

Quasiclassical frustration

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We study the dissipative properties of a harmonic oscillator subject to two independent heat baths, one of which couples to its position and the other one to its momentum. This model describes a large spin impurity in a ferromagnet. We find that some effects of the two heat baths partially cancel each other. Most notably, oscillations may remain underdamped for arbitrarily strong coupling. This effect is a direct consequence of the mutually conjugate character of position and momentum. For a single dissipative bath coupled linearly to both position and momentum, no underdamped regime is possible for strong coupling. The dynamics of purity loss for one and two wave packets is also investigated.

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The harmonic oscillator provides a scenario where the physics of a quantum particle coupled to a heat bath can be investigated analytically.^{1–3} Thus its study may shed light on physical effects that have been found in less tractable models. Recently, Novais *et al.*⁴ have investigated, by means of perturbative renormalization group methods, the equilibrium dynamics of a quantum magnetic impurity in a ferromagnet and have found that the symmetric coupling of the magnon bath to the x and y spin components (z being the magnetization direction) effectively decreases the strength of the coupling. For this property they have coined the term *quantum frustration*, since it may be interpreted as the inability of the spin bath to simultaneously measure two noncommuting observables such as the x and y components (S_x and S_y) of the impurity spin. An important consequence is that spin coherence may be longer lived than it would if only one of the spin components were coupled to the dissipative bath. This effect could be relevant for quantum information, as it would provide a mechanism for quantum spins to remain coherent for long times.

Here we investigate the effect of quantum frustration in a different but exactly tractable physical system, namely, a harmonically oscillator coupled separately, through its position and momentum, to two independent oscillator heat baths.⁵ In the symmetric limit, this model describes the behavior of a large spin magnetic impurity in a ferromagnet. Another instance of a physical system that interacts, through position and momentum, with two different baths is a Josephson junction, where the relative particle number (proportional to the electric dipole) couples to the radiation field while the relative phase interacts with the quasiparticle field.⁶ Those two environments have different spectral properties. To investigate a more symmetric coupling, we consider a harmonic oscillator whose position and momentum interact linearly with two independent Ohmic baths.⁷ We find that the two baths do cancel in some but not all respects. A degree of cancellation is revealed by the persistence of underdamped oscillations for arbitrarily strong dissipation provided that the two baths couple with comparable strength. Quantum purity is weakened by the presence of a symmetric second bath

although its decay is slowed. Since a symmetrically damped harmonic oscillator behaves like a large spin in a ferromagnet, we refer to such a mixed situation as *quasiclassical frustration*.

We investigate the following model Hamiltonian:⁶

$$H = V_q(q + \delta q) + V_p(p + \delta p) + \sum_k \omega_{qk} a_{qk}^\dagger a_{qk} + \sum_k \omega_{pk} a_{pk}^\dagger a_{pk}, \quad (1)$$

$$\delta q = ig_p(p + \delta p) \sum_k C_{pk}(a_{pk}^\dagger - a_{pk}),$$

$$\delta p = ig_q(q + \delta q) \sum_k C_{qk}(a_{qk}^\dagger - a_{qk}), \quad (2)$$

where $[q, p] = i$ and g_q, g_p are sufficiently well-behaved functions. For general g_q, g_p the two equations in Eq. (2) cannot be decoupled without generating interactions between the two baths. However, if one of the coupling functions, say g_p , equals 1, then $[\delta p, \delta q] = 0$ and it becomes possible to remove both fluctuating contributions from the potential terms through the unitary transformations $U_p = \exp(ip\delta q)$ and $U_q = \exp[i\int^{q+\delta q} \delta p(q') dq']$. One arrives at the Hamiltonian

$$H = V_q(q) + \sum_k \omega_{qk} \left| a_{qk} + \frac{C_{qk}}{\omega_{qk}} \int^q dq' g_q(q') \right|^2 + V_p(p) + \sum_k \omega_{pk} \left| a_{pk} + \frac{C_{pk}}{\omega_{pk}} p \right|^2, \quad (3)$$

where the shorthand notation $|a|^2 \equiv a^\dagger a$ has been used.

