Heat bath approach to Landau damping and Pomeranchuk quantum critical points

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(Received 9 August 2005; revised manuscript received 27 September 2005; published 14 November 2005)

We study the problem of the damping of collective modes close to a Pomeranchuk quantum critical point in a Fermi liquid. In analogy with problems in dissipative open quantum systems, we derive the Landau damping of a Fermi liquid by integrating out a macroscopic number of degrees of freedom from a generating functional. Being a reformulation of the linearized Boltzmann equation, this approach reproduces well-known results from the theory of Fermi liquids. We also study the Bethe-Salpeter equations within the Landau theory and discuss the implications of these results on quantum phase transitions of the Pomeranchuk type and its dynamical exponent, z. We apply our results to the electronic nematic instability and find $z=3$ in the collisionless limit.

DOI: 10.1103/PhysRevB.72.195104 PACS number(s): 71.10.Ay, 71.10.Hf, 71.10.Pm

I. INTRODUCTION

Recently there has been interest in the so-called quantum critical Pomeranchuk instability where the interactions in a fermion system are strong enough to make the Fermi surface soft and ultimately unstable toward a spontaneous deformation.[1] These instabilities have been proposed to be behind exotic electronic phases such as electronic liquid crystals in cuprate superconductors[2,3] and fractional quantum Hall effect,[4,5] metamagnetic transitions in electronic systems such as $Sr_3Ru_2O_7$,[6] electronic systems close to van Hove singularities,[7] "hidden order" in URu_2Si_2,[8,9] a mechanism for generation of spin-orbit coupling,[10] and even phase transitions in interacting quantum dots.[11] The Pomeranchuk instability can take place both on a lattice and in the continuum[2,12,13] and is driven by attractive quasiparticle-quasiparticle interactions. These interactions in Fermi liquid theory are parametrized by the Landau parameters, $F_m^{s,a}$, where s (a) stands for the symmetric (antisymmetric) spin channel and $m=0,1,\ldots$. for the angular momentum (we focus on the case of the two-dimensional isotropic liquid although the three-dimensional case is completely analogous). Pomeranchuk showed that if some $F_m^{s,a}$ becomes too attractive ($F_m^{s,a}<-1$ in two dimensions) at $T=0$, then the Fermi surface becomes unstable and cannot sustain collective oscillations. The simplest example of such a transition is a ferromagnetic (or Stoner) instability that occurs when $F_0^a<-1$, which is characterized by a diverging magnetic susceptibility, χ [$\chi \propto 1/(1+F_0^a)$]. A similar instability with a diverging compressibility, $\mathcal{K}$, can be achieved by pulling on a liquid to put it into a negative pressure region when $F_0^s<-1$ [$\mathcal{K} \propto 1/(1+F_0^s)$].[14] There is also the possibility that the instability takes place in a higher angular momentum channel. For example, a deformation into an elliptical Fermi surface can be achieved by having a strong enough attraction in the d channel, that is, $F_2^s<-1$. The broken symmetry phase in this case has been proposed to be an *electronic nematic* because of the nomenclature in the classical theory of liquid crystals where an analogous phase exists. The properties of the broken symmetry phase have been found to be very peculiar with marked non-Fermi liquid properties.[2] Although Fermi liquid theory cannot predict what is the nature of the broken symmetry phase, it can be used to study the approach to the instability from the Fermi liquid (quantum disordered) side of the transition (see Fig. 1).

One of the main characteristics of a Pomeranchuk quantum critical point is that in its proximity the excitations are not of the simple quasiparticle type, but rather become heavily damped because of the soft nature of the Fermi surface. This is so because the Fermi surface allows for a diverging number of low energy particle-hole pairs to accompany an excited quasiparticle. Presumably, a similar situation holds for most of the single-particle excitations in the broken symmetry phase,[2] although, the Fermi liquid theory is not able to access that phase directly.

A natural way to study the collective modes of an interacting electron fluid is via Fermi surface bosonization. The method of bosonization has been very successful in providing a framework for describing the breakdown of the normal Fermi liquid in one-dimensional (1D) systems, where instead the Luttinger liquid emerges as the "standard" model.[15] In

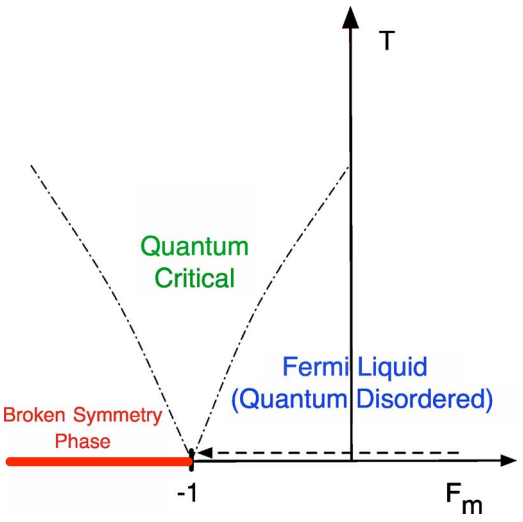

FIG. 1. (Color online) Phase diagram of a quantum critical Pomeranchuk instability of order m. Vertical axis: temperature T; horizontal axis: Landau parameter F_m. The dashed arrow line shows the direction of approach to the quantum critical point used in this work.

