Solitons and quasielectrons in the quantum Hall matrix model

T. H. Hansson, J. Kailasvuori, and A. Karlhede

Fysikum, Stockholm University, AlbaNova University Center, SE - 106 91 Stockholm, Sweden

R. von Unge

Department of Theoretical Physics, Masaryk University, 611 37 Brno, Czech Republic

We show how to incorporate fractionally charged quasielectrons in the finite quantum Hall matrix model. The quasielectrons emerge as combinations of BPS solitons and quasiholes in a finite matrix version of the noncommutative ϕ^4 theory coupled to a noncommutative Chern-Simons gauge field. We also discuss how to properly define the charge density in the classical matrix model, and calculate density profiles for droplets, quasiholes and quasielectrons.

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I. INTRODUCTION

During the last few years a new class of models for the fractional quantum Hall (QH) effect has emerged. The basic construction is due to Susskind,[1] who observed that the Laughlin states at filling fraction $\nu=1/(2k+1)$ are naturally described by a noncommutative Chern-Simons (CS) theory, or equivalently, by an infinite matrix model with the Lagrangian,

$$L_0 = \frac{eB}{2}\, \mathrm{Tr}\{(\dot{X}^a - i[X^a,\hat{a}_0]_m)\epsilon_{ab}X^b + 2\theta\hat{a}_0\}, \qquad (1)$$

where $X^a(t)$, $a=1,2$, and $\hat{a}_0(t)$ are Hermitian matrices—the latter being a Lagrange multiplier imposing the matrix commutator constraint, $[X^1,X^2]=i\theta$. The area θ that enters L_0 is the noncommutativity parameter, and B is the transverse magnetic field.

As it stands, this model has only a single state, since the solution to the constraint, which can only be satisfied by infinite matrices, is unique (up to gauge transformations). This reflects that the theory is topological and thus has no excitations when defined on an infinite plane.[22]

The parameter θ can be interpreted as an area per particle, giving the unique state a constant density $\rho=1/2\pi\theta$. Modifying the constraint by hand, one finds other solutions corresponding to fractionally charged quasielectrons and quasiholes.[1]

In an important development, Polychronakos extended the model by supplementing (1) with the Lagrangian,

$$L_b = \Phi^\dagger(i\partial_0 - \hat{a}_0)\Phi, \qquad (2)$$

where Φ is a complex bosonic N-vector.[2] The X^a's in (1) are now Hermitian $N\times N$ matrices, and the constraint is changed to

$$[X^1,X^2] = i\theta - \frac{i}{eB}\Phi\Phi^\dagger = i\theta\left(1 - \frac{1}{\kappa}\Phi\Phi^\dagger\right), \qquad (3)$$

where $\kappa=eB\theta>0$ is the so-called level number. It is striking that this finite QH matrix model (QHMM) already at the classical level describes several key features of the quantum Hall system:

(1) In the presence of a rotationally invariant confining potential, the ground state is a finite size circular "droplet" with a constant bulk density $\bar{\rho}$ depending on the level number.

(2) The excitation spectrum is consistent with that of a QH droplet. In particular there are quasihole states in the bulk and gapless quasielectron-quasihole states at the edge.

(3) In the absence of a potential, there is a set of degenerate low density states corresponding to single particles in the lowest Landau level, at well separated positions in the plane.

In particular note that the presence of quasielectron and quasihole excitations takes this description beyond that of a classical incompressible fluid, and we shall see below how the model also describes how QH droplets are formed from well separated particles in a strong magnetic field.

Quantizing (1), and assuming the underlying matrix degrees of freedom to be fermionic, Susskind showed that the density is quantized at the Laughlin fractions $\nu=\bar{\rho}/\rho_0 =1/(2k+1)$, where k is integer and $\rho_0=eB/2\pi$ (when $\hbar=1$) is the density of states in a single Landau level.[23] At a technical level, it was also shown in Ref. 2 that in the presence of a quadratic potential, there is an exact mapping of the QH matrix model onto the Calogero model, both in the classical and the quantum case. This mapping yields explicit expressions for both energy levels and wave functions.[3]

In a previous paper we extended the QHMM model further, by constructing a class of conserved charges and accompanying currents, thus allowing for a coupling to an external electromagnetic field.[4] We then went on to calculate low momentum response functions in the classical model, in particular:

(1) The ground state density, being the response to a constant electric potential, A_0.

(2) The quantum Hall response σ_H.

(3) The response to a weak and slowly varying external $\vec{B}$ field.

The results were all in agreement with the known properties of the Laughlin states.

In spite of these successes there are several basic aspects of QH physics which are not incorporated in the finite matrix

model given by (1) and (2). Most significantly,

(a) There is no unambiguous definition of density.

(b) There are no quasielectron solutions.

(c) There is no natural way to introduce spin and/or multilayer degrees of freedom analoguous to the usual description in terms of multicomponent CS fields.[5]

(d) There is no generalization to fractions other than the Laughlin ones.

In this paper we address the first two points. Before turning to the technicalities, we will give some general comments to the above list, and also briefly discuss the status of the quantized QH matrix model.

In Ref. 4 we constructed a class of conserved currents for the classical matrix model. As we will discuss below, there are three natural conditions on the charge density operator: it should be non-negative, satisfy the classical version of the sine-algebra characteristic of the lowest Landau level, and have the correct limit for particles separated much further than the magnetic length. Unfortunately, we have not found any definition that satisfies all these demands.

The absence of quasielectrons in the noncommutative theories is related to the existence of a minimal area for each particle. A clue as to how to get quasielectrons is given in Susskind's original paper where they emerged from an ad hoc change of the constraint. In a recent paper, Bak *et al.* showed how the addition of a noncommutatve scalar field ϕ provides a dynamical version of this mechanism, and gives a model with soliton solutions with charge density larger than ρ_0.[6] In Sec. III we shall construct the corresponding finite matrix model.

The problem of spin, (or pseudospin corresponding to, e.g., a multilayer index) derives from the restricted nature of noncommutative gauge theories—U(N) is the only allowed gauge group.[7] The standard multicomponent CS Lagrangians employed to describe spin and pseudospin, as well as the general classification of abelian QH liquids given by Wen,[5] are all based on the gauge group U(1)^k.

The problem of finding non-Laughlin states, as already mentioned, is superficially the same as for spin—there is no noncommutative version of the standard multicomponent CS theories. One would hope that the spin problem could be addressed by introducing fermionic degrees of freedom and couple them in a judicious way to the bosonic matrices. We have tried several such schemes, but none turned out to be a convincing solution. Neither do we know of any construction for generating non-Laughlin states, and the initial hope that the matrix theory would provide a new and more powerful framework for the classification of QH liquids has so far been elusive.

Thus, turning to quantum theory, there is no matrix model where the density is quantized to other fractions than the Laughlin ones (except for the trivial case of direct sums), and in particular there is no way to get the experimentally prominent Jain series,[8,9] $\nu = n/(2pn \pm 1)$). At a technical level the quantized QHMM is hard to handle since the current and density operators are mathematically very complicated objects. This means that although the quantum states of the model are known via the mapping from the Calogero model,[2] it is not possible to calculate density profiles. Even for the simplest case of two by two matrices the manipulation of

exponentials of matrices with (quantum) noncommuting elements is very difficult.

We already stressed the pros and cons of the *classical* matrix model, and the aim of this paper is to extend this model to allow for quasielectrons and also to find a density operator that can describe quasiparticle and edge profiles consistent with what is known about the QH system. As we shall see, this endeavor has been rather successful, at least on a qualitative level. Within an extended finite QH matrix model, we can describe QH droplets, exponentially falling edges, quasihole and quasielectron excitations.

The paper is organized as follows. In the next section we discuss the ambiguities in the definition of the density operator and give the arguments in favor of our special choice. We then calculate density profiles for droplets, and quasiparticles and compare with what is expected from other approaches such as CS mean field theory, and Laughlin wave functions. In Sec. III we first show how to incorporate densities larger than $\bar{\rho}$ by adding a scalar field to the finite matrix model. The resulting theory has soliton solutions with integer charges, and quasielectrons can be constructed by adding holes on top of these solitons. We give explicit expressions for the solutions and calculate the density profiles which are again compared with alternative descriptions. In the last section we summarize our results and contrast the classical matrix model approach with the standard classical commutative CS description. Some technical points about the positivity of the density operator and possible alternative definitions are given in an appendix.

