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Semi-Meissner state and neither type-I nor type-II superconductivity in multicomponent superconductors

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Traditionally, superconductors are categorized as type I or type II. Type-I superconductors support only Meissner and normal states, while type-II superconductors form magnetic vortices in sufficiently strong applied magnetic fields. Recently there has been much interest in superconducting systems with several species of condensates, in fields ranging from condensed matter to high energy physics. Here we show that the classification into types I and II is insufficient for such multicomponent superconductors. We obtain solutions representing thermodynamically stable vortices with properties falling outside the usual type-I/type-II dichotomy, in that they have the following features: (i) Pippard electrodynamics, (ii) interaction potential with long-range attractive and short-range repulsive parts, (iii) for an n-quantum vortex, a nonmonotonic ratio $E(n)/n$ where $E(n)$ is the energy per unit length, (iv) energetic preference for nonaxisymmetric vortex states, "vortex molecules." Consequently, these superconductors exhibit an emerging first order transition into a "semi-Meissner" state, an inhomogeneous state comprising a mixture of domains of two-component Meissner state and vortex clusters.

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The formation of vortices in type-II superconductors subjected to a magnetic field is one of the most remarkable phenomena occurring in condensed matter. In all type-II superconductors (i.e., superconductors where the GL parameter, which is the ratio of the magnetic field penetration length to the coherence length,[1] is $\kappa > 1/\sqrt{2}$) these vortices share a set of properties: an n-quantum vortex is unstable with respect to decay into n one-quantum vortices, two vortices have purely repulsive interaction, and invasion of vortices under normal conditions is a second order phase transition characterized by a critical value of the external magnetic field H_{c1}.

Vortices as solutions of the GL equations also exist formally in a type-I superconductor ($\kappa < 1/\sqrt{2}$), but these vortices are thermodynamically unstable. The special case $\kappa = 1/\sqrt{2}$ is also very interesting since, at this value of κ, vortices do not interact.[2,3] One should note, however, that in real life systems the situation is more complicated; experiments[4] show that in certain materials with $\kappa \approx 1/\sqrt{2}$ there might exist a tiny attractive force between vortices at a certain distance. Such an interaction was reproduced in a modified one-component GL model with additional terms in the regime[5] $\kappa \approx 1/\sqrt{2}$. We should also mention a long-range van der Waals-type vortex attraction in layered systems produced by thermal fluctuations or disorder.[6]

Besides superconductivity, the vortex concept has a direct counterpart in high energy physics, called the Nielsen-Olesen string. Such strings have been considered in cosmology[7] where they are expected to form during a symmetry breaking phase transition in the early universe. There also exists a similar division of semilocal cosmic strings in the Higgs doublet model into types I and II.[8]

Recently, multicomponent superconducting systems have attracted increasing interest in areas ranging through metallic superconductors, hydrogen in extreme conditions, and color superconductivity in dense QCD.[9–11] Below we consider a generic two-component superconductor (TCS), showing that it allows a novel type of thermodynamically stable vortices, whose electrodynamics is of Pippard type with respect to one of the order parameters and which have nonmonotonic interaction energy for a wide range of parameters as an intrinsic feature. We will show that, as a result of this, such a TCS displays very unconventional magnetic properties which have no counterparts in single-component systems, and do not fall into either the standard type-I or type-II classes.

In the simplest case, the TCS (related to the two-Higgs model[12]) can be described by the following GL energy density:

$$F = \frac{\hbar^2}{4m_1}\left|\left(\nabla + i\frac{2e}{\hbar c}\mathbf{A}\right)\Psi_1\right|^2 + \frac{\hbar^2}{4m_2}\left|\left(\nabla + i\frac{2e}{\hbar c}\mathbf{A}\right)\Psi_2\right|^2 + V(|\Psi_{1,2}|^2) + \eta[\Psi_1^*\Psi_2 + \Psi_2^*\Psi_1] + \frac{\mathbf{H}^2}{8\pi}, \quad (1)$$

