Nonequilibrium thermodynamics of unsteady superfluid turbulence in counterflow and rotating situations

D. Jou[1] and M. S. Mongiovì[2]

[1]*Departament de Física, Universitat Autònoma de Barcelona 08193 Bellaterra, Catalonia, Spain*

[2]*Dipartimento di Metodi e Modelli Matematici, Università di Palermo c/o Facoltà di Ingegneria, Viale delle Scienze, 90128, Palermo, Italy*

(Received 17 December 2004; revised manuscript received 6 July 2005; published 21 October 2005)

The methods of nonequilibrium thermodynamics are used in this paper to relate an evolution equation for the vortex line density L, describing superfluid turbulence in the simultaneous presence of counterflow and rotation, to an evolution equation for the superfluid velocity $\mathbf{v}_s$, in order to be able to describe the full evolution of $\mathbf{v}_s$ and L, instead of only L. Two alternative possibilities are analyzed, related to two possible alternative interpretations of a term coupling the effects of the counterflow and rotation on the vortex tangle, and which imply some differences between situations where counterflow and rotation vectors are parallel or orthogonal to each other. One arrives to a modified Gorter-Mellink equation with new terms dependent on the angular speed. Finally, two proposals to describe the effects of anisotropy of the vortex tangle on the dynamical equations for $\mathbf{v}_s$ and L are examined.

DOI: 10.1103/PhysRevB.72.144517 PACS number(s): 67.40.Vs, 47.37.+q, 47.27.−i, 05.70.Ln

I. INTRODUCTION

The use of nonequilibrium thermodynamics for the analysis of unsteady superfluid turbulence[1–5] has revealed useful to explore its evolution equations and to suggest some experiments to discriminate between different microscopic interpretations leading to different macroscopic equations for the evolution of counterflow superfluid turbulence. A full description of this well-known phenomenon would require an evolution equation for the (averaged) vortex line density L describing the vortex tangle and another equation for the evolution of the (averaged) counterflow velocity $\mathbf{V}=\mathbf{v}_n-\mathbf{v}_s$ ($\mathbf{v}_n$ and $\mathbf{v}_s$ being the (averaged) velocities of the normal and superfluid components), which is related to the (averaged) heat flux $\mathbf{q}$ (as $\mathbf{q}=\rho_s T s \mathbf{V}$, ρ_s the density of the superfluid component, T the temperature, s the entropy).

The more subtle characteristics of the process, that differ from the averaged quantities L, $\mathbf{V}$, $\mathbf{v}_n$, and $\mathbf{v}_s$, and which may establish a link between the rotational of the local velocity and the vorticity, do not participate in this macroscopic description. Thus, for instance, an average homogeneous heat flux (or $\mathbf{V}$) produces, beyond some critical value, a complex mesh of vortex lines, whose local detailed description requires a statistical analysis. The macroscopic descriptions of this problem directly explore the relation between the macroscopic averages of L and $\mathbf{V}$. Though, up to now, most of the experiments in this field are carried out under a constant value of the counterflow velocity $\mathbf{V}$, some specific situations where the simultaneous variation in $\mathbf{V}$ and L may arise are, for instance: (a) letting $\mathbf{V}$ change in a periodic way and studying the effect of the frequency of this change on the time variation of L (b) cutting down suddenly the heat supply to the superfluid and studying the simultaneous decay of $\mathbf{V}$ (which will not be instantaneous) and of L. In both situations, the vortices will not follow the instantaneous value of $\mathbf{V}$, but the rate of change of variation of $\mathbf{V}$ will have an influence on the instantaneous value of L. An analysis of such unsteady situations is certainly challenging for a more complete understanding of the interactions between the counterflow and the vortex formation and destruction.

Another point studied in the present paper is the interaction between rotation, counterflow and vortex formation. The most known experiment on simultaneous rotation and counterflow is the apparatus of Swanson *et al.*[6] in which rotation and heat flow are parallel to each other. It would be easy to make them antiparallel, by simply rotating the container in the opposite sense, and this would reveal features which are not seen if only the parallel situation is studied. Furthermore, it would be easy to have a situation where the heat flux and the rotation vector are neither parallel nor antiparallel: for instance, one could incorporate a thin heat conductor along the rotation axis and keep it at a temperature higher than that of the wall: in this way, one would have a controllable radial heat flux in addition to the usual longitudinal heat flux. This would make that the local heat flux were not locally parallel to the rotation vector. Though here we are interested in averaged values of the vortex line density, rather than in a detailed local formulation, we could consider the average over angular sections around the axis, which should exhibit the consequences of this lack of parallelism between both vectors. An other experiment in which heat flux is orthogonal to the rotating axis, which have a very different geometry, is that performed by Yarmchuk and Glaberson[7] which will be discussed in Sec. VI.

In this paper, we will carry out an analysis of these situations, by combining nonequilibrium thermodynamics and a previous equation[8] we proposed for the interaction between counterflow and rotation (when they are parallel to each other). Even in this situation, we outline two different possible extensions of our former equation to the situation where counterflow and rotation are not parallel to each other. We explore the restrictions of Onsager-Casimir reciprocity relations in both cases, and such an analysis let us obtain two alternative versions for the microscopic force between the counterflow and the rotating vortices. Thus, though seemingly formal, our paper does suggest new experiments and emphasizes on their possible microscopic significance.

The evolution equation for L under constant values of $\mathbf{V}$ has been explored for many years. In summary, neglecting the influence of the walls, such an equation is the well-known Vinen's equation for the evolution of the vortex line density L:[9,10]

$$\frac{dL}{dt} = \alpha V L^{3/2} - \beta\kappa L^2, \tag{1.1}$$

with V the absolute value of the counterflow velocity, $\kappa = h/m$ the quantum of rotation (m the mass of the ^{4}He atom and h Planck's constant) and α and β dimensionless parameters.

In general conditions, the velocity $\mathbf{V}$ could also change with time, and therefore a full description of the problem would require to know an evolution equation for $\mathbf{V}$. Instead, one can write an equation for the superfluid velocity $\mathbf{v}_s$, linked to the counterflow velocity by the relation $\mathbf{v}_s = \mathbf{v} - (\rho_n/\rho)\mathbf{V}$ ($\mathbf{v}$ being the velocity of the mixture). Of course, the evolution equations for L and for $\mathbf{v}_s$ would be strongly coupled with each other, as $\mathbf{V}$ influences the vortex tangle, which modifies on its turn the velocity. The analysis of such evolution equations may be undertaken from several perspectives, amongst them macroscopic nonequilibrium thermodynamics, which guarantees their consistency with the second law.

Vinen's equation has been obtained from different microscopic models, differing in their interpretations of the annihilation of the vortices: that of Feynman-Vinen[9–11] and that of Schwarz.[12–14] In the Feynman-Vinen model this process is attributed to the transformation of their energy into heat through the breaking of small vortex rings into thermal excitations. In Schwarz's model the annihilation of the vortices is interpreted as a return of their energy to the kinetic energy of the main flow, rather than to its internal energy. Both interpretations yield the same Eq. (1.1) for L, but they lead, as it will be seen, to different predictions for the equation for $\mathbf{v}_s$.

From the point of view of nonequilibrium thermodynamics it is interesting to consider the problem of simultaneous evolution of $\mathbf{v}_s$ and L and their possible couplings. In Refs. 1, 5, and 15, Nemirowskii *et al.* have shown that application of Onsager-Casimir reciprocity relations to this problem leads to an evolution equation for $\mathbf{v}_s$ going beyond the so-called Gorter-Mellink force in unsteady situations; in particular they showed that different predictions for the sign of the additional coupling term are obtained according to which microscopic interpretation is used. Such couplings suggest experiments which would indicate which interpretation is the most suitable one.

The aim of the present paper is to extend this kind of analysis to a more general range of phenomena, simultaneously including not only counterflow turbulence but also the ordered array of vortices arising when the superfluid is submitted to a rotation. The point under consideration has much current interest because of the increasing experimental and theoretical activity in situations combining rotation and counterflow,[6–8,16–21] where the basic set of equations is still not settled out.

The plan of the paper is the following one. In Sec. II we give a sketch of the Nemirowskii analysis, which sets the framework we will use. In Secs. III and IV we analyze two different possible interpretations of a term coupling counterflow and rotation and its consequences on the evolution equation for $\mathbf{v}_s$. In Sec. V we discuss two descriptions of the anisotropy of the vortex tangle and its effects on the dynamical equations for L and $\mathbf{v}_s$. In the final section we perform a qualitative comparison with experiments of the predictions of our two interpretations in the case of parallel and orthogonal counterflow and rotation and we present some simple microscopic arguments on the possible role of the relative direction of counterflow and rotation.

II. BRIEF REVIEW OF COUNTERFLOW THERMODYNAMIC ANALYSIS

Here we briefly review the essential lines of the thermodynamic analysis of the evolution equations for L and $\mathbf{v}_s$ for a description of counterflow turbulence in unsteady states, as presented in Refs. 1, 5, and 15, and whose ideas will be extended in the next sections.