II. PARTICLES, DROPLETS, AND QUASIHOLES

In this section we shall study the density profiles of various solutions of the finite classical matrix model. These solutions were all found by Polycronakos,[2] who also determined gross characterizations such as the radius and mean density of the QH droplet, and the charge of the quasihole. To calculate the profiles, we must first give a definition of the density operator. As stressed in the introduction, our choice, although not unique, gives profiles in good agreement with those obtained by other methods.

A. The density operator

In Ref. 4 we constructed a class of conserved currents for the classical matrix model. The general form of the charge and current was given by

$$\rho(\vec{y}, t) = \text{Tr}[\hat{\delta}(y^a - X^a(t))], \tag{4}$$

$$\vec{j}(\vec{y}, t) = \text{Tr}[(\dot{\vec{X}} - i[\vec{X}, \hat{a}_0]_m)\hat{\delta}(y^a - X^a(t))],$$

where $\hat{\delta}(y^a - X^a)$ is a matrix-valued kernel. The general form of this kernel follows from symmetry considerations and current conservation,

$$\hat{\delta}(y^a - X^a) = \int \frac{d^2k}{(2\pi)^2} f(k_a(y^a - X^a), \epsilon_{ab}k^a(y^b - X^b), \theta k^2) \tag{5}$$
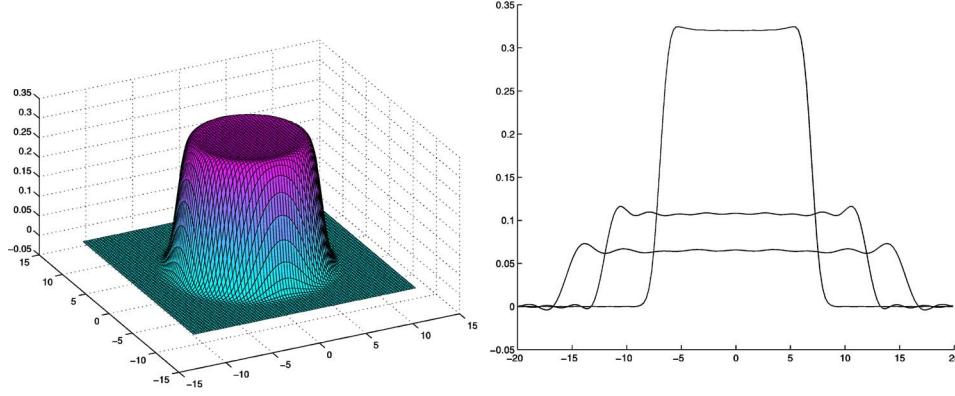
and a rather natural guess is

FIG. 1. (Color online) To the left $\rho(\vec{x})$ for $\nu=1$ as given by the density operator (7) and the droplet solution (12) with $N=50$. We use dimensionless units given by $2\ell^2=2\nu\theta=1$. To the right the density $\rho(\vec{x})$ in profile along the x-axis is given for $\nu=1,\frac{1}{3},\frac{1}{5}$, respectively. All states have roughly constant bulk densities, but note the increasing wiggles, and the building up of a rim at the edge, for lower ν.[12] Note also the small violations in positivity at the edge.

$$\hat{\delta}_W(y^a - X^a) = \int \frac{d^2k}{(2\pi)^2} e^{ik_a(y^a - X^a)} g(\theta k^2), \qquad (6)$$

where $g(0)=1$. [The special case $g(x)=1$ is known as Weyl-ordering.] For almost diagonal matrices, appropriate for widely separated particles at positions $\vec{x}_n = \vec{X}_{nn}$, this corresponds to $\rho_{\vec{k}} = g(\theta k^2) \Sigma_n e^{-i\vec{k}\cdot\vec{x}_n}$. Clearly g can be thought of as a form factor, and for the particular choice $g(x) = e^{-(x/d)^2}$ we have Gaussian "blobs" which are very suggestive of maximally localized one-particle wave-functions in the LLL. With this motivation we shall use

$$\rho_{\vec{k}} = e^{-(\ell^2 k^2/2)} \, \text{Tr} \, e^{-i\vec{k}\cdot\vec{X}}, \qquad (7)$$

where the length d was chosen as to match the rms radius $\sqrt{\langle r^2 \rangle} = \sqrt{2}\,\ell \equiv (2\hbar/eB)^{1/2}$ which is the appropriate value for an isolated electron in the LLL.

Before proceeding to use the formula (7) to calculate density profiles, we mention some problems with this definition. The first, and most severe, is that $\rho(\vec{x})$ is not positive definite on the space of matrices satisfying the constraint (3). This is not obvious, but can be shown by numerical calculations which also indicate that this is mainly a problem for very small systems, typically $N<10$, and also gets more severe with lower ν. For moderately large N, very small violations of positivity is seen in typical density profiles such as the "droplet" solution shown in Fig. 1 for $N=50$. For extreme cases, such as $N=2$, the violation of positivity is large, as shown in the Appendix. We have not been able to show that the definition (7) gives a positive definite density in the limit $N\to\infty$, although our numerics appears to support this possibility.

The situation is less favorable for other ordering prescriptions. So will for instance antiordering, defined by

$$\hat{\delta}_{ao}(z - Z, \bar{z} - Z^\dagger) = \int \frac{d^2k}{(2\pi)^2} e^{(i\bar{k}/2)(z-Z)} e^{(ik/2)(\bar{z}-Z^\dagger)} \tilde{g}(\theta k^2), \qquad (8)$$

where again $\tilde{g}(0)=1$, give strongly fluctuating profiles, and large negative values for the density even for rather large

N.[10] By going outside the class of density operators that can be written on the form (4), i.e., as a trace of a matrix kernel, one can define a positive definite density operator with the correct limiting behavior for separated particles. This construction, which essentially involves taking the square root of a delta function, has, however, other shortcomings. Technical details are given in the Appendix.

A second problem is that we would expect the Fourier components of the quantum mechanical density operator to satisfy the following commutation relation,

$$[\rho_{\vec{k}}, \rho_{\vec{p}}]_{QM} = 2i \sin\left(\frac{\ell^2}{2}(\vec{k} \times \vec{p})\right) e^{(\ell^2/2)\vec{k}\cdot\vec{p}} \rho_{\vec{k}+\vec{p}}, \qquad (9)$$

which is the sine algebra pertinent to the density operator projected onto the LLL. We have not been able to find any definition of the density that satisfies (9) except for $\theta=0$, where the antiordering is known to be correct. The claim in Ref. 4 that a particular quantum reordering of (8) satisfies (9) for $N=2$ is erroneous. [24] Actually, by studying the classical limit we can show that there is no quantum ordering of neither the matrix Weyl ordered nor antiordered density operators that satisfies (9).

For readers familiar with the string theory literature, the following comment might be of interest. In string theory one can show that the Weyl ordered expression for the density, corresponding to $g(\theta k^2)=1$ in (6), gives the density of the lower dimensional RR-charged D-branes. This follows since (6) is nothing but the Seiberg-Witten map for the noncommutative field strength which implies that it couples to the Ramond-Ramond forms in precisely the correct way to act as a source of the corresponding RR-charge.[11] In our case there is no such reason to use Weyl-ordering to define the density of particles and we may modify this expression as long as it respects the symmetries of the problem. Note, however, that our choice (7) coincides with Weyl-ordering for the $k=0$ component corresponding to the total charge.