At this point it is important to specify what we mean by coupling to position or momentum. Those are the particle variables to which the environment couples as a set of otherwise independent harmonic oscillators. This is the case, e.g., in (3), where the bath oscillators would remain independent if q and p were c numbers. On the contrary, this is not the case in (1), where, due to the nonlinear character of V_q

and V_p , the bath oscillators do interact with each other.¹⁰ A popular example is that of a charged particle interacting with the photon field.⁶ When the velocity-coupling model is adopted, it must be accompanied by a diamagnetic term that contains an interaction between photons. By contrast, through a canonical transformation, the photon field interacts with the particle position without an explicit interphoton interaction. Thus, within the precise convention we propose here, the electromagnetic field couples to the position of a charged particle.

We focus on the case of a harmonic oscillator [$V_q(q) = \omega_q q^2/2$, $V_p(p) = \omega_p p^2/2$] with $g_q = g_p = 1$

$$H = \frac{\omega_q}{2} q^2 + \sum_k \omega_{qk} \left| a_{qk} + \frac{C_{qk}}{\omega_{qk}} q \right|^2 + \frac{\omega_p}{2} p^2 + \sum_k \omega_{pk} \left| a_{pk} + \frac{C_{pk}}{\omega_{pk}} p \right|^2. \quad (4)$$

This is the model of dissipative coupling that displays the highest degree of symmetry between q and p . Hence it is interesting to study frustration in a physical system other than a quantum spin in a ferromagnet. The two baths of independent harmonic oscillators are described by the spectral densities

$$J_n(\omega) = 2 \sum_k |C_{nk}|^2 \delta(\omega - \omega_{nk}), \quad n = p, q. \quad (5)$$

We assume a power-law behavior at $\omega=0$ and write $J_n(\omega) = 2\gamma_n \omega^{\alpha_n} / (\omega_{\text{ph}}^{\alpha_n - 1} \pi)$, where the introduction of the frequency ω_{ph} renders the coupling constants γ_n dimensionless. Moreover, large cutoff frequencies Ω_n are assumed to exist for both environments.

The Hamiltonian (4) is equivalent to that of a magnetic impurity of large spin $\hbar l$ embedded in a ferromagnetic environment, with the correspondence $q \sim S_x / \hbar \sqrt{2l}$ and $p \sim S_y / \hbar \sqrt{2l}$ in the limit of large spins.⁸ Such large spins can be found in, e.g., magnetic particles.⁹

Eliminating the bath variables, the Heisenberg equations of motion for q and p are obtained,

$$\begin{aligned} \dot{q}(t) &= \omega_p p(t) + \int^t ds K_p(t-s) \dot{p}(s) + F_p(t), \\ -\dot{p}(t) &= \omega_q q(t) + \int^t ds K_q(t-s) \dot{q}(s) + F_q(t), \end{aligned} \quad (6)$$

where $K_n(t) \equiv \int_0^\infty J_n(\omega) \cos(\omega t) d \ln \omega$ and $F_n(t) = \sum_k C_{nk} a_{nk} \exp(-i\omega_{nk} t) + \text{H.c.}$ In Fourier space, Eq. (6) reads

$$[\tilde{J}_q(\omega) - \omega_q] q + i\omega p = F_q, \quad (7)$$

$$-i\omega q + [\tilde{J}_p(\omega) - \omega_p] p = F_p, \quad (8)$$

where $\tilde{J}_n(\omega)$ is the symmetrized Riemann transform

$$\tilde{f}(\omega) = \omega^2 \mathcal{P} \int_0^\infty \frac{f(\omega')}{\omega'(\omega'^2 - \omega^2)} d\omega' - i \operatorname{sgn}(\omega) \frac{\pi}{2} f(|\omega|). \quad (9)$$