Luttinger liquids there are no well-defined quasiparticles and the electron breaks down into charge and spin collective modes. It is now well understood that the anomalous behavior in one dimension is mainly due to the restricted phase space. Indeed, there exists Pomeranchuk instabilities also in 1D which lead to a breakdown of the Luttinger liquid. What happens is that either the charge-velocity or spin-velocity vanishes at the transition. Since the charge (spin) susceptibility goes as the inverse of the charge (spin) velocity,[15] the signature of the instability is a diverging susceptibility, just as in the higher dimensional case. Thus, the Pomeranchuk instability is a general concept, independent of the dimensionality of the system under interest. The higher dimensional bosonization approach was developed many years ago[16–21] and reproduces the known results from the theory of Fermi liquids, especially the physics of collective modes and particle-hole excitations which are described in terms of boson coherent states. The basic reason for this is that both theories describe fermion systems dominated by forward scattering correctly, and this is what the Landau Fermi liquid is all about.[22]

For problems such as those involving classical limits, it is a good idea to use a description in terms of bosons since they can have classical analogs whereas fermions cannot. In particular, when one is studying phase transitions in the language of a Landau-Ginzburg expansion the order parameter is a classical construction which in the quantum language should be described by bosons. One can generally trade a fermion system with density-density interactions to one involving bosons by using a Hubbard-Stratanovich transformation and then integrate out the fermions from the path integral that describes the system. This is the standard device to study quantum critical theories in Fermi systems.[23–25] In such approaches to quantum critical phenomena, dissipative terms are generated in the order parameter effective action, the so-called Landau damping. However, there have been some arguments about whether it is viable to integrate out all of the fermions from the system in this way, or if some properties of the system are lost due to the gapless nature of the electrons in a Fermi liquid. In particular, there are particle-hole-like excitations with arbitrarily low energies which must be treated with care since they may control the critical behavior of a Fermi system. Nevertheless, the integration of high energy degrees of freedom as usually occurs in renormalization group (RG) approaches should give sensible results. Obviously, the particle-hole continuum-type excitations do not disappear in a faithful bosonic reformulation of the system, although their nature might be disguised. When some of these modes are in resonance with the collective excitations, the result is Landau damping. This occurs only for modes with a velocity smaller than the Fermi velocity.

In this paper, we show that the Landau damping problem can be recast in terms of a trace over bosonic modes. In particular, the modes associated with the high angular momentum oscillations of the Fermi surface are traced out, since the collective excitations we are interested in are concentrated in the low angular momentum modes. Furthermore, using the Bethe-Salpeter equation in the context of the collision integral, we can study the lifetime of quasiparticles and collective modes in a Fermi liquid close to a Pomeranchuk quantum critical point. Moreover, we discuss the dynamical critical exponent of the same quantum critical point. The paper is organized as follows. In Sec. II, we show how Landau damping as described above can be reproduced within a simple purely bosonic reformulation of the Fermi liquid theory. Here the damping comes about from the coupling to a bath of harmonic oscillators similarly to what happens in dissipative quantum mechanics. In Sec. III, we study the collision integral in the language of Fermi liquid theory, focusing on the behavior near the critical point. This turns out to be important since collisions set the lifetime of the quasiparticles, and as one approaches the critical point, the lifetime decays rapidly. We conclude in Sec. IV with a discussion of the implications of these phenomena on the theory close to the critical point. In an appendix, we present some mathematical details.

II. LANDAU DAMPING FROM A BOSON BATH

The phenomenon of Landau damping is conventionally thought of as the damping of a collective mode due to "a *coherent* interaction of the collective mode with those particles which "surf-ride" on the crests of the running wave."[26] In this section, we show explicitly how Landau damping can also be viewed as the dissipation of a mode due to its coupling to a set of independent oscillators, i.e., a "boson bath." Here the origin of these bosons are particle-hole excitations of the fermion system itself. This approach is similar in spirit to theories of dissipation in quantum mechanics.[27,28]

Let us take as our starting point, the Landau kinetic equation that describes the evolution of the quasiparticle distribution function in the spin symmetric channel, $n_{\vec{p}}(\vec{r},t)$[26]

$$\frac{\partial n_{\vec{p}}(\vec{r},t)}{\partial t} + \nabla_p \epsilon_{\vec{p}}(\vec{r},t)\cdot\nabla_r n_{\vec{p}}(\vec{r},t) - \nabla_r \epsilon_{\vec{p}}(\vec{r},t)\cdot\nabla_p n_{\vec{p}}(\vec{r},t) = I[n_{\vec{p}'}], \tag{1}$$

where $\epsilon_{\vec{p}}(\vec{r},t)$ is the quasiparticle energy and $I[n_{\vec{p}'}]$ is the collision integral ($\vec{p}$ is the quasiparticle momentum, $\vec{r}$ is the position in real space, and t is time).[29] Since the physics goes on close to the Fermi surface, it is convenient to change variables to $\nu_{\vec{p}}(\vec{r},t)$ which we define by writing the deviation from the ground state values of quasiparticle distribution as $\delta n_{\vec{p}}(\vec{r},t) = -(\partial n_p^0/\partial\epsilon_p)\nu_p(\vec{r},t)$. Physically $\nu_p(\vec{r},t)$ measures the local deformation of the Fermi surface. Upon a Fourier transform in space, the linearized kinetic equation for $\nu_p(\vec{r},t)$ becomes

$$\frac{\partial \nu_{\vec{p}}(\vec{q},t)}{\partial t} + i\vec{v}_{\vec{p}}\cdot\vec{q}[\nu_{\vec{p}}(\vec{q},t) + \delta\epsilon_{\vec{p}}(\vec{q},t)] = I'[\nu_{\vec{p}'}], \tag{2}$$

where $\vec{v}_{\vec{p}}$ is the Fermi velocity at the momentum $\vec{p}$. Again the notation