To summarize, we have no *a priori* reason to choose (6) rather than e.g. antiordering, or in fact any other ordering in the general class (5). Similarly, there is no theoretical moti-
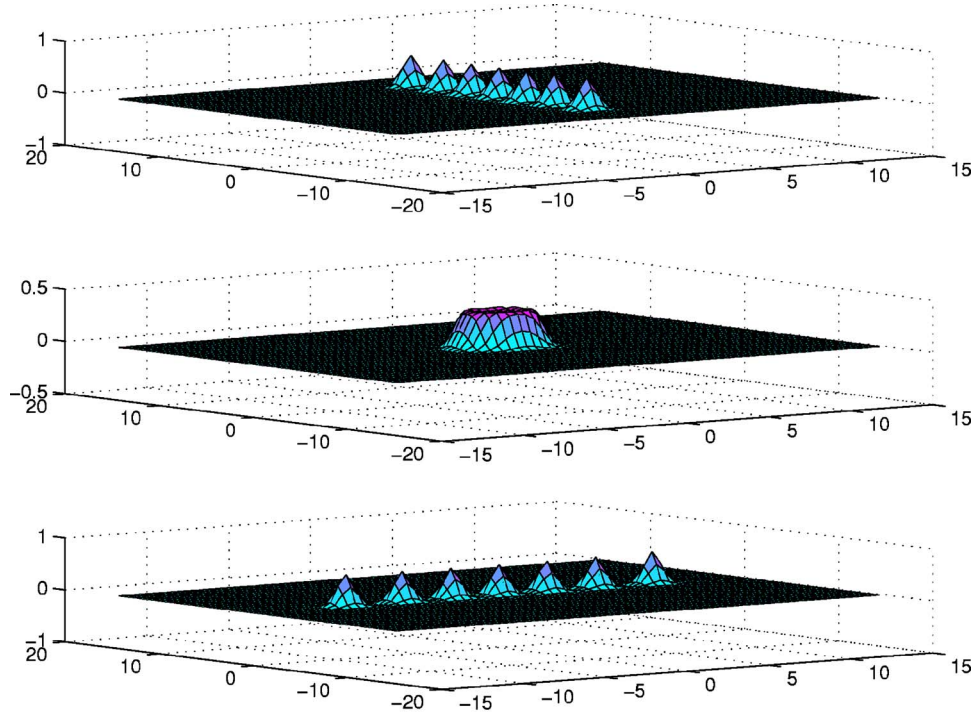
FIG. 2. (Color online) The compression of seven quantum Hall particles as discusssed in the text. Note the perfect circular symmetry of the maximally compressed state in the middle picture.

vation for taking any particular $g(x)$. Instead our choice (7) is phenomenologically motivated, and its usefulness will be demonstrated in the rest of this paper.

B. From particles to droplets

For the gauge choice $\Phi^\dagger = \sqrt{eB\theta}(11\cdots1)$, the constraint (3) is solved by the following matrices:[2]

$$X^1_{mn} = x_m \delta_{mn},\tag{10}$$

$$X^2_{mn} = y_m \delta_{mn} - \frac{i\theta}{x_m - x_n}(1 - \delta_{mn}),$$

where we (arbitrarily) chose to diagonalize the hermitian matrix X^1. For widely separated x_i's, the off-diagonal terms, that are responsible for the "θ-repulsion," are small, and the diagonal elements can be interpreted as the coordinates of the particles. More generally, we can think of the (gauge invariant) eigenvalues of the matrices X_i as particle coordinates x_i and y_i. Note, however, that there is no unambiguous way to pair these eigenvalues to position coordinates for the particles.

Another convenient gauge choice is $\Phi^\dagger = \sqrt{N\kappa}(0\cdots01)$ and introducing the dimensionless complex coordinates $Z \equiv 1/\sqrt{2\theta}(X^1 + iX^2)$, the constraint takes the form

$$[Z, Z^\dagger] = 1 - N|N-1\rangle\langle N-1|,\tag{11}$$

where the bra-ket notation refers to an oscillator basis as explained in e.g. Ref. 7. In the large N limit, this is the usual ladder operator algebra, and the effect of the boundary field is only at the "edge" of the matrix. It is thus natural to seek a solution for Z similar to the lowering operator a in the n-representation. One finds,

$$Z = \sum_{n=0}^{N-1} \sqrt{n}|n-1\rangle\langle n|.\tag{12}$$

We will refer to this as the droplet solution. By a U(N)-transformation it can be put on the form (10), with $y_m=0$ (since the matrix elements in Z are real) and almost equidistantly spaced x_m:s, which are the (gauge invariant) eigenvalues of the hermitian combination $(Z+Z^\dagger)/2$.[25]

Using the choice (7) for $\rho_{\vec{k}}$, we can calculate the corresponding x-space density profile $\rho(\vec{x})$, which is shown in Fig. 1 for $\nu=1, \frac{1}{3}, \frac{1}{5}$ and $N=50$. The lower ν is, the more pronounced is the up-shooting rim at the edge. Excluding a circular segment containing the rim, the distribution is very well fitted by the formula

$$\rho(r, \theta) = \frac{\rho_0}{2}\left(1 - \tanh\frac{r - r_0}{\beta\ell}\right)\tag{13}$$

with $r_0 \approx 0.99\sqrt{2\theta N} = 0.99\ell\sqrt{2N/\nu}$ for all three ν and $\beta \approx 1.05, 0.92, 0.82$ for $\nu=1, \frac{1}{3}, \frac{1}{5}$ and $N=50$, respectively. This is consistent with the expectation of a constant bulk density and a very rapid fall-off at the edge over a distance of the order of the magnetic length.[26]

In Fig. 2 we illustrate how a droplet is formed when several, initially well separated, particles approach each other. The middle figure is a density plot of the droplet solution (12) for seven particles. From this solution we extracted the eigenvalues x_m, and then generated a set of solutions of the form (10) by scaling the x_m's by a common factor, λ. The top
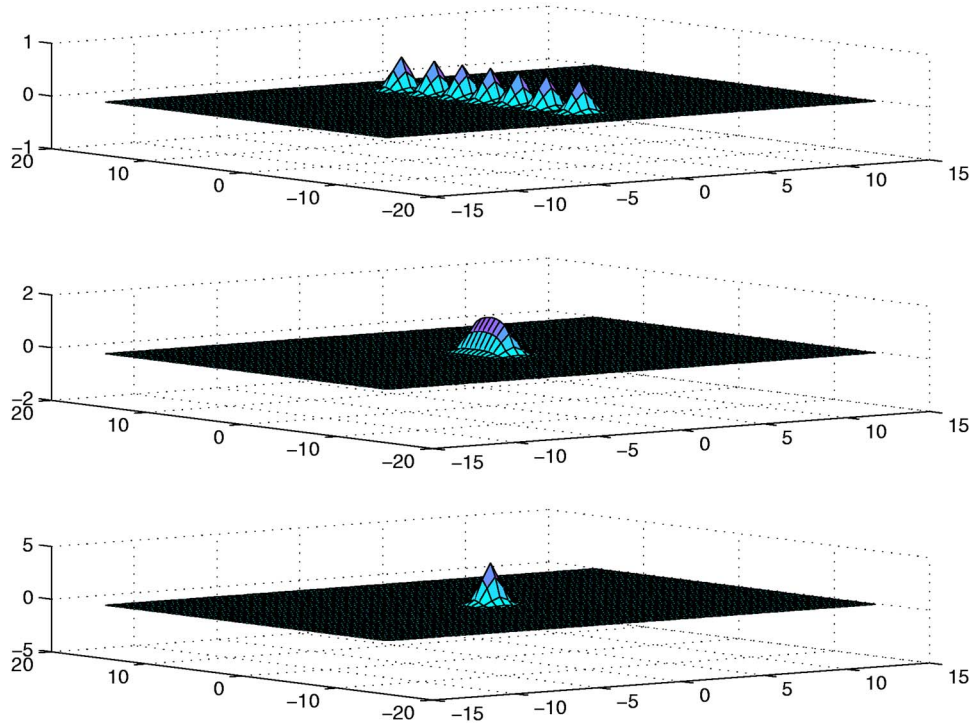
FIG. 3. (Color online) As in Fig. 2 but with $\theta=0$. No droplet is formed and the maximally compressed state in the last picture is just seven superimposed Gaussians.

figure shows the density for $\lambda=5$, corresponding to particles well separated on the x-axis. In the limit of large λ, the x_m's are simply the coordinates of the particles. The bottom picture is for $\lambda=1/5$. Because of the θ-repulsion the particles cannot be compressed further than the droplet, and the result is instead particles separated along the conjugate y-direction. That this effect is entirely due to the finite value of θ is demonstrated in Fig. 3, which is identical to Fig. 2, but with the off-diagonal θ-repulsion terms in (10) set to zero [still keeping the same form factor $g(\theta k^2)$]. In this case no circular droplet is formed and the maximally compressed state is simply an overlap of the individual Gaussian distributions.

