more than one renormalization procedure. We shall now describe the generalized orthogonal approximation (GOA) for the integrals appearing in the anisotropic cases.

In order to accomplish the task of calculating the critical indices at least up to the two-loop level, we need a minimal set of Feynman diagrams to work with. The one-, two-, and three-loop integrals we shall need to determine are given by the following expressions:

$$I_2=\int \frac{d^{d-\Sigma_{n=2}^{L} m_n} q \Pi_{n=2}^{L} d^{m_n} k_{(n)}}{\left\{\left[\sum_{n=2}^{L}(k_{(n)}+K'_{(n)})^2\right]^n+(q+P)^2\right\}\left(\sum_{n=2}^{L}(k_{(n)}^2)^n+q^2\right)}, \tag{A1}$$

$$I_3=\int \frac{d^{d-\Sigma_{n=2}^{L} m_n} q_1 d^{d-\Sigma_{n=2}^{L} m_n} q_2 \Pi_{n=2}^{L} d^{m_n} k_{1(n)} \Pi_{n=2}^{L} d^{m_n} k_{2(n)}}{\left[q_1^2+\sum_{n=2}^{L}(k_{1(n)}^2)^n\right]\left[q_2^2+\sum_{n=2}^{L}(k_{2(n)}^2)^n\right]} \times \frac{1}{\left\{(q_1+q_2+P)^2+\left[\sum_{n=2}^{L}(k_{1(n)}+k_{2(n)}+K'_{(n)})^2\right]^n\right\}}, \tag{A2}$$

$$I_4 = \int \frac{d^{d-\Sigma_{n=2}^{L} m_n} q_1 d^{d-\Sigma_{n=2}^{L} m_n} q_2 \Pi_{n=2}^{L} d^{m_n} k_{1(n)} \Pi_{n=2}^{L} d^{m_n} k_{2(n)}}{\left[q_1^2 + \sum_{n=2}^{L} (k_{1(n)}^2)^n \right] \left\{ (P-q_1)^2 + \sum_{n=2}^{L} [(K'_{(n)} - k_{1(n)})^2]^n \right\}} \times \frac{1}{\left[q_2^2 + \sum_{n=2}^{L} (k_{2(n)}^2)^n \right] \left\{ (q_1 - q_2 + p_3)^2 + \sum_{n=2}^{L} [(k_{1(n)} - k_{2(n)} + k'_{3(n)})^2]^n \right\}}, \tag{A3}$$

$$I_5 = \int \frac{d^{d-\Sigma_{n=2}^{L} m_n} q_1 d^{d-\Sigma_{n=2}^{L} m_n} q_2 d^{d-\Sigma_{n=2}^{L} m_n} q_3 \Pi_{n=2}^{L} d^{m_n} k_{1(n)}}{\left[q_1^2 + \sum_{n=2}^{L} (k_{1(n)}^2)^n \right] \left[q_2^2 + \sum_{n=2}^{L} (k_{2(n)}^2)^n \right] \left[q_3^2 + \sum_{n=2}^{L} (k_{3(n)}^2)^n \right]} \frac{\Pi_{n=2}^{L} d^{m_n} k_{2(n)} \Pi_{n=2}^{L} d^{m_n} k_{3(n)}}{\left\{ (q_1 + q_2 - p)^2 + \sum_{n=2}^{L} [(k_{1(n)} + k_{2(n)} - k'_{(n)})^2]^n \right\}} \times \frac{1}{\left\{ (q_1 + q_3 - p)^2 + \sum_{n=2}^{L} \left[(k_{1(n)} + k_{3(n)} - k'_{(n)})^2 \right]^n \right\}}. \tag{A4}$$

We stress that in the above expressions P, p_3, and p are external momenta perpendicular to the various competing axes, whereas $K'_{(n)}$, $k'_{3(n)}$, and $k'_{(n)}$ are external momenta characterizing the nth competing axes, respectively. For arbitrary values of the external momenta, these integrals can be calculated by making use of approximations very similar to those first developed to describe second character Lifshitz points.[8] As a matter of fact, the generalized dissipative approximation was formerly used to compute the critical exponents out of the anomalous dimension and correlation length exponents corresponding to space directions perpendicular to the competing axes for this model. Indeed, the generalized orthogonal approximation was figured out using a similar analogy. Nevertheless, since the dissipative approximation cannot approach the isotropic cases, we shall not describe it in this paper. Instead, we shall make use of the generalized orthogonal approximation, for it can approach both anisotropic and isotropic behaviors.

Certain useful identities will be derived in order to solve the integrals above. Let us proceed to find them out. Our starting point is the integral derived in Ref. 8, namely

$$\int_{-\infty}^{\infty} dx_1 \cdots dx_m \exp[-a(x_1^2 + \cdots + x_m^2)^n] = \frac{1}{2n} \Gamma\left(\frac{m}{2n}\right) a^{-m/2n} S_m. \tag{A5}$$

After choosing $r^2 = x_1^2 + \cdots + x_m^2$, one can take $y = r^n$ to write the last equation in the form

$$\int_0^{\infty} dy y^{m/n-1} \exp(-ay^2) = \frac{1}{2} a^{-m/2n} \Gamma\left(\frac{m}{2n}\right). \tag{A6}$$

The integral

$$\int_{-\infty}^{\infty} \exp[-ax^{2k} - b(x+x_0)^{2k}] dx \tag{A7}$$

cannot be solved exactly for all k. In fact, for generic $k \geqslant 2$ it has no elementary primitive. Nevertheless, one can select only the homogeneous function of a by using the following approximation. First, make the change of variables $y = x^k$. Second perform the approximation $(x+x_0)^k \cong x^k + x_0^k$. Thus this integral becomes

$$\int_{-\infty}^{\infty} \exp[-ax^{2k} - b(x+x_0)^{2k}] dx = \exp(-by_0^2) \frac{2}{k} \int_0^{\infty} \exp[-(a+b)y^2 - 2byy_0] y^{1/k-1} dy. \tag{A8}$$

