Hubbard model with infinite-range attractive interaction

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The ground state and low-temperature properties of the Hubbard model with infinite-range attractive interaction in the presence of a magnetic field are considered. This model is exactly solvable in any dimension and is mapped onto an ideal gas of three species of exclusons. The system is found to behave as a noninteracting Fermi gas whenever the correlation function $C(\mathbf{k})=\langle n_{\mathbf{k}\uparrow}n_{\mathbf{k}\downarrow}\rangle-\langle n_{\mathbf{k}\uparrow}\rangle\langle n_{\mathbf{k}\downarrow}\rangle$ is negligible, otherwise it displays nontrivial correlated behavior which manifests as fractional statistical effects. The corresponding effective Hamiltonians and free energies governing the distinct regimes are presented. In one dimension, our results are analyzed in light of those derived using the on-site attractive Hubbard chain.

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I. INTRODUCTION

The Hubbard model has been widely used to describe strongly correlated electron systems, particularly itinerant electron magnetism, metal-insulator transition, and superconductivity.[1,2] In its original version[3] the model describes electrons hopping between neighboring sites of a lattice and subject to a on-site Coulomb repulsive interaction. However, several variants of this original version have been proposed, including attractive[4] and long-range[2] interactions, and some extreme limits such as infinite dimensional systems,[5] infinite-range hopping,[6,7] and infinite-range interaction.[8–14] Despite that these extreme limits are physically unattainable, they allow exact solutions from which nontrivial features may be derived thus providing insights into more realistic models.

Here we focus our attention on the case of infinite-range interaction.[8] Its original motivation was the proposal[9] of an entropy suitable in describing the thermodynamics of strongly correlated electrons in a narrow band near a Mott metal-insulator transition (MIT), i.e., a spin liquid, by excluding double occupancy in $\mathbf{k}$ space. Later on, a Hamiltonian, both in the reciprocal[10] and real space,[8] was proposed whose associated entropy recovers the spin liquid one[9] in the infinite-U limit. A special feature of this model is to provide non-Fermi-liquid behavior through pure forward scattering mechanisms[15,16] and a very simple analytical approach to a MIT.[8,11,17–19] Further, it has been shown[15,17,19] that this model is mapped onto an ideal gas of three species of particles (exclusons) obeying fractional exclusion statistics,[20] which manifests itself on the behavior of the several physical properties investigated. One should also remark that, in the context of g on statistical mechanics with possible application to Mott insulators, it was emphasized[21] that "what is needed is local repulsion in momentum space, which could arise directly from a long-range force or indirectly through correlation effects."

In this work we extend these investigations to include the case of an attractive infinite-range interaction. In Sec. II we present the model and its exclusion representation in the presence of an external magnetic field. Section III is devoted to discuss the ground state and low-temperature properties using a correlation function in $\mathbf{k}$ space related to the fractional species of the model. Although the model is soluble in any dimension d, here we specialize in the one-dimensional case for which explicit analytical results are provided and discussed in light of equivalent ones derived based on the on-site attractive Hubbard chain (AHC).[22–25] In Sec. IV we discuss fractional statistics effects on the low-temperature specific heat. Finally in Sec. V we summarize our findings and present some concluding remarks.

II. MODEL AND EXCLUSION REPRESENTATION IN A FIELD

The system we study is a Hubbard-like Hamiltonian with infinite-range interaction[8] in the presence of an external magnetic field $\mathbf{H}=H\hat{z}$:

$$\mathcal{H}=-t\sum_{\langle i,j\rangle\sigma}c_{i\sigma}^{\dagger}c_{j\sigma}+\frac{U}{N}\sum_{j_1,j_2,j_3,j_4}\delta_{j_1+j_3,j_2+j_4}c_{j_1\uparrow}^{\dagger}c_{j_2\uparrow}c_{j_3\downarrow}^{\dagger}c_{j_4\downarrow} -\frac{g\mu_B H}{2}\sum_i(n_{i\uparrow}-n_{i\downarrow})-(\mu+U/2)\sum_{i\sigma}n_{i\sigma}, \qquad (1)$$

where $c_{i\sigma}$ ($c_{i\sigma}^{\dagger}$) are electron anihilation (creation) operators, $n_{i\sigma}=c_{i\sigma}^{\dagger}c_{i\sigma}$ is the number operator, and $\sigma=\uparrow,\downarrow$. The first term describes the hopping of electrons on a d-dimensional hypercubic lattice of N sites and $\langle i,j\rangle$ denotes the nearest neighbors. The second term describes an infinite-range interaction, which selects the sites that conserve the center of mass of the two particles in the scattering process through the restriction $\delta_{j_1+j_3,j_2+j_4}$. The third term is the Zeeman interaction, where g is the gyromagnetic factor and μ_B is the Bohr magneton. Finally, the last term represents a coupling to a reservoir of particles, where μ is the chemical potential such that half-filled bands correspond to $\mu=0$ at zero temperature. Fourier transformed, this model has a simple diagonal representation in $\mathbf{k}$ space:[8–10,12,13]

$$\mathcal{H} = \sum_{\mathbf{k},\sigma} (\varepsilon_{\mathbf{k}} - \mu - U/2) c^{\dagger}_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma} + U \sum_{\mathbf{k}} n_{\mathbf{k}\uparrow} n_{\mathbf{k}\downarrow} - \frac{g\mu_B H}{2} \sum_{\mathbf{k}} (n_{\mathbf{k}\uparrow} - n_{\mathbf{k}\downarrow}), \quad (2)$$

where $\varepsilon_{\mathbf{k}} = -t\Sigma_{\langle 0,j\rangle} e^{i\mathbf{k}\cdot\mathbf{r}_j}$ and $\Delta = 4td$ is the bandwidth. We remark that the diagonality of $\mathcal{H}$ in $\mathbf{k}$ space does not imply necessarily that the physics of the model is trivial,[19] as shown below.