C. The quasihole solution

The droplet solution (12) can readily be modified to describe a quasihole, i.e., a state where the density close to the origin is depleted compared to the droplet state. Polychronakos found

$$Z = \sqrt{q}|N\rangle\langle 0| + \sum_{n=1}^{N-1} \sqrt{n+q}|n-1\rangle\langle n|, \qquad (14)$$

where $0<q$ corresponding to a shift in the eigenvalues of the radius operator

$$R^2 = (X^1)^2 + (X^2)^2 = 2\theta \sum_{n=0}^{N-1}\left(n+\frac{1}{2}+q\right)|n\rangle\langle n|, \qquad (15)$$

with the amount q relative to the original droplet. By inserting (14) into (7), we get the distribution $\rho(\vec{x})$ shown in Fig. 4. Here the charge of the quasihole, q, is a free parameter, but it

becomes quantized in the quantum matrix model.[2]

Although there is a clear charge deficit at the origin, the matrix model does not reproduce the complete expulsion of the electrons characteristic of the Laughlin quasiholes. A more detailed comparison is made in Fig. 5, where we show the cumulative integrated charge $Q(R)=\pi\int_0^R dr^2 \rho(r)$ for the Laughlin quasihole (left)[13] and the matrix model quasihole (14) (right). We also calculated the root mean square radius for a quasihole of charge νe in a state with filling fraction ν numerically with the results $\sqrt{\langle r^2\rangle}/\ell=2.6,1.4,1.1$ for $\nu=1,\frac{1}{3},\frac{1}{5}$, respectively. This can be compared with the vortex solution of the mean field composite boson model (see, e.g., Ref. 14), where the vortex has $\langle r^2\rangle=\nu 2.5\ell^2$.

III. BPS SOLITONS IN THE FINITE MATRIX MODEL

We already mentioned that the finite matrix model defined by (1) and (2) does not allow for quasielectron solutions. On the other hand, such solutions can be found if the constraint is modified by hand. In the infinite matrix model we can take

$$[Z,Z^\dagger] = 1 + q|0\rangle\langle 0|, \qquad (16)$$

which describes a quasihole at the origin for $q>0$ and a quasielectron for $q<0$. To have dynamical quasielectrons a constraint of this type has to appear as one of the equations of motion. Such a construction, based on a noncommutative version of the Jackiw-Pi model,[15] was given by Bak *et al.* in Ref. 6. We first briefly review their work, and then show how to construct a corresponding finite matrix model. This will require both a modification of the action for the noncommutative scalar field, ϕ, and a coupling between ϕ and the boundary field Φ.
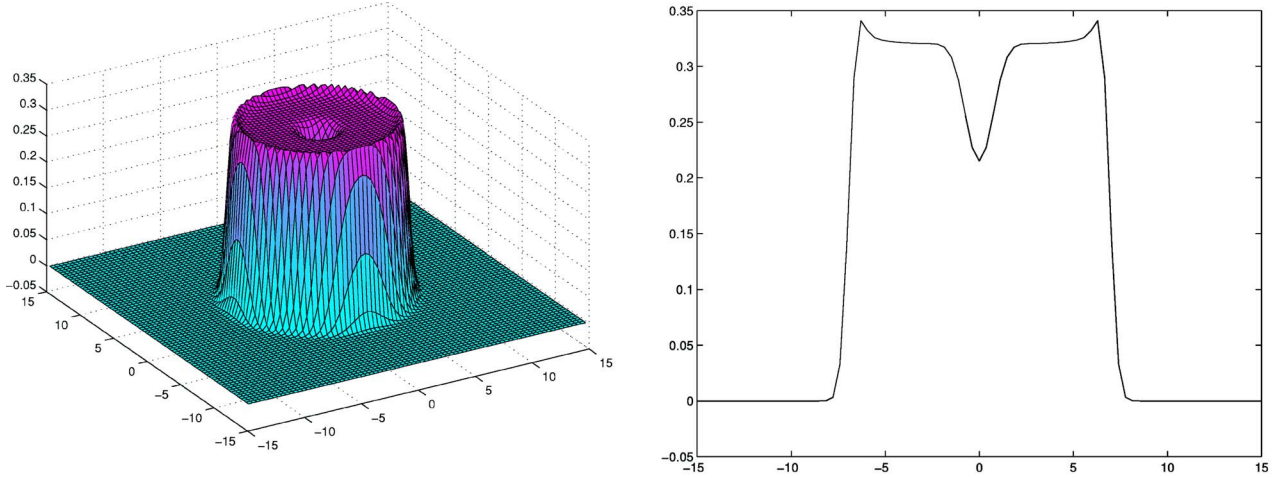
FIG. 4. (Color online) The density distribution $\rho(\vec{x})$ and its profile given by (7) for (14) for $N=50$, $\nu=q=1/3$.

In terms of the complex covariant position operators $Z = (1/\sqrt{2\theta})(X^1+iX^2)$ and $Z^\dagger$, and the CS level number κ, the noncommutative CS Lagrangian (1) takes the form,

$$L_{CS} = \frac{i\kappa}{2} \text{Tr}(Z^\dagger D_0 Z - Z D_0 Z^\dagger) + \kappa \, \text{Tr} \, a_0, \quad (17)$$

to which Bak *et al.* added a noncommutative ϕ^4 Lagrangian[27]

$$L_s = \text{Tr}\left\{ \phi^\dagger i D_0 \phi - \frac{1}{2m} D_i \phi (D_i \phi)^\dagger - \frac{\lambda}{2\theta} (\phi^\dagger \phi)^2 \right\}. \quad (18)$$

Here ϕ is a matrix field in the fundamental representation, i.e., it transforms as $\phi \rightarrow U\phi$ under the gauge transformation U. The covariant derivatives are defined by $D_\mu = \partial_\mu + i\hat{a}_\mu$ and act on ϕ as

$$D_0 \phi = \partial_0 \phi + i\hat{a}_0 \phi,$$

$$D_i \phi = \frac{i}{\theta} \epsilon_{ij} [\hat{x}_j, \phi] + i\hat{a}_i \phi = \frac{i}{\theta} \epsilon_{ij} [X_j \phi - \phi \hat{x}_j], \quad (19)$$

where $\hat{x}_i$ are noncommuting coordinates, $[\hat{x}_i, \hat{x}_j] = i\epsilon_{ij}\theta$. The corresponding derivatives are given by $\partial_i = (i/\theta)\epsilon_{ij}[\hat{x}^j, \bullet]$, and the matrices X_i, defining the actual state, are related to the noncommutative gauge potential via

$$X_i = \hat{x}_i - \theta \epsilon_{ij} \hat{a}_j, \quad (20)$$

i.e., $\hat{a}_i$ parametrizes the deviation from the ground state solution $[X^1, X^2] = i\theta$.