where $\Psi_\alpha = |\Psi_\alpha| e^{i\phi_\alpha}$ and $V(|\Psi_{1,2}|^2) = \Sigma_{\alpha=1,2} - b_\alpha|\Psi_\alpha|^2 + c_\alpha/2|\Psi_\alpha|^4$. In (1), vortices with phase winding in only one field have logarithmically divergent energy per unit length if $\eta = 0$, and linearly divergent if $\eta \neq 0$.[9] Here we do not consider the effect of thermal fluctuations, so we can, without loss of generality, restrict attention to vortices with $\phi_1 - \phi_2 = const$ in the case $\eta = 0$. These have finite energy per unit length. Moving η from zero will merely change the core sizes of such vortices, so this is a trivial extension. The same applies to other possible additional potential terms in (1).

Equation (1) possesses three characteristic length scales: two coherence lengths $\xi_\alpha = \hbar/\sqrt{4m_\alpha b_\alpha}$ and the magnetic field penetration length $\lambda = (c/\sqrt{8\pi}e)[|\bar{\Psi}_1|^2/m_1 + |\bar{\Psi}_2|^2/m_2]^{-1/2}$,

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where $|\bar{\Psi}_\alpha| = \sqrt{b_\alpha / c_\alpha}$. It will also be convenient to define length parameters $\lambda_\alpha = c\sqrt{m_\alpha}/(\sqrt{8\pi}e|\bar{\Psi}_\alpha|)$, and thermodynamic critical magnetic fields for the *individual* condensates, $H_{ct(\alpha)} = \Phi_0/(2\sqrt{2}\pi\xi_\alpha\lambda_\alpha)$. When the individual condensate Ψ_1 is of type II, we denote its first and the second critical magnetic fields as $H_{c1(1)} = \Phi_0/(4\pi\lambda_1^2)[\ln(\lambda_1/\xi_1)+0.08]$ and $H_{c2(1)} = \Phi_0/(2\pi\xi_1^2)$, where Φ_0 is the magnetic flux quantum.

Let us consider the magnetic properties of a TCS in several regimes. In the simplest cases, when $\xi_1 \approx \xi_2 \gg \lambda$ and when $\xi_1 \approx \xi_2 \ll \lambda$, the magnetic properties of the TCS parallel those of single-component type-I and type-II superconductors correspondingly. However, we find that if one of the condensates has $\lambda_1/\xi_1 < 1/\sqrt{2}$ while the other has $\lambda_2/\xi_2 > 1/\sqrt{2}$, the TCS in an external field has a much richer phase diagram.

In the case when $\xi_1 \ll \lambda_1$ and $\xi_2 \gg \lambda_2$, and $H_{ct(2)}$, the thermodynamic critical magnetic field for the type-I condensate Ψ_2, is much higher than $H_{c2(1)}$ for the type-II condensate Ψ_1, the system undergoes a first order transition from a normal state immediately into a TCS state. This is because, at fields higher than $H_{ct(2)}$, the system does not allow nontrivial solutions of the linearized GL equation for Ψ_1, while, when the field is lowered below $H_{ct(2)}$, there appears a transition immediately into a TCS state. In this state the magnetic field is screened in the bulk of the sample largely due to surface current Ψ_2, and nothing can preclude the appearance of the second condensate Ψ_1 in the bulk of the sample, even if the applied field is much larger than $H_{c2(1)}$.