$$\delta\epsilon_{\vec{p}} = \sum_{\vec{p}'} f_{\vec{p}\vec{p}'}\,\delta n_{\vec{p}'} \tag{3}$$

is standard and describes the interaction (parametrized by $f_{\vec{p}\vec{p}'}$) among excited quasiparticles. For an isotropic liquid in

two dimensions, $\vec{v}_{\vec{p}}$ is parallel to $\vec{p}$, and since the magnitude of p is nearly equal to the Fermi momentum p_F, we can label $\vec{p}$ by an angle θ_p (measured from the direction of $\vec{q}$). Then, the interaction $f_{\vec{p}\vec{p}'}$ only depends on the angle between the momenta and is convenient to expand in Fourier components because of the circular symmetry as follows:

$$\nu(\theta) = \sum_m \nu_m e^{im\theta}, \tag{4}$$

$$f(\theta,\theta') = \sum_m f_m e^{im(\theta-\theta')}, \tag{5}$$

where $f_m = f_{-m}$. The sum over m should be cut off at some integer of order $E_F/\Lambda \equiv N_\Lambda \gg 1$, where E_F is the Fermi energy and Λ is some energy scale below which the Landau theory is valid. Projecting onto the Fourier components and setting the collision integral to zero for the time being, we arrive at

$$\frac{\partial \nu_m(\vec{q},t)}{\partial t} + \frac{iv_F q}{2}[(1+F_{m-1})\nu_{m-1}(\vec{q},t) + (1+F_{m+1})\nu_{m+1}(\vec{q},t)] = 0. \tag{6}$$

Here v_F is the Fermi velocity and the $F_n = N(0)f_n$ are the interaction parameters normalized with the density of states at the Fermi energy, $N(0)$. Note that these equations are linear in the time derivatives and couple the time derivatives of odd components to the even components and vice versa. Thus the equations respect time reversal invariance implying nondissipative dynamics.

To make the connection with the more familiar simple oscillator, we eliminate the odd components in favor of a second time derivative.[30] With the notations

$$a_m(\vec{q},t) = \sqrt{1+F_m}\,\nu_m(\vec{q},t), \tag{7}$$

$$A_m = \frac{1}{4}(1+F_m)(2+F_{m-1}+F_{m+1}), \tag{8}$$

$$B_m = \frac{1}{4}\sqrt{1+F_{m-1}}(1+F_m)\sqrt{1+F_{m+1}}, \tag{9}$$

the equations of motion become

$$\frac{\partial^2 a_m(\vec{q},t)}{\partial t^2} + (v_F q)^2[A_m a_m(\vec{q},t) + B_{m-1}a_{m-2}(\vec{q},t) + B_{m+1}a_{m+2}(\vec{q},t)] = 0. \tag{10}$$

These equations of motion are also generated from Euler-Lagrange equations and the classical Lagrangian density

$$\mathcal{L} = \sum_m \frac{\partial a^*_{2m}(\vec{q},t)}{\partial t}\frac{\partial a_{2m}(\vec{q},t)}{\partial t} - (v_F q)^2 \sum_{m,n} a^*_{2m}(\vec{q},t)V_{mn}a_{2n}(\vec{q},t), \tag{11}$$

where

$$V_{mn} = \delta_{m,n}A_{2m} + \delta_{m-1,n}B_{2m-1} + \delta_{m+1,n}B_{2m+1}. \tag{12}$$

Notice that (11) describes a set of coupled harmonic oscillators coupled by "springs" with spring constants given by $V_{m,n}$. This result shows the clear collective nature of the Fermi surface modes.

We can also encode the evolution of the system with the methods of path integrals[31] in the real time partition function

$$Z = \int Da^* Da e^{iS}, \tag{13}$$

where we also perform a Fourier transform in time

$$S = \sum_{\vec{q},m,n} \int d\omega a^*_{2m}(\vec{q},\omega)\tilde{V}_{mn}a_{2n}(\vec{q},\omega) \tag{14}$$

with

$$\tilde{V}_{mn} = (v_F q)^2[s^2\delta_{mn} - V_{mn}], \tag{15}$$

$$s = \omega/(v_F q). \tag{16}$$

It is clear from these results that the propagator of the collective modes is essentially given by the inverse of the coupling matrix $\tilde{V}_{m,n}$, that is, $\tilde{V}^{-1}_{m,n}$. Therefore, the eigenmodes of $\tilde{V}$ are the collective excitations of the system and its eigenenergies determine the frequency of oscillations of these modes.

We can gain further insight into the problem by splitting the action (14) into two decoupled sectors ($S = S^+ + S^-$) by using the symmetric and antisymmetric combinations

$$a^\pm_m = \frac{1}{\sqrt{2}}(a_m \pm a_{-m}), \tag{17}$$

defined for $m \geq 0$. These correspond to the longitudinal (+) and transverse (−) modes in the sense of having even and/or odd parity with respect to the line defined by $\vec{q}$. Notice that the transverse modes have no density (i.e., $m=0$) component.