In summary, Nemirowskii *et al.*[5,15] consider for the entropy density s of the superfluid in the presence of vortex lines a differential form which may be written as

$$T\frac{ds}{dt} = -\rho_s \mathbf{V}\frac{d\mathbf{v}_s}{dt} + \epsilon_V \frac{dL}{dt}, \tag{2.1}$$

with

$$-\rho_s\mathbf{V} \equiv \frac{\partial u}{\partial \mathbf{v}_s}, \quad \epsilon_V \equiv \frac{\partial u}{\partial L} = \frac{\rho_s\kappa^2}{4\pi}\ln\left[\frac{1}{a_0 L^{1/2}}\right], \tag{2.2}$$

being u the internal energy density and ϵ_V the contribution to the internal energy per unit length of the vortex line (a_0 is the dimension of the vortex core, which is very small, of the order of one Å). According to the formalism of nonequilibrium thermodynamics one may obtain evolution equations for $\mathbf{v}_s$ and L by writing $d\mathbf{v}_s/dt$ and dL/dt in terms of their conjugate forces $-\rho_s\mathbf{V}$ and ϵ_V, including coupling terms between each other, in the matrix form

$$\begin{bmatrix} \dfrac{d\mathbf{v}_s}{dt} \\ \dfrac{dL}{dt} \end{bmatrix} = L\begin{bmatrix} -\dfrac{\rho_n}{\rho\rho_s}A\mathbf{U} & \pm\dfrac{\alpha}{\rho_s}\dfrac{\mathbf{V}}{|V|}L^{1/2} \\ -\dfrac{\alpha}{\rho_s}\dfrac{\mathbf{V}}{|V|}L^{1/2} & -\kappa\dfrac{\beta}{\epsilon_V}L \end{bmatrix}\begin{bmatrix} -\rho_s\mathbf{V} \\ \epsilon_V \end{bmatrix}, \tag{2.3}$$

where $\mathbf{U}$ is the unit matrix.

Here and in the following, the average line density L is defined as $L = (1/\Lambda)\int d\xi$, where ξ is the arc length along the vortices and the integral is taken along all vortices in the sample volume Λ. Also $\mathbf{V}$ and $\mathbf{v}_s$ are averaged velocities and do not coincide with the local counterflow and superfluid velocities. As usual in the study of turbulent phenomena, the more subtle characteristics of the process, that differ from the averaged quantitities L, $\mathbf{V}$, and $\mathbf{v}_s$ do not participate in this macroscopic description.

In Eq. (2.3) the second equation has been written to recover Vinen's equation (1.1). In the first one, which was the aim of Nemirowskii research, A is a friction coefficient whereas the second term (for which we keep a double sign to discuss the ambiguity related to it) comes in a natural way from the Onsager-Casimir reciprocity relation. In Feynman-Vinen view, L is a scalar quantity which does not change under time reversal, unlike the superfluid velocity $\mathbf{v}_s$ which changes sign. According to Onsager-Casimir, this leads to antisymmetry of crossed coefficients thus leading to the + sign. In Schwarz view, L possesses vectorial properties and it would change on time reversal, just like the superfluid velocity. This leads to the symmetry of the kinetic coefficients in the matrix in Eq. (2.3), i.e., to the − sign in the upper right-hand term.

Now, we focus our attention on the equation for $d\mathbf{v}_s/dt$, given by the first line in Eq. (2.3). which is

$$\frac{d\mathbf{v}_s}{dt} = \frac{\rho_n}{\rho} AL\mathbf{V} \pm \frac{\alpha}{\rho_s}\frac{\mathbf{V}}{V} L^{3/2}\epsilon_V. \tag{2.4}$$

The second term does not depend on the modulus of $\mathbf{V}$, but only on its direction, and it is then called a "dry friction" force in analogy with the force acting in the friction between two solids. This coupling between $d\mathbf{v}_s/dt$ and ϵ_V arises naturally in the scheme of classical irreversible thermodynamics, and its sign depends on the interpretation of L, as it has been stressed.

In Gorter-Mellink law, L is supposed to be given by its steady-state value, which, according with Eq. (1.1) is

$$L = \frac{\alpha^2}{\beta^2\kappa^2} V^2, \tag{2.5}$$

and Eq. (2.4) may be written as

$$\frac{d\mathbf{v}_s}{dt} = \left[\frac{\rho_n}{\rho} A \frac{\alpha^2}{\beta^2\kappa^2} \pm \frac{\alpha^{5/2}\epsilon_V}{\rho_s(\beta\kappa)^{3/2}}\right] V^2\mathbf{V} \equiv A' V^2\mathbf{V}, \tag{2.6}$$

with A' a coefficient, dependent on temperature T, defined by Eq. (2.6), i.e., by the combination of quantities appearing in the second term of Eq. (2.6), and it is related to the so-called Gorter-Mellink force between normal fluid and the vortex tangle, proportional to V^3. However, in the complete model (2.3), L is not always given by Eq. (2.5) and the two terms in Eq. (2.6) may behave in different ways, according to Eq. (2.4), in unsteady states.

Thus, application of nonequilibrium thermodynamics yields an evolution equation for $\mathbf{v}_s$ with new terms, which are not present in the most intuitive and simple version of the theory, based on a friction force, just the first term in Eq. (2.4). The sign of the new term depends on the microscopic interpretation. The final decision will depend on the consistency with experiments.[15]

III. EVOLUTION EQUATION FOR L IN COUNTERFLOW IN ROTATING CONTAINERS: POSSIBLE INTERPRETATIONS OF THE COUPLING TERM

In Ref. 8, we have proposed for the evolution of L in the presence of $\mathbf{V}$ and Ω the following phenomenological generalization of Vinen's equation:

$$\frac{dL}{dt} = -\beta\kappa L^2 + [\alpha_1 V + \beta_2\sqrt{\kappa\Omega}]L^{3/2} - \left[\beta_1\Omega + \beta_4\frac{V\sqrt{\Omega}}{\sqrt{\kappa}}\right]L, \tag{3.1}$$

where terms dependent on Ω (the absolute value of the angular velocity $\mathbf{\Omega}$) appear, which are not present in Eq. (1.1). Here, we explore how the terms in Ω influence the evolution equation for L. In particular we will pay a special attention to the term $V\sqrt{\Omega}$ which plays an especially relevant role in our equation (3.1), because it describes the nonlinear coupling between rotation and counterflow, whose effects are non additive, as is known by experiments.[6] The status of this term must be clearly understood, because, for the moment, we are still lacking for a microscopic interpretation for it, in spite that it accounts for current experimental observations.[8]

One could consider two possible alternatives: in the first one $V\sqrt{\Omega}$ depends on the angle between $\mathbf{V}$ and $\mathbf{\Omega}$, i.e., on the scalar product $\mathbf{V}\cdot\Omega$, in the second one it does not depend on the angle between these two vectors, but only on their absolute values. As $\mathbf{V}$ is a polar vector and $\mathbf{\Omega}$ an axial vector, a mathematically consistent version of Eq. (3.1), which contains both these alternatives, is

$$\begin{aligned}\frac{dL}{dt} = &-\beta\kappa L^2 + L^{3/2}\left[\alpha_1\hat{\mathbf{V}}\cdot\mathbf{U}\cdot\mathbf{V} + \frac{\beta_2}{\sqrt{\kappa\Omega}}\hat{\mathbf{\Omega}}\cdot\mathbf{U}\cdot\mathbf{\Omega}\right] \\ &- L\left[\beta_1\hat{\mathbf{\Omega}}\cdot\mathbf{U}\cdot\mathbf{\Omega} + \frac{\beta_4}{\sqrt{\kappa\Omega}}\Omega\cdot(a_1\hat{\mathbf{V}}\hat{\mathbf{\Omega}} + a_2\hat{\mathbf{\Omega}}\hat{\mathbf{V}})\cdot\mathbf{V}\right],\end{aligned} \tag{3.2}$$

$\mathbf{U}$ being the second order unit tensor, with $a_1+a_2=1$ and where $\hat{\mathbf{V}}\hat{\mathbf{\Omega}}$ and $\hat{\mathbf{\Omega}}\hat{\mathbf{V}}$ are the diadic products between $\hat{\mathbf{V}}$ and $\hat{\mathbf{\Omega}}$, being $\hat{\mathbf{V}}$ and $\hat{\mathbf{\Omega}}$ the unit vectors parallel to $\mathbf{V}$ and $\mathbf{\Omega}$. In particular, if $a_1=a_2=1/2$, the tensor $a_1\hat{\mathbf{V}}\hat{\mathbf{\Omega}}+a_2\hat{\mathbf{\Omega}}\hat{\mathbf{V}}$, responsible for the coupling between $\mathbf{V}$ and $\mathbf{\Omega}$, is symmetric.

In the situation which has been studied theoretically and experimentally up to now, namely, a container rotating around its axis and heated from the base, where $\mathbf{\Omega}$ and $\mathbf{V}$ are parallel to each other, the two interpretations $a_1=1$, $a_2=0$ and $a_1=0$, $a_2=1$ are equivalent, but these two alternatives will lead to different results in other physically interesting situations, as for instance a cylindrical container rotating around its axis and heated radially along it, in which case $\mathbf{V}$ would be radial (i.e., $\mathbf{V}$ perpendicular to $\mathbf{\Omega}$), a situation for which, to our knowledge, there are no experimental analyses neither numerical simulations.

IV. SIMULTANEOUS ROTATION AND COUNTERFLOW: NONLINEAR COUPLING

Up to here, we have restricted ourselves to discuss the results of (Ref. 8) and to summarize the main ideas of the analysis of Nemirowskii *et al.*[1,5,15] In this section, we follow the general lines of their work to study the evolution equations of superfluid turbulence in simultaneous presence of counterflow and rotation.