Now consider the regime occurring when $\lambda_1/\xi_1 > 1/\sqrt{2}$; $\lambda_2/\xi_2 < 1/\sqrt{2}$; $\xi_2 > \lambda$. Then a vortex solution should have an extended core associated with the condensate Ψ_2 which exceeds the penetration length. (The key question: whether such vortices can be thermodynamically stable will be answered below.) For a TCS, the vortex energy consists of the energies of the cores, the kinetic energy of the Meissner current, and the magnetic field energy. The magnetic field energy and the kinetic energy of the screening current are given by $F_m = (1/8\pi)\int d^3x\mathbf{H}^2 + (1/8\pi)\int d^3x\lambda_{\text{eff}}^2(\text{curl }\mathbf{H})^2$. Here we stress that, in the present case, $|\Psi_2(\mathbf{x})|^2$ varies slowly over the London penetration length λ. The magnetic field is screened at a distance from the core which is smaller than ξ_2. This means that only a depleted density of Cooper pairs of the condensate Ψ_2 participates in the screening of the magnetic field. Thus, one cannot use the London penetration length λ, but should introduce an effective penetration length $\lambda_{\text{eff}} = [1/\lambda_1^2 + 1/\tilde{\lambda}_2^2(\mathbf{x})]^{-1/2}$ where $\tilde{\lambda}_2(\mathbf{x}) = c\sqrt{m_2}/(\sqrt{8\pi}e|\Psi_2(\mathbf{x})|) > \lambda_2 = c\sqrt{m_2}/(\sqrt{8\pi}e|\bar{\Psi}_2|)$. In the case when $\lambda_1 \ll \xi_2$ and λ_1 is much smaller than the Pippard length of Ψ_2 we have $\lambda_{\text{eff}} \approx \lambda_1$ which corresponds to the situation when the magnetic field is screened mostly by condensate Ψ_1. On the other hand, in the case when $\xi_2 \gg \lambda_1 \gg \lambda_2 \gg \xi_1$ the magnetic field can be screened at the scale of the Pippard penetration length $\lambda = \lambda_2^P \approx (\lambda_2^2\xi_2)^{1/3}$ of the condensate Ψ_2 (this expression is valid when $\lambda_2^P \ll \lambda_1$). In the above expression for F_m, we cut off integrals at the distance ξ_1 from the center of the core in order to obtain an estimate of the vortex energy with logarithmic accuracy. Then the energy per unit length of a one-flux-quantum vortex is

$$E \approx \left(\frac{\Phi_0}{4\pi\lambda_{\text{eff}}}\right)^2 \ln\frac{\lambda_{\text{eff}}}{\xi_1} + \mathcal{V}_{c1} + \mathcal{V}_{c2}, \qquad (2)$$

where $\mathcal{V}_{c\alpha}$ are the energies of the cores per unit length which are of order of magnitude of (core size)$\times$(condensation energy). The estimate of the core energy can also be expressed as

$$\mathcal{V}_{c\alpha} \approx \frac{\pi\xi_\alpha^2 H_{ct(\alpha)}^2}{8\pi} = \frac{\Phi_0^2}{8\pi}\frac{e^2|\Psi_\alpha|^2}{c^2 m_\alpha} = \frac{1}{4}\left(\frac{\Phi_0}{4\pi}\right)^2\frac{1}{\lambda_\alpha^2}. \qquad (3)$$

Consequently, the energy of two cores is $\mathcal{V}_{c1}+\mathcal{V}_{c2} \approx (\Phi_0/8\pi\lambda)^2$. Let us now assume that such vortices are thermodynamically stable. (Below we demonstrate numerically that it is indeed the case.) Then a straightforward calculation of the field H_{c1}^0 at which it becomes energetically favorable to let a *single* vortex into the superconductor gives

$$H_{c1}^0 \approx \frac{\Phi_0}{4\pi}\left[\frac{1}{\lambda_{\text{eff}}^2}\ln\frac{\lambda_{\text{eff}}}{\xi_1}\right] + \frac{\Phi_0}{16\pi}\left[\frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^2}\right]. \qquad (4)$$