Formally, there can be an infinite number of Landau parameters F_m that parametrize the interactions in a Fermi liquid. In practice, however, only a few Landau parameters are taken into account.[29] In the case of the Pomeranchuk instability, we can concentrate on a single Landau parameter, that is, the most singular, since all the other parameters only provide trivial renormalizations of the Fermi liquid properties. Without lack of generality, let us assume that there are M Landau parameters in the description of a quantum liquid, that is, the F_m are zero for $m \geq M+1$. In this case, the matrix $\tilde{V}_{m,n}$ in (15) for $m,n \geq M+1$ (we call it the high-M block) can be written in the longitudinal and transverse sectors in terms of blocks of the form

$$\begin{pmatrix} s^2-\frac{1}{2} & -\frac{1}{4} & 0 & 0 & \cdots \\ -\frac{1}{4} & s^2-\frac{1}{2} & -\frac{1}{4} & 0 & \\ 0 & -\frac{1}{4} & s^2-\frac{1}{2} & -\frac{1}{4} & \\ 0 & 0 & -\frac{1}{4} & s^2-\frac{1}{2} & \\ \vdots & & & & \ddots \end{pmatrix}. \tag{18}$$

The matrix (18) (assumed to have size $N\times N$) has orthonormal eigenvectors of the form

$$\vec{\varphi}(k)=\sqrt{\frac{2}{N+1}}(\sin(k),\sin(2k),\sin(3k),\ldots,\sin(Nk))^T, \tag{19}$$

with respective eigenvalues $\lambda(s,k)=s^2-[1+\cos(k)]/2$ (the allowed values for the k's are $k=\pi n/(N+1)$, where $n=1\ldots N$). The block (18) is coupled to the mode M via two off-diagonal terms in $V_{m,n}$. Schematically, we have

$$\begin{pmatrix} & & & \vdots & \vdots & \\ & \text{low-}M\text{ block} & & 0 & 0 & \cdots \\ & & & B_{2M+1} & 0 & \cdots \\ \cdots & 0 & B_{2M+1} & & & \\ \cdots & 0 & 0 & \text{high-}M\text{ block} & & \\ & \vdots & \vdots & & & \end{pmatrix}. \tag{20}$$

Using the diagonalized form of the block matrix, it is not hard to integrate out the high-M modes. The end result is to change

$$A_M \to A_M+\delta A_M, \tag{21}$$

in the low-M block of $\tilde{V}_{mn}$, where

$$\delta A_M = 2B_{2M+1}^2 G(s) \tag{22}$$

and

$$G(s)=\frac{1}{N+1}\sum_k \frac{\sin^2(k)}{s^2-[1+\cos(k)]/2}. \tag{23}$$

From this result, we see that $G(s)$ as a function of complex s is analytic in the upper and lower half plane. It has a set of poles on the real axis between $s=-1$ and $s=1$. In the limit of $N\to\infty$, the poles become dense resulting in a branch cut. By taking $N\to\infty$, $G(s)$ can be expressed as an integral

$$G(s)=\int_{-\pi}^{\pi}\frac{dk}{\pi}\frac{\sin^2(k)}{2s^2-1-\cos(k)}. \tag{24}$$

Taking s to lie away from the branch cut, one can turn the integral into a contour integral around the unit circle centered at the origin. The resulting integral is easily evaluated for $|s|>1$ by the residue theorem and can be analytically continued to $|s|<1$ with the result

$$G(s)=-2+4s^2-4\sqrt{s^4-s^2}. \tag{25}$$

So that for real s and $|s|>1$,

$$G(s)=-2+4s^2-4|s|\sqrt{s^2-1}, \tag{26}$$

while for real s and $|s|<1$,

$$G(s\pm i0^+)=-2+4s^2\mp 4is\sqrt{1-s^2}. \tag{27}$$

Hence, the final result of integrating out the high angular momentum modes is the introduction of a shift in the Landau parameters with the introduction of an imaginary part when $|s|=|\omega|/(qv_F)<1$, which is nothing but the Landau damping. By only keeping the low-M modes has the time reversal invariance been broken.

Since what we have done is a reformulation of the Landau theory, it should give the same result as the conventional approach. A simple check amounts to taking only $F_0\neq 0$ and integrate out all modes except the zeroth one in the longitudinal sector, that is, $M=0$. The well-known zero sound mode should then come out as the zero of the only remaining matrix element (which is equivalent to having a pole in the propagator). Explicitly, we obtain

$$\frac{s^2}{2}-\frac{A_0+2\delta A_0}{2}=0. \tag{28}$$

Substituting the expression for A_0, B_1, and $G(s)$, we get an undamped solution for $F_0>0$ with

$$|s|=\frac{1+F_0}{\sqrt{(1+F_0)^2-F_0^2}}, \tag{29}$$

which is the dispersion of the zero mode in two dimensions, e.g., see Ref. 19.

III. LIFETIME EFFECTS CLOSE TO A POMERANCHUCK QUANTUM CRITICAL POINT

In this section, we investigate the effect of the proximity of a Pomeranchuk quantum critical point on the single particle and collective modes of a Fermi liquid. Using the collision integral formalism, we study the quasiparticle lifetime via the Bethe-Salpeter equation close to the quantum critical point. We show that the critical point associated with the quantum fluctuations change substantially the quasiparticle lifetime without affecting the damping of the collective modes at low angular momentum.

A. Single-particle states

We follow the usual procedure and study the lowest order process, i.e., quasiparticle-quasiparticle scattering. Then the expression for the collision integral becomes[29]

$$\begin{aligned} I[n_{\vec{p}}] = 2\pi \sum_{\vec{p}_2\sigma_2} {\sum_{\vec{p}_3\sigma_3}}' \sum_{\vec{p}_4\sigma_4} & |\langle 34|t|12\rangle|^2 \\ & \times \delta_{\vec{p}_1+\vec{p}_2,\vec{p}_3+\vec{p}_4}\delta_{\sigma_1+\sigma_2,\sigma_3+\sigma_4}\delta(\epsilon_1+\epsilon_2-\epsilon_3-\epsilon_4) \\ & \times [n_3n_4(1-n_2)(1-n_1)-n_1n_2(1-n_3)(1-n_4)], \end{aligned} \tag{30}$$

which describe scattering processes in and out of the state $\vec{p}$ which is labeled 1. The spin information can be encoded into the the definition of the matrix element $|\langle t\rangle|^2$ as described in Ref. 29, so we drop the spin information from now on (or assume a spin-independent scattering amplitude).