In the exclusion representation, the total energy of the model has the expected simple additive form,[26] $E = \langle \mathcal{H} \rangle = \Sigma_{\mathbf{k},\alpha} \tilde{\varepsilon}_{\mathbf{k}\alpha} \tilde{n}_{\mathbf{k}\alpha}$, with the corresponding species energies (thereafter named excluson α, $\alpha = 1$, 2 and 3) given by[19]

$$\tilde{\varepsilon}_{\mathbf{k}1} = \varepsilon_{\mathbf{k}} - \mu - U/2 - g\mu_B H/2, \quad (3)$$

$$\tilde{\varepsilon}_{\mathbf{k}2} = \varepsilon_{\mathbf{k}} - \mu - U/2 + g\mu_B H/2, \quad (4)$$

$$\tilde{\varepsilon}_{\mathbf{k}3} = 2(\varepsilon_{\mathbf{k}} - \mu). \quad (5)$$

It can also be shown, using this formalism,[19,26] that the fractional species thermal averages read[19]

$$\tilde{n}_{\mathbf{k}1} = \langle n_{\mathbf{k}\uparrow} \rangle - \langle n_{\mathbf{k}\uparrow} n_{\mathbf{k}\downarrow} \rangle = \frac{w_{\mathbf{k}2} w_{\mathbf{k}3}}{(1 + w_{\mathbf{k}1})(1 + w_{\mathbf{k}2})(1 + w_{\mathbf{k}3})}, \quad (6)$$

$$\tilde{n}_{\mathbf{k}2} = \langle n_{\mathbf{k}\downarrow} \rangle - \langle n_{\mathbf{k}\uparrow} n_{\mathbf{k}\downarrow} \rangle = \frac{w_{\mathbf{k}3}}{(1 + w_{\mathbf{k}2})(1 + w_{\mathbf{k}3})}, \quad (7)$$

$$\tilde{n}_{\mathbf{k}3} = \langle n_{\mathbf{k}\uparrow} n_{\mathbf{k}\downarrow} \rangle = \frac{1}{1 + w_{\mathbf{k}3}}, \quad (8)$$

where $w_{\mathbf{k}1} = e^{\beta\tilde{\varepsilon}_{\mathbf{k}1}}$, $w_{\mathbf{k}2} = (1 + w_{\mathbf{k}1}) e^{\beta(\tilde{\varepsilon}_{\mathbf{k}2} - \tilde{\varepsilon}_{\mathbf{k}1})} = e^{\beta\tilde{\varepsilon}_{\mathbf{k}2}} + e^{\beta(\tilde{\varepsilon}_{\mathbf{k}2} - \tilde{\varepsilon}_{\mathbf{k}1})}$ and $w_{\mathbf{k}3} = (1 + w_{\mathbf{k}2}) e^{\beta(\tilde{\varepsilon}_{\mathbf{k}3} - \tilde{\varepsilon}_{\mathbf{k}2})} = e^{\beta\tilde{\varepsilon}_{\mathbf{k}3}} + e^{\beta(\tilde{\varepsilon}_{\mathbf{k}3} - \tilde{\varepsilon}_{\mathbf{k}2})} + e^{\beta(\tilde{\varepsilon}_{\mathbf{k}3} - \tilde{\varepsilon}_{\mathbf{k}1})}$, and $\beta = 1/k_B T$ is the inverse temperature. In equilibrium $w_{\mathbf{k}\alpha} = D_{\mathbf{k}\alpha}/N_{\mathbf{k}\alpha}$ satisfies the Wu-Haldane distribution,[26] with $D_{\mathbf{k}\alpha} = G_{\mathbf{k}\alpha} - \Sigma_{\mathbf{k}'\alpha'} g_{\mathbf{k}\mathbf{k}';\alpha\alpha'} N_{\mathbf{k}'\alpha'}$, $\tilde{n}_{\mathbf{k}\alpha} = N_{\mathbf{k}\alpha}/G_{\mathbf{k}\alpha}$, where $G_{\mathbf{k}\alpha}$ is the number of available quantum states in the absence of particles, $N_{\mathbf{k}\alpha}$ is the corresponding number of particles, and the Haldane's matrix $g_{\mathbf{k}\mathbf{k}';\alpha\alpha'}$ defines the statistical interaction:[19]

$$g_{\mathbf{k}\mathbf{k}';\alpha\alpha'} = \delta_{\mathbf{k}\mathbf{k}'} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad \alpha, \alpha' = 1,2,3. \quad (9)$$

Finally, the grand-canonical free energy reads:[19,26]

$$\Omega(T,H,\mu) = -\frac{1}{\beta} \sum_{\mathbf{k},\alpha} \ln(1 + w^{-1}_{\mathbf{k}\alpha}) = \frac{1}{\beta} \sum_{\mathbf{k}} \ln(1 - \tilde{n}_{\mathbf{k}1} - \tilde{n}_{\mathbf{k}2} - \tilde{n}_{\mathbf{k}3}) = -\frac{1}{\beta} \sum_{\mathbf{k}} \ln(1 + e^{-\beta\tilde{\varepsilon}_{\mathbf{k}1}} + e^{-\beta\tilde{\varepsilon}_{\mathbf{k}2}} + e^{-\beta\tilde{\varepsilon}_{\mathbf{k}3}}). \quad (10)$$

It is clear that for $U = 0$ the above expression reduces to the free Fermi case [note, in addition, that Eq. (9) does not depend on U]. Then, the exclusion representation is redundant since $\tilde{n}_{\mathbf{k}3}$ factorizes, i.e., $\tilde{n}_{\mathbf{k}3} = \langle n_{\mathbf{k}\uparrow} \rangle \langle n_{\mathbf{k}\downarrow} \rangle$, and it is then possible to choose $g_{\mathbf{k}\mathbf{k}';\alpha\alpha'} = \delta_{\mathbf{k},\mathbf{k}'} \delta_{\alpha,\alpha'}$, with $\alpha = \uparrow$ and $\downarrow$ only, as expected. However, neglecting exponentially small terms at low temperatures and for finite values of the attractive interaction U, in the next section we shall arrive at this same conclusion when the magnetic field is larger than a certain critical value.