Our aims are to find the evolution equation for $\mathbf{v}_s$ consistent with Eq. (3.1) and to explore some topics related with the anisotropy of the tangle, which are not dealt neither in Vinen's equation (1.1), which assumes an isotropic tangle, neither in our proposal (3.1). In the present section we will analyze the nonlinear coupling, keeping our work as parallel as possible with Nemirowskii one, and in Sec. V we will consider the anisotropy.

First of all, it is important to recall that in the presence of pure rotation (which produces an ordered array of vortex lines parallel to the rotation axis), the evolution equation of $\mathbf{v}_s$ has the form[22]

$$\frac{d\mathbf{v}_s}{dt} + 2\mathbf{\Omega} \times \mathbf{v}_s + \mathbf{i}_0 = -\frac{\rho_n}{\rho} B \hat{\mathbf{\Omega}} \times \mathbf{\Omega} \times \mathbf{V} - \frac{\rho_n}{\rho} B' \mathbf{\Omega} \times \mathbf{V}, \tag{4.1}$$

where $\mathbf{i}_0$ is the inertial force and B and B' the Hall-Vinen dimensionless coefficients describing the interaction between the normal fluid and the vortex lines. Both these coefficients depend in a complicated manner on the temperature.[23]

If we take into account that, in this case, $L=\tilde{\alpha}|\Omega|$ (with $\tilde{\alpha}=2/\kappa$), this equation may be rewritten in terms of L as

$$\frac{D\mathbf{v}_s}{dt} = -L\frac{\rho_n}{\rho}\frac{B}{\tilde{\alpha}}\left(\hat{\mathbf{\Omega}} \times \hat{\mathbf{\Omega}} \times \mathbf{V} + \frac{B'}{B}\hat{\mathbf{\Omega}} \times \mathbf{V}\right). \tag{4.2}$$

where we have denoted for simplicity of notation

$$\frac{D\mathbf{v}_s}{dt} = \frac{d\mathbf{v}_s}{dt} + 2\mathbf{W} \cdot \mathbf{\Omega} \cdot \mathbf{v}_s + \mathbf{i}_0, \tag{4.3}$$

with $\mathbf{W}$ the totally antisymmetric third-order tensor such that $\mathbf{W} \cdot \mathbf{\Omega} \cdot \mathbf{v}_s = \mathbf{\Omega} \times \mathbf{v}_s$.

In the presence of an isotropic contribution of the vortex tangle, due to the simultaneous presence of the counterflow, an additional term of the form (2.4) should be included, and further an additive contribution $\mathbf{F}_{\text{coupl}}=\mathbf{F}_c(\mathbf{V},\mathbf{\Omega})$ due to couplings between counterflow, rotation and superfluid velocity, in a way similar to that presented in Sec. II:

$$\frac{D\mathbf{v}_s}{dt} = (1-b)\frac{\rho_n}{\rho}AL\mathbf{V} - bL\frac{\rho_n}{\rho}\frac{B}{\tilde{\alpha}}\left(\hat{\mathbf{\Omega}} \times \hat{\mathbf{\Omega}} \times \mathbf{V} + \frac{B'}{B}\hat{\mathbf{\Omega}} \times \mathbf{V}\right) \pm \frac{\alpha}{\rho_s}L^{3/2}\epsilon_V \hat{\mathbf{V}} + \mathbf{F}_c(\mathbf{V},\mathbf{\Omega}). \tag{4.4}$$

Here b is a parameter related to the anisotropy of vortex lines, describing the relative weight of the array of vortex lines parallel to $\mathbf{\Omega}$ and the isotropic tangle: when $b=0$ we recover an isotropic tangle and when $b=1$ the ordered array. In the pure isotropic limit, we are thus left with the $(\rho_n/\rho)AL\mathbf{V}$ contribution in the first term of Eq. (4.4).

Observing that for an isotropic tangle it results $A=2B/3\tilde{\alpha}$,[23] and introducing the tensor

$$\Pi \equiv (1-b)\frac{2}{3}\mathbf{U} + b\left(\mathbf{U} - \hat{\mathbf{\Omega}}\hat{\mathbf{\Omega}} + \frac{B'}{B}\mathbf{W} \cdot \hat{\mathbf{\Omega}}\right), \tag{4.5}$$

Eq. (4.4) can be written

$$\frac{D\mathbf{v}_s}{dt} = \frac{\rho_n}{\rho}L\frac{B}{\tilde{\alpha}}\Pi \cdot \mathbf{V} \pm \frac{\alpha}{\rho_s}\frac{\mathbf{V}}{V}L^{3/2}\epsilon_V + \mathbf{F}_c(\mathbf{V},\mathbf{\Omega}). \tag{4.6}$$

The tensor (4.5) provides a description of some aspects of the anisotropy of the tangle in presence of counterflow and rotation, as we will see in more detail in Sec. V.

We want to mention, however that, in Ref. 24, we have proposed to describe the superfluid turbulence in the framework of extended thermodynamics,[25] using as independent variables the heat flux $\mathbf{q}$ and a vorticity tensor $\mathbf{P}_\omega$ associated to the vortex line. In such a case, the time derivative of the entropy density s_{EIT} has the form

$$T\frac{ds_{\text{EIT}}}{dt} = \vec{\gamma} \cdot \frac{d\mathbf{q}}{dt} + \underline{\underline{\epsilon}}^V : \frac{d\mathbf{P}_\omega}{dt}, \tag{4.7}$$

where $\vec{\gamma}$ and $\underline{\underline{\epsilon}}^V$ are the variables conjugated to $\mathbf{q}$ and $\mathbf{P}_\omega$. As it has been mentioned in the Introduction, $\mathbf{V}$ is clearly related to $\mathbf{q}$. Concerning $\mathbf{P}_\omega$, it has the form[24,26]

$$\mathbf{P}_\omega = \kappa L[\lambda\langle \mathbf{U} - \mathbf{s}'\mathbf{s}' \rangle + \lambda' \langle \mathbf{W} \cdot \mathbf{s}' \rangle], \tag{4.8}$$

$\mathbf{s}'$ being the unit vector tangent to the vortices, λ and λ' dimensionless functions of T and ρ, such that

$$\frac{\lambda'}{\lambda} = \frac{B'}{B} \tag{4.9}$$

and brackets denote macroscopic average. As we will see in more detail in Sec. V, tensor $\mathbf{P}_\omega$ is linked to the tensor Π defined by Eq. (4.5) by the relation $\mathbf{P}_\omega = \kappa L \lambda \Pi$.

Our aim here is to include Eqs. (3.2) and (4.6) into a common thermodynamic framework, and study possible couplings between them. Thus, as well as in Sec. II, we will consider the evolution equations for $\mathbf{v}_s$ and L and we will write $d\mathbf{v}_s/dt$ and dL/dt in terms of $-\rho_s\mathbf{V}$ and ϵ_V, in order to recover the generalized Eq. (3.1) for dL/dt and Eq. (4.6) plus some possible coupling for $d\mathbf{v}_s/dt$, in analogy with Sec. II.

A. Interpretation of $V\sqrt{\Omega}$ as $(1/\sqrt{\Omega})\mathbf{\Omega} \cdot \hat{\mathbf{V}}\hat{\mathbf{\Omega}} \cdot \mathbf{V}$ [$a_1=1$, $a_2=0$ in equation (3.2)]

In this subsection, we interpret the term $V\sqrt{\Omega}$ as $(1/\sqrt{\Omega})\mathbf{\Omega} \cdot \hat{\mathbf{V}}\hat{\mathbf{\Omega}} \cdot \mathbf{V} = V\sqrt{\Omega}\cos^2\theta$. As one sees, this term depends on the angle θ between $\mathbf{V}$ and $\mathbf{\Omega}$, reducing to zero when $\mathbf{V}$ is perpendicular to $\mathbf{\Omega}$.

Recall that in Eq. (3.1) we wrote $V\sqrt{\Omega}$ because in the experiments we wanted to describe by means of it, $\mathbf{V}$ and $\mathbf{\Omega}$ where parallel to each other, whereas a situations with the angle between $\mathbf{V}$ and $\mathbf{\Omega}$ different from zero, are not known by us. In this spirit, we write in the second line of the following system (4.10), the equation for L in the form given in Eq. (3.2) (with $a_1=1$ and $a_2=0$) and by means of Onsager-Casimir reciprocity we build up the second part of the evolution equation for $\mathbf{v}_s$, as was outlined in Sec. II. The result is

$$\begin{bmatrix} \dfrac{D\mathbf{v}_s}{dt} \\ \dfrac{dL}{dt} \end{bmatrix} = L \begin{bmatrix} -\dfrac{\rho_n}{\rho\rho_s}\dfrac{B}{\tilde{\alpha}}\Pi & -\dfrac{\beta_4}{\rho_s\sqrt{\kappa\Omega}}(\hat{\mathbf{\Omega}}\cdot\hat{\mathbf{V}})\mathbf{\Omega} + \dfrac{\alpha_1}{\rho_s}\hat{\mathbf{V}}L^{1/2} \\ \dfrac{\beta_4}{\rho_s\sqrt{\kappa\Omega}}(\hat{\mathbf{\Omega}}\cdot\hat{\mathbf{V}})\mathbf{\Omega} - \dfrac{\alpha_1}{\rho_s}\hat{\mathbf{V}}L^{1/2} & -\dfrac{1}{\epsilon_V}(\beta\kappa L - \beta_2\sqrt{\kappa\Omega}L^{1/2} + \beta_1\Omega) \end{bmatrix} \begin{bmatrix} -\rho_s\mathbf{V} \\ \epsilon_V \end{bmatrix}, \tag{4.10}$$

according to the form (4.6) of the evolution equation for $\mathbf{v}_s$ in the simultaneous presence of the ordered vortex array produced by rotation and the disordered tangle produced by the counterflow.