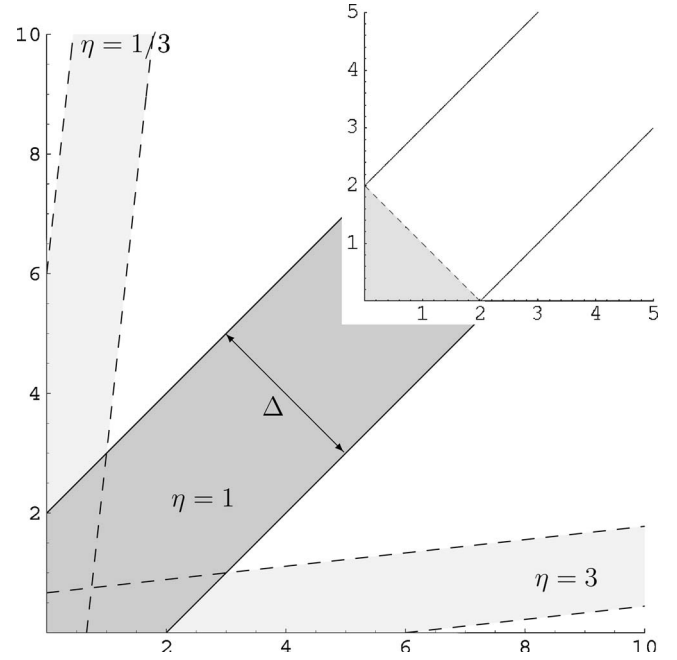


FIG. 1. Stripes of underdamped oscillations in the (γ_q, γ_p) plane for three different values of the parameter η , for q and p coupled to different baths. Inset: the shaded region denotes underdamping for a single bath coupled to q and p .

The oscillation modes are given by the zeros of the polynomial $(\omega_0 \equiv \sqrt{\omega_q \omega_p})$

$$\chi^{-1}(\omega) = \omega_0^2 - \omega^2 - \omega_q \tilde{J}_p(\omega) - \omega_p \tilde{J}_q(\omega) + \tilde{J}_q(\omega) \tilde{J}_p(\omega), \quad (10)$$

where $\chi(\omega)$ is the generalized susceptibility.

We further assume that the two heat baths are Ohmic: $J_n(\omega) = 2\gamma_n \omega / \pi$. For $\Omega_n \rightarrow \infty$, this implies $\tilde{J}_n(\omega) \rightarrow i\gamma_n \omega$. Then the eigenfrequencies are given by

$$\omega_0^2 - i(\omega_q \gamma_p + \omega_p \gamma_q) \omega - (1 + \gamma_q \gamma_p) \omega^2 = 0, \quad (11)$$

the solutions being

$$\omega_{\pm} = \frac{\omega_0}{(1 + \gamma_q \gamma_p)^{1/2}} (-i\kappa \pm \sqrt{1 - \kappa^2}), \quad (12)$$

$$\kappa \equiv \frac{\gamma_q \omega_p + \gamma_p \omega_q}{2\omega_0(1 + \gamma_q \gamma_p)^{1/2}}. \quad (13)$$

The transition from $\kappa < 1$ to $\kappa > 1$ marks the crossover from underdamped to overdamped oscillations. The condition $\kappa < 1$ requires (criterion A)

$$|\gamma_q \omega_p - \gamma_p \omega_q| < 2\omega_0. \quad (14)$$

The underdamped region satisfying (14) lies in a stripe of width $\Delta = 4\eta(1 + \eta^4)^{-1/2}$ with $\eta = (\omega_q / \omega_p)^{1/2}$, limited by the graphs of the functions $f(\gamma_q) = (\gamma_q \pm 2\eta)\eta^{-2}$. The stripes of underdamped oscillations in the (γ_q, γ_p) plane are plotted in Fig. 1 for $\eta = 1/3, 1, 3$. A remarkable consequence is that, given a value of, e.g., γ_q , one may drive the system from the overdamped to the underdamped regime by increasing γ_p .