Next, perform another change of variables, namely, $y' = y + by_0/(a+b)$. We then obtain

$$\int_{-\infty}^{\infty} \exp[-ax^{2k} - b(x+x_0)^{2k}] dx = \exp\left(-\frac{ab}{a+b} y_0^2\right) \frac{2}{k} \times \left\{ \int_0^{\infty} \exp[-(a+b)y'^2] \left(y' - \frac{by_0}{a+b} \right)^{1/k-1} dy' - \int_0^{by_0/(a+b)} \exp[-(a+b)y'^2] \left(y' - \frac{by_0}{a+b} \right)^{1/k-1} dy' \right\}. \tag{A9}$$

Since the integrals are convergent, we can make use of the

approximation $(y'-b/2a)^{1/k-1}=y'^{1/k-1}+\cdots$. The ellipsis stands for the remaining terms that will be subtracted from the last integral, producing a type of error function. The original integral is then approximated by its leading contribution, i.e.,

$$\int_{-\infty}^{\infty}\exp[-ax^{2k}-b(x+x_0)^{2k}]dx \cong \exp\left(-\frac{ab}{a+b}x_0^{2k}\right)\frac{1}{k}\Gamma\left(\frac{1}{2k}\right)(a+b)^{-1/2k}. \quad \text{(A10)}$$

It can be easily shown that the generalization for the m sphere yields

$$\int_{-\infty}^{\infty}\exp\{-a(x_1^2+\cdots+x_m^2)^k - b[(x_1+x_{01})^2+\cdots+(x_m+x_{0m})^2]\}^k dx_1\cdots dx_m \cong \exp\left(-\frac{ab}{a+b}x_0^{2k}\right)\frac{1}{2k}S_m\Gamma\left(\frac{m}{2k}\right)(a+b)^{-m/2k}. \quad \text{(A11)}$$

We found it appropriate to name this approximation the generalized orthogonal approximation, for it is a natural generalization of that discussed in the usual second character case.[8]

Let us now start our calculation of loop integrals given by Eqs. (A1)–(A4). We can calculate the one-loop integral using two Schwinger parameters. Using the formula derived above, the integration over quadratic momenta can be shown to be given by

$$I_2=\frac{1}{2}S_{(d-\sum_{n=2}^{L}m_n)}\Gamma\left(\frac{(d-\sum_{n=2}^{L}m_n)}{2}\right)\int_0^\infty\int_0^\infty d\alpha_1 d\alpha_2 \times\exp\left(-\frac{\alpha_1\alpha_2P^2}{\alpha_1+\alpha_2}\right)(\alpha_1+\alpha_2)^{-(d-\sum_{n=2}^{L}m_n)/2}\int(\Pi_{n=2}^{L}d^{m_n}k_{(n)}) \times\exp\left(-\alpha_1\sum_{n=2}^{L}(k_{(n)}^2)^n-\alpha_2\sum_{n=2}^{L}[(k_{(n)}+K'_{(n)})^2]^n\right). \quad \text{(A12)}$$

We can now expand the argument of the last exponentials. Notice that now we have a product of independent integrals corresponding to the momentum components along arbitrary competing subspaces. Those integrals cannot be performed analytically. If we use the homogeneity property of the remaining integrals in the arbitrary competing external momenta scales, we can simplify the calculation by utilizing the generalized orthogonal approximation. In fact, taking $(k_{(n)}+K'_{(n)})^n\cong k_{(n)}^n+K'^n_{(n)}$ and using Eq. (A11), we obtain

$$\int d^{m_n}k_{(n)}\exp[-\alpha_1k_{(n)}^{2n}-\alpha_2(k_{(n)}+K'_{(n)})^{2n}] = \frac{1}{2n}S_{m_n}\times(\alpha_1+\alpha_2)^{-m_n/2n}\exp\left(-\frac{\alpha_1\alpha_2K'^{2n}_{(n)}}{\alpha_1+\alpha_2}\right)\Gamma\left(\frac{m_n}{2n}\right). \quad \text{(A13)}$$

Therefore we can write the integral as

$$I_2=\frac{1}{2}S_{(d-\sum_{n=2}^{L}m_n)}\Gamma\left(\frac{(d-\sum_{n=2}^{L}m_n)}{2}\right)\times\left(\Pi_{n=2}^{L}\frac{S_{m_n}\Gamma\left(\frac{m_n}{2n}\right)}{2n}\right)\int_0^\infty\int_0^\infty d\alpha_1 d\alpha_2 \times\exp\left(-\frac{\alpha_1\alpha_2\{P^2+\sum_{n=2}^{L}[(K'_{(n)})^2]\}^n}{\alpha_1+\alpha_2}\right) \times(\alpha_1+\alpha_2)^{-\{d-\sum_{n=2}^{L}[(n-1)m_n/n]\}/2}. \quad \text{(A14)}$$