As in (2.4), the new term not contained in the previous evolution equation for $\mathbf{v}_s$ is the coupling term between $d\mathbf{v}_s/dt$ and ϵ_V in the matrix in Eq. (4.10). The ambiguity present in that equation is omitted, because, for the sake of simplicity, we assume here the Feynman-Vinen microscopic interpretation, leading to an antisymmetric matrix. Accordingly, if we write Eq. (3.2) as

$$\frac{dL}{dt} = -\beta\kappa L^2 + [\alpha_1 V + \beta_2\sqrt{\kappa\Omega}]L^{3/2} - \left[\beta_1\Omega + \frac{\beta_4}{\sqrt{\kappa\Omega}}\frac{(\mathbf{\Omega}\cdot\mathbf{V})^2}{\Omega V}\right]L, \tag{4.11}$$

the corresponding evolution equation of $\mathbf{v}_s$ is

$$\frac{d\mathbf{v}_s}{dt} + 2\mathbf{\Omega}\times\mathbf{v}_s + \mathbf{i}_0 = L\frac{\rho_n}{\rho}\frac{B}{\tilde{\alpha}}\Pi\cdot\mathbf{V} + L\frac{\epsilon_V}{\rho_s}\left(\alpha_1\hat{\mathbf{V}}L^{1/2} - \frac{\beta_4}{\sqrt{\kappa\Omega}}(\hat{\mathbf{\Omega}}\cdot\hat{\mathbf{V}})\mathbf{\Omega}\right). \tag{4.12}$$

Comparing with Eq. (4.6) we see that in this interpretation is

$$\mathbf{F}_{\mathrm{coupl}} = -L\frac{\epsilon_V}{\rho_s}\frac{\beta_4}{\sqrt{\kappa\Omega}}(\hat{\mathbf{\Omega}}\cdot\hat{\mathbf{V}})\mathbf{\Omega}. \tag{4.13}$$

This term is the one corresponding to the $\mathbf{\Omega}\cdot\hat{\mathbf{V}}\hat{\mathbf{\Omega}}\cdot\mathbf{V}$ contribution in Eq. (4.10). It is collinear with $\mathbf{\Omega}$, depends on the angle between $\mathbf{V}$ and $\mathbf{\Omega}$, vanishing in particular when $\mathbf{\Omega}$ is orthogonal to $\mathbf{V}$.

The presence of this term may be linked to the axial mutual friction force, present in rotating helium II, and evidenced when the direction of propagation of the second sound is not orthogonal with respect to the vortex lines. Indeed, careful measurements by Snyder and Putney[27] and by Mathieu, Plaçais, and Simon[28] have evidenced for some axial mutual friction effects not completely understood (see also Ref. 23). These effects may be due to the presence, in the tensor (4.5) of a term of the type $\hat{\mathbf{\Omega}}\hat{\mathbf{\Omega}}$. In this case, tensor Π should be written

$$\Pi = (1-b)\frac{2}{3}\mathbf{U} + b\left[\left(1-\frac{B''}{B}\right)(\mathbf{U}-\hat{\mathbf{\Omega}}\hat{\mathbf{\Omega}}) + 2\frac{B''}{B}\hat{\mathbf{\Omega}}\hat{\mathbf{\Omega}} - \frac{B'}{B}\mathbf{W}\cdot\hat{\mathbf{\Omega}}\right], \tag{4.14}$$

which gives rise to a force, parallel to $\mathbf{\Omega}$ dependent on the counterflow velocity $\mathbf{V}$:

$$\mathbf{F}_{\mathrm{axial}} = 2bL\frac{B''}{\tilde{\alpha}}\frac{\rho_n}{\rho}(\mathbf{V}\cdot\hat{\mathbf{\Omega}})\hat{\mathbf{\Omega}}. \tag{4.15}$$

We think that the coupling between $\mathbf{V}$ and $\mathbf{\Omega}$ may be responsible also of a small dry-friction effect parallel to $\mathbf{\Omega}$ and leading to the latter term in Eq. (4.12).

B. Interpretation of $V\sqrt{\Omega}$ as $V\sqrt{\Omega}=(1/\sqrt{\Omega})\mathbf{\Omega}\cdot\hat{\mathbf{\Omega}}\hat{\mathbf{V}}\cdot\mathbf{V}$ [($a_1=0$, $a_2=1$ in Eq. (3.2)]

In this subsection we study the consequences of assuming that the term $V\sqrt{\Omega}$ in Eq. (3.1) does not depend on the relative angle θ between the vectors $\mathbf{V}$ and $\mathbf{\Omega}$. In this case we will write instead of Eq. (4.10)

$$\begin{bmatrix} \dfrac{D\mathbf{v}_s}{dt} \\ \dfrac{dL}{dt} \end{bmatrix} = L \begin{bmatrix} -\dfrac{\rho_n}{\rho\rho_s}\dfrac{B}{\tilde{\alpha}}\Pi & \dfrac{1}{\rho_s}\left[\alpha_1 L^{1/2} - \beta_4\dfrac{\sqrt{\Omega}}{\sqrt{\kappa}}\right]\hat{\mathbf{V}} \\ -\dfrac{1}{\rho_s}\left[\alpha_1 L^{1/2} - \beta_4\dfrac{\sqrt{\Omega}}{\sqrt{\kappa}}\right]\hat{\mathbf{V}} & -\dfrac{1}{\epsilon_V}[\beta\kappa L - \beta_2\sqrt{\kappa\Omega}L^{1/2} + \beta_1\Omega] \end{bmatrix} \begin{bmatrix} -\rho_s\mathbf{V} \\ \epsilon_V \end{bmatrix}. \tag{4.16}$$

As in Eq. (4.10), we have written the second line of Eq. (4.16) in order to reproduce Eq. (3.1) [or, equivalently Eq. (3.2) with $a_1=0$ and $a_2=1$] and we have required the matrix to follow the Onsager-Casimir symmetry. Then, if the equation for dL/dt is Eq. (3.1), the equation for $d\mathbf{v}_s/dt$ is different from Eq. (4.12). Indeed it is

$$\frac{d\mathbf{v}_s}{dt} + 2\mathbf{\Omega} \times \mathbf{v}_s + \mathbf{i}_0 = L\frac{\rho_n}{\rho}\frac{B}{\tilde{\alpha}}\Pi \cdot \mathbf{V} + L\frac{\epsilon_V}{\rho_s}\left(\alpha_1 L^{1/2} - \frac{\beta_4}{\sqrt{\kappa}}\sqrt{\Omega}\right)\hat{\mathbf{V}}. \tag{4.17}$$

In this case the new term, corresponding to the $V\sqrt{\Omega}$ contribution in Eq. (3.1), has the form

$$\mathbf{F}_{\text{coupl}} = -\beta_4 L\frac{\epsilon_V}{\rho_s}\sqrt{\frac{\Omega}{\kappa}}\hat{\mathbf{V}}, \tag{4.18}$$

it is parallel to $\mathbf{V}$, but does not depend on the modulus of $\mathbf{V}$, and is therefore a dry friction term as the one of Nemirovskii.

We recall now that in Ref. 8 we have found, in the steady state (L and $\mathbf{V}$ constant), that the solution of the equation for L in Eq. (4.16) is

$$L^{1/2} = \frac{\beta_4}{\alpha_1}\sqrt{\frac{\Omega}{\kappa}} \quad \text{for } 0 < V < V_{c2}, \tag{4.19}$$

$$L^{1/2} = \frac{\alpha_1}{\beta\kappa}(V - V_{c2}) + \frac{\beta_4}{\alpha_1}\sqrt{\frac{\Omega}{\kappa}} \quad \text{for } V > V_{c2}, \tag{4.20}$$

while the critical value V_{c2} of the velocity V, which characterizes the transition to a turbulent disordered tangle, found in Ref. 8 is, in agreement with experimental observations,

$$V_{c2} = \left[2\frac{\beta_4}{\alpha_1} - \frac{\beta_2}{\beta}\right]\sqrt{\Omega\kappa}. \tag{4.21}$$

Substituting Eqs. (4.19) and (4.20) in the off-diagonal term in the matrix in Eq. (4.16), one obtains the following expression for the dry friction force:

$$\mathbf{F}_{\text{dry}} = \frac{\epsilon_V}{\rho_s}\left[\alpha_1 L^{1/2} - \beta_4\frac{\sqrt{\Omega}}{\sqrt{\kappa}}\right]\hat{\mathbf{V}} \equiv 0 \quad \text{for } V < V_{c2}, \tag{4.22}$$

$$\mathbf{F}_{\text{dry}} = \frac{\epsilon_V}{\rho_s}\left[\alpha_1 L^{1/2} - \beta_4\frac{\sqrt{\Omega}}{\sqrt{\kappa}}\right]\hat{\mathbf{V}} \equiv \frac{1}{\rho_s}\frac{\alpha_1}{\beta\kappa}(V - V_{c2})\hat{\mathbf{V}} \quad \text{for } V > V_{c2}. \tag{4.23}$$

As a consequence, the dry-friction force is absent for $V < V_{c2}$ (and in pure rotation, too) and it is equal to Eq. (4.23) for $V > V_{c2}$, when the array produced by the rotation becomes a disordered nonisotropic tangle. Indeed, in a steady state (L and Ω constant), Eq. (4.17), would take the expression

$$\frac{D\mathbf{v}_s}{dt} = L\frac{\rho_n}{\rho}\frac{B}{\tilde{\alpha}}\Pi \cdot \mathbf{V} \tag{4.24}$$

with L expressed by Eq. (4.19), for $V < V_{c2}$, and

$$\frac{D\mathbf{v}_s}{dt} = L\frac{\rho_n}{\rho}\frac{B}{\tilde{\alpha}}\Pi \cdot \mathbf{V} + \frac{\epsilon_V}{\rho_s}\frac{\alpha_1}{\beta\kappa}(V - V_{c2})\hat{\mathbf{V}}, \tag{4.25}$$

with L expressed by Eq. (4.20), for $V > V_{c2}$. Summarizing, for $V < V_{c2}$ the dry-friction force is absent, while, for $V > V_{c2}$, when the array of rectilinear vortex lines becomes a disordered tangle, an additional term collinear with $\mathbf{V}$ appears. Thus V_{c2} indicates the threshold not only of the vortex line dynamics but also of the friction acting on the velocity $\mathbf{v}_s$ itself; this seems logical, as both variables are mutually related, in general terms.