III. GROUND STATE AND LOW-TEMPERATURE PROPERTIES

The physics of the model (2) can be completely clarified by studying the correlation function

$$C(T,H,\varepsilon_{\mathbf{k}} - \mu; U) = \langle n_{\mathbf{k}\uparrow} n_{\mathbf{k}\downarrow} \rangle - \langle n_{\mathbf{k}\uparrow} \rangle \langle n_{\mathbf{k}\downarrow} \rangle = \tilde{n}^2_{\mathbf{k}3} e^{\beta\tilde{\varepsilon}_{\mathbf{k}3}} (1 - e^{\beta U}) = \frac{1 - e^{\beta U}}{\{2\cosh[\beta(\varepsilon_{\mathbf{k}} - \mu)] + 2e^{\beta U/2} \cosh(\beta g\mu_B H/2)\}^2}, \quad (11)$$

where use of Eqs. (3)–(8) was made. Its physical relevance in the present context is made clear by rewriting Eq. (10) in the suggestive form:

$$\Omega(T,H,\mu) = \frac{1}{\beta} \sum_{\mathbf{k}} \ln[(1 - \langle n_{\mathbf{k}\uparrow} \rangle)(1 - \langle n_{\mathbf{k}\downarrow} \rangle) + C(\mathbf{k})], \quad (12)$$

from which it is obvious that it differs from the expression of the grand-canonical free energy of a noninteracting Fermi gas by the presence of $C(\mathbf{k})$ only. It turns out that whenever $C(\mathbf{k})$ is exponentially small, i.e., $\tilde{n}_{\mathbf{k}3} \simeq \langle n_{\mathbf{k}\uparrow} \rangle \langle n_{\mathbf{k}\downarrow} \rangle$, the system behaves as a noninteracting Fermi gas, otherwise it will be shown that the electron correlation effects manifest themselves as fractional statistical effects in the context of the Haldane-Wu formalism.

It is well known[27–29] that the on-site repulsive and attractive Hubbard models satisfy the duality relation:

$$\Omega(T,H,\mu + U/2; U) = \Omega[T, 2\mu/g\mu_B, (g\mu_B H - U)/2; -U] + N[(g\mu_B H - U)/2 - \mu].$$

However, this is not verified in the case of infinite-range interaction. Indeed, using Eq. (10), we obtain

$$\Omega(T,H,\mu + U/2; U) = N\left[\frac{g\mu_B H - U}{2} - \mu\right] + \Omega[T, 2\mu/g\mu_B, (g\mu_B H - U)/2; -U] + \frac{1}{\beta} \sum_{\mathbf{k}} \ln g(T,H,\mu; U), \quad (13)$$

where

$$g(T,H,\mu;U)=\left\{\frac{C(T,H,\varepsilon_{\mathbf{k}}-\mu;U)}{C[T,(2\varepsilon_{\mathbf{k}}/g\mu_B)-H,\mu;U]}\right\}^{1/2}. \quad (14)$$

Note that the extra contribution involves the correlation function $C(T,H,\varepsilon_{\mathbf{k}}-\mu;U)$ in a nontrivial manner and nullifies for $U=0$, as expected. Therefore, it is more effective to study the repulsive and attractive cases separately.

In the repulsive case and $k_BT \ll U$, Eq. (11) reduces to $[C(T,H,\varepsilon_{\mathbf{k}}-\mu;U)\equiv C(\mathbf{k})]$

$$C(\mathbf{k})=\frac{-1}{\{2e^{-\beta U/2}\cosh[\beta(\varepsilon_{\mathbf{k}}-\mu)]+2\cosh(\beta g\mu_B H/2)\}^2}+\mathcal{O}(e^{-\beta U}). \quad (15)$$

Therefore, in zero field and $T=0$, $C(\mathbf{k})$ is of $\mathcal{O}(1)$ only if the joint condition: $\varepsilon_{\mathbf{k}}-\mu-U/2\leqslant 0$ *and* $\varepsilon_{\mathbf{k}}-\mu+U/2\geqslant 0$ is satisfied. It then implies in a fractional occupation of the one-particle distribution function, i.e., $\langle n_{\mathbf{k}\sigma}\rangle_{T=0}=\frac{1}{2}$ in the region of $\mathbf{k}$ space where double occupancy is not allowed, and in the occurrence of non-Fermi liquid thermodynamic behavior[19] [see Eq. (40) and Fig. 6 in Ref. 19]. Low-temperature effects have also been carefully analyzed,[19] including contributions around the "generalized" Fermi surface, i.e., $\varepsilon_{\mathbf{k}}=\mu\pm U/2$. In particular, for $2k_BT > g\mu_B H (H\to 0)$, the Pauli spin susceptibility is subdominant in all metallic phases, and a Curie-type of response appears. However, for $2k_BT\ll g\mu_BH$, $C(\mathbf{k})=\mathcal{O}(e^{-\beta g\mu_B H})$ and, therefore, a fermionic description suffices at low temperatures. Indeed, by replacing $n_{\mathbf{k}\uparrow}n_{\mathbf{k}\downarrow}\to n_{\mathbf{k}\downarrow}$ [consistent with $\tilde{n}_{\mathbf{k}2}$ of $\mathcal{O}(e^{-\beta g\mu_B H})$] in Eq. (2), we find the effective Hamiltonian

$$\mathcal{H}_{eff}=\sum_{\mathbf{k},\sigma}(\varepsilon_{\mathbf{k}}-\mu)c^{\dagger}_{\mathbf{k}\sigma}c_{\mathbf{k}\sigma}-\frac{(g\mu_BH+U)}{2}\sum_{\mathbf{k}}(n_{\mathbf{k}\uparrow}-n_{\mathbf{k}\downarrow}). \quad (16)$$

Note that in this regime the interaction $(U/g\mu_B)$ plays the role of an effective magnetic field and the system is now described by an effective fermionic statistical matrix: $g^{eff}_{\mathbf{k}\mathbf{k}';\alpha\alpha'}=\delta_{\mathbf{k},\mathbf{k}'}\delta_{\alpha,\alpha'}$ $(\alpha,\alpha'=\uparrow,\downarrow)$. In fact, we can obtain, using Eq. (10),

$$\Omega(T,H,\mu)=\Omega_0[T,H+(U/g\mu_B),\mu]+\mathcal{O}(e^{-\beta g\mu_BH}), \quad (17)$$

where

$$\Omega_0(T,H,\mu)=-\frac{1}{\beta}\sum_{\mathbf{k}}\ln(1+e^{-\beta(\varepsilon_{\mathbf{k}}-\mu-g\mu_BH/2)}+e^{-\beta(\varepsilon_{\mathbf{k}}-\mu+g\mu_BH/2)}+e^{-2\beta(\varepsilon_{\mathbf{k}}-\mu)}) \quad (18)$$

is the free energy of the noninteracting Fermi gas; $n_{\mathbf{k}\sigma}$ are Fermi distribution functions with $H_{eff}=[H+(U/g\mu_B)]$.