Take $x=\alpha_1\{P^2+\Sigma_{n=2}^{L}[(K'_{(n)})^2]^n\}$ and $y=\alpha_2\{P^2+\Sigma_{n=2}^{L}[(K'_{(n)})^2]^n\}$. Then, define $v=x/(x+y)$. Consequently, the parametric integrals can be performed with this change of variables. Using the identity

$$\Gamma(a+bx)=\Gamma(a)[1+bx\psi(a)+O(x^2)] \quad \text{(A15)}$$

and expressing everything in terms of the ϵ_L parameter leads to the following expression for I_2:

$$I_2=\frac{1}{2}\left[S_{(d-\sum_{n=2}^{L}m_n)}\Gamma\left(2-\sum_{n=2}^{L}\frac{m_n}{2n}\right)\left(\Pi_{n=2}^{L}\frac{S_{m_n}\Gamma\left(\frac{m_n}{2n}\right)}{2n}\right)\right] \times\left[1-\frac{\epsilon_L}{2}\psi\left(2-\sum_{n=2}^{L}\frac{m_n}{2n}\right)\right]\Gamma\left(\frac{\epsilon_L}{2}\right) \times\int_0^1 dv\left(v(1-v)\left\{P^2+\sum_{n=2}^{L}[(K'_{(n)})^2]^n\right\}\right)^{-\epsilon_L/2}. \quad \text{(A16)}$$

This is a homogeneous function (with the same homogeneity degree) in $(P,K'_{(n)})$. The factor

$$\left[S_{(d-\sum_{n=2}^{L}m_n)}\Gamma\left(2-\sum_{n=2}^{L}\frac{m_n}{2n}\right)\left(\Pi_{n=2}^{L}\frac{S_{m_n}\Gamma\left(\frac{m_n}{2n}\right)}{2n}\right)\right]$$

is going to be absorbed in a redefinition of the coupling constant after performing each loop integral and shall be omitted henceforth. We can follow two routes from the last equation. The first one is to perform the v integral in terms of products of the Euler Γ functions. It will be useful in the calculation of higher-loop integrals since we need the subdiagrams of this type in order to compute the complete integral. Then, we obtain

$$I_2(P,K'_{(n)})=\left(P^2+\sum_{n=2}^{L}[(K'_{(n)})^2]^n\right)^{-\epsilon_L/2}\frac{1}{\epsilon_L}(1+h_{m_L}\epsilon_L). \quad \text{(A17)}$$

This is appropriate to calculate the critical exponents only using normalization conditions, but we would like the solution in a form suitable for minimal subtraction as well. The

second possibility convenient for both types of renormalization schemes is to expand the last integral as

$$\int_0^1 dv\left[v(1-v)\left(P^2+\sum_{n=2}^{L}[(K'_{(n)})^2]^n\right)\right]^{-\epsilon_L/2}=1-\frac{\epsilon_L}{2}L(P,K'_{(n)}), \tag{A18}$$

where

$$L(P,K'_{(n)})=\int_0^1 dv\,\ln\left[v(1-v)\left(P^2+\sum_{n=2}^{L}[(K'_{(n)})^2]^n\right)\right]. \tag{A19}$$

Hence this integral reads

$$I_2(P,K'_{(n)})=\frac{1}{\epsilon_L}\left(1+(h_{m_L}-1)\epsilon_L-\frac{\epsilon_L}{2}L(P,K'_{(n)})\right), \tag{A20}$$

where $h_{m_L}=1+\{\psi(1)-\psi[2-\Sigma_{n=2}^{L}(m_n/2n)]\}/2$. Notice that whenever $m_3=\cdots=m_L=0$, $h_{m_2}=[i_2]_m$, and the usual anisotropic Lifshitz critical behavior is trivially obtained from this more general competing situation. This form is convenient for the renormalization using minimal subtraction. Instead, for normalization conditions we have

$$I_{2SP_1}=\cdots=I_{2SP_L}=\frac{1}{\epsilon_L}(1+h_{m_L}\epsilon_L), \tag{A21}$$

since $L(SP_1=\cdots=SP_L)=-2$, with $SP_1\equiv(P^2=1,K'_{(n)}=0),\ldots,$ $SP_L\equiv(P=0,(K'_{(L)})^2=1)$.

The simplifying condition $(k_{(n)}+K'_{(n)})^n=k_{(n)}^n+K'^n_{(n)}$ for the one-loop integral can be generalized to the higher-loop graphs. It is translated in the statement that *the loop momenta characterizing a certain competition subspace in a given bubble (subdiagram) do not mix to all loop momenta not belonging to that bubble*. The simplest practical application of this principle can be viewed in the calculation of the "sunset" two-loop integral I_3 contributing to the two-point function

$$I_3=\int\frac{d^{d-\sum_{n=2}^{L}m_n}q_1 d^{d-\sum_{n=2}^{L}m_n}q_2\Pi_{n=2}^{L}d^{m_n}k_{1(n)}\Pi_{n=2}^{L}d^{m_n}k_{2(n)}}{\left[q_1^2+\sum_{n=2}^{L}(k_{1(n)}^2)^n\right]\left[q_2^2+\sum_{n=2}^{L}(k_{2(n)}^2)^n\right]}\times\frac{1}{\left\{(q_1+q_2+P)^2+\left[\sum_{n=2}^{L}(k_{1(n)}+k_{2(n)}+K'_{(n)})^2\right]^n\right\}}. \tag{A22}$$

Defining $K''_{(n)}=k_{1(n)}+K'_{(n)}$ and using the condition $k_{2(n)}K''_{(n)}=0$, one can solve the integral over q_2, $k_{2(n)}$ first, picking out only the homogeneous part of each individual integral. The remaining parametric integrals contains the divergence (pole in ϵ_L) and can be solved as before. Using Eq. (A17), we obtain

$$I_3(P,K')=\frac{1}{\epsilon_L}(1+h_{m_L})\times\int\frac{d^{d-\sum_{n=2}^{L}m_n}q_1\Pi_{n=2}^{L}d^{m_n}k_{1(n)}}{\left[q_1^2+\sum_{n=2}^{L}(k_{1(n)}^2)^n\right]\left\{(q_1+P)^2+\sum_{n=2}^{L}[(k_{1(n)}+K'_{(n)})^2]^n\right\}^{\epsilon_L/2}}. \tag{A23}$$

