Theory of disordered flux-line liquids

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(Received 30 January 2004; revised manuscript received 3 February 2005; published 4 October 2005)

We study the equilibrium statics and nonequilibrium driven dynamics of flux-line liquids in the presence of a random pinning potential. Under the assumption of replica symmetry, we find in the static case using a replica Gaussian variational method that the only effect of disorder is to increase the tilt modulus and the confining "mass" of the internal modes of the flux lines, thus decreasing their thermal wandering. In the nonequilibrium driven case, we derive the long-scale, coarse-grained equation of motion of the vortices in the presence of disorder, which, apart from new Kardar-Parisi-Zhang nonlinearities, has the same form as the equation of motion for unpinned vortices, with renormalized coefficients. We also compute the structure factor of a disordered vortex liquid, and show that the disorder contributes an additive, Lorentzian squared term, to the structure factor of the center of mass mode, otherwise leaving the functional form of terms describing the internal modes unchanged. The expression of the static structure factor derived within our approach is consistent both with experimental data and with the standard theory of elasticity of vortices in high temperature superconductors.

DOI: 10.1103/PhysRevB.72.144502 PACS number(s): 74.20.De, 74.25.Qt

I. INTRODUCTION

During the past 15 years, the study of the properties of flux-line liquids in high-temperature superconductors (HTSC's) has been one of the most active areas of research in vortex phenomenology. Yet, despite an impressive body of literature, both experimental and theoretical, and despite a relatively good understanding of the overall behavior and macroscopic properties of vortex liquids, it seems that the important question of the actual microscopic correlations of flux-line trajectories inside such liquids has not been fully understood yet. Indeed, of the several theoretical approaches that have been used to study the properties of liquid-vortex matter, one particular approach, which has had a rather strong impact on our present understanding of flux liquids in HTSC's, is boson mapping, developed by several authors,[1–6] which is based on the observation[7] that there is a formal mapping between the partition function of a three-dimensional system of interacting flux lines and the imaginary-time partition function of quantum bosons in two dimensions. Although boson mapping is ultimately used to find density-density correlation functions and does not contain, after coarse graining, any detailed information about flux-line trajectories, it has been argued,[2] based on the behavior of the structure factor derived in this and other hydrodynamic approaches,[8] that flux lines wander throughout the sample in a random-walk-like fashion, much in the same way as in a hypothetical "ideal gas" of noninteracting vortices. This implies, in particular, that the internal fluctuations of flux lines have an average spatial extent which diverges with the sample thickness L.

The above interpretation, and in fact the whole hydrodynamic approach to flux-line liquids, suffers from a number of inconsistencies which have been pointed out and discussed in detail in two recent papers by the author.[9,10] In these two articles, the author has proposed an approach to study three-dimensional flux-line liquids in type-II superconductors which, instead of the density, uses the actual conformation variables of vortices as the fundamental dynamical variables of the flux-line system. This approach, which makes contact with the standard theory of classical fluids, is based on the separation of the dynamical variables of flux lines into center-of-mass (c.m.) and internal modes and on the observation that the repulsive interactions between flux lines must lead to a certain degree of confinement of the internal modes, whose fluctuations are shown to be bounded and to no longer diverge with the sample thickness.[9,10] This picture is obviously in contradiction with the results of the boson mapping of Refs. 2–4.

In this paper we wish to generalize the methodology developed in these previous studies (Refs. 9 and 10) to study the statics and dynamics of vortex liquids in the presence of a random pinning potential. In the static case, we shall show in particular that disorder leads to an enhancement of the tilt modulus and the confining "mass" of the internal modes of flux lines, thereby reducing their thermal wandering. These static results are then generalized to the dynamics of driven vortex liquids in the presence of a random pinning potential. In contrast to earlier studies of this system, in which the density was used as the fundamental dynamical variable, in our approach flux-line trajectories, which are the true dynamical variables of the system, are used throughout. This enables us to derive the coarse-grained, large-scale equation of motion of vortices in the flux liquid in the presence of disorder, in close analogy with earlier work on driven, disordered flux lattices.[11–13] We also present a detailed derivation of the hydrodynamics of flux-lines, and show that the general analytic form of the structure factor $S(\mathbf{r},z)$ remains unchanged, except for a Lorentzian squared additive contribution of disorder to the structure factor *of the c.m. mode* of flux lines, in disagreement with previous boson mapping[3,4] and other hydrodynamic formulations.[8]