On the other hand, in the attractive case, for $H<H_C$ and $2k_BT\ll g\mu_B(H_C-H)$, Eq. (11) reduces to

$$C(\mathbf{k})=\frac{1}{[2\cosh\beta(\varepsilon_{\mathbf{k}}-\mu)]^2}+\mathcal{O}(e^{-\beta g\mu_B(H_C-H)/2}), \quad (19)$$

where $H_C=|U|/(g\mu_B)$ is the critical field that breaks a pair of electrons with opposite spins bounded by the energy $|U|$ in any dimension (i.e., a free "Cooper pair" with equal momenta in $\mathbf{k}$ space, here identified as excluson 3). It is also interesting to note that this critical field, which is independent of the band filling $n=N_e/N$ (N_e is the total number of electrons), coincides with the first term of the expansion of the corresponding H_C in the strong-coupling limit $(|U|\to\infty)$ of the on-site AHC [see Eq. (49), Ref. 23]. The low-temperature physics is now derived from the effective Hamiltonian

$$\langle\mathcal{H}_{eff}\rangle=\sum_{\mathbf{k}}\tilde{\varepsilon}_{\mathbf{k}3}\tilde{n}_{\mathbf{k}3}, \quad (20)$$

describing a spinless free Fermi gas, $g^{eff}_{\mathbf{k}\mathbf{k}';\alpha\alpha'}=\delta_{\mathbf{k},\mathbf{k}'}\delta_{\alpha,3}$, with free "Cooper pairs" as composite particles [note that the energy $\tilde{\varepsilon}_{\mathbf{k}3}$, Eq. (5), does not depend on H, and so Pauli's response is nullified]. Equation (20) can be inferred from Eq. (2) by replacing $n_{\mathbf{k}\uparrow},n_{\mathbf{k}\downarrow}\to n_{\mathbf{k}\uparrow}n_{\mathbf{k}\downarrow}$ [consistent with $\tilde{n}_{\mathbf{k}1}$ and $\tilde{n}_{\mathbf{k}2}$ of $\mathcal{O}(e^{-\beta g\mu_B(H_C-H)/2})$ and $\mathcal{O}(e^{-\beta g\mu_B(H_C+H)/2})$, respectively]. In this regime $\tilde{n}_{\mathbf{k}3}$ obeys a fermionic distribution (therefore, even at $T=0$ there is no condensation of the Cooper pairs in agreement with the prediction for the on-site singlet pairing correlation function[8]):

$$\tilde{n}_{\mathbf{k}3}=\frac{1}{e^{\beta\varepsilon_{\mathbf{k}3}}+1}+\mathcal{O}(e^{-\beta g\mu_B(H_C-H)/2}), \quad (21)$$

and the system exhibits correlations at the Fermi surface $\varepsilon_{\mathbf{k}}=\mu$, consistent with $C(\mathbf{k})$ of $\mathcal{O}(1)$ in Eq. (19). Note that from Eq. (10), $\Omega(T,H,\mu)=-(1/\beta)\Sigma_{\mathbf{k}}\ln(1+e^{-\beta\tilde{\varepsilon}_{\mathbf{k}3}})+\mathcal{O}(e^{-\beta g\mu_B(H_C-H)/2})$, in agreement with Eq. (20).

For $H>H_C$, insight can be gained in our analysis by rewriting Eq. (2) in the form

$$\mathcal{H}=\sum_{\mathbf{k},\sigma}(\varepsilon_{\mathbf{k}}-\mu)c^{\dagger}_{\mathbf{k}\sigma}c_{\mathbf{k}\sigma}+|U|\sum_{\mathbf{k}}(n_{\mathbf{k}\downarrow}-n_{\mathbf{k}\uparrow}n_{\mathbf{k}\downarrow})-\frac{g\mu_B(H-H_C)}{2}\sum_{\mathbf{k}}(n_{\mathbf{k}\uparrow}-n_{\mathbf{k}\downarrow}). \quad (22)$$

Similarly to the repulsive case, where $H_C=0$, in the present case $C(\mathbf{k})=\mathcal{O}(e^{-\beta g\mu_B(H-H_C)})$ and $\tilde{n}_{\mathbf{k}2}=\mathcal{O}(e^{-\beta g\mu_BH})$ for $H>H_C$ and $2k_BT\ll g\mu_B(H-H_C)$. So, in view of Eqs. (12) and (22), the thermodynamic properties of the model are those of a noninteracting Fermi gas with a shift H_C in the magnetic field. Again we have $g^{eff}_{\mathbf{k}\mathbf{k}';\alpha\alpha'}=\delta_{\mathbf{k},\mathbf{k}'}\delta_{\alpha,\alpha'}$ $(\alpha,\alpha'=\uparrow,\downarrow)$. Indeed, from Eq. (10) one obtains

$$\Omega(T,H,\mu)=\Omega_0(T,H-H_C,\mu)+\mathcal{O}(e^{-\beta g\mu_B(H-H_C)}), \quad (23)$$

as expected. This mapping allows us to conclude that for the half-filled band case and $T=0$, the system undergoes a field-induced MIT at the critical field $H_{QC}=(\Delta/g\mu_B)+H_C$ [in a noninteracting Fermi gas it occurs at $H_{QC}=(\Delta/g\mu_B)$]. This transition is in the same universality class of a spinless free Fermi gas[19,30,31] (see Fig. 2 and discussion in the end of this section). Yet to illustrate the H_C shift, we now consider the one-dimensional magnetization, $M(T,H,\mu) = g\mu_B S_z(T,H,\mu) = -(\partial\Omega/\partial H)_{T,\mu}$, as a function of H and n, where