Using Feynman parameters, integrating the loop momenta along with the remaining parametric integrals, and expanding the resulting Γ functions in ϵ_L we find

$$I_3(P,K')=\left(P^2+\sum_{n=2}^{L}K_n'^{2n}\right)\frac{-1}{8\epsilon_L}\times\left(1+2h_{m_L}\epsilon_L-\frac{3}{4}\epsilon_L-2\epsilon_L L_3(P,K'_{(n)})\right), \tag{A24}$$

where

$$L_3(P,K')=\int_0^1 dx(1-x)\ln\left[\left(P^2+\sum_{n=2}^{L}K_n'^{2n}\right)x(1-x)\right]. \tag{A25}$$

At the symmetry points SP_n, it can be rewritten as

$$I_{3SP_1}=\cdots=I_{3SP_L}=\frac{-1}{8\epsilon_L}\left(1+2h_{m_L}\epsilon_L+\frac{5}{4}\epsilon_L\right). \tag{A26}$$

From the above equation we can derive the expressions

$$I'_{3SP_1}\left(\equiv\frac{\partial I_{3SP_1}}{\partial P^2}\right)=\cdots=I'_{3SP_L}\left(\equiv\frac{\partial I_{3SP_2}}{\partial K'^{2L}_{(L)}}\right)=\frac{-1}{8\epsilon_L}\left(1+2h_{m_L}\epsilon_L+\frac{1}{4}\epsilon_L\right). \tag{A27}$$

To complete our description of the $1PI$ two-point vertex parts, consider the integral

$$I_5=\int\frac{d^{d-\sum_{n=2}^{L}m_n}q_1 d^{d-\sum_{n=2}^{L}m_n}q_2 d^{d-\sum_{n=2}^{L}m_n}q_3\Pi_{n=2}^{L}d^{m_n}k_{1(n)}}{\left[q_1^2+\sum_{n=2}^{L}(k_{1(n)}^2)^n\right]\left[q_2^2+\sum_{n=2}^{L}(k_{2(n)}^2)^n\right]\left[q_3^2+\sum_{n=2}^{L}(k_{3(n)}^2)^n\right]}\frac{\Pi_{n=2}^{L}d^{m_n}k_{2(n)}\Pi_{n=2}^{L}d^{m_n}k_{3(n)}}{\left\{(q_1+q_2-p)^2+\sum_{n=2}^{L}[(k_{1(n)}+k_{2(n)}-k'_{(n)})^2]^n\right\}}$$
$$\times\frac{1}{\left\{(q_1+q_3-p)^2+\sum_{n=2}^{L}[(k_{1(n)}+k_{3(n)}-k'_{(n)})^2]^n\right\}}, \tag{A28}$$

which is the three-loop diagram contributing to the two-point vertex function. Incidentally, there is a symmetry in the dummy loop momenta $q_2\rightarrow q_3$ and $k_{2(n)}\rightarrow k_{3(n)}$. Concerning the integrations either over q_2, $k_{2(n)}$ or q_3, $k_{3(n)}$, we use the condition $[k_{2(n)}+(k_1-K')]^n=k^n_{2(n)}+(k_{1(n)}-K'_{(n)})^n$ when the integration is performed over k_2 as well as $[k_{3(n)}+(k_{1(n)}-K'_{(n)})]^n=k^n_{3(n)}+(k_1-K'_{(n)})^n$ when the integral over k_3 is realized. The two internal bubbles, which are represented by the integrals over $(q_2,k_{2(n)})$ and $(q_3,k_{3(n)})$, respectively, are actually the same, resulting in $I_2((q_1-P),(k_{1(n)}-K'_{(n)}))$. Next take $P\rightarrow -P$, $K'_{(n)}\rightarrow -K'_{(n)}$. Using a Feynman parameter and proceeding in close analogy to the calculation of I_3, we find

$$I_5(P,K'_{(n)})=\left(P^2+\sum_{n=2}^{L}K'^{2n}_n\right)\frac{-1}{6\epsilon_L^2}[1+3h_{m_L}\epsilon_L-\epsilon_L-3\epsilon_L L_3(P,K'_{(n)})]. \tag{A29}$$

At the symmetry points $SP_1,\ldots,SP_L$ one obtains

$$I'_{5SP_1}\left(\equiv\frac{\partial I_{5SP_1}}{\partial P^2}\right)=\cdots=I'_{5SP_L}\left(\equiv\frac{\partial I_{5SP_2}}{\partial K'^{2L}_{(L)}}\right)=\frac{-1}{6\epsilon_L^2}\left(1+3h_{m_L}\epsilon_L+\frac{1}{2}\epsilon_L\right). \tag{A30}$$

Finally we compute one of the two-loop diagrams contributing to the four-point function, namely

$$I_4=\int\frac{d^{d-\sum_{n=2}^{L}m_n}q_1 d^{d-\sum_{n=2}^{L}m_n}q_2\Pi_{n=2}^{L}d^{m_n}k_{1(n)}\Pi_{n=2}^{L}d^{m_n}k_{2(n)}}{\left[q_1^2+\sum_{n=2}^{L}(k_{1(n)}^2)^n\right]\left\{(P-q_1)^2+\sum_{n=2}^{L}[(K'_{(n)}-k_{1(n)})^2]^n\right\}}$$
$$\times\frac{1}{\left[q_2^2+\sum_{n=2}^{L}(k_{2(n)}^2)^n\right]\left\{(q_1-q_2+p_3)^2+\sum_{n=2}^{L}[(k_{1(n)}-k_{2(n)}+k'_{3(n)})^2]^n\right\}}. \tag{A31}$$