Defining the current

$$J_i = \frac{-i}{2m}[(D_j\phi)\phi^\dagger - \phi(D_j\phi)^\dagger], \quad (21)$$

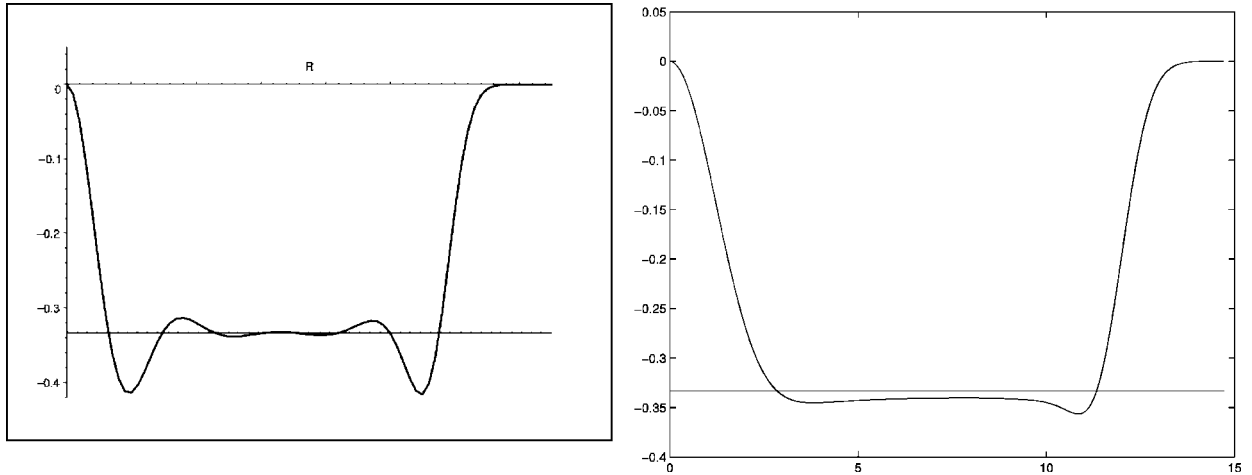the Hamiltonian can, after some algebra, be written as

FIG. 5. To the left, Monte Carlo calculation (Ref. 13) of the difference $Q_{\text{hole}}(R) - Q_{\text{droplet}}(R)$ in the cumulative integrated charge between the Laughlin quasihole density distribution and the ground state density distribution for $N=50$, $\nu=1/3$. The horizontal line corresponds to the charge difference $1/3$. The radial distance is in units of $\sqrt{2}\ell$ and the charge in units of e. To the right, the same distribution calculated from the matrix model.

$$H = \mathrm{Tr}\, \frac{1}{2m}(D_1\phi - iD_2\phi)(D_1\phi - iD_2\phi)^\dagger - \frac{\epsilon_{ij}}{2}\,\mathrm{Tr}\, D_i J_j$$

$$- \frac{1}{2m\theta}\,\mathrm{Tr}\big([Z,Z^\dagger] - 1 + \lambda m\phi\phi^\dagger\big)\phi\phi^\dagger. \qquad (22)$$

Assume that the solution is regular enough for the covariant derivative of the current to integrate to zero. Also, set $\lambda = 1/m\kappa$. Then the last term will vanish because of the constraint

$$[Z,Z^\dagger] = 1 - \frac{1}{\kappa}\phi\phi^\dagger. \qquad (23)$$

The Hamiltonian reduces to the first term which is quadratic and equals zero for $(D_1\phi - iD_2\phi) = 0$. This choice of parameters corresponds to the theory being of the Bogomol'nyi-Prasad-Sommerfield (BPS) form.[6,15] The complete set of BPS equations is given by

$$\hat{a}_0 = \frac{1}{2\kappa\theta}\phi\phi^\dagger, \qquad (24)$$

$$D_1\phi - iD_2\phi = 0,$$

$$[Z,Z^\dagger] = 1 - \frac{1}{\kappa}\phi\phi^\dagger,$$

and it is easy to check that any solution of these is also a solution to the full time-independent equations of motion corresponding to $L_{CS} + L_s$.

We now turn to finite matrices. Because of the coupling $\sim \mathrm{Tr}(\hat{a}_0\phi\phi^\dagger)$ the Gauss law constraint became (23), which can be satisfied by finite matrices. Of course, it is then no longer possible to have $[\hat{x}^1, \hat{x}^2] = i\theta[z, z^\dagger] = i\theta$, where $z = (1/\sqrt{2\theta})(\hat{x}^1 + i\hat{x}^2)$. Instead we let

$$[\hat{z}, \hat{z}^\dagger] = M = 1 - N|N-1\rangle\langle N-1|, \qquad (25)$$

with

$$\hat{z} = \sum_{n=1}^{N-1} \sqrt{n}|n-1\rangle\langle n|. \qquad (26)$$

Now the Hamiltonian can no longer be written on BPS form (22), but by adding the term,

$$L_M = \frac{1}{2m\theta}\,\mathrm{Tr}(+\phi M\phi^\dagger - \phi\phi^\dagger), \qquad (27)$$

it is a matter of algebraic manipulations to show that the Hamiltonian corresponding to $L = L_{CS} + L_s + L_M$ is again of the BPS type.

It is not hard to verify that the model we just defined has droplet solutions, and topological solitons of the type found by Bak et al.[6] There are, however, no quasihole solutions. This can be remedied by also adding a Polychronakos-type boundary field, which has the additional advantage that the sector where the scalar field is not excited becomes identical to the original finite QH matrix model. Our final Lagrangian now reads

$$L = L_{CS} + L_s + L_M + L_\Phi \qquad (28)$$

where the boundary Lagrangian is given by

$$L_\Phi = \Phi^\dagger iD_0\Phi - \frac{\lambda}{2\theta}\Phi^\dagger\phi\phi^\dagger\Phi \qquad (29)$$

yielding the Gauss law constraint

$$[Z,Z^\dagger] = 1 - \frac{1}{\kappa}\phi\phi^\dagger - \frac{1}{\kappa}\Phi\Phi^\dagger. \qquad (30)$$

The last term in (29) was added to allow the Hamiltonian to have a BPS form almost identical to (22), but with the BPS equation (23) replaced by (30). The remaining BPS equations are unchanged. This completes the derivation of the extended finite QH matrix model, which is a finite matrix version of the conformal Chern-Simons-Higgs model introduced by Jackiw and Pi.[15]

We now turn to a discussion of the solutions of this model. First note that all solutions discussed in Sec. II, i.e., the isolated particles (10), the droplet (12), and the quasihole (14), can all be taken over unchanged if we set $\phi=0$. For nonzero ϕ we will have two new types of solutions corresponding to waves and solitons. The latter, which will provide the basic building block for the quasielectrons, are the most interesting, but we first briefly discuss the former.

A. Collective modes

For our model to give a realistic description of the QH system it is important that the collective wavelike solutions in the bulk are gapped. This is certainly expected from the analogy with the continuum model, but should nevertheless be established in the matrix model context. Let us first consider the case of a constant density of ϕ particles, $\tilde{\rho}$, (not to be confused with the constant density of electrons $\bar{\rho} = 1/2\pi\theta$ represented by the solution $\hat{Z} = \hat{z}, \phi = 0$) in the infinite matrix model of Bak et al.[6] The mean field solution that we want to expand about is given as an expansion in the density $\tilde{\rho}$

$$Z = \left(1 - \frac{\tilde{\rho}}{2\kappa}\right)z + \mathcal{O}(\tilde{\rho}^2),$$

$$\phi = \sqrt{\tilde{\rho}}\big(\hat{1} + \mathcal{O}(\tilde{\rho})\big), \qquad (31)$$

$$a_0 = -\frac{\lambda}{2\theta}\tilde{\rho} + \mathcal{O}(\tilde{\rho}^2),$$

which solves the full equations of motion to first order in $\tilde{\rho}$.