$$S_z(T,H,n) = \frac{1}{2}\sum_{\mathbf{k}} (\langle n_{\mathbf{k}\uparrow}\rangle - \langle n_{\mathbf{k}\downarrow}\rangle) = \frac{1}{2}\sum_{\mathbf{k}} \tilde{n}_{\mathbf{k}1} + \mathcal{O}(e^{-\beta g\mu_B(H_C+H)/2}), \quad (24)$$

in which μ is eliminated in favor of n. One should stress that Eq. (24) is valid in any regime of T, H, and n. We find for $0 < n \leqslant 1$ the following results: $S(T=0,H,n) \equiv S_z(T=0,H,n)/N_e = 0$ for $0 \leqslant H \leqslant H_C$, while for $H_C < H \leqslant H_S$ [where the saturation field, H_S, is defined by Eq. (27) below] we have

$$S(T=0,H,n) = \frac{1}{2\pi n}\left\{\arccos\left[e(n,H) - \frac{g\mu_B(H-H_C)}{\Delta}\right] - \arccos\left[e(n,H) + \frac{g\mu_B(H-H_C)}{\Delta}\right]\right\} = S_0(T=0,H-H_C,n), \quad H_C < H \leqslant H_S, \quad (25)$$

where

$$e(H,n) = \left\{\cos^2\left(\frac{n\pi}{2}\right) - \left[\frac{g\mu_B(H-H_C)}{\Delta}\right]^2 \cot^2\left(\frac{n\pi}{2}\right)\right\}^{1/2}, \quad (26)$$

$g\mu_B S_0(T=0,H,n)$ is the magnetization of the noninteracting Fermi gas and we have used $\mu(H_C < H \leqslant H_S, n) = -(\Delta/2)e(H,n)$. In this regime of fields, the magnetization increases continuously and saturates $(S=\frac{1}{2})$ at the saturation field

$$H_S(n,|U|) = H_C + \frac{\Delta}{2g\mu_B}(1-\cos n\pi), \quad 0 < n \leqslant 1. \quad (27)$$

One should note that in the on-site AHC both S and H_S depend on U in a more complex manner[23,24] and, in the strong-coupling limit, $H_S(n,|U|\to\infty) = H_C - (\Delta/g\mu_B)\cos n\pi$ [see Eq. (5.1), Ref. 24]. In Fig. 1 we plot the magnetization per particle, Eq. (25), as a function of the dimensionless magnetic field H/H_C at several band fillings n and for $|U|/\Delta = 1.25$. In this scale for the magnetic field, the onset of the magnetization occurs at the point $H/H_C = 1$ for all curves. For small n, the magnetization per particle rapidly saturates in response to a small variation of the magnetic field around H_C of the order of $\Delta H = H_S - H_C \simeq n^2\pi^2\Delta/(4g\mu_B)(n \ll 1)$, as derived from Eq. (27), and approaches to a steplike singularity in the limit $n \to 0$. With the exception of this case, all magnetization curves start with a finite slope (Pauli susceptibility on a chain as shown below). These figures should be compared with similar plots reported in Refs. 23 and 24, $0 < n < 1$, and with an early calculation by Takahashi[27] at half-filling for the on-site AHC. Note that, as in the on-site

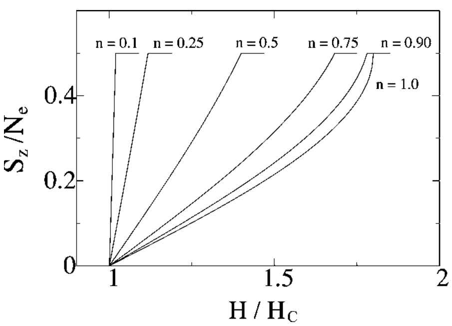

FIG. 1. Ground state magnetization per particle in units of $g\mu_B$, $S = S_z/N_e$, versus H/H_C at several band fillings n and $|U|/\Delta = 1.25$. All curves start at H_C and saturate at H_S where $S = \frac{1}{2}$.

AHC,[27] at $n=1$ the magnetization curve display an infinite slope at saturation. However, in the on-site case the onset points slightly differ due to the more complex behavior of $H_C(n,|U|)$.[23,24] Using Eq. (24), the zero-temperature spin susceptibility right above H_C is given by

$$\chi(T=0,H_C^+,n) = \lim_{H\to H_C^+}\lim_{T\to 0}\frac{g\mu_B}{N}\left(\frac{\partial S_z}{\partial H}\right) = \frac{(g\mu_B)^2}{\pi\Delta\sin(n\pi/2)} = \chi_P, \quad (28)$$

where χ_P is Pauli susceptibility on a chain, i.e, $M(T,H,n) = \chi_P(H-H_C) + \ldots, 2k_BT \ll g\mu_B(H-H_C)$.

We shall now investigate the very interesting behavior of the system in the vicinity of $H=H_C$, under the condition $2k_BT > g\mu_B|H-H_C|$. Therefore, the ground state susceptibility must be calculated by fixing $H=H_C$ and taking the limit $T\to 0$ afterwards. Before doing so, we would like to discuss some important features that characterize this behavior. First, we note that, for $2k_BT > g\mu_B|H-H_C|$, we find up to $\mathcal{O}(e^{-\beta g\mu_B(H+H_C)/2})$:

$$C(\mathbf{k}) \simeq \frac{1}{[2\cosh\beta(\varepsilon_{\mathbf{k}}-\mu) + e^{\beta g\mu_B(H-H_C)/2}]^2}, \quad (29)$$

$$\tilde{n}_{\mathbf{k}1} \simeq \frac{e^{\beta g\mu_B(H-H_C)/2}}{2\cosh\beta(\varepsilon_{\mathbf{k}}-\mu) + e^{\beta g\mu_B(H-H_C)/2}}, \quad (30)$$

$$\tilde{n}_{\mathbf{k}2} = \mathcal{O}(e^{-\beta g\mu_B(H+H_C)/2}), \quad (31)$$

$$\tilde{n}_{\mathbf{k}3} \simeq \frac{e^{-\beta(\varepsilon_{\mathbf{k}}-\mu)}}{2\cosh\beta(\varepsilon_{\mathbf{k}}-\mu) + e^{\beta g\mu_B(H-H_C)/2}}. \quad (32)$$