C. General case

In the general case (a_1 and a_2 both different from zero, $a_1 + a_2 = 1$) there is a superposition of the effects described in Secs. IV A and IV B. It is easily seen that in this case the evolution equation for L is Eq. (3.2) while the evolution equation for $\mathbf{V}$ is

$$\frac{d\mathbf{v}_s}{dt} + 2\mathbf{\Omega} \times \mathbf{v}_s + \mathbf{i}_0 = L\frac{\rho_n}{\rho}\frac{B}{\tilde{\alpha}}\Pi \cdot \mathbf{V} + L\frac{\epsilon_V}{\rho_s}\left[\alpha_1 L^{1/2}\hat{\mathbf{V}} - \frac{\beta_4}{\sqrt{\kappa\Omega}}(a_1\hat{\mathbf{V}}\hat{\mathbf{\Omega}} + a_2\hat{\mathbf{\Omega}}\hat{\mathbf{V}}) \cdot \mathbf{\Omega}\right]. \tag{4.26}$$

In this case, the coupling term $\mathbf{F}_{\text{coupl}}$ assumes the most general expression

$$\mathbf{F}_{\text{coupl}} = -L\frac{\epsilon_V}{\rho_s}\frac{\beta_4}{\sqrt{\kappa\Omega}}[a_1(\mathbf{\Omega} \cdot \hat{\mathbf{V}})\hat{\mathbf{\Omega}} + a_2(\mathbf{\Omega} \cdot \hat{\mathbf{\Omega}})\hat{\mathbf{V}}]. \tag{4.27}$$

The term with a_1 is collinear with $\mathbf{\Omega}$ and depends on the angle within $\mathbf{V}$ and $\mathbf{\Omega}$ while the term with a_2 implies a reduction of the force on $\mathbf{v}_s$ related to $\sqrt{\Omega}$ and it would be collinear with $\mathbf{V}$ instead than with $\mathbf{\Omega}$.

We are not aware of any previous proposal of a contribution such as the term (4.27); if, for example, we consider the experiment described in Ref. 6, where $\mathbf{V}$ and $\mathbf{\Omega}$ are collinear, this new term indicates that an isotropic tangle rotating with angular speed $\mathbf{\Omega}$ would exert a smaller force on $\mathbf{v}_s$ than the same non-rotating tangle. We do not have for the moment a microscopic interpretation for this contribution, though its related term in Eq. (3.2) describes well the experimental results.

V. SIMULTANEOUS ROTATION AND COUNTERFLOW: ANISOTROPY OF THE TANGLE

The assumption of an isotropic tangle (referred to the orientational distribution of the vector tangent to the vortex lines) is often satisfactory in pure counterflow analyses, but it is not so in the presence of rotation. Indeed, in pure rotation the vortices form an ordered array, with the vortex lines parallel to the rotation axis. In simultaneous presence of rotation and counterflow, there appears an interesting interplay between the ordering tendency of the rotation and the disordering tendency of counterflow, which bears some analogy with the ordering tendency of an external magnetic field on a system of magnetic dipoles and a disordering tendency expressed by absolute temperature.[8,21,29]

Therefore, the question of anisotropy is inescapable in the study of simultaneous counterflow and rotation. There is indeed much current interest in this point.[16–21,29,30] Here we do not aim to solve this delicate problem, but we want to take advantage of the thermodynamic formalism to explore how the anisotropy would influence the dynamical equations for L and $\mathbf{v}_s$.

A. Vectorial approach

To begin our analysis, we mention an interesting recent proposal by Lipniacki[16] to generalize Vinen's equation to an anisotropic tangle. In his proposal, based on previous works on Schwarz,[12–14] the anisotropy is described by a vector $\mathbf{I}$, related to the vortex tangle structure by

$$\mathbf{I} = \frac{\langle \mathbf{s}' \times \mathbf{s}'' \rangle}{\langle |s''| \rangle}. \tag{5.1}$$

Here $\mathbf{s}(\xi,t)$ describes the vortex lines, with ξ the length along the vortices; the primes indicate differentiation with respect to ξ in such a way that $\mathbf{s}'$ is directed along the local tangent of the vortex and $\mathbf{s}''$ points towards the local center of curvature. In the localized induction approximation, this vector $\mathbf{I}$ is proportional to the self-induced velocity of a given point of the filament. The angular brackets stand for the average over the total vortex length of the tangle. In a vortex loop, $\mathbf{s}' \times \mathbf{s}''$ is parallel to the axis of the loop and it tends to orientate parallel to $\mathbf{V}$. According to Lipniacki, Vinen's equation should be generalized as

$$\frac{dL}{dt} = \alpha \mathbf{V} \cdot \mathbf{I} L^{3/2} - \beta \kappa L^2. \tag{5.2}$$

The origin of Lipniacki proposal (5.2) may be found in the microscopic analysis of vortex dynamics by Schwarz,[12–14] where an equation analogous to Eq. (5.2) is derived. Usually, the term in $\mathbf{I}$ is included in the α coefficient of Vinen's equation. In the localized induction approximation, the microscopic evolution for the line length of a vortex tangle satisfies the equation

$$\frac{\partial L}{\partial t} = \int \alpha [\mathbf{V} \cdot (\mathbf{s}' \times \mathbf{s}'') - \tilde{\beta} |s''|^2] d\xi; \tag{5.3}$$

the integral is carried out along all the vortex lines in the unit volume. Thus the scalar product $\mathbf{V} \cdot (\mathbf{s}' \times \mathbf{s}'')$ appears in a natural way in the microscopic form of Vinen's equation.

In fact, according to Lipniacki, the Vinen's equation would be recovered when $\mathbf{s}' \times \mathbf{s}''$ is always parallel to the external field $\mathbf{V}$. It is in this situation that vortex lines tend to be elongated in such a way they increase the total vortex length per unit volume. If $\mathbf{s}' \times \mathbf{s}''$ is not everywhere parallel to $\mathbf{V}$ there would be, accordingly with Schwarz and Lipniacki, a reduction in the vortex production term. Note the fact that we are talking about an isotropic situation, whereas Lipniacki refers to total anisotropy; this is so because we are referring to the distribution of different vectors, namely, $\mathbf{s}'$ and $\mathbf{s}' \times \mathbf{s}''$ respectively.

Equation (5.2) and the corresponding equation for $d\mathbf{v}_s/dt$ may be written in a tensorial form analogous to Eq. (2.3) as

$$\begin{bmatrix} \dfrac{d\mathbf{v}_s}{dt} \\ \dfrac{dL}{dt} \end{bmatrix} = L \begin{bmatrix} -\dfrac{\rho_n}{\rho\rho_s} A \mathbf{U} & \pm \alpha \dfrac{1}{\rho_s} L^{1/2} \mathbf{I} \\ -\dfrac{\alpha}{\rho_s} L^{1/2} \mathbf{I} & -\kappa \dfrac{\beta}{\epsilon_V} L \end{bmatrix} \begin{bmatrix} -\rho_s \mathbf{V} \\ \epsilon_V \end{bmatrix}. \tag{5.4}$$

The evolution equation for $\mathbf{v}_s$ would then be, according to the first line of Eq. (5.4):

$$\frac{d\mathbf{v}_s}{dt} = -\frac{\rho_n}{\rho} A L \mathbf{V} \pm \frac{\alpha}{\rho_s} \epsilon_V L^{3/2} \mathbf{I}. \tag{5.5}$$

Thus the anisotropy of $\mathbf{s}' \times \mathbf{s}''$ in the vortex production term would modify too the friction force acting on $\mathbf{V}$. This is logical, because the vortices are produced by the friction of the normal fluid. It must be noted that the scalar coefficients themselves, as A and α, could now become a function of the anisotropy parameter defined, for instance, by $\hat{\mathbf{I}} \cdot \hat{\mathbf{V}}$ (whose value is 1 for a tangle in which $\mathbf{s}' \times \mathbf{s}''$ is always parallel to $\mathbf{V}$).