To make progress, we linearize the collision integral in the deviations δn_i from (local) equilibrium n_i^0. Moreover, we employ the relaxation time approximation, which amounts to keeping only the term proportional to δn_1

$$I = -\frac{1}{\tau_{\vec{p}}}\delta n_{\vec{p}}, \tag{31}$$

where

$$\frac{1}{\tau_{\vec{p}}} = 2\pi \sum_{\vec{p}_2} {\sum_{\vec{p}_3,\vec{p}_4}}' |\langle 34|t|12\rangle|^2 \delta_{\vec{p}_1+\vec{p}_2,\vec{p}_3+\vec{p}_4}\delta(\epsilon_1+\epsilon_2-\epsilon_3-\epsilon_4) \times [n_3^0 n_4^0(1-n_2^0)+n_2^0(1-n_3^0)(1-n_4^0)]. \tag{32}$$

One can do better than the relaxation time approximation, but it is not necessary for our purposes.[29]

Now we introduce the momentum transfer $\vec{q}$ and the energy transfer ω via the identity $\delta(\epsilon_1+\epsilon_2-\epsilon_3-\epsilon_4)=\int d\omega\,\delta(\epsilon_1-\epsilon_3-\omega)\,\delta(\epsilon_2-\epsilon_4+\omega)$. We also specialize to nearly forward scattering. This is an approximation in three dimensions where it amounts to neglecting out of plane scattering. In two dimensions, it is essentially exact because of momentum conservation and the reduced phase space. This means that we restrict the sum over $\vec{q}$ to $q<q_c$, where $q_c \ll p_F$ is some cutoff momentum. Dropping the superscripts on the occupation numbers for clarity, we then get

$$\frac{1}{\tau} = \frac{2\pi}{[1-n(\epsilon)]}\int d\omega \sum_{\vec{q}}\sum_{\vec{p}'} |\langle 34|t|12\rangle|^2 \times \delta[v_F q\cos(\theta_p)-\omega]\delta[v_F q\cos(\theta_{p'})-\omega] \times [1-n(\epsilon-\omega)]n(\epsilon')[1-n(\epsilon'+\omega)]. \tag{33}$$

This form is convenient when $\epsilon>0$, for $\epsilon<0$, one can instead exchange $n\leftrightarrow 1-n$ everywhere. The end result is even in ϵ.

To proceed, we need an expression for the matrix element. This one can get from the solution of the Bethe-Salpeter equation, which describes repeated scattering of a quasiparticle-quasihole pair

$$t_{\vec{p}\vec{p}'}(\vec{q},\omega+i\eta) = f_{\vec{p}\vec{p}'} - \sum_{\vec{p}''\neq\vec{p}'} f_{\vec{p}\vec{p}''}\frac{\vec{q}\cdot\nabla_{p''}n_{p''}^0}{\omega+i\eta-\vec{q}\cdot\vec{v}_{\vec{p}''}} t_{\vec{p}''\vec{p}'}(\vec{q},\omega+i\eta).$$

The formalism for solving the Bethe-Salpeter equation in three dimensions can be found in Ref. 32, for example. Here we outline the procedure in two dimensions.

First, one expands t in Fourier components

$$t_{\vec{p}\vec{p}'}(\vec{q},\omega+i\eta) = \sum_{m,m'} t_{m,m'}(\vec{q},\omega+i\eta)e^{i(m\theta_p - m'\theta_{p'})}, \tag{34}$$

where the angles θ are measured with respect to $\vec{q}$. A matrix equation for $t_{m,m'}$ can be derived from the orthogonality between the Fourier components

$$(1+F_m)t_{m,m'} = \delta_{m,m'}f_m + F_m\sum_n K(n-m)t_{n,m'}, \tag{35}$$

where

$$K(n) = \int_0^{2\pi}\frac{d\theta}{2\pi}\frac{s}{s-\cos(\theta)}e^{in\theta}, \tag{36}$$

and

$$s = \frac{\omega+i\eta}{v_F q}. \tag{37}$$

Explicit expressions for $K(n)$ can be found in Ref. 19. Given the values of $\{f_n\}$, one can solve (35) to get $t_{m,m'}$.

Close to a Pomeranchuk quantum critical point when one of the Landau parameters reaches a critical value, we can keep only the relevant Landau parameter (see Fig. 1). In this paper, we study the "nematic" critical point and choose to keep only $F_2=F_{-2}$ nonzero, in which case it is straightforward to solve for t. The only nonzero components are

$$t_{2,2} = t_{-2,-2} = \frac{1+F_2-F_2K(0)}{D}f_2, \qquad t_{2,-2} = t_{-2,2} = \frac{F_2K(4)}{D}f_2, \tag{38}$$

where

$$D = [1+F_2-F_2K(0)]^2 - [F_2K(4)]^2. \tag{39}$$

Now that we have an explicit expression for $\langle t\rangle$ we can go back to the expression for the lifetime in (33), turn the sums into integrals and perform the angular integrals with the result

$$\frac{1}{\tau} = \frac{N(0)L^2}{(2\pi v_F)^2[1-n(\epsilon)]}\int d\omega[1-n(\epsilon-\omega)] \times \int d\epsilon' n(\epsilon')[1-n(\epsilon'+\omega)]\int_{|\omega|/v_F}^{q_c}\frac{dq}{q}|t(s,q)|^2, \tag{40}$$

where

$$|t(s,q)|^2 = \frac{16}{1-s^2}[1-8s^2(1-s^2)]|t_{2,2}+t_{2,-2}|^2 + 512s^4(1-s^2)(|t_{2,2}|^2+|t_{2,-2}|^2). \tag{41}$$

The first term in (42) gives the dominating contribution (from small s) so we only keep it in what follows. Notice that due to the integration limits on q in (40) we have $s\leqslant 1$ and the singularity near $s=1$ is removed by the behavior of $|t_{2,2}+t_{2,-2}|^2$.