Recall that $P=p_1+p_2$, p_i $(i=1,\ \ldots,\ 3)$ are external momenta perpendicular to the competing axes. On the other hand, $K'_{(n)}=k'_{1(n)}+k'_{2(n)}$, and $k'_{i(n)}$ $(i=1,\ \ldots,\ 3)$ are the external momenta along arbitrary competition directions. We can integrate first over the bubble $(q_2,k_{2(n)})$. It is convenient to choose Schwinger parameters in the calculation. Then, we use two Feynman parameters and solve for the loop momenta to obtain the following parametric form:

$$I_4=\frac{1}{2}f_m(\epsilon_L)\frac{\Gamma(\epsilon_L)\Gamma\left(2-\sum_{n=2}^{L}\frac{m_n}{2n}-\frac{\epsilon_L}{2}\right)S_{(d-\sum_{n=2}^{L}m_n)}}{\Gamma\left(\frac{\epsilon_L}{2}\right)}\left(\Pi_{n=2}^{L}\frac{\Gamma\left(\frac{m_n}{2n}\right)S_{mn}}{2n}\right)\int_0^1 dy\, y(1-y)^{1/2\epsilon_L-1}\int_0^1 dz\Bigg[yz(1-yz)\left(P^2+\sum_{n=2}^{L}K'^{2n}_{(n)}\right)$$
$$+y(1-y)\left(p_3^2+\sum_{n=2}^{L}k'^{2n}_{3(n)}\right)-2yz(1-y)\left(p_3P+\sum_{n=2}^{L}(-1)^n k'^{n}_{3(n)}K'^{n}_{(n)}\right)\Bigg]^{-\epsilon_L}. \tag{A32}$$

The integral over y is singular at $y=1$ when $\epsilon_L=0$. Replace the value $y=1$ inside the integral over z,[1,8] integrate over y, and expand the Gamma functions in ϵ_L. This implies that

$$I_4 = \frac{1}{2\epsilon_L^2}\left(1 + 2\ h_{m_L}\epsilon_L - \frac{3}{2}\epsilon_L - \epsilon_L L(P,K')\right). \quad \text{(A33)}$$

This form is particularly suitable for the renormalization procedure using minimal subtraction. For the purpose of normalization conditions, the value of this integral at the symmetry points discussed before is given by

$$I_{4SP_1} = \cdots = I_{4SP_L} = \frac{1}{2\epsilon_L^2}\left(1 + 2\ h_{m_L}\epsilon_L + \frac{1}{2}\epsilon_L\right). \quad \text{(A34)}$$

The method proposed here is equivalent to a different regularization procedure to calculate Feynman integrals whose propagators have any combination of even powers of momenta. We can define the measure of the m_n-dimensional sphere in terms of a half integer measure. In fact, taking $k=p^{2n}$, one has $d^{m_n}k \equiv d^{m_n/2n}p = (1/2n)p^{m_n/2n-1}dpd\Omega_{m_n}$. Hence the approximation required to solve the integrals results that the new "measure" $d^{m_n/2n}p$ is invariant under translations $p'=p+a$. This is a simple generalization of the same property valid for the usual m_2-fold Lifshitz behaviors.

APPENDIX B: ISOTROPIC DIAGRAMS IN THE GENERALIZED ORTHOGONAL APPROXIMATION

The computation of the Feynman integrals using the generalized orthogonal approximation is simpler in the isotropic cases, since there is only one subspace to be integrated over. At the Lifshitz point $\delta_{0n}=\tau_{nn'}=0$ and solely the $2L$th power of momentum appears in the propagator for the case of Lth character isotropic critical point. The isotropic analogous of the one-loop integral contributing to the four-point vertex part is

$$I_2 = \int \frac{d^{m_n}k}{[(k+K')^2]^n (k^2)^n}. \quad \text{(B1)}$$

We can use two Schwinger parameters and the orthogonality condition $(k+K')^n \cong k^n + K'^n$, resulting in the expression

$$I_2(K') = \int d^{m_n}k \int_0^\infty \int_0^\infty d\alpha_1 d\alpha_2 e^{-(\alpha_1+\alpha_2)(k^2)^n} e^{-2\alpha_2 K'^n k^n} e^{-\alpha_2 (K'^2)^n}. \quad \text{(B2)}$$

Turning to polar spherical coordinates, take $r^2 = k_1^2 + \cdots + k_{m_n}^2$. Making the transformation $k^n = p$ the volume element becomes $d^{m_n}k = (1/n)p^{m_n/n-1}dpd\Omega_{m_n} \equiv d^{m_n/n}p$. The former integral with a nth power of momenta changes to a quadratic integral over p. After discarding the infinite terms which change the measure $d^{m_n/n}k$ under the translation $y'=y+b/2a$, only the leading contribution is picked out and we have

$$\int d^{m_n}k e^{-a(k^2)^n - bk^n} = \int d^{m_n/n}p e^{-ap^2 - bp} \cong a^{-m_n/2n} e^{b^2/4a} \frac{1}{2n}\Gamma\left(\frac{m_n}{2n}\right) S_{m_n}. \quad \text{(B3)}$$

When this result is replaced into the expression of $I_2(K')$, we get

$$I_2(K') = \frac{S_{m_n}}{\epsilon_n}\left(1 - \frac{\epsilon_n}{2n}[1 + L(K'^2)]\right). \quad \text{(B4)}$$

Henceforth we absorb the factor of S_{m_n} in this integral through a redefinition of the coupling constant and shall do so after performing each loop integral for arbitrary vertex parts. Note that this absorption factor is different from that arising in the anisotropic case in the limit $d \to m_n = 4n - \epsilon_n$. Since the geometric angular factor coming from the anisotropic cases becomes singular in the above isotropic limit the attempt of extrapolating from one case to another is not valid, at least within the framework of the ϵ_L expansion presented in this work. This is a further technical evidence that the isotropic and anisotropic cases have to be tackled differently. Thus

$$I_2(K') = \frac{1}{\epsilon_n}\left(1 - \frac{\epsilon_n}{2n}[1 + L(K'^2)]\right). \quad \text{(B5)}$$