B. Tensorial approach

A different way to describe the anisotropy of the tangle (referred now to the orientational distribution of the vector $\mathbf{s}'$ tangent to the vortex lines) is to use a full tensor related to the tensor $\mathbf{P}_\omega$ introduced in Ref. 24 [see Eq. (4.8)]. Considering only the symmetric part, an explicit possibility is to use the tensor Π^s defined by

$$\Pi^s = \langle \mathbf{U} - \mathbf{s}'\mathbf{s}' \rangle. \tag{5.6}$$

Indeed, when the tangle is isotropic (referred to the $\mathbf{s}'$ distribution) one has

$$\Pi^s = \langle \mathbf{U} - \mathbf{s}'\mathbf{s}' \rangle = \frac{2}{3}\mathbf{U}, \tag{5.7}$$

whereas, in the rotation case, the tangent vector $\mathbf{s}'$ becomes $\mathbf{s}' = \hat{\mathbf{\Omega}}$ and Π^s takes the form

$$\Pi^s = \mathbf{U} - \hat{\mathbf{\Omega}}\hat{\mathbf{\Omega}}. \tag{5.8}$$

Note that here, when we refer to an isotropic tangle, we take into account only an isotropic distribution of the tangent vectors $\mathbf{s}'$, but we do not refer to $\mathbf{s}' \times \mathbf{s}''$, in contrast with Lipniacki's approach.

Under the simultaneous presence of counterflow and rotation, which is the situation we are interested in, Π will have the form introduced in Eq. (4.5), and

$$\Pi^s = \frac{2}{3}(1-b)\mathbf{U} + b(\mathbf{U} - \hat{\mathbf{\Omega}}\hat{\mathbf{\Omega}}). \tag{5.9}$$

Tensor (5.9) provides a very intuitive description of some especially relevant aspects of the global geometry of vortex tangle in simultaneous presence of counterflow and rotation. In particular, it has the feature that the second part of Eq. (5.9) is a projector in the direction perpendicular to $\mathbf{\Omega}$, in such a way that if $\mathbf{V}$ is parallel to $\mathbf{\Omega}$, as it is in usual experiments, the second contribution from Eq. (5.9) will vanish and the product $\Pi^s \cdot \mathbf{V}$, will simply be

$$\Pi^s \cdot \mathbf{V} = \frac{2}{3}(1-b)\mathbf{V}. \tag{5.10}$$

In Eq. (5.9) b will be a function of V and Ω, in general. A possibility could be, for instance,

$$b(L,\Omega) = \frac{\Omega'^2}{\Omega'^2 + V'^2}, \tag{5.11}$$

with $\Omega' = \Omega/\Omega_c$ and $V' = V/V_c$, Ω_c and V_c being the critical values of Ω and V at which the laminar state ($L=0$) becomes unstable. Equation (5.11) has, in fact, the required limits $b=1$ for high rotation and $b=0$ for high V. This proposal is only meant as an illustration, rather than a well confirmed expression.

Finally, we write system (4.10) by taking into account the influence of the anisotropy of vortex tangle. Following the thermodynamic formalism used along this paper we may write, in the interpretation of $V\sqrt{\Omega}$ as the product $(1/\sqrt{\Omega})\Omega \cdot \hat{\mathbf{V}}\hat{\mathbf{\Omega}} \cdot \mathbf{V}$, discussed in Sec. IV A,

$$\begin{bmatrix} \dfrac{D\mathbf{v}_s}{dt} \\ \dfrac{dL}{dt} \end{bmatrix} = L \begin{bmatrix} -\dfrac{\rho_n}{\rho\rho_s}\dfrac{B}{\tilde{\alpha}}\Pi & +\dfrac{1}{\rho_s}\left[\alpha_1 L^{1/2}\mathbf{I} - \dfrac{\beta_4}{\sqrt{\Omega\kappa}}\dfrac{\mathbf{\Omega}\cdot\mathbf{V}}{\Omega V}\mathbf{\Omega}\right] \\ -\dfrac{1}{\rho_s}\left[\alpha_1 L^{1/2}\mathbf{I} - \dfrac{\beta_4}{\sqrt{\Omega\kappa}}\dfrac{\mathbf{\Omega}\cdot\mathbf{V}}{\Omega V}\mathbf{\Omega}\right] & -\dfrac{1}{\epsilon_V}[\beta\kappa L - \beta_2\sqrt{\kappa\Omega}L^{1/2} + \beta_1\Omega] \end{bmatrix} \begin{bmatrix} -\rho_s\mathbf{V} \\ \epsilon_V \end{bmatrix}. \tag{5.12}$$

These equations may be written as

$$\frac{dL}{dt} = -\beta\kappa L^2 + [\alpha_1\mathbf{V}\cdot\mathbf{I} + \beta_2\sqrt{\kappa\Omega}]L^{3/2} - \left[\beta_1\Omega + \frac{\beta_4}{\sqrt{\kappa\Omega}}\frac{(\mathbf{\Omega}\cdot\mathbf{V})^2}{\Omega V}\right]L, \tag{5.13}$$

$$\frac{d\mathbf{v}_s}{dt} + 2\mathbf{\Omega}\times\mathbf{v}_s + \mathbf{i}_0 = L\frac{\rho_n}{\rho}\frac{B}{\tilde{\alpha}}\Pi\cdot\mathbf{V} + L\left[\frac{\epsilon_V}{\rho_s}\left(\alpha_1 L^{1/2}\mathbf{I} - \frac{\beta_4}{\sqrt{\Omega\kappa}}(\hat{\mathbf{\Omega}}\cdot\hat{\mathbf{V}})\mathbf{\Omega}\right)\right]. \tag{5.14}$$

To write the expressions corresponding to Eqs. (5.13) and (5.14), using the interpretation in Sec. IV B is straightforward, so that we will not do it here, to avoid unnecessary repetitions.

VI. CONCLUSIONS

We have examined the joint evolution equations for the counterflow velocity $\mathbf{V}$ (in fact for $\mathbf{v}_s$) and the vortex line density L from the perspective of irreversible thermodynamics. Starting from an evolution equation for L, the formalism of irreversible thermodynamics has been used to obtain a consistent evolution equation for $\mathbf{v}_s$. Such equation would be needed to describe general unsteady situations where both L and $\mathbf{V}$ (or $\mathbf{v}_s$) change with time. At present, most of the counterflow experiments use an imposed value of $\mathbf{V}$ and study the subsequent evolution of the vortex line density of the tangle.

We follow the ideas set out by Nemirowskii *et al.*[1,5,15] in an analysis of pure counterflow situation. Their main issue was, besides the obtention of an evolution equation for $\mathbf{V}$, the discussion of an ambiguity in a coupling term, related to two possible microscopic interpretations of the mechanism of vortex lines annihilation. Instead, our analysis has dealt with a more general situation, incorporating simultaneous counterflow and rotation. In our case, we have explored an ambiguity related to the interpretation of a term coupling the effects of counterflow and rotation, and which accounts for the fact that effects of both phenomena are not merely additive, as it is known from experiments. We have outlined two different interpretations of these terms and we have obtained the corresponding evolution equations for $\mathbf{V}$, for each for them.

Thus, the present formal analysis could reveal at full its potential interest when experiments involving different directions of $\mathbf{V}$ and $\mathbf{\Omega}$ will be performed. Furthermore, our analysis shows [in Eqs. (4.24) and (4.25)] a discontinuity in the friction force when V exceeds a threshold value, consistent with a discontinuity in the geometrical features of the vortex lines.

To have a more microscopic understanding of the interest to study the geometry of the tangle not only in situations where $\mathbf{\Omega}$ and $\mathbf{V}$ are parallel to each other but also when they have opposite sign, we may follow the simplified stability analysis of an isolated helical vortex line proposed by Tsubota *et al.*[21] The vortex motion is governed by the equation[14]

$$\frac{d\mathbf{s}}{dt} = \mathbf{v}_{sl} + \alpha\mathbf{s}'\times(\mathbf{v}_n - \mathbf{v}_{sl}) - \alpha'\mathbf{s}'\times[\mathbf{s}'\times(\mathbf{v}_n - \mathbf{v}_{sl})], \tag{6.1}$$

where $\mathbf{v}_{sl}=\mathbf{v}_s+\mathbf{v}_i$ is the "local superfluid velocity," sum of the superfluid velocity at large distance from any vortex line and of the "self-induced velocity," a flow due to any other vortex, including other parts of the same vortex, induced by the curvature of all these lines; this latter contribution, at a point s_0 on the line, is given by the Biot-Savart law; in the local-induction approximation we can write

$$\mathbf{v}_i = \nu[\mathbf{s}' \times \mathbf{s}'']_{s=s_0}, \quad \text{with } \nu = \frac{\kappa}{4\pi}\ln\left(\frac{1}{|\mathbf{s}''|a_0}\right), \tag{6.2}$$

with $\kappa=|\vec{\kappa}|$ and a_0 the dimension of the vortex core, of the dimension of one Å.

To describe the vortex motion in the presence of rotation and counterflow, it is need to generalize Eq. (6.1) to a rotating frame. Tsubota *et al.*[21] have found that in a rotating vessel the evolution equation (6.1) of vortex line must be modified a

$$\frac{d\mathbf{s}}{dt} = \mathbf{v}_{sl} + \alpha\mathbf{s}' \times (\mathbf{v}_n - \mathbf{v}_{sl}) - \alpha'\mathbf{s}' \times [\mathbf{s}' \times (\mathbf{v}_n - \mathbf{v}_{sl})] + \dot{\mathbf{s}}_{\rm rot} + \mathbf{v}_{\rm rot}, \tag{6.3}$$

where $\dot{\mathbf{s}}_{\rm rot}$ is the velocity of the vortex caused by the rotation and $\mathbf{v}_{\rm rot}$ is the superflow induced by the rotating vessel (for an explicit expression of these two contributes see Ref. 21).