However this characteristic field strength cannot be interpreted as the first critical magnetic field as in type-II superconductors. In fact, in contrast to the type-II regime, invasion of vortices into a superconductor in the regime in question should be accompanied by a magnetization jump and be of first order. This is because the vortex has an extended core at length scale $\xi_2 > \lambda_{\text{eff}}$ which gives rise to attraction between vortices $U \propto K_0(\sqrt{2}r/\xi_2)$ for $r > \lambda_{\text{eff}}$, where K_0 is the Bessel function. The attraction originates in winning in condensation energy in the condensate Ψ_2 when outer cores overlap. As we shall see below, the interaction potential has a repulsive part at a shorter length scale of the order of λ_{eff}. Consequently, a lattice of vortices with a spacing determined by the minimum of the interaction potential is preferred over a system of widely separated vortices. So the energy of a system of n vortices in this regime is minimized when vortices spontaneously form *lattice clusters* with overlapping outer cores. We should observe that H_{c1}^0 in the present situation is larger than the thermodynamic critical magnetic field of the type-I condensate, $H_{ct(2)} = \Phi_0/(2\sqrt{2}\pi\xi_2\lambda_2)$. We stress that this does not mean that at the field of (4) the condensate Ψ_2 is completely depleted, however. This is because, when the applied field is close to H_{c1}^0, the field is mostly screened by the supercurrent of the condensate Ψ_1, which circulates along the sample's edge. Thus the vortex system is dilute, or, more precisely, the intervortex distance is only determined by the effective attraction. *So in contrast to the usual type-I* and *type-II behaviors, the TCS in this regime displays a first order transition into an inhomogeneous state consisting of clusters of vortices, where the order parameter Ψ_2 is depleted due to the overlap of outer cores*. So superconductivity in these vortex "droplets" is dominated by the order parameter Ψ_1. Since the vortex density depends on the applied field, while the intervortex distance is determined by the nonmonotonic interaction potential, there should be present, besides these clusters of vortices, domains of two-component superconductivity in the vortexless Meissner state. We call this the *semi-Meissner state*. The transition into the semi-Meissner state may be viewed as a vortex matter analog of the condensation of water droplets or a sublimation process in classical physics, with the external field playing the role of "pressure." At a higher value of an external field the system will transition from the semi-Meissner state to the regular Abrikosov lattice. This transition might be rather complicated because of the features of the vortex interaction potential. That is, the standard argument in favor of triangular lattice symmetry no longer holds. Different lattice symmetries for a given density of vortices are characterized by different numbers of neighbors and different nearest-neighbor distances, so one might construct a sequence of transitions between lattices of different symmetries as a function of external field strength. Again an analogy between the external field strength and the role of pressure in classical physics might be invoked.

The key question regarding the existence of this transition and the semi-Meissner state is whether vortices in this regime are stable not only topologically but also thermodynamically. Below we find an answer to this numerically.

For the purposes of numerical simulation of this system, it is convenient to rescale the fields. Let $\mathsf{A}=(2e/\hbar c)\mathbf{A}$, $\psi_\alpha = e/c\sqrt{(8\pi/m_\alpha)}\Psi_\alpha$. Further, we introduce some new parameters: $\mu_\alpha^2=(2\pi e^2 c^2/\hbar^2)m_\alpha^2 c_\alpha$, $u_\alpha^2=(4b_\alpha/\pi e^2 c^2 m_\alpha c_\alpha)$, so that the effective potential becomes $V_\alpha=(\mu_\alpha^2/8)(u_\alpha^2-|\psi_\alpha|^2)^2$. Then the free energy density in the case $\eta=0$ is

$$\frac{16\pi e^2}{\hbar^2 c^2}F = \frac{1}{2}|\nabla \times \mathsf{A}|^2 + \sum_{\alpha=1,2}\frac{1}{2}|(\nabla + i\mathsf{A})\psi_\alpha|^2 + V_\alpha. \quad (5)$$

In terms of the new parameters, the three natural length scales are the inverse masses of the photon and the two Higgs bosons, that is, $\lambda=(u_1^2+u_2^2)^{-1/2}$, $\xi_\alpha/\sqrt{2}=(\mu_\alpha u_\alpha)^{-1}$. This system supports radially symmetric solutions

$$\mathsf{A} = r^{-1}a(r)(-\sin\theta,\cos\theta), \qquad \psi_\alpha = \sigma_\alpha(r)e^{ni\theta}, \quad (6)$$

where a and σ_α are real and satisfy the boundary conditions $a(0)=\sigma_\alpha(0)=0$, $a(\infty)=-n$, $\sigma_\alpha(\infty)=u_\alpha$. This ansatz reduces the field equations to an ODE system,