The suitable symmetry point $(K'^2)^n = 1$ useful for the purpose of normalization conditions leads to the following simple outcome:

$$I_2(K') = \frac{1}{\epsilon_n}\left(1 + \frac{\epsilon_n}{2n}\right). \quad \text{(B6)}$$

The next step is the evaluation of the integral

$$I_3 = \int \frac{d^{m_n}k_1 d^{m_n}k_2}{[(k_1+k_2+K')^2]^n (k_1^2)^n (k_2^2)^n}, \quad \text{(B7)}$$

Integrate first over k_2. Take $K''=k_1+K'$ and use the condition $(k_2+K'')^n \cong k_2^n + K''^n$ to obtain

$$I_3 = \frac{1}{\epsilon_n}\left(1 + \frac{\epsilon_n}{2n}\right) \int \frac{d^{m_n}k_1}{\{[(k_1+K')^2]^n\}^{\epsilon_n/2n} (k_1^2)^n}. \quad \text{(B8)}$$

Utilizing a Feynman parameter, we can integrate over k_1. After the expansion $m_n = 4n - \epsilon_n$ is done inside the argument of the resulting Γ functions and using the expression

$$\int \frac{d^{m_n/n}q}{(q^2 + 2kq + m^2)^\alpha} \cong \frac{1}{2n}\frac{\Gamma\left(\frac{m_n}{2n}\right)\Gamma\left(\alpha - \frac{m_n}{2n}\right)(m^2 - k^2)^{m_n/2n-\alpha} S_{m_n}}{\Gamma(\alpha)}, \quad \text{(B9)}$$

the integral I_3 can be found to be

$$I_3 = -\frac{(K'^2)^n}{8n\epsilon_n}\left[1 + \epsilon_n\left(\frac{1}{4n} - \frac{2}{n}L_3(K'^2)\right)\right]. \quad \text{(B10)}$$

At the symmetry point, this reduces to

$$I_3 = -\frac{1}{8n\epsilon_n}\left(1+\frac{9}{4n}\epsilon_n\right), \tag{B11}$$

implying that

$$\left.\frac{\partial I_3}{\partial (K'^2)^n}\right|_{SP} = I_3' = -\frac{1}{8n\epsilon_{4n}}\left(1+\frac{5}{4n}\epsilon_{4n}\right). \tag{B12}$$

The three-loop integral I_5 is given by

$$I_5 = \int \frac{d^{m_n}k_1 d^{m_n}k_2 d^{m_n}k_3}{[(k_1+k_2+K')^2]^n[(k_1+k_3+K')^2]^n (k_1^2)^n (k_2^2)^n (k_3^2)^n}, \tag{B13}$$

where we took for convenience the redefinition $K' \to -K'$. The integrals over k_2 and k_3 are the same. Thus following analogous steps and employing the same reasoning as in the calculation of I_3 we get to

$$I_5 = -\frac{(K'^2)^n}{6n\epsilon_n^2}\left[1+\epsilon_n\left(\frac{1}{2n}-\frac{3}{n}L_3(K'^2)\right)\right]. \tag{B14}$$

At the symmetry point, the following expression follows trivially:

$$\left.\frac{\partial I_5}{\partial (K'^2)^n}\right|_{SP} = I_5' = -\frac{1}{6n\epsilon_n^2}\left(1+\frac{2}{n}\epsilon_n\right). \tag{B15}$$

The two-loop integral I_4 in the isotropic behavior is

$$I_4 = \int \frac{d^{m_n}k_1 d^{m_n}k_2}{(k_1^2)^n[(K'-k_1)^2]^n (k_2^n)[(k_1-k_2+k_3')^2]^n}, \tag{B16}$$

where $K' = k_1' + k_2'$. The integration can be done along the same lines of the computation performed for its anisotropic counterpart. It is straightforward to show that

$$I_4(K'^2) = \frac{1}{2\epsilon_n^2}\left(1-\frac{\epsilon_n}{2n}[1+2L(K'^2)]\right). \tag{B17}$$

At the symmetry point the integral can be rewritten in the form

$$I_4(K'^2=1) = \frac{1}{2\epsilon_n^2}\left(1+\frac{3\epsilon_n}{2n}\right). \tag{B18}$$

As was shown above, these results are a natural generalization of those originally developed for the second character Lifshitz points. It can be checked that all integrals reduce to the usual ϕ^4 values for $n=1$ and reproduce the results from Ref. 8 in the case $n=2$.

APPENDIX C: ISOTROPIC INTEGRALS IN THE EXACT CALCULATION

An interesting feature of the isotropic case is that it can be calculated exactly. We now proceed to yield the exact solution to the Feynman diagrams without performing approximations:

$$I_2 = \int \frac{d^{m_n}k}{[(k+K')^2]^n (k^2)^n}. \tag{C1}$$

Using a Feynman parameter and making the continuation $d = m_n = 4n - \epsilon_n$ we get

$$I_2(K') = \frac{\Gamma\left(2n-\frac{\epsilon_n}{2}\right)\Gamma\left(\frac{\epsilon_n}{2}\right)S_{m_n}}{2\Gamma(n)\Gamma(n)}\left(\frac{\Gamma(n)\Gamma(n)}{\Gamma(2n)}-\frac{\epsilon_n}{2}L_n(K')\right), \tag{C2}$$

where $L_n(K')$ is given by

$$L_n(K') = \int_0^1 dx x^{n-1}(1-x)^{n-1}\ln[x(1-x)K'^2]. \tag{C3}$$

This integral is the analogous of the integral $L(K')$ appearing in the orthogonal approximation. Here it depends explicitly on n, and that is the reason we have included a subscript in it to emphasize this dependence. The integration over x together with the ϵ_n expansion of the Gamma functions results in

$$I_2(K') = \frac{S_{m_n}}{\epsilon_n}\left[1-\frac{\epsilon_L}{2}\left(\psi(2n)-\psi(1)+\frac{\Gamma(2n)}{\Gamma(n)\Gamma(n)}L_n(K')\right)\right]. \tag{C4}$$