TABLE I. Condition for coherent dynamics according to three different criteria (left column, labeled *A*, *B*, *C*), for three particular cases (upper row). The symmetric limit includes the assumption $\eta = 1$. The general case is given in main text.

		symmetric	only γ_q	only γ_p
$\kappa < 1$	(A)	always	$\gamma_q < 2\eta$	$\gamma_p < 2\eta^{-1}$
$D_q(\omega)$ has peak	(B)	$\gamma < \sqrt{3}$	$\gamma_q < \sqrt{2}\eta$	always
$ \text{Im } \omega_{\pm} < \text{Re } \omega_{\pm} $	(C)	$\gamma < 1$	$\gamma_q < \sqrt{2}\eta$	$\gamma_p < \sqrt{2}\eta^{-1}$

For $\eta=1$, the oscillator is underdamped if $\gamma_q \approx \gamma_p$, i.e., if the couplings to the two baths are of comparable strength. When $\gamma_q = \gamma_p \equiv \gamma$ the dimensionless parameter κ becomes,

$$\kappa = \gamma(1 + \gamma^2)^{-1/2} < 1. \quad (15)$$

Thus, in the fully symmetric case, the oscillator remains underdamped for *all* values of the coupling strength. This is in contrast with the case of one noise ($\gamma_p=0$), which requires $\gamma_q < 2\eta$ to be underdamped. Conversely, if $\gamma_q=0$, the condition is $\gamma_p < 2\eta^{-1}$. The inset of Fig. 1 shows the underdamped region for an oscillator coupling through q and p to a *single* heat bath.⁸ Surprisingly, the behavior is qualitatively different in that underdamping is always lost for sufficiently large coupling.

Another interesting quantity is $D_q(\omega) \equiv \text{Im } \chi_{qq}(\omega)/\omega$, where $\chi_{qq}(\omega)$ is the Fourier transform of $\langle [q(t), q(0)] \rangle$. For Ohmic environments,

$$D_q(\omega) = \frac{\gamma_q \omega_p^2 + \gamma_p (1 + \gamma_q \gamma_p) \omega^2}{[(1 + \gamma_q \gamma_p) \omega^2 - \omega_0^2]^2 + (\gamma_q \omega_p + \gamma_p \omega_q)^2 \omega^2}. \quad (16)$$

Following Ref. 4, we may view the presence of a peak in $D_q(\omega)$ as a signature for the existence of coherent dynamics (criterion *B*). This occurs for

$$\gamma_q^3 \omega_p^4 < (2\gamma_q \omega_p^2 + \gamma_p \omega_0^2) \omega^2. \quad (17)$$

In the symmetric case, this translates into $\gamma < \sqrt{3}$. For an oscillator coupled to a single bath through its position ($\gamma_p=0$), the requirement is $\gamma_q < \sqrt{2}\eta$. Finally, if $\gamma_q=0$, $D_q(\omega)$ always displays a peak.¹¹

A third possible condition for the existence of coherent dynamics is that, in Eq. (12),

$$|\text{Im } \omega_{\pm}| < |\text{Re } \omega_{\pm}|. \quad (18)$$

(criterion *C*). This yields

$$\gamma_q^2 \omega_p^2 + \gamma_p^2 \omega_q^2 < 2\omega_0^2. \quad (19)$$

In the symmetric case, the condition (19) becomes $\gamma < 1$. For $\gamma_p=0$, it becomes $\gamma_q < \sqrt{2}\eta$, while for $\gamma_q=0$, it reads $\gamma_p < \sqrt{2}\eta^{-1}$.