$$a'' - \frac{a'}{r} - (n+a)(\sigma_1^2+\sigma_2^2) = 0,$$

$$\sigma_\alpha'' + \frac{\sigma_\alpha'}{r} - (n+a)^2\frac{\sigma_\alpha}{r^2} + \frac{\mu_\alpha^2}{2}\sigma_\alpha(u_\alpha^2-\sigma_\alpha^2) = 0, \quad (7)$$

solutions of which may be found by means of a shooting method similar to that used in Ref. 13 (see remark[14]). Figures 1(a)–1(c) were generated using this scheme, employing a fourth-order Runge-Kutta method with variable r step for the numerical integration, and with $r_0=0.01$, $r_1=2$, $r_\infty=8$. The regime of most interest is the one where $\xi_1/\sqrt{2}<\lambda<\xi_2/\sqrt{2}$. To explore the dynamical issues of interest, we would like to make the disparity between ξ_1 and ξ_2 as extreme as possible. There is a numerical limit to the disparity of length scales we can achieve (see remark[15]), but the phenomena of interest can be demonstrated within that limit.

Figure 1(a) shows the profile functions of a single two-component vortex in the neither type-I nor type-II regime, where the electrodynamics is of Pippard type in respect of component Ψ_2. Note the disparity in healing lengths of the two condensate fields. Figure 1(b) shows the energy of a single two-component vortex as a function of ξ_2, with λ and ξ_1, fixed, normalized by E_0, the energy of a *one-component* vortex with the same λ and $\xi=\xi_1$. Figure 1(c) shows the energy per vortex of $n=1$, $n=2$ and $n=3$ cocentered vortices at fixed ξ_1, ξ_2, λ, as a function of $|\bar{\Psi}_1|/|\bar{\Psi}_2|$, normalized by E_0, the energy of a single one-component vortex with the same λ and $\xi=\xi_1$. When $|\bar{\Psi}_1|=|\bar{\Psi}_2|$, the $n=1$ vortex is favored, but as $|\bar{\Psi}_1|/|\bar{\Psi}_2|$ increases, first the $n=2$ vortex then the $n=3$ vortex become energetically favored. This is a very interesting property since such an effect has no counterpart in type-II or type-I one-gap superconductors ($N=1$ Abelian Higgs model). We should note that the $n=2,3$ cocentered solutions are unstable with respect to formation of nonaxisymmetric n-quantum vortex molecules, as we shall now see.

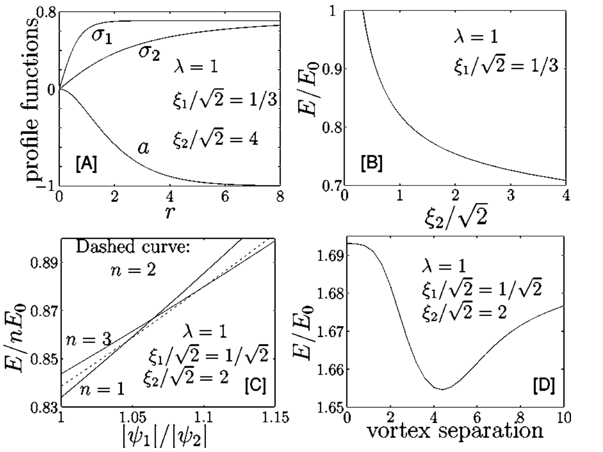

FIG. 1. (a) Numerical solution for the neither type-I nor type-II vortex. (b) The energy of a double-core vortex as a function of the disparity of the two coherence lengths. (c) Nonmonotonic ratio of energy per unit length to number of flux quanta for axisymmetric n-quantum vortices. (d) Nonmonotonic interaction potential for double-core vortices.

To compute the interaction energy of a vortex pair [shown in Fig. 1(d)], one must go beyond the radially symmetric ansatz and resort to a lattice minimization method (see remark[16]). Note that the intervortex force is attractive at long range but has a repulsive core, as predicted. The only stable static two-vortex we find in this regime has nonzero vortex separation and broken axial symmetry. For an isolated one-quantum vortex, we find numerically that when $\xi_1 \ll \lambda \ll \xi_2$ and λ_1 is much smaller than the Pippard length of the condensate Ψ_2, the vortex energy per unit length tends asymptotically to the energy of a vortex in a single condensate Ψ_1 plus the core energy of the vortex of condensate Ψ_2. Noting that the thermodynamical critical magnetic field of Ψ_1 is proportional to $(\xi_1\lambda_1)^{-1}$, while the core energy of the vortex in Ψ_2 is proportional to λ_2^{-2}, this proves the thermodynamical stability of the neither type-I nor type-II vortices whose electrodynamics is of Pippard type, and the existence of the semi-Meissner state.