As before, we absorb the factor S_{m_n} in a redefinition of the coupling constant. We need to do this for each loop integral. Thus

$$I_2(K') = \frac{1}{\epsilon_n}\left[1-\frac{\epsilon_n}{2}\left(\psi(2n)-\psi(1)+\frac{\Gamma(2n)}{\Gamma(n)\Gamma(n)}L_n(K')\right)\right]. \tag{C5}$$

This is a useful result for doing minimal subtraction. Defining the quantity $D(n) = 1/2\psi(2n) - \psi(n) + 1/2\psi(1)$, at the symmetry point $K'^2 = 1$ the integral turns out to be

$$I_{2SP} = \frac{1}{\epsilon_n}[1+D(n)\epsilon_n]. \tag{C6}$$

Now, let us calculate the integral I_3. As before, take $K'' = k_1 + K'$ and solve for the internal bubble k_2 using Feynman parameters solving the momentum independent integrals over the Feynman parameters and expanding the Gamma functions in ϵ_n; we end up with

$$I_3 = K'^{2n}(-1)^n \frac{\Gamma^2(2n)}{4\Gamma(3n)\Gamma(n+1)}\frac{1}{\epsilon_n}\left[1+\epsilon_n\left(B_n-\frac{L_{3n}(K')}{A_n}\right)\right], \tag{C7}$$

where

$$A_n = \frac{\Gamma(2n)\Gamma(n)}{\Gamma(3n)}, \tag{C8}$$

$$B_n = D(n) - \frac{1}{2}\sum_{p=1}^{2n-1}\frac{1}{p} + \sum_{p=1}^{n}\frac{1}{p} + \frac{\sum_{p=0}^{2n-1}\frac{(2n-1)!(-1)^{p+1}}{2p!(2n-1-p)!(n+p)^2}}{A_n}, \tag{C9}$$

$$L_{3n}(K') = \int_0^1 dx x^{2n-1}(1-x)^{n-1}\ln[x(1-x)K'^2]. \tag{C10}$$

Again, this integral depends explicitly on n and should be compared with its counterpart arising in the orthogonal approximation. We then learn that for massless propagators with arbitrary power of momenta, the external momentum dependent part of the Feynman integrals generalizes the standard ϕ^4 in the manner prescribed above.

At the symmetry point, the integral I_3 simplifies to the following expression:

$$I_{3SP} = (-1)^n\frac{\Gamma(2n)^2}{4\Gamma(n+1)\Gamma(3n)\epsilon_n}\left[1+\left(D(n)+\frac{3}{4}+\frac{1}{n}\right)\epsilon_n\right] \times\left[1+\frac{3\epsilon_n}{2}\left(\sum_{p=3}^{3n-1}\frac{1}{p}-\sum_{p=2}^{2n-1}\frac{1}{p}\right)+\frac{\epsilon_n}{2}\sum_{p=1}^{n-1}\frac{1}{n-p}\right]. \tag{C11}$$

Notice that while the first term inside the parentheses contributes for arbitrary values of n, the last factor of $O(\epsilon_n)$ in the brackets are corrections which contribute solely for $n \geqslant 2$. (A similar feature will also take place in the calculation of I_4 and I_5.) Therefore taking the derivative with respect to K'^{2n} at the symmetry point produces the result

$$I'_{3SP} = (-1)^n\frac{\Gamma(2n)^2}{4\Gamma(n+1)\Gamma(3n)\epsilon_n}\left[1+\left(D(n)+\frac{3}{4}\right)\epsilon_n\right] \times\left[1+\frac{3\epsilon_n}{2}\left(\sum_{p=3}^{3n-1}\frac{1}{p}-\sum_{p=2}^{2n-1}\frac{1}{p}\right)+\frac{\epsilon_n}{2}\sum_{p=1}^{n-1}\frac{1}{n-p}\right]. \tag{C12}$$

We now calculate the three-loop integral I_5. Proceeding analogously, we can show that it has the solution

$$I_5 = K'^{2n}(-1)^n\frac{\Gamma^2(2n)}{3\Gamma(3n)\Gamma(n+1)}\frac{1}{\epsilon_n^2}\left[1+\epsilon_n\left(C_n-\frac{3L_{3n}(K')}{2A_n}\right)\right], \tag{C13}$$

where

$$C_n = 2D(n) - \frac{1}{2}\sum_{p=1}^{2n-1}\frac{1}{p} + \frac{3}{2}\sum_{p=1}^{n}\frac{1}{p} + \frac{\sum_{p=0}^{2n-1}\frac{(2n-1)!(-1)^{p+1}}{p!(2n-1-p)!(n+p)^2}}{A_n}. \tag{C14}$$

At the symmetry point this result gets simplified. Taking the derivative with respect to the external momenta we get

$$I'_{5SP} = (-1)^n\frac{\Gamma(2n)^2}{3\Gamma(n+1)\Gamma(3n)\epsilon_n^2}[1+(2D(n)+1)\epsilon_n] \times\left[1+2\epsilon_n\left(\sum_{p=3}^{3n-1}\frac{1}{p}-\sum_{p=2}^{2n-1}\frac{1}{p}\right)+\epsilon_n\sum_{p=1}^{n-1}\frac{1}{n-p}\right]. \tag{C15}$$

Notice that the $O(\epsilon_n)$ terms inside the brackets gives a nonvanishing contribution only for $n \geqslant 2$.