In Table I, the coherence signatures for the main three particular cases are summarized. Criteria *A* and *C* are symmetric in q and p , but not criterion *B*. The general trend (particularly clear if one considers *A* and *B*) is that, starting from a single dissipative bath coupled to, e.g., q , the introduction of a second bath that couples to p with the same

spectrum and comparable strength favors coherent and underdamped dynamics. For example, an oscillator with $\eta=1$ that is driven from $\gamma_p=0$ to $\gamma_p=\gamma_q=\gamma$, with γ_q fixed at a value $\sqrt{2} < \gamma_q < \sqrt{3}$, will cross over from incoherent to coherent behavior under both criteria *A* and *B*.

We have noted the striking result that, in the symmetric case, the oscillator is underdamped for all values of γ . However, criteria *B* and *C* indicate that the oscillator lies deep in the underdamped region only if γ is small. Such a limitation is also patent in the (possible) maximum of $D_q(\omega)/D_q(0)$ as well as in the ratio $|\text{Re } \omega_{\pm}|/|\text{Im } \omega_{\pm}| = \gamma^{-1}$. Both quantities stay well above unity only if γ is small. A related point is that, as $\gamma \rightarrow \infty$, the ratio $D_q(\omega)/D_q(0)$ does not saturate but rather decays as $\omega_0^2/\gamma^2 \omega^2$ for nonzero ω .

An oscillator initially prepared in a pure coherent state that at $t=0$ begins to interact linearly with an oscillator bath is described by a reduced density ρ that remains Gaussian at all times. Then the purity $\mathcal{P}(t) \equiv \text{Tr}(\rho^2)$ is given by

$$\mathcal{P}^{-2}(t) = 4\langle q^2 \rangle \langle p^2 \rangle. \quad (20)$$

At long times, $\langle q^2 \rangle$ and $\langle p^2 \rangle$ reach their equilibrium values, which contain contributions from both baths as well as hybrid terms that vanish if any of the two baths disappear.⁶ For simplicity, we focus on the zero temperature, weak coupling case. Then,

$$\langle q^2 \rangle = \frac{1}{2\eta} - \frac{\gamma_q}{2\eta^2} + \gamma_p \left(\ln \frac{\Omega_p}{\omega_0} - \frac{1}{2} \right) + \mathcal{O}(\gamma_q, \gamma_p), \quad (21)$$

and similarly for $\langle p^2 \rangle$. The two baths have opposite effects on $\langle q^2 \rangle$, but for $\Omega_n \gg \omega_0$ the logarithmic divergence from the p -coupled bath overwhelms the squeezing of q favored by the q -coupled environment. An impure mixture results

$$\begin{aligned} \mathcal{P}_{\infty}^{-2} &\simeq 1 + \frac{2\gamma_q}{\eta} \left(\ln \frac{\Omega_p}{\omega_0} - 1 \right) + 2\gamma_p \eta \left(\ln \frac{\Omega_q}{\omega_0} - 1 \right) + \mathcal{O}(\gamma_q, \gamma_p) \\ &\gg 1. \end{aligned} \quad (22)$$

The dynamical evolution of the purity $\mathcal{P}(t)$ is cumbersome in the general case, but it becomes tractable in the symmetric problem. We find ($\Omega_q = \Omega_p = \Omega$)

$$\mathcal{P}(t) \simeq \begin{cases} e^{-\Omega t}, & 0 \leq t \leq \Omega^{-1} \\ \mathcal{P}_{\infty} \left[1 + \frac{2\gamma}{(1 + \gamma^2)^{3/2}} \frac{e^{-t/\tau}}{t^2} \right], & \Omega^{-1} \ll t \rightarrow \infty, \end{cases} \quad (23)$$

with $\tau^{-1} = \gamma\omega_0/(1 + \gamma^2) = |\text{Im } \omega_{\pm}|$. The fast decay on the scale of Ω^{-1} comes from the choice of decoupled initial conditions¹² and from the concurrence of two baths.¹³ For $t \gg \Omega^{-1}$, the system evolves slowly towards equilibrium. The divergence of τ for $\gamma \rightarrow \infty$ might be interpreted as robustness against purity loss. However, it should be noted that such a slowing down merely reflects a dilation of all time scales with increased friction. For instance, $|\text{Re } \omega_{\pm}| = \omega_0/(1 + \gamma^2)$ vanishes even faster.