Helical waves are vortex-wave modes for which the wave vector is along the rotation axis. We consider in particular a helical deformation of wave vector k and amplitude ε, where $\varepsilon \ll k^{-1}$. Ignoring the nonlocal contribution, the line moves with the local self-induced velocity $\mathbf{v}_i$ defined in Eq. (6.2). This velocity is perpendicular to the undisturbed line and to the displacement vector from the undisturbed line to the point considered. Each vortex line element therefore executes motion about the undisturbed line in a circle of radius $R \approx \varepsilon$ and with a frequency $\omega=|\mathbf{v}_i|/\varepsilon$ in sense opposite to the sense of the velocity field.

We note that the straight vortex lines formed in rotating helium have vorticity $\vec{\kappa}$ parallel to the angular velocity $\mathbf{\Omega}$ of the vessel. As a consequence, helical waves are circularly polarized vortex waves in which each vortex line element executes circular motion in a plane perpendicular to the axis of rotation in opposite sense to the rotation of the vessel: if the rotation of the vessel is righthanded ($\mathbf{\Omega}=\Omega\hat{\mathbf{z}}$ with Ω positive) the helical waves are left-handed waves.

Now we assume $\mathbf{\Omega}=\Omega\hat{\mathbf{z}}$ and we follow the simplified analysis proposed by Tsubota *et al.*[21] They assume the helical vortex line given by

$$\mathbf{s} = (\varepsilon\cos\phi, \varepsilon\sin\phi, z), \tag{6.4}$$

where $\phi=kz-\omega t$ and $\varepsilon(t)\ll 1$. It easily follows that $\mathbf{s}'=(-k\varepsilon\sin\phi, k\varepsilon\cos\phi, 1)$ and $\mathbf{s}''=(-k^2\varepsilon\cos\phi, -k^2\varepsilon\sin\phi, 0)$. We assume a counterflow velocity given by $\mathbf{V}=(0,0,V)$ (V positive or negative).

Neglecting (as in Ref. 21) the small friction coefficient α' and the two additional terms due to the rotation, Eq. (6.3) simplifies as

$$\frac{d\mathbf{s}}{dt} = \mathbf{v}_s + \nu\mathbf{s}' \times \mathbf{s}'' + \alpha\mathbf{s}' \times (\mathbf{V} - \mathbf{v}_i). \tag{6.5}$$

Neglecting term of second order in ε we obtain

$$\mathbf{v}_s = -(\rho_n/\rho)(0,0,V) = (0,0,v_s), \tag{6.6}$$

$$\mathbf{v}_i = \nu\mathbf{s}' \times \mathbf{s}'' = \nu k^2\varepsilon(\sin\phi, -\cos\phi, 0), \tag{6.7}$$

$$\alpha\mathbf{s}' \times (\mathbf{V} - \mathbf{v}_i) = \alpha\varepsilon(kV - \nu k^2)(\cos\phi, \sin\phi, 0). \tag{6.8}$$

Substituting in Eq. (6.5), we obtain the following set of equations:

$$\begin{cases} [\dot{\varepsilon} - \alpha\varepsilon(kV - \nu k^2)]\cos\phi + [\varepsilon(\nu k^2 - \omega)]\sin\phi = 0, \\ [\dot{\varepsilon} - \alpha\varepsilon(kV - \nu k^2)]\sin\phi + [\varepsilon(\nu k^2 - \omega)]\cos\phi = 0, \\ \dot{z} - v_s = 0, \end{cases} \tag{6.9}$$

which yields

$$z = v_s t + z_0 = -\frac{\rho_n}{\rho}Vt + z_0, \ \omega = \nu k^2, \tag{6.10}$$

and leads to the following equation for ϵ:

$$\frac{d\varepsilon}{dt} = \alpha[kV - \nu k^2]\varepsilon. \tag{6.11}$$

Substituting Eq. (6.10) in Eq. (6.4) one obtains

$$\mathbf{s} = [\varepsilon\cos(kv_s - \omega)t, \varepsilon\sin(kv_s - \omega)t, v_s t + z_0]. \tag{6.12}$$

This helix goes towards higher values of z if it results that

$$v_s > 0, \tag{6.13}$$

i.e., if $\mathbf{V}$ is opposite of the z axis, and is left handed if it results that

$$kv_s - \omega = kv_s - \nu k^2 = -k\frac{\rho_n}{\rho}V - \nu k^2 < 0. \tag{6.14}$$

Summarizing, if $kV=\nu k^2$ (for example if k and V are both positive), Eq. (6.11) has the solution ε=cost; as a consequence, for this particular value of V helical waves are present, with $\omega=\nu k^2$ and $k=\sqrt{V/\nu}$. As ε is time independent, these waves are stable. In this case it results $\phi=-\{[(\rho_s+2\rho_n)/\rho]\nu k^2\}t$. This wave is left handed and goes toward lower values of z. The only difference, if V and k are chosen both negative, is that the left-handed helical wave goes toward higher values of z.

We suppose now $kV \neq \nu k^2$. In this case the stationary solution of Eq. (6.11) is $\varepsilon=0$. This solution will be stable if the growth rate $\sigma=kV-\nu k^2$ of ε will be negative, i.e., if $kV < \nu k^2$. When $\mathbf{V}$ is parallel to the propagation direction of the helical wave and k is chosen positive (plus sign in the term kV), the growth rate of ε is $\sigma=kV-\nu k^2$ in such a way that for a given $\mathbf{V}$, waves with sufficiently high wave vector k will increase their amplitude. However, for $\mathbf{V}$ opposite to the propagation direction of the wave, the growth rate of ε will be always negative, and the helical perturbations of the rectilinear vortex will disappear within a short time. For opposite values of k, the opposite will hold. If we repeat the same simplified calculation with $\mathbf{V}$ orthogonal to the z axis, we see that, in this case, the helical perturbations to straight vortex lines always decrease with time, in such a way that the straight vortices are stable. Thus the relative direction of $\mathbf{\Omega}$ and $\mathbf{V}$ could have a role on the geometry of the tangle. The present microscopic argument, however, is too simplified to lead to definite predictions, because it ignores the interactions between vortices and the global characteristics of the flow. Thus, more detailed microscopic analyses should be done and, above all, more experiments.

In this context, two situations would be particularly interesting: to study the differences in the situations with $\mathbf{\Omega}$ and $\mathbf{V}$ parallel and antiparallel, and a situation with $\mathbf{V}$ orthogonal to $\mathbf{\Omega}$, as in a superfluid in a rotating annulus with higher temperature on the walls of the internal cylinder, thus producing a radial counterflow[31] or in a rotating cylinder with a thin hot wire along the rotation axis.

It is worthwhile to compare the predictions of Sec. IV A with those of IV B in a particular situation. We consider a rotating cylinder filled with superfluid helium, in which an axial heat flux or a radial heat flux is imposed (the latter may be produced by a thin hot wire along the rotation axis; in more realistic terms, one should study the flow between two concentric cylinders, with the inner one being hotter than the outer one, for instance). The values of the numerical coefficients obtained in Ref. 8 are sufficient to make a qualitative comparison between the predictions of Sec. IV A with those of Sec. IV B.

To do so, we recall that the solution of Eq. (3.1) for $L^{1/2}$ in terms of $\mathbf{V}$ and $\mathbf{\Omega}$ (to which both interpretations in Secs. IV A and IV B reduce when $\mathbf{V}$ and $\mathbf{\Omega}$ are parallel to each other) is furnished by Eqs. (4.19) and (4.20), with a critical velocity V_{c2} given by Eq. (4.21). For $V<V_{c2}$, $L^{1/2}$ depends on $(\Omega/k)^{1/2}$ but not on V, whereas for $V>V_{c2}$, $L^{1/2}$ increases linearly with V. Experimentally, a critical velocity $V_{c1}<V_{c2}$ also appears, in which a small step in the value of $L^{1/2}$ is found, as discussed at length in Ref. 8. Comparison with the experimental data by Swanson, Barenghi, and Donnelly[6] for superfluid turbulence in a rotating cylinder with $\mathbf{V}$ and $\mathbf{\Omega}$ parallel to each other, yields $\alpha_1/\beta\kappa=47\ \mathrm{cm}^{-2}\,\mathrm{s}$, $\beta_4/\alpha_1=1.43$ and $\beta_2/\alpha_3=2.68$. Therefore Eq. (4.20) becomes

$$L^{1/2}=47V+1{,}25\sqrt{\Omega/\kappa}\ \mathrm{cm}^{-1}\ \text{for}\ V>V_{c2}, \qquad (6.15)$$

where the counterflow velocity V is expressed in cm/s. Assume, now, a situation in which $\mathbf{V}$ is perpendicular to $\mathbf{\Omega}$. In this case, the coupling term vanishes in Sec. IV A, thus yielding an effective value $\beta_4=0$ in Eqs. (3.1) and (4.19)–(4.21). In contrast, in Sec. IV B the coupling term is independent on the angle between $\mathbf{V}$ and $\mathbf{\Omega}$. Thus, according to Sec. IV B, the behavior of $L^{1/2}$ in terms of V and Ω would be the same for $\mathbf{V}$ and $\mathbf{\Omega}$ parallel or perpendicular to each other. Instead, according to Sec. IV A, the behavior would be

$$L^{1/2}=47V+2{,}68\sqrt{\Omega/\kappa}\ \mathrm{cm}^{-1}\ \text{for all}\ V, \qquad (6.16)$$

indeed, in this case the critical counterflow velocity V_{c2} becomes negative.