We then expand (17) and (18) to quadratic order around the mean field solution, and then use the polar decomposition $\phi = UP$ which is valid for an arbitrary square matrix. Here U is a unitary and P is a positive semidefinite Hermitian matrix.[16] By a U(N) gauge transformation, we can now remove the U(N) phase U from the field ϕ. As a result we find that the kinetic term of what remains of ϕ becomes a total derivative and this field thus becomes a Lagrange multiplier enforcing a constraint relating the fluctuations of the gauge

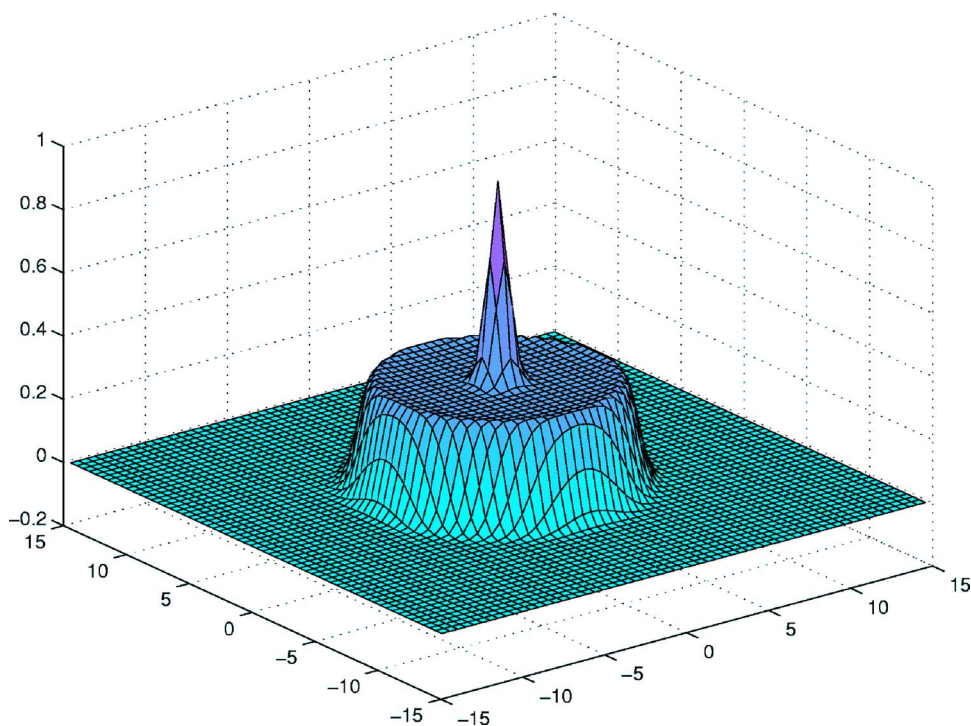FIG. 6. (Color online) The density distribution of a soliton of charge -1 for $N=50$, $\nu=1$.

field and ϕ. The result is that we have moved the entire dynamics from the scalar field to the gauge field Z. This is the noncommutative version of going to unitary gauge in Ginzburg-Landau Chern-Simons (GLCS) theory. The resulting Lagrangian reads,

$$L_{CS}+L_{\phi}=i\kappa\,\mathrm{Tr}\,a^{\dagger}\dot{a}-\frac{\tilde{\rho}}{2m}\,\mathrm{Tr}(aa^{\dagger}+a^{\dagger}a)+\cdots,\quad(32)$$

in terms of $\hat{a}\equiv i(\hat{z}-Z)=\sqrt{\theta/2}(\hat{a}_1+i\hat{a}_2)$, with $\hat{a}_i$ given by (20). The dots indicate commutator terms corresponding to spatial derivatives, as well as potential terms and terms of higher order in $\tilde{\rho}$. There is also a constraint equation that relates density fluctuations to the noncommutative gauge field. Just as in the commutative case, the Lagrangian (32) has the form of a harmonic oscillator, and consequently exhibits a gap at $\omega_c=\tilde{\rho}/\kappa m$. This has the natural interpretation as the Kohn mode at the cyclotron frequency of the ϕ particles.

For solutions with vanishing background density $\tilde{\rho}$—the simplest case being that of a soliton considered below—there will be zero modes corresponding to translations. In the full model (28) there will also be gapless edge modes. We have not analyzed these more complicated cases, but we think that the above demonstration of the similarity between the commutative and noncommutative models strongly suggests that the latter will not develop any gapless modes not found in the former.

B. Solitons and quasielectrons

In Ref. 6, Bak *et al.* found noncommutative counterparts of the self dual vortex solutions due to Jackiw and Pi. These correspond to a quantized flux, and, as will be clear from the explicit expressions given below, they carry unit electric charge. Since flux is quantized, one cannot have fractionally charged quasielectrons in the model by Bak *et al.*, but in our finite matrix model there is a natural construction in terms of a soliton combined with a quasihole.

1. The charge -1 soliton

Using (19) one can derive

$$D_1\phi-iD_2\phi=-\sqrt{\frac{2}{\theta}}(Z^{\dagger}\phi-\phi\hat{z}^{\dagger}).\quad(33)$$

A soliton of charge -1 centered at the origin is now given by the following expression:

$$Z=\sum_{n=2}^{N-1}\sqrt{n-1}|n-1\rangle\langle n|,$$

$$\Phi=\sqrt{(N-1)\kappa}|N-1\rangle,\quad(34)$$

$$\phi=\sqrt{\kappa}|0\rangle\langle 0|.$$

It is easy to verify by direct substitution that this indeed is a solution to the BPS equations corresponding to the full QHMM (28). In Fig. 6 we show a density plot of this solution for $N=50$.

2. The quasielectron

In analogy with the soliton solution (34), we can now try to construct a fractional quasi-electron based on the Ansatz, $Z=\sum_{n=1}^{N-1}\sqrt{n+q}|n-1\rangle\langle n|$ with $-1<q<0$. Note, however, that this implies $Z|0\rangle\neq 0$ so the only option seems to be ϕ
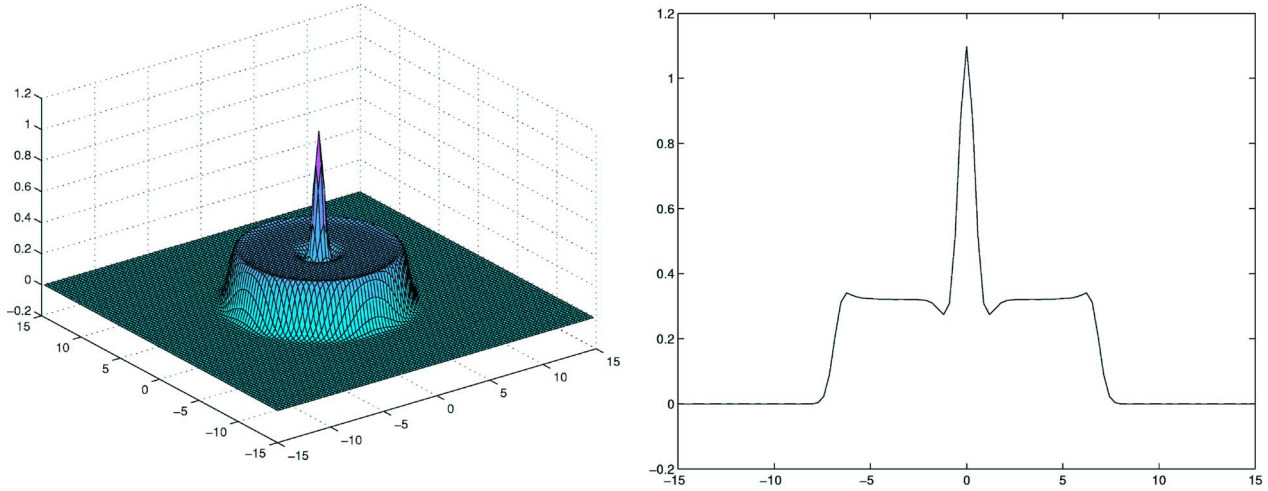
FIG. 7. (Color online) The density distribution of a charge $-\frac{1}{3}$ quasielectron composed of a charge $+\frac{2}{3}$ quasihole on top of a charge -1 soliton for $N=50$, $\nu=1/3$.

$=\sqrt{(N+q)\kappa}|N-1\rangle\langle0|$ and $\Phi=\sqrt{-q\kappa}|0\rangle$. Such a solution would, however, not reduce to the ground-state (12) as $q\to0$.

An obvious alternative construction, alluded to above, is to add a quasihole to the soliton, i.e., to combine the solutions (34) and (14). This amounts to finding a q-dependent modification of Z for which the constraint is still q-independent. The solution is

$$Z=\sqrt{q}|N-1\rangle\langle1| + \sum_{n=1}^{N-1}\sqrt{n-1}|n-1\rangle\langle n|,$$

$$\Phi=\sqrt{N\kappa}|N-1\rangle, \qquad (35)$$

$$\phi=\sqrt{\kappa}|0\rangle\langle0|,$$

for which we still have $Z|0\rangle=0$ and hence $\bar{D}\phi=0$. Again the BPS equations can be verified by direct substitution. In Fig.

7 we show for $N=50$, $\nu=1/3$ a $q=2/3$ hole sitting on the top of a soliton of charge -1 to produce a charge $-1/3$ quasielectron. The small dip at the foot of the peak has also been seen in numerical studies of QH wave functions,[13] see Fig. 8.