We have also investigated the decay of the purity $\mathcal{P}(t)$ when at $t=0$ the system is prepared in a linear superposition of two coherent states centered at $q = \pm a/2$ with zero average

momentum.¹⁴ For the symmetric case, we find⁸

$$P(t) = \frac{\mathcal{P}(t)}{2} \left\{ 1 + \frac{\cosh^2 \left\{ \frac{a^2}{4} \left[\phi(t)\mathcal{P}(t) - \frac{1}{2} \right] \right\}}{\cosh^2(a^2/8)} \right\}, \quad (24)$$

where $\phi(t)$ is a complicated function that evolves from $\phi(0)=1$ to $\lim_{t \rightarrow \infty} \phi(t)=0$, and the single wave-packet purity $\mathcal{P}(t)$ is given in (23). As expected, $P(t)/\mathcal{P}(t) \rightarrow 1$ as $a \rightarrow 0$, and $P(t)/\mathcal{P}(t) \rightarrow 1/2$ as $a \rightarrow \infty$.

Interestingly, the structure of (24) is such that, as time passes and $\phi(t)\mathcal{P}(t)$ evolves from 1 to 0, the ratio $P(t)/\mathcal{P}(t)$ starts at unity, as corresponds to a pure state, then decreases and finally, at long times, goes back to unity. When a is large $P(t)/\mathcal{P}(t)$ decays rapidly on a time scale $\sim 1/4a^2\gamma$ to $1/2$. There it stays for a time that increases with distance as $\sim \gamma^{-1} \ln a$. Afterwards it returns to one. The ratio $1/2$ can be rightly interpreted as resulting from the incoherent mixture of the two wave packets. Thus it comes as a relative surprise that $P(t)/\mathcal{P}(t)$ becomes unity again at long times, as if coherence among the two wave packets were eventually recovered. The physical explanation lies in the ergodic character of the long time evolution, with both wave packets evolving towards the equilibrium configuration described in Eqs. (20)–(22). Once the two initially separate wave packets begin to overlap, they regain mutual coherence. Due to the symmetry of the problem, a similar result would have been obtained if the oscillator had started from a superposition of two coherent states located in the same region of real space but with different average values of the momentum.

In summary, we have found features that are reminiscent of an effective particle-bath decoupling, such as the persistence of underdamped oscillations for arbitrarily large values of γ in the case of a symmetric oscillator and the slowing down of purity decay for a Gaussian wave packet. Another feature is that two initially separate wave packets regain relative coherence at long times because they recombine. The situation is reminiscent of the quantum frustration exhibited by a magnetic quantum impurity albeit in a more limited form.⁴ The main difference between the two problems is the dimensionality of the particle Hilbert space. Evolving in the continuum, the quantum oscillator can be considerably degraded by the effect of the environment, as shown in (23). It is only when the two wave packets recombine because of ergodicity that relative purity is recovered. By contrast, the spin-1/2 magnetic impurity lives in a two-dimensional space. The only possible effect of the environment is to flip the spin. Thus, in any representation, two initially orthogonal states quickly overlap and tend to preserve mutual coherence. The net result is an increased decoupling from a symmetrically dissipative environment. The requirement of low dimensionality suggests that strong frustration is a genuinely quantum effect.

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¹⁰An example is the electromagnetic field coupled to a charge position (Ref. 6). The alternative velocity-coupling model requires a diamagnetic term that is quadratic in the photon field.

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¹³For $\gamma_p=0$, the time scale of relaxation is γ_q^{-1} and $\mathcal{P}(t)$ has no initial slip.

¹⁴The decay of coherence of such a system in the presence of one noise has been studied in Ref. 3 but without explicit attention to the purity evolution.