In fact, the numerical values in Eqs. (6.15) and (6.16) are mainly indicative rather than an exact derivation, since the values of the parameters could be influenced by the geometry of the system and by the degree of anisotropy of the tangle, which is not known in sufficient detail nowadays. However, the main qualitative differences are expected to hold. They are (a) $L^{1/2}$ depends on $\mathbf{V}$ for all values of the outwards radial counterflow velocity, because in this case V_{c2} becomes negative, (b) the slope of $L^{1/2}$ with respect to V is higher in Eq. (6.5) than in Eq.(6.4). Thus, the expressions studied in this paper are not merely formal, but they have testable consequences. Notice that both Secs. IV A and IV B predict that the situation with $\mathbf{V}$ and $\mathbf{\Omega}$ counterparallel to each other would be the same as if they are parallel to each other, because Sec. IV B does not depend on the relative direction of $\mathbf{V}$ and $\mathbf{\Omega}$ and Sec. IV A depends on the square of the angle they are forming. The differences in both situations cannot be found in the value of L, but in the expressions (4.13) and (4.18) of $\mathbf{F}_{\mathrm{coupl}}$ which in Eq. (4.18) is always opposite to $\mathbf{V}$, while in Eq. (4.13) depends on the angle between $\mathbf{V}$ and $\mathbf{\Omega}$.

Another situation to which apply Eq. (3.1) could be the experiments carried out by Yarmchuk and Glaberson.[7] In their work they arranged a pair of horizontal parallel glass plates to form a closed channel of rectangular cross section closed at one end with a heater nearby, and open at the other end to the liquid helium bath. The channel investigated was of large aspect ratio, the length being 0.5 mm, the width 1.4 cm, and the length 5.5 cm. The channel is rotated about a vertical axis orthogonal to the direction of the heat flux. In this way the counterflow velocity $\mathbf{V}$ is orthogonal to angular velocity $\mathbf{\Omega}$ of the sample. Temperature and chemical gradients where obtained as a function of heater power and rotation speed. They found a linear regime, in which the temperature and chemical gradients increase as the rotation speed increases, and a critical heater power q_{c2} (to which correspond a critical counterflow velocity V_{c2}) associated with the onset of turbulent regime, which increase as the rotation speed increased, becoming proportional to $\sqrt{\Omega}$ as Ω gets large. Notwithstanding the very different geometry, these results allow us to affirm that the coefficient a_1 in Eq. (3.2) is different from zero. To establish whether the coefficient a_2 can be chosen equal to zero, i.e. if we must choose the interpretation in Sec. IV B, further experiments must be made.

We have indicated that the two macroscopic interpretations considered in our paper, namely, Sec. IV A and IV B lead to different expressions for the force coupling the heat flow and the rotating mesh of vortices: namely, we are respectively led to Eqs. (4.13) and (4.18). Thus, in spite that our analysis is of macroscopic origin, it stimulates the consideration of the microscopic forces between heat flow and rotating vortices. Thus, even if the ambiguity mentioned in connection with the interpretation of the coupling term could be removed, it would remain the problem of providing a microscopic interpretation for this term, which describes the influence of a rotating vortex tangle on the friction force acting on $\mathbf{v}_s$. In our equation, we find that such friction force would be reduced in the presence of a rotation with respect to the friction in a nonrotating tangle with the same vortex line density. In a more detailed analysis, the relation between the rotational of the local velocity and the vorticity should be taken into account, as it is done, for instance, in the analysis of vortex lines in rotating cylinders[32] and in a new formulation of the Hall-Vinen-Bekarevich-Khalatnikov equations.[33]

Finally, we have considered two ways of incorporating the degree of anisotropy of the tangle, which arises because rotation tends to produce an anisotropy array of vortex lines, parallel to the direction $\mathbf{\Omega}$, whereas counterflow tends to produce an isotropy tangle. One of these procedures is proposed by Lipniacki and is based on the scalar product of a vector $\mathbf{I}$ and the velocity $\mathbf{V}$. Our proposal is to use a full tensor Π describing the superposition of the contribution of a fully isotropic tangle and of a completely ordered array of vortices parallel to Ω, the relative weight being a non linear function of V and Ω. We have examined how both descriptions of anisotropy influence the evolution equations for L and $\mathbf{V}$.

ACKNOWLEDGMENTS

We acknowledge the support of the Acción Integrada España-Italia (Grant No. S2800082F HI2004-0316 of the Spanish Ministry of Science and Technology and grant IT2253 of the Italian MIUR). D.J. acknowledges the financial support from the Dirección General de Investigación of the Spanish Ministry of Education under Grant No. BFM 2003-06033 and of the Direcció General de Recerca of the Generalitat of Catalonia, under Grant No. 2001 SGR-00186. M.S.M acknowledges the financial support from MIUR under grant "Nonlinear mathematical problems of wave propagation and stability in models of continuous media" and by "Fondi 60%" of the University of Palermo.

[1] S. K. Nemirovskii and V. V. Lebedev, Zh. Eksp. Teor. Fiz. 1729 (1983) [Sov. Phys. JETP **57**, 1009 (1983)].

[2] K. Yamada, S. Kashiwamura, and K. Miyake, Physica B **154**, 318 (1989).

[3] J. A. Guerst, Physica B **154**, 327 (1989).

[4] J. A. Guerst, Physica A **183**, 279 (1992).

[5] S. K. Nemirovskii, G. Stamm, and W. Fiszdon, Phys. Rev. B **48**, 7338 (1993).

[6] C. E. Swanson, C. F. Barenghi, and R. J. Donnelly, Phys. Rev. Lett. **50**, 190 (1983).

[7] E. J. Yarmchuk and W. I. Glaberson, Phys. Rev. Lett. **41**, 564 (1978).

[8] D. Jou and M. S. Mongiovì, Phys. Rev. B **69**, 094513 (2004).

[9] W. F. Vinen, Proc. R. Soc. London, Ser. A **240**, 493 (1957).

[10] W. F. Vinen, Proc. R. Soc. London, Ser. A **243**, 400 (1958).

[11] R. P. Feynman, in *Progress in Low Temperature Physics*, edited by C. J. Gorter (1955), Vol. 1, p. 17.

[12] K. W. Schwarz, Phys. Rev. Lett. **49**, 283 (1982).

[13] K. W. Schwarz, Phys. Rev. B **31**, 5782 (1985).

[14] K. W. Schwarz, Phys. Rev. B **38**, 2398 (1988).

[15] S. K. Nemirovskii and W. Fiszdon, Rev. Mod. Phys. **67**, 37 (1995).

[16] T. Lipniacki, Phys. Rev. B **64**, 214516 (2001).

[17] C. F. Barenghi, S. Hulton, and D. C. Samuels, Phys. Rev. Lett. **89**, 275301 (2002).

[18] T. Araki, M. Tsubota, and C. F. Barenghi, Physica B **329–333**,226 (2003).

[19] M. Tsubota, T. Araki, and C. F. Barenghi, Phys. Rev. Lett. **90**, 205301 (2003).

[20] A. P. Finne T. Araki, R. Blaauwgeers, V. B. Eltsov, N. B. Kopnin, M. Krusius, L. Skrbek, M. Tsubota, and G. E. Volovik, Nature **424**, 1022 (2003).

[21] M. Tsubota, C. F. Barenghi, T. Araki, and A. Mitani, Phys. Rev. B **69**, 134515 (2004).

[22] H. E. Hall and W. F. Vinen, Proc. R. Soc. London, Ser. A **238**, 215 (1956).

[23] R. J. Donnelly, *Quantized Vortices in Helium II* (Cambridge University Press, Cambridge, 1991).

[24] D. Jou, G. Lebon, and M. S. Mongiovì, Phys. Rev. B **66**, 224509 (2002).

[25] D. Jou, J. Casas-Vázquez, and G. Lebon, *Extended Irreversible Thermodynamics* (Springer-Verlag, Berlin, 2001).

[26] M. S. Mongiovì, R. A. Peruzza, and D. Jou, in *Recent Results in Devel. Physics*, (Transworld Research Network Publishing, Kerala, India, 2004), Vol. 5, pp. 1033–1056.

[27] H. A. Snyder and Z. Putney, Phys. Rev. **150**, 110 (1966).

[28] P. Mathieu, B. Plaçais, and Y. Simon, Phys. Rev. B **29**, 2489 (1984).

[29] R. J. Donnelly, J. Phys.: Condens. Matter **11**, 7783 (1999).

[30] *Quantized Vortex Dynamics and Superfluid Turbulence*, edited by C. F. Barenghi, R. J. Donnelly, and W. F. Vinen (Springer, Berlin, 2001).

[31] L. Campbell, Phys. Rev. B **36**, 6773 (1987).

[32] I. M. Khalatnikov, *An Introduction to the Theory of Superfluidity* (Benjamin, New York, 1965).

[33] D. D. Holm, in *Quantized Vortex Dynamics and Superfluid Turbulence*, edited by C. F. Barenghi, R. J. Donnelly, and W. F. Vinen (2001), p. 114.