Note that this solution does not reduce to the ground state as $q\to1$. This is not necessarily a drawback, since it might be interpreted as the limiting case of a small exciton, i.e., an overlapping state of an electron and a hole. Finally we should mention that we have not investigated the stability of our quasielectron solution, so we cannot be sure that it will not decay into a soliton and a quasihole. Although such a calculation amounts to a straightforward small oscillation analysis, it is algebraically complicated, and also of limited interest since we would in any case have a stabilizing Coulomb interaction in a more detailed model.

IV. SUMMARY AND DISCUSSION

To summarize, we have argued for a particular expression for the charge density in the *classical* QH matrix models, and
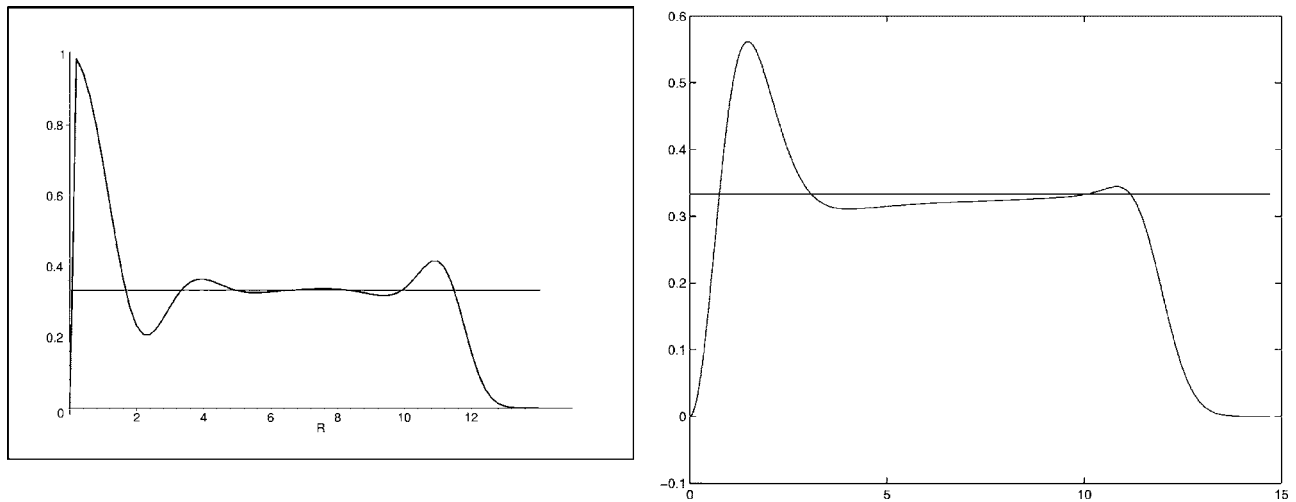

FIG. 8. The same plots as in Fig. 5, but for a Jain quasielectron (Ref. 13). Note the characteristic dip at the foot to the right of the quasielectron.

shown that with this definition, the various solutions corresponding to separated particles, droplets and quasiholes are reproduced in reasonable agreement with standard treatments based on wave functions and GLCS mean field solutions. We furthermore extended the model to incorporate densities higher than that of the ground state, and found quasielectron solutions. Again the profiles were in good agreement with those found from explicit wave functions.

In this connection it is fair to ask what has been gained by the classical QH matrix model as compared to the usual effective low energy CS theory of the form,

$$\mathcal{L} = \epsilon^{\mu\nu\lambda} \frac{1}{4\pi\nu} a_\mu \partial_\nu a_\lambda - \epsilon^{\mu\nu\lambda} \frac{1}{2\pi} A_\mu \partial_\nu a_\lambda - j_\mu a^\mu, \quad (36)$$

where j_μ is the quasiparticle current. This effective theory is topological, has a unique state on an infinite plane and can be derived from the GLCS theory by expanding around a mean field.[28] We shall now briefly contrast this approach to that of the classical QH matrix model. That the two theories are closely connected is clear from Susskind's original formulation of the QH matrix model as a noncommutative CS theory described by,

$$\mathcal{L} = \frac{1}{4\pi\nu} \epsilon^{\mu\nu\lambda} \left(a_\mu \star \partial_\nu a_\lambda + \frac{2i}{3} a_\mu \star a_\nu \star a_\lambda \right), \quad (37)$$

where the Moyal star product $\star$ is defined with a noncommutative parameter $\theta = (2\pi\bar\rho)^{-1}$. This is a purely topological theory, consistent with the infinite matrix model having a unique state with a constant density $\bar\rho$. Quasielectrons and quasiholes can be introduced by hand in the infinite matrix model by changing the constraint. In the commutative CS theory this corresponds to adding delta function sources. Here we see the first advantage of the matrix model in that it gives a size $\sim\theta$ to the quasiparticles.[29]

Adding the boundary field Φ to the matrix model allows for a plethora of states not described by the usual CS approach. Defining the latter on a manifold with a boundary gives edge degrees of freedom corresponding to chiral Luttinger liquids, but there are no excitations inside the bulk nor outside the droplet. The basic reason is that the edge excitations in the CS theory can be understood as hydrodynamic modes of an incompressible liquid, while the matrix model allows for density fluctuations in the fluid itself. From this it is also clear that no questions regarding density profiles or effective sizes of quasiparticles can be addressed in the framework of pure CS theory.

There is an asymmetry between quasielectrons and quasiholes in the matrix model, since there is a maximal density given by the noncommutative parameter θ. This was the basic reason that forced us to introduce a new field, ϕ, to describe quasielectrons, while quasiholes were present already in the model based on only a CS field X^i and the boundary field Φ needed to "absorb" the anomaly. Such an asymmetry is present also in other descriptions of the QH effect. For instance, in the wave function approach Laughlin's quasihole wave function is essentially unique, while there are several quite different approaches to the quasielectron state.[13] The introduction of a new field raises questions about the correct

counting of degrees of freedom. The finite matrix model without any extra field describes N particles, but with a phase space repulsion giving a maximum density $\sim 1/\theta$. As we have shown, the extra field relaxes the maximum density constraint in a way consistent with QH phenomenology, but one might worry that we have at the same time introduced additional unphysical (gapped) excitations in the high energy part of the spectrum. We have not investigated this problem any further.

To summarize, there are some aspects of QH physics that is more easily described in the conventional low energy CS framework, notably the classification of Abelian QH liquids using a generalization of (36) involving many gauge fields, developed by Wen.[5] The QH matrix model, on the other hand, allows for a more detailed analysis of density profiles and a dynamical description of quasielectrons and quasiholes.

There are several detailed questions left open concerning the details of the classical QHMM, and the quantum theory is to a great extent unexplored territory. If we might venture a guess, we would, however, say that if the noncommutative approach to QH physics is to provide any essential new physical insights one has either to find ways to generalize the quantum models—with the aim of understanding the hierarchy and/or the Jain states—or to find some quantitative use for the classical description. The mere fact that a classical model can do so well in describing a strongly interacting system in the extreme quantum regime is in itself intriguing, and it might be quite interesting to extend the model to include disorder and study possible phase transitions.

APPENDIX A: MORE ON DENSITY OPERATORS

In this appendix we first demonstrate that the definition of $\rho(\vec{x})$ implied by (7) gives negative values for certain configurations satisfying the constraint (3). We then give two alternative definitions of $\rho(\vec{x})$ which are positive, but have other difficulties.

1. The Weyl-ordered density is not positive

For small electron numbers and small filling fractions one can find many solutions for where the density becomes negative. In some cases it just about becomes negative, but for other solutions the violation of positivity is big, as can be seen in Fig. 9.

That the density is sometimes negative can, for special cases, also be established analytically to lowest order in θ.