This article is organized as follows. In Sec. II, we use a simple perturbative analysis to study the effect of disorder on vortex liquids. This perturbative analysis is then refined in Sec. III where we use the replica Gaussian variational method[14] to properly include the effect of the relatively strong thermal fluctuations which are a common characteris-

tic of HTSC's. In Sec. IV, we construct an action formulation for the dynamics of flux lines in a vortex liquid. Then, before considering the nonequilibrium case of a driven disordered flux liquid, we shall first be interested in the equilibrium dynamics of flux liquids, which we will investigate in quite some detail in Sec. V. Such an investigation is not only a natural step toward the more complex disordered case, but is also necessary for the developments to follow, since one needs to correctly specify the near-equilibrium dynamics of the unpinned interacting liquid in order to be able to tackle the out-of-equilibrium driven, disordered case. In Sec. VI, we derive the coarse-grained dynamics of driven, disordered flux liquids in the limit of high drives before deriving the structure factor of pinned flux liquids in Sec. VII. Section VIII contains a discussion of our results along with our conclusions.

II. FLUX-LINE LIQUID IN THE PRESENCE OF DISORDER: PERTURBATIVE ANALYSIS

We thus consider a flux-line liquid in $d=d_\perp+1$ dimensions (we use $d_\perp=2$ in all explicit calculations), in the presence of an external pinning potential. Our starting point is the Hamiltonian[1,2]

$$H=\sum_{i=1}^{N}\int dz\left\{\frac{1}{2}K\left(\frac{d\mathbf{r}_i}{dz}\right)^2+\frac{1}{2}\sum_{j(\neq i)}V(\mathbf{r}_i(z)-\mathbf{r}_j(z))\right\}+\sum_{i=1}^{N}\int dz V_d(\mathbf{r}_i(z),z), \tag{1}$$

where the $d_\perp$-dimensional vector $\mathbf{r}_i(z)$ parametrizes the trajectory of the ith flux line as it traverses the superconducting sample, K is the tilt modulus of the flux lines, $V(r)=2\varepsilon_0 K_0(r/\lambda)$ is the interaction potential between flux line elements at equal height, and $V_d(\mathbf{r},z)$ is a random pinning potential. In the above expression of $V(r)$, λ is the London penetration depth in the (ab) planes, K_0 is a modified Bessel function,[15] and $\varepsilon_0=(\phi_0/4\pi\lambda)^2$, where $\phi_0=hc/2e$ is the flux quantum (here h is Planck's constant, e the absolute value of the electronic charge and c the speed of light in vacuum).[16] In Eq. (1) and all equations below, the origin of heights is taken to be at the center of the sample and all z integrals are taken from $-L/2$ to $L/2$ (L is the sample thickness). For simplicity, we shall consider that the probability distribution of V_d is Gaussian, with zero mean and variance:

$$\langle V_d(\mathbf{r},z)V_d(\mathbf{r}',z')\rangle=\Delta(\mathbf{r}-\mathbf{r}',z-z'). \tag{2}$$

We next consider the canonical partition function of this system (here T is temperature, and we use units such that Boltzmann's constant $k_B=1$),

$$Z=\int\prod_{i=1}^{N}[d\mathbf{r}_i(z)]e^{-H/T}, \tag{3}$$

and average over the disorder by introducing p replicas of the above system and making use of the well-known replica trick,

$$\overline{\ln Z}=\lim_{p\to 0}\frac{\overline{Z^p}-1}{p}, \tag{4}$$

upon which we obtain the following, disorder-averaged Hamiltonian:

$$\bar{H}=\sum_{a=1}^{p}\sum_{i=1}^{N}\int dz\frac{1}{2}\left\{K\left(\frac{d\mathbf{r}_i^a}{dz}\right)^2+\sum_{j(\neq i)}V(\mathbf{r}_i^a(z)-\mathbf{r}_j^a(z))\right\}-\frac{1}{2T}\sum_{a,b=1}^{p}\sum_{i,j=1}^{N}\int dzdz'\Delta(\mathbf{r}_i^a(z)-\mathbf{r}_j^b(z');z-z'), \tag{5}$$

where the superscripts $(a,b,\ldots)$ label replicas. In what follows, it will prove useful to write the flux-line position at height z, $\mathbf{r}_i(z)$, as the sum

$$\mathbf{r}_i(z)=\mathbf{r}_{0i}+\mathbf{u}_i(z), \tag{6}$$

where $\mathbf{r}_{