Therefore, since $\tilde{n}_{\mathbf{k}2}$ is exponentially small and taking $H=H_C$, $E=\langle\mathcal{H}\rangle$ calculated using Eq. (22) reduces to the first term. However, under the above condition, $\langle n_{\mathbf{k}\uparrow}\rangle$ and $\langle n_{\mathbf{k}\downarrow}\rangle$ are given by

$$\langle n_{\mathbf{k}\uparrow}\rangle \simeq \frac{e^{-\beta(\varepsilon_{\mathbf{k}}-\mu)} + e^{\beta g\mu_B(H-H_C)/2}}{2\cosh\beta(\varepsilon_{\mathbf{k}}-\mu) + e^{\beta g\mu_B(H-H_C)/2}}, \quad (33)$$

$$\langle n_{\mathbf{k}\downarrow}\rangle \simeq \frac{e^{-\beta(\varepsilon_{\mathbf{k}}-\mu)}}{2\cosh\beta(\varepsilon_{\mathbf{k}}-\mu)+e^{\beta g\mu_B(H-H_C)/2}}, \qquad (34)$$

i.e., the interacting ↑ and ↓ electrons give rise to quasiparticles that do not obey neither the Fermi nor the Wu-Haldane distribution functions. The true low-temperature elementary excitations of the system, however, are the exclusons 1 and 3 with distribution functions given by Eqs. (30) and (32). Indeed, this issue is clarified by the discussion presented in Sec. II of Ref. 19 for the general case involving the mapping of the Hubbard model with infinite-range interaction onto a free gas with three species of exclusons. In the present case, since excluson 2 is exponentially small, the effective nondiagonal statistical matrix is reduced to

$$g^{eff}_{\mathbf{kk}';\alpha\alpha'} = \delta_{\mathbf{k},\mathbf{k}'}\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}, \quad \alpha,\alpha' = 1,3, \qquad (35)$$

obtained from Eq. (9) by eliminating both the second row and column associated with excluson 2. The matrix $\Lambda_{\mathbf{kk}';\alpha\alpha'}$ [see Eq. (21) of Ref. 19] connecting the two referred representations, i.e., quasiparticles ↑ and ↓ and exclusons 1 and 3, reads:

$$\Lambda_{\mathbf{kk}';\alpha\alpha'} = -\delta_{\mathbf{k},\mathbf{k}'}\begin{pmatrix}1 & -1\\ 0 & 1\end{pmatrix}, \quad \alpha,\alpha' = 1,3, \qquad (36)$$

which implies that $\tilde{N}_{\mathbf{k}\alpha}=-\Sigma_{\mathbf{k}'\alpha'}\Lambda_{\mathbf{kk}';\alpha\alpha'}N_{\mathbf{k}'\alpha'}$ and $\tilde{\varepsilon}_{\mathbf{k}\alpha}=-\Sigma_{\mathbf{k}'\alpha'}\varepsilon_{\mathbf{k}'\alpha'}(\Lambda^{-1})_{\mathbf{k}'\mathbf{k};\alpha'\alpha}$, where the tilde refers to elementary excitations obeying the Wu-Haldane distribution, whereas the other representation has energy in a diagonal form but do not obey the exclusion statistics. Here, at $H=H_C$: $\tilde{\varepsilon}_{\mathbf{k}\alpha=1,3}=[\varepsilon_{\mathbf{k}1}=\varepsilon_{\mathbf{k}}-\mu,\varepsilon_{\mathbf{k}3}=2(\varepsilon_{\mathbf{k}}-\mu)]$ and $\varepsilon_{\mathbf{k}\alpha=\uparrow,\downarrow}=[\varepsilon_{\mathbf{k}\uparrow}=\varepsilon_{\mathbf{k}}-\mu,\varepsilon_{\mathbf{k}\downarrow}=\varepsilon_{\mathbf{k}}-\mu]$. Therefore, the exclusion representation is evident by the fact that at $H=H_C$, thermal excitations above the Fermi surface, i.e., $\varepsilon_{\mathbf{k}}=\mu$, involving spin ↑ quasiparticles alone are possible, but excitations involving the spin ↓ quasiparticles occur in pairs with the spin ↑ electrons in the same orbital due to correlations at the Fermi surface [see Eq. (7) with $\tilde{n}_{\mathbf{k}2}\simeq 0$]. This is corroborated by noting that in this regime the free energy can be written in terms of exclusons 1 and 3

$$\Omega(T,H,\mu) = -\frac{1}{\beta}\sum_{\mathbf{k}}\ln(1+e^{-\beta\varepsilon_{\mathbf{k}1}}+e^{-\beta\varepsilon_{\mathbf{k}3}}) + \mathcal{O}(e^{-\beta g\mu_B(H+H_C)/2}), \qquad (37)$$

whereas the expression involving the ↑ and ↓ quasiparticles contains a separated contribution from correlations at the Fermi surface [see, e.g., Eq (12)]:

$$\Omega(T,H,\mu) = \Omega_0(T,H-H_C,\mu) - \frac{1}{\beta}\sum_{\mathbf{k}}\ln\left\{1-\frac{e^{-\beta g\mu_B(H-H_C)/2}}{2\cosh\beta(\varepsilon_{\mathbf{k}}-\mu)+2\cosh\left[\frac{\beta g\mu_B(H-H_C)}{2}\right]}\right\} + \mathcal{O}(e^{-\beta g\mu_B(H+H_C)/2}). \qquad (38)$$

One should note that this contrasts with the regime $2k_BT \ll g\mu_B(H-H_C)$ and $C(\mathbf{k})$ is exponentially small [see Eq. (23)], in which case both ↑ and ↓ free electrons can be independently excited, although in equilibrium $\tilde{n}_{\mathbf{k}2}$ is still exponentially small due to the presence of the magnetic field.