Usually, i.e., for a not-too-strong attractive interaction, it is a good approximation to take t to only depend on s. Close to the critical point, however, one must also include a q dependence in t, explicitly we take

$$1 + F_2 = \delta + \kappa(q/q_c)^2, \tag{42}$$

where $\kappa \propto \partial^2 F_2/\partial q^2$. Now we can trade the integral over q to one over s. The integral is then dominated by the small-s contribution and it is possible to approximate the integral by a scaling argument. For small s, we have

$$t_{2,2} + t_{2,-2} = \frac{f_2}{1 + F_2 - F_2[K(0) + K(4)]},$$

$$\approx \frac{f_2}{\delta + \kappa(q/q_c)^2 + 2iF_2 s},$$

$$\approx \frac{f_2 s^2}{\delta s^2 + (\omega/\omega_c)^2 - 2is^3}. \tag{43}$$

Using (43), the q integral in (40) can be written as

$$16|f_2|^2 \int_{|\omega|/v_F q_c}^{1} ds \frac{s^3}{[\delta s^2 + (\omega/\omega_c)^2]^2 + 4s^6}. \tag{44}$$

For $\delta \to 0$ (that is, at the quantum critical point), we scale $s^3 \propto \omega^2$ so that the integral becomes $\propto \omega^{-4/3}$. Performing the remaining integrals at $T=0$, one gets

$$\frac{1}{\tau} \propto |\epsilon|^{2/3}, \tag{45}$$

leading to a very anomalous energy dependence of the quasiparticle lifetime.[2,12] For finite δ (away from the quantum critical point, inside of the quantum disordered regime), we should recover the standard Fermi liquid result. By scaling $s \propto \omega$, we get

$$\frac{1}{\tau} \propto \epsilon^2 \ln(\omega_c/\epsilon)^2 + \text{const} \times \epsilon^2, \tag{46}$$

which is the Fermi liquid result.[12] The above results can be summarized by the scaling form of the quasiparticle lifetime that can be obtained using the contour integral trick of Pethick[32] that we give in the appendix. The general result for the quasiparticle lifetime is

$$\frac{1}{\tau} = \text{const} \times \epsilon^2 + 2\omega_c^2(\epsilon/\omega_c)^{2/3} F\left[\left(\frac{3}{\delta}\right)^3 \left(\frac{\epsilon}{\omega_c}\right)^2\right], \tag{47}$$

where $F(\xi)$ is a scaling function such that $F(\xi) \approx 1$ for $\xi \gg 1$ and $F(\xi) \approx -\xi^{2/3} \log(\xi)$ for $\xi \ll 1$, reproducing the results of Eqs. (45) and (46). A similar crossover formula (for the imaginary part of the self-energy) in the case of a ferromagnetic quantum critical point was found by Chubukov *et al.*[33,34] Notice that our results can be generalized for any kind of Pomeranchuk quantum critical point given the critical Landau parameter. The result of this lowest order approximation is thus that close to the nematic critical point in two dimensions (2D) the lifetime of the quasiparticle dominates over the energy and the concept of the usual quasiparticle is not well defined. To see the real fate of the quasiparticle, one must include higher order scattering processes or use a more microscopic theory. Nevertheless, the calculation in this section signals the breakdown of normal Fermi liquid behavior close to a Pomeranchuk quantum critical point.

Studies by other groups of similar models also find a breakdown of Fermi liquid theory at the critical point. The $\epsilon^{2/3}$ behavior in two dimensions has also been found before in related systems, e.g., see Refs. 2, 12, and 34–36.

B. Collective modes

To calculate the relaxation time for collective modes, one must keep in mind that inside the linearized collision integral there are terms of the form $\delta n_{\vec{p}} - \delta n_{\vec{p}+\vec{q}}$ where the second term is neglected in the usual relaxation time approximation. The existence of the second term makes collisions less effective in the damping of collective modes. More explicitly, linearizing the collision integral (30) for the ν's in (2), we get

$$I'[\nu] = -\frac{2\pi}{1 - n(\epsilon_1)} \sum_{\vec{p}_2} {\sum_{\vec{p}_3,\vec{p}_4}}' |\langle 34|t|12\rangle|^2 \times \delta_{\vec{p}_1+\vec{p}_2,\vec{p}_3+\vec{p}_4} \delta(\epsilon_1 + \epsilon_2 - \epsilon_3 - \epsilon_4) \times n_2(1 - n_3)(1 - n_4)[\nu_1 - \nu_3 + \nu_2 - \nu_4]. \tag{48}$$

Expanding the ν's in Fourier components, as in Sec. II, and projecting the collision integral, we get

$$\frac{1}{\tau_m} = m^2 \frac{1}{\tau'}, \tag{49}$$

where $1/\tau'$ is given by (40) with the substitution $\int dq \to \int dq q^2/2p_F^2$. To get this result, we have neglected the term $\nu_2 - \nu_4$ since this term is typically of the same size as the first (i.e., $\nu_1 - \nu_3$) and just gives a different prefactor in the calculation of $1/\tau$ of order 1. More important is the expansion of $\cos[m(\theta + q/p_F)]$ in powers of mq/p_F, which is only valid for $m \ll p_F/q_c \approx N_\Lambda$. The terms with odd powers of q vanish upon performing the angle averages.