In conclusion, superconductivity in multicomponent systems has recently attracted much interest in the physics of condensed matter and beyond. Here we show that, in contrast to ordinary superconductors, multicomponent systems allow for thermodynamically stable vortices even in the Pippard regime. Moreover, their magnetic properties in a certain range of parameters do not allow one to classify such a superconductor as type-II or type-I. Rather, it should legitimately be placed in a separate class. Such a type of superconductivity should be relevant for a variety of systems. For example, it is well known that disparity of coherence lengths occurs naturally in two-band superconductors, e.g., MgB_2 and $Mg_{1-x}Al_xB_2$, while type-II superconductors not belonging to the regime we consider in this paper have significant disparity in coherence lengths.[17] Analogous situations might appear in mixtures of condensates with different pairing symmetries. Another candidate for this type of superconductivity is the projected liquid metallic state of hydrogen[10] where this regime is expected to be realized under certain conditions. A similar situation might also occur in the color superconducting state in quark matter.[11] Certain features of the considered state should be preserved and might be experimentally accessed for vortex stacks[18] in layered systems when one layer is type II and another is strongly type I.

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[14] Since the system is singular at $r=0$ and $r=\infty$, we shoot forwards from $r=r_0$, small, and backwards from $r=r_\infty$, large, applying a matching condition at an intermediate radius r_1. The shooting parameters are the Taylor coefficients s_i for the small r fields, and q_i, the magnetic dipole and scalar monopole charges of the large r fields (Ref. 13): small r, $a(r)=s_0r^2+\cdots$; $\sigma_\alpha(r)=s_\alpha r^n+\cdots$ large r, $a(r)=-n+q_0rK_1(\sqrt{u_1^2+u_2^2}r)+\cdots$, $\sigma_\alpha(r)=u_\alpha-q_\alpha K_0(\mu_\alpha u_\alpha r)+\cdots$. The problem is thus reduced to zeroing an $\mathbb{R}^6$ valued function (the mismatch at r_1 of $a,\sigma_1,\sigma_2,a',\sigma_1',\sigma_2'$) on $\mathbb{R}^6$, which may be solved by a Newton-Raphson method.

[15] When ξ_2 is large, the field ψ_2 has a long range, so it approaches its vacuum value only slowly. We must thus choose r_∞ to be large. But then if ξ_1 is small, $u_1-\sigma_1(r_\infty)$ is tiny, so large changes in q_1 are required to control the shots. Hence, the Newton-Raphson scheme is close to singular, and the numerical method becomes unstable.

[16] The idea is to fix a value of the vortex separation ρ, then minimize $E=\int_{\mathbb{R}^2}F$ subject to the constraint that $\psi_\alpha(-\rho/2,0)=\psi_\alpha(\rho/2,0)=0$, $\alpha=1,2$. We chose to place the system on a regular spatial lattice (spacing $\delta x=\delta y=h$) with forward differences replacing the partial derivatives occurring in F, then to solve the constrained gradient flow equation for E using the Euler method with time step δt, until the configuration converged to a minimum (up to numerical tolerance, that is, until the energy loss per time step is less than E_{tol}). Figure 1(d) was produced using this scheme with $h=0.1$ and $\delta t=0.01$ on a 201×101 grid, with $E_{tol}=10^{-5}$, which should be compared with a typical two-vortex energy (in natural units) of 2π. Again, practical considerations limit the length-scale disparity we can handle. If ξ_1 is small, we need h to be small so as to resolve the core structure of the ψ_1 field. But then if ξ_2 is large, we need a very big grid to ensure that the ψ_2 field experiences no significant boundary effects.

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