To conclude, let us calculate the integral I_4. The integral over the bubble k_2 can be solved directly by taking the effective external momenta as $K''=-k_1-k'_3$. It has $I_2(K'')$ as a subdiagram. Using the information obtained in calculating I_2 and working out the details we find the intermediate result:

$$I_4 = f_n(\epsilon_n)\frac{\Gamma\left(2n-\frac{\epsilon_n}{2}\right)\Gamma(\epsilon_n)}{2\Gamma(n)^2\Gamma\left(\frac{\epsilon_n}{2}\right)}\int_0^1 dy y^{2n-1}(1-y)^{\epsilon_n/2-1} \times\int_0^1 dz[z(1-z)]^{n-1}[yz(1-yz)K'^2+y(1-y)k_3'^2 - 2yz(1-y)K'k'_3]^{-\epsilon_n}. \tag{C16}$$

The situation resembles that in the calculation of I_4 using the orthogonal approximation. Again, set $y=1$ in the integral over z and carry out the integral over y independently. Performing the integral over y, we find

$$I_4 = f_n(\epsilon_n)\frac{\Gamma\left(2n-\frac{\epsilon_n}{2}\right)\Gamma(\epsilon_n)\Gamma(2n)}{2\Gamma(n)^2\Gamma\left(2n+\frac{\epsilon_n}{2}\right)} \times\int_0^1 dz[z(1-z)]^{n-1}[z(1-z)K'^2]^{-\epsilon_n}. \tag{C17}$$

Then for the purposes of minimal subtraction, it can be expressed as

$$I_4 = \frac{1}{2\epsilon_n^2}\left[1+\left(D(n)-1-\frac{\Gamma(2n)L_n}{\Gamma(n)^2}\right)\epsilon_n-\epsilon_n\sum_{p=1}^{2n-2}\frac{1}{2n-p}\right]. \tag{C18}$$

We emphasize that the last $O(\epsilon_n)$ term in this expression only contributes for $n \geqslant 2$. At the symmetry point, we find

$$I_{4SP} = \frac{1}{2\epsilon_n^2}\left[1+[D(n)+1]\epsilon_n + \epsilon_n\left(\sum_{p=1}^{2n-2}\frac{1}{2n-p}-2\sum_{p=1}^{n-1}\frac{1}{n-p}\right)\right]. \tag{C19}$$

Once again, the last two terms of $O(\epsilon_n)$ containing the sums in the above expression correspond to corrections in the case $n \geqslant 2$. Since the corrections are absent when starting from scratch for the $n=1$ case, by neglecting them in the above expressions it can be easily checked that all of these integrals reduce to the values of the ordinary $\lambda\phi^4$ in the limit $n \to 1$. We learn that a general feature of the exact calculation is that higher loop integrals generally receive further contributions to the subleading singularities for $n \geqslant 2$.

In order to make contact with the more concrete case of the usual second character Lifshitz point obtained by Hornreich, Luban, and Shtrikman,[3] we discuss the particular $n=2$ case next.

a. $n=2$ case

Let us now analyze the Feynman integrals involved in the calculation of the critical indices of the ordinary second character Lifshitz critical behavior. This is merely a particular case of the most general isotropic CECI model discussed in the previous subsection. Nevertheless, the discussion of this particular case is useful when comparing with the original previous result obtained by Hornreich, Luban, and Shtrikman about three decades ago. Needless to say, both results agree for the exponents η_{L4} and ν_{L4}, which in the notation of Sec. IV correspond to η_2 and ν_2. Moreover, our treatment permits us to obtain two interesting results for this behavior: the results for η_2 are extended including corrections up to $O(\epsilon_2^3)$ whereas the remaining exponents are obtained through the complete set of scaling relations derived in Ref. 7 up to $O(\epsilon_2^2)$.

Since the calculation was already indicated in the last subsection, we simply quote the results. For calculating the exponents using minimal subtraction the most appropriate form of the integrals are given by

$$I_2(K') = \frac{1}{\epsilon_L}\left(1 - \frac{11\epsilon_L}{12} - 3\epsilon_L L_2(K')\right), \tag{C20}$$

$$I_3 = K'^4 \frac{3}{80\epsilon_L}\left[1 - \epsilon_L\left(\frac{17}{120} + 20L_{32}(K')\right)\right], \tag{C21}$$

$$I_5 = K'^4 \frac{1}{20\epsilon_L^2}\left[1 - \epsilon_L\left(\frac{7}{60} + 30L_{32}(K')\right)\right], \tag{C22}$$

$$I_4 = \frac{1}{2\epsilon_L^2}\left[1 - \left(\frac{23}{12} + 6L_{12}(K'^2)\right)\epsilon_L\right]. \tag{C23}$$

For the use of normalization conditions, however, it is convenient expressing these integrals at their symmetry point. Instead of calculating I_3 and I_5 at the symmetry point, we need their derivatives with respect to K'^4 at the symmetry point. Thus we find

$$I_{2SP} = \frac{1}{\epsilon_L}\left(1 - \frac{\epsilon_L}{12}\right). \tag{C24}$$

$$I'_{3SP} = \frac{3}{80\epsilon_L}\left(1 + \frac{131}{120}\epsilon_L\right), \tag{C25}$$

$$I'_{5SP} = \frac{1}{20\epsilon_L^2}\left(1 + \frac{26}{15}\epsilon_L\right), \tag{C26}$$

$$I_4 = \frac{1}{2\epsilon_L^2}\left[1 - \frac{1}{4}\epsilon_L\right]. \tag{C27}$$

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