2. Alternative density operators

We now give two alternative definitions of the density operator that are both non-negative. The starting point is the
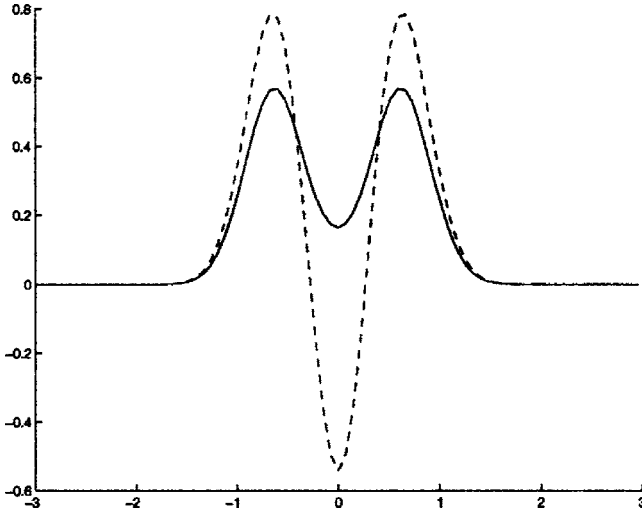
FIG. 9. Density profiles of the droplet solution (12) for $N=2$, $\nu=1/5$. The definition (7) is given by the broken line and the positive definite definition (A7) by the full line. Both definitions give a circularly symmetric distribution. The density (A7) integrates to a total particle number of about 1.66.

non-relativistic density operator for N point particles,

$$\rho(\vec{x}) = \sum_{n=1}^{N} \delta(\vec{x} - \vec{x}_n). \tag{A1}$$

If $\vec{x}_n$ are taken as quantum operators, this is also the first quantized density operator in the $\vec{x}$ representation. The operator (A1) is by construction non-negative, since it is a sum of positive operators. In momentum space, the operator (A1) takes the form

$$\rho(\vec{k}) = \sum_{n=1}^{N} e^{-i\vec{k}\cdot\vec{x}_n}. \tag{A2}$$

It is well known that quantum mechanical particles in the lowest Landau level are described by the following density operator:

$$\rho_L(\vec{k}) = \sum_{n=1}^{N} e^{-(i/2)\bar{k}z_n} e^{-(i/2)k\bar{z}_n} \tag{A3}$$

with $z_n = x_n^1 + i x_n^2$ and $[z_m, \bar{z}_n] = 2\ell\,\delta_{mn}$. In $\vec{x}$ space this becomes,

$$\rho_L(\vec{x}) = \frac{1}{2\pi\ell^2} \sum_{n=1}^{N} e^{-[(\vec{x}-\vec{x}_n)^2/2\ell^2]} = \sum_{n=1}^{N} \delta_\ell(\vec{x}-\vec{x}_n), \tag{A4}$$

where δ_ℓ can be thought of as a regularized delta function. Again the operator (A4) is positive by construction.

With these preliminaries, we now present two possible definitions of $\rho(\vec{x})$ in the finite matrix model that are manifestly positive. The most obvious idea is to try to extract a set of N particle positions, $\vec{x}_n$ from the matrices X^i and simply plug these into a formula of the type (A4). In this case we are of course free to use any positive definite profile function for the particles, but by choosing exactly (A4) we ensure that the

profile of a single particle in the matrix model is identical to that of an electron in the lowest Landau level. The problem of defining coordinates in the QH matrix model was discussed in a paper by Karabali and Sakita.[17] They showed that taking the eigenvalues of the complex matrix Z as particle positions,[30] correctly reproduced the low momentum part of the Laughlin wave function, while the short distance part was distorted—the characteristic $|z_i - z_j|^{2\kappa}$ behavior of the two particle correlation was softened to a lower power. We would thus expect that a density operator defined by (A4) and the coordinates proposed in Ref. 17 in spite of being positive, would have difficulties in describing the profiles studied in this paper, which vary rapidly on the order of a magnetic length. Since the construction is very indirect, we also do not have any closed expression for the density and current similar to (4).

Another possibility is based on expressing the operator (A4) as a square of an operator, thus making the positivity manifest:

$$\rho_L(\vec{x}) = (\rho_L(\vec{x})^{1/2})^2, \tag{A5}$$

where

$$\rho_L(\vec{x})^{1/2} \approx \frac{1}{\sqrt{2\pi\ell^2}} \sum_{n=1}^{N} e^{-[(\vec{x}-\vec{x}_n)^2/4\ell^2]} \approx \sum_{n=1}^{N} \delta_\ell^{1/2}(\vec{x}-\vec{x}_n), \tag{A6}$$

which is a good approximation when the particles are far apart. Going to Fourier space, where the square of the distribution become a convolution integral, we are led to the following proposal for the density operator:

$$\rho_{\text{pos}}(\vec{p}) = \frac{2\ell^2}{\pi} \int d^2k\, e^{-\ell^2[(\vec{p}-\vec{k})^2+k^2]} \text{Tr}[e^{-i(\vec{p}-\vec{k})\cdot\vec{X}}] \text{Tr}[e^{-i\vec{k}\cdot\vec{X}}]. \tag{A7}$$

The corresponding $\rho_{\text{pos}}(\vec{x})$ is positive by construction, and it is easy to show that for widely separated particles, where the matrices become almost diagonal, the profile reproduces the one given by (A5) and (A6). When the particles come closer this is no longer true. Figure 9 shows the droplet solution with the old definition (7) shown by a broken line, and the definition (A7) by a solid line. Other examples, like for particles further apart, show again that if the density becomes negative, (A7) repairs that. The basic problem of that definition is that it is not normalized, i.e., $\int d^2x \rho_{\text{pos}}(\vec{x}) \neq N$. This can of course be remedied by a renormalization, but difficulties remain.

Note that the definition (A7) is not in the general class (4) since it involves the product of two traces rather than a single trace over a matrix kernel. This in particular means that our construction of a conserved current is no longer valid, but more importantly, that Pandora's box is opened—Why should we restrict ourselves to the product of two traces? Why not several, or perhaps even an infinite series?

In summary, we have given alternative constructions of the density operator which are manifestly non-negative.

There are, however, other difficulties related to these proposals, and we have no reason to believe that they would provide a better description than (7) that we used in the main text of the paper.

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[22] On a manifold with nontrivial topology one expects, in analogy with the commutative case, a ground state degeneracy in the *quantum* matrix model.

[23] In his original paper, Susskind argued, based on the underlying fermion statistics, that the level number κ is quantized as an odd integer so that the classical relation $\kappa = eB\theta$, or $\rho = 1/2\pi\theta = \rho_0/\kappa$, implies that the density is quantized at the Laughlin fractions. This argument was later corrected by Polychronakos (Ref. 2) and by Hellerman and Susskind (Ref. 18). It turns out that in the *quantum* matrix model the relation between density and level number is shifted to $\nu = 1/(1+\kappa)$ because of quantum fluctuations. There is, however, also a correction to the relation between the level number and statistics that was used in Ref. 1. These effects in fact cancel and leave the original result of Susskind unchanged. A topological argument for the quantization of κ to any integer was given by Nair and Polychronakos (Ref. 19).

[24] The error is corrected in arXiv: cond-mat/0304271v2.

[25] These eigenvalues are actually the roots of the Nth order Hermite polynomial as pointed out by Polychronakos in Ref. 2.

[26] Taking the logarithm of the density shows that the tail in fact falls faster than the exponential in Eq. (13). A better fit is given by $e^{-((r-r_0)/\beta\ell)^2}$. However, fitting of just the tail is much more uncertain than is a fitting where also the plateau is used.

[27] Note that in their paper the ϕ^4-term has a positive coefficient, which might lead one to worry about the positive definiteness of the energy. However, the constraint cancels the ϕ^4-term and leaves a positive definite Hamiltonian. The same thing happens with a negative coefficient as in our case.

[28] Since the quantum GLCS theory is a direct rewriting of the microscopic theory (Ref. 20) (using a singular gauge transformation), it is an interesting open question whether the noncommutative action could be obtained by a more sophisticated mean field approach.

[29] By introducing further terms in the expansion around the mean field in the commutative CS theory, one generates terms $\sim b^2$, where b is the CS magnetic field. Such a term will give a size to the quasiparticle, as discussed in e.g., Ref. 21.

[30] All but a set of measure zero of the complex matrices Z can be diagonalized as $Z = XEX^{-1}$.