For these modes, the simple scaling evaluation of the integrals gives

$$\frac{1}{\tau_m} \propto \left(\frac{m}{N_\Lambda}\right)^2 \epsilon^{4/3}, \tag{50}$$

for $\delta \to 0$ and

$$\frac{1}{\tau_m} \propto \left(\frac{m}{N_\Lambda}\right)^2 \epsilon^2, \tag{51}$$

for finite δ. From this we can conclude that the collisionless approximation is good for self-driven modes ($\omega \approx \epsilon$) when $(m/N_\Lambda)^2 \ll 1$. Note that this implies that it makes sense to talk about, e.g., the $M=2$ mode also in the critical regime. The high-M modes are more sensitive to low-q scattering. For these it is not possible to relate $\cos[m(\theta + q/p_F)]$ in any simple way by $\cos(m\theta)$, and hence the single particle result above should be a good approximation. This means that the high-M modes are heavily damped in the critical regime. For intermediate M, we expect a crossover between the two lim-

iting cases. Also, note that in all cases (single particle or collective), there is a prefactor of $1/(k_F v_F) \approx 1/\epsilon_F$ and the correct powers of ω_c to make the dimensions correct.

IV. QUANTUM CRITICAL REGIME—DISCUSSION

Recently, Yang[13] has argued in favor of a dynamical critical exponent of $z=2$ for the Pomeranchuk quantum critical point in 2D. This is in disagreement with, e.g., the results of Ref. 2. The claim that the other noncritical modes just renormalize the couplings in the critical theory is not obvious. In particular, the low-energy particle-hole fermion excitations which are reproduced within the bosonic formulation as linear combinations of the high angular components are neglected in Ref. 13. The calculation in Sec. II illustrates the effect of these modes in the collisionless regime and in a simpler setting than that of multidimensional bosonization, which requires a complicated Bogoliubov transformation to achieve similar results.[20]

Taking only F_0, F_1, and $F_2 \neq 0$, we can get the equations for the transverse and longitudinal modes from the result of Sec. II. If we integrate out all the modes that we are not interested in (i.e., $M \geqslant 3$), the resulting equations for the modes are

$$s_-^2 - A_2 - \delta A_2 = 0, \tag{52}$$

for the transverse $(-)$ mode and

$$\begin{vmatrix} (s_+^2 - A_0)/2 & -B_1 \\ -B_1 & s_+^2 - A_2 - \delta A_2 \end{vmatrix} = 0, \tag{53}$$

for the longitudinal $(+)$ mode. From these equations, one can extract the leading scaling behavior to be $s_-^2 \sim [\delta + \kappa (q/q_c)^2]/4$ and $s_+ \sim -i[\delta + \kappa (q/q_c)^2/2]$, assuming that only the d channel goes critical. So the transverse mode has a dynamical critical exponent of $z=2$ and the longitudinal $z=3$ in agreement with Refs. 2 and 37. The longitudinal mode, being slower, dominates the critical behavior.

As we found in Sec. III, the collisionless behavior is only a good approximation for low-M modes in the critical regime. The question that arises is what would be the effect of collisions on the modes that participate in the Landau damping. The coupling of the $M=2$ mode to the rest of the system still results in a damped motion. Generally, one can integrate out the modes with $M \geqslant 3$ with the same result as above in (52) and (53). The difference is that we no longer have an explicit expression for δA_2, since the approximations of Sec. III break down. However, we still must have $\delta A_2 = 2B_3^2 U(q,\omega)$, where $U(q,\omega)$ is a function such that $U(q,0) = U(s=0) = -2$. The calculation of $U(q,\omega)$ is beyond of the scope of this work. Nevertheless, if $U(q,\omega)$ is known, the longitudinal mode equation becomes

$$s_+^2 - \frac{1}{8}\left[\delta + \kappa\left(\frac{q}{q_c}\right)^2\right] \times [U(q,\omega) - U(q,0)] = 0. \tag{54}$$

If we assume that $U(q,\omega) - U(q,0) \sim -i\omega^\alpha/q$ in the $\omega \to 0$ limit, the dynamical critical exponent of the mode is $z = 3/(2-\alpha)$. This result allows us to put bonds on the value of the dynamical exponent since we expect $2/3 \leqslant \alpha \leqslant 1$ in such a way that $9/4 \leqslant z \leqslant 3$. One can speculate that the modes that are responsible for the damping of the critical mode, having very low energy, are weakly damped since there is not much for them to decay into because of the small phase space. This would hint toward $z=3$ even when collisions are taken into account. Another argument in favor of $z=3$ is that the collisionless limit is a good approximation when the mean free path is much longer than the wavelength, which translates into $1/\tau \ll q v_F$.[26] Taking $1/\tau \sim \omega^{2/3}$ and $\omega \sim q^3$, the inequality is clearly satisfied as $q \to 0$, indicating the validity of the collisionless limit.