We now proceed to calculate the ground state susceptibility at $H=H_C$, using $\mu(T=0,H=H_C,n)=-(\Delta/2)\cos(n\pi/2)$, which can be performed using the two above referred representations leading to the same result:

$$\chi(T=0,H=H_C,n) = \lim_{T\to 0}\lim_{H\to H_C}\chi(T,H,n) = \chi_P\int_{-\infty}^{+\infty}\frac{\cosh x}{(1+2\cosh x)^2}dx = \frac{(4\pi-3\sqrt{3})}{9\sqrt{3}}\chi_P, \qquad (39)$$

which amounts to almost 50% of χ_P. Note that Eq. (39) must be evaluated under the condition $2k_BT > g\mu_B|H-H_C|$, which contrasts with the previous calculation in Eq. (28). In fact, using Eq. (24) and either Eqs. (33) and (34) or Eq. (30), we find

$$M(T,H,n) = A\left(\frac{k_BT}{g\mu_B}\right) + \left(\frac{4\pi-3\sqrt{3}}{9\sqrt{3}}\right)\chi_P(H-H_C) + \ldots, \quad 2k_BT > g\mu_B|H-H_C|, \qquad (40)$$

where

$$A = \lim_{T\to 0}\frac{M(T,H=H_C,n)}{(k_BT/g\mu_B)} = \chi_P\int_{-\infty}^{+\infty}\frac{dx}{2\cosh x+1} = \frac{2\pi}{3\sqrt{3}}\chi_P. \qquad (41)$$

To conclude this section, in Fig. 2 we display the one-dimensional ground state phase diagram $[(H/H_C),n]$, which illuminates the features exhibited by the model. I denotes the Mott-insulating phase and QCP is the quantum critical point. Regime (2) is limited above by the curve $H_S(n)/H_C$ and separated from regime (1) by the line $H/H_C=1$. In regimes (3) and (4) the magnetization is field-saturated. With the exception of regime (1), which is populated by excluson 3 only,

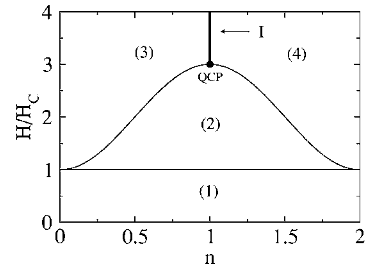

FIG. 2. Ground state phase diagram H/H_C versus n for $|U|/\Delta=0.5$. In regime (1) $H/H_C<1$ and the system is populated by exclusons 3 only. In regime (2) $\uparrow$ and $\downarrow$ **k** states coexist and are limited above by the curve $H_S(n)/H_C$. In regimes (3) and (4), the magnetization is field-saturated and connected by the particle-hole symmetry. The QCP is found at $n=1$ and $H=H_s(n=1)=H_{QC}$, and line I denotes the Mott-insulating phase.

this phase diagram is analogous to the repulsive U-induced MIT (Ref. 19) because of the correspondence $U\leftrightarrow g\mu_B(H-H_C)$ [implying the correspondence $\rho_{min}\leftrightarrow H_S(n)$; see Eq. (29) of Ref. 19] between repulsive and attractive cases, respectively. These correspondences can be understood by taking the limits $\lim_{H\to 0}\lim_{T\to 0}$ of Eq. (17) and further comparison with the $T\to 0$ limit of Eq. (23). It works because both ρ_{min}, in the repulsive case, and H_S, in the attractive one, signal the end of double occupancy in **k** space for $0<n\leqslant 1$. However, in the former case both $\uparrow$ and $\downarrow$ **k** states coexist with fractional occupation,[19] whereas in the latter the system is field-saturated.

IV. LOW-TEMPERATURE SPECIFIC HEAT

It is very instructive to investigate the effect of the fractional statistics using the grand-canonical specific heat

$$C_\mu = -T\left(\frac{\partial^2\Omega}{\partial T^2}\right)_{\mu,H}. \tag{42}$$

In fact, from Eq. (10) we can find an expression for C_μ in terms of the mean occupation numbers of the excluson species as follows:

$$\begin{aligned}C_\mu(T,H,\mu) = k_B\beta^2\sum_{\mathbf{k}}\,[&\tilde{\varepsilon}_{\mathbf{k}1}^2\tilde{n}_{\mathbf{k}1}(1-\tilde{n}_{\mathbf{k}1}) + \tilde{\varepsilon}_{\mathbf{k}2}^2\tilde{n}_{\mathbf{k}2}(1-\tilde{n}_{\mathbf{k}2})\\ &+ \tilde{\varepsilon}_{\mathbf{k}3}^2\tilde{n}_{\mathbf{k}3}(1-\tilde{n}_{\mathbf{k}3}) - 2\tilde{\varepsilon}_{\mathbf{k}1}\tilde{\varepsilon}_{\mathbf{k}2}\tilde{n}_{\mathbf{k}1}\tilde{n}_{\mathbf{k}2}\\ &- 2\tilde{\varepsilon}_{\mathbf{k}1}\tilde{\varepsilon}_{\mathbf{k}3}\tilde{n}_{\mathbf{k}1}\tilde{n}_{\mathbf{k}3} - 2\tilde{\varepsilon}_{\mathbf{k}2}\tilde{\varepsilon}_{\mathbf{k}3}\tilde{n}_{\mathbf{k}2}\tilde{n}_{\mathbf{k}3}].\end{aligned} \tag{43}$$

Comparison of Eq. (43) with the expression for C_μ of an ideal Fermi gas makes it obvious the presence of mutual statistical interactions between the three species of the model.

We shall now discuss the low-temperature specific heat in the attractive case for $H<H_C$. In low fields, Eq. (43) reduces to

$$\begin{aligned}C_\mu(T,H<H_C,\mu) &= k_B\beta^2\sum_{\mathbf{k}}\tilde{\varepsilon}_{\mathbf{k}3}^2\tilde{n}_{\mathbf{k}3}(1-\tilde{n}_{\mathbf{k}3}),\\ &+ \mathcal{O}(e^{-\beta g\mu_B(H_C-H)/2}),\end{aligned} \tag{44}$$

as expected from Eq. (20). Here $\tilde{n}_{\mathbf{k}3}$ is given by Eq. (21). We can then obtain the linear specific-heat coefficient,

$$\gamma(H,\mu) = \lim_{T\to 0}\frac{C_\mu(T,H,\mu)}{NT}, \tag{45}$$

thus probing the thermal response of the distinct metallic regimes. Since $\gamma(H,n)=\gamma(H,\mu)+\lim_{T\to 0}[(\partial S/\partial\mu)_{T,H}(\partial\mu/\partial T)_{H,n}]$ and the last term nullifies, where $S(T,H,\mu)$ is the entropy, we can use $\mu(T=0,H\leqslant H_C,n)=-(\Delta/2)\cos(n\pi/2)$ to obtain

$$\gamma(H<H_C,n) = \frac{\pi k_B^2}{3\Delta\,\sin(n\pi/2)} = \frac{1}{4}\gamma_0(H=0,n), \tag{46}$$

where $\gamma_0(H=0,n)$ is the linear specific-heat coefficient of the zero field noninteracting Fermi gas on a chain. Note that the abrupt change of $\gamma(H,n)$ as the interaction is turned on, i.e., $\lim_{|U|\to 0}\lim_{H\to 0}\gamma(H,n)\neq\gamma_0(H=0,n)$, can be interpreted as a fractional statistics effect since only excluson 3 is thermally activated as $H\to 0$ before the interaction is switched off. We point out that in the on-site AHC $\gamma(H<H_C,n)$ is also H independent,[25] although it depends on U such that $\gamma(H=0,n;|U|\to 0)=(1/2)\gamma_0(H=0,n)$.[32] On the other hand, at $H=H_C$ the system attains a correlated regime where exclusons 1 and 3 coexist. In this case, C_μ reads

$$\begin{aligned}C_\mu(T,H=H_C,\mu) = k_B\beta^2\sum_{\mathbf{k}}\,[&\tilde{\varepsilon}_{\mathbf{k}1}^2\tilde{n}_{\mathbf{k}1}(1-\tilde{n}_{\mathbf{k}1})\\ &+ \tilde{\varepsilon}_{\mathbf{k}3}^2\tilde{n}_{\mathbf{k}3}(1-\tilde{n}_{\mathbf{k}3}) - 2\tilde{\varepsilon}_{\mathbf{k}1}\tilde{\varepsilon}_{\mathbf{k}3}\tilde{n}_{\mathbf{k}1}\tilde{n}_{\mathbf{k}3}],\\ &+ \mathcal{O}(e^{-\beta g\mu_B H_C}),\end{aligned} \tag{47}$$

where $\tilde{n}_{\mathbf{k}1}$ and $\tilde{n}_{\mathbf{k}3}$ are given by Eqs. (30) and (32) at $H=H_C$. We then find

$$\begin{aligned}\gamma(H=H_C,n) &= \frac{3\gamma_0(H=0,n)}{\pi^2}\int_{-\infty}^{+\infty}\frac{x^2(2+\cosh x)}{(1+2\cosh x)^2}dx\\ &= \frac{2\gamma_0(H=0,n)}{3},\end{aligned} \tag{48}$$

which can also be obtained by using Eqs. (38) and (42). However, for $H>H_C$ and $T\to 0$, we obtain $\gamma(H,n)=\gamma_0(H-H_C,n)$, in agreement with the discussion of the previous section, which nullifies in the insulating phase.

V. CONCLUDING REMARKS

In conclusion, we have studied the Hubbard model with infinite-range attractive interaction in the presence of a magnetic field. The central point of this paper has been the study of the low-temperature properties of the model, such as the magnetic susceptibility and specific heat, in connection with its exclusion representation. This investigation is evident by the existence of a critical field H_C which sets an energy scale in the study of the thermodynamic behavior of the system, where $g\mu_B H_C=|U|$ is the energy necessary to break a bounded pair of electrons with opposite spins at $T=0$. We also showed that whenever the correlation function $C(\mathbf{k})=\langle n_{\mathbf{k}\uparrow}n_{\mathbf{k}\downarrow}\rangle-\langle n_{\mathbf{k}\uparrow}\rangle\langle n_{\mathbf{k}\downarrow}\rangle=\mathcal{O}(1)$ the system displays nontrivial correlated behavior which manifests as fractional statistical effects in the context of the Haldane-Wu formalism. In fact, for $H < H_C$ and $2k_BT \ll g\mu_B(H-H_C)$ the system behaves as a spinless free Fermi gas governed by the effective statistical matrix $g^{eff}_{\mathbf{kk}';\alpha\alpha'}=\delta_{\mathbf{k},\mathbf{k}'}\delta_{\alpha,3}$, i.e., only exclusons 3 are thermally excited. In the vicinity of $H=H_C$ both the field and thermal responses must be treated carefully. Indeed, for $2k_BT > g\mu_B|H-H_C|$, the low-temperature properties should be derived using the effective nondiagonal statistical matrix $g^{eff}_{\mathbf{kk}';\alpha\alpha'}=\delta_{\mathbf{k},\mathbf{k}'}\binom{1\ 1}{0\ 1}$, $\alpha,\alpha'=1,3$, i.e., excitations of exclusons 1 and 3 must be properly considered. The above-mentioned cases contrast with the regime $2k_BT \ll g\mu_B(H-H_C)$, in which case $C(\mathbf{k})$ is exponentially small and the system behaves as a free Fermi gas, i.e., $g^{eff}_{\mathbf{kk}';\alpha\alpha'}=\delta_{\mathbf{k},\mathbf{k}'}\delta_{\alpha,\alpha'}$ $(\alpha,\alpha'=\uparrow,\downarrow)$.

In closing, one should stress that, even when $C(\mathbf{k}) = \mathcal{O}(1)$, the properties of interest have been calculated using both the fermionic approach, in which correlated $\uparrow$ and $\downarrow$ quasiparticles are employed, and the exclusion representation in which only statistical interations play a role.

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