A questionable thing with the whole approach close to the critical point is whether the calculations are self-consistent. The simplest particle-particle scattering process gave a lifetime that is proportional to $\epsilon^{2/3}$. The same result can be obtained from a one-loop calculation of the imaginary part of the fermion self-energy, which describes similar physics. Taking this seriously would invalidate the quasiparticle picture and, hence, make the whole ground for the calculation questionable. Notice, however, that although the quasiparticle description seems to break down, the low angular momentum collective spectrum remains essentially unchanged [except by the damping of the collective modes given by the subleading contributions (50) and (51)].

One further circumstantial evidence for a breakdown of the quasiparticle near the quantum critical point comes from the theory of multidimensional bosonization.[20] The calculations are tractable for an isotropic interaction, the connection to the notations in this paper is $Ng = F_0$. The Bogoliubov transformation used to diagonalize the system connects different patches i and l of the Fermi surface according to the matrices

$$\mathcal{M}_{il} = \delta_{il} + \frac{1}{N}\frac{Ng}{1+Ng}\frac{\sqrt{s_i s_l}}{s_l - s_i},$$
$$\mathcal{N}_{il} = -\frac{1}{N}\frac{Ng}{1+Ng}\frac{\sqrt{s_i s_l}}{s_l + s_i}, \tag{55}$$

where $s_i = \cos(\theta_i)$. Away from the critical point, the second term in $\mathcal{M}$ represents the dressing of the particle by interactions, and $\mathcal{N}$ is always small. Close to the critical point, however, the factor $Ng/(1+Ng) = F_0/(1+F_0)$ blows up when $F_0 \to -1$ and a broad dressing cloud dominates the excitations, also $\mathcal{N}$ is no longer negligible and is an important part of the excitation. Moreover the expression for the quasiparticle residue $Z = \exp\{-\frac{1}{4}(F_0/1+F_0)^2\}$ shows that it vanishes exponentially as the transition is approached. The calculation for the d-channel Pomeranchuk transition is much more complicated because the kernel of the integral equation is no longer a product kernel. We expect similar conclusions, however. Because the Fermi liquid theory breaks down close to the critical point, a calculation within the Fermi liquid theory itself does not make much sense there, except for indicating its own breakdown. The correct description is not in terms of the standard Fermi liquid theory, and it would be very interesting to know what it is.

In summary, we have presented an alternative approach for the study of Landau damping in Fermi liquids based on the trace over high angular momentum modes. We find that this approach reproduces well-known results in the literature and allows for the study of the quasiparticles and collective modes close to a Pomeranchuk quantum critical point. Although our approach is general and can be used to study any instability of the Pomeranchuk type, we focused on the problem of the electronic nematic that has been discussed in the literature recently.[2,13,38–40] We found that the quasiparticle lifetime becomes anomalously short at the quantum critical point leading to a breakdown of the Fermi liquid description.

ACKNOWLEDGMENTS

This work was supported through NSF Grant No. DMR-0343790.

APPENDIX: CONTOUR INTEGRAL

The formula for the inverse lifetime can be written (at $T=0$ and $\epsilon>0$ for simplicity)

$$\frac{1}{\tau}=\frac{1}{2\pi v_F k_F}\int_0^{\epsilon} d\omega\omega\left[-c_s^{<}+N^2(0)\int_0^1\frac{ds}{s}|t(s,q)|^2\right]$$
$$\equiv\frac{1}{2\pi v_F k_F}\int_0^{\epsilon} d\omega\omega(-c_s^{<}+I_s). \tag{A1}$$

To get this we have added and subtracted the contribution of the $N^2(0)$ times the s-integral down to zero, this is approximately

$$c_s^{<}=16\int_0^{|\omega|/v_F q_c} ds\frac{s^3}{[\delta s^2+(\omega/\omega_c)^2]^2+4s^6}$$
$$=16\int_0^1 dx\frac{x^3}{[\delta x^2+\kappa]^2+4x^6\left(\frac{\omega}{v_F q_c}\right)^2}, \tag{A2}$$

which to leading order is a constant (for finite κ), note that we set $F_2=-1$ everywhere where it is not dangerous to do so.

Now we can follow Pethick[32] and turn the integral in I_s into two contour integrals, the difference being that in our case the two contours have different orientation

$$I_s=-\frac{2N(0)}{i}\oint_c\frac{ds}{s^2}\frac{1}{\sqrt{1-s^2}}\frac{1}{1+4s^2(s^2-1)}(t_{2,2}+t_{2,-2}), \tag{A3}$$

using the fact that the integral is dominated by small values of s, one can rewrite it as

$$I_s=-8\mathcal{P}\int_0^{\infty} dz\frac{1}{\delta z^2-(\omega/\omega_c)^2+2z^3}, \tag{A4}$$

after a deformation of the contour to the imaginary axis. Scaling $z=\delta x/6$, we get

$$I_s=-16\left(\frac{3}{\delta}\right)^2\mathcal{P}\int_0^{\infty} dx\frac{1}{x^3+3x^2-4Y}, \tag{A5}$$

where $Y=(3/\delta)^3(\omega/\omega_c)^2$. Finally, performing the remaining integral, we obtain

$$\frac{1}{\tau}=\frac{1}{2\pi v_F k_F}\left[-c_s^{<}\frac{\epsilon^2}{2}\right.$$
$$\left.+2\epsilon^2\left(\frac{\omega_c}{\epsilon}\right)^{4/3}\frac{1}{\xi^{1/3}}\mathcal{P}\int_1^{\infty} dx\ln\left[\frac{x^3-3x+2(1-2\xi)}{x^3-3x+2}\right]\right], \tag{A6}$$

which is of the desired scaling form with $\xi=(3/\delta)^3(\epsilon/\omega_c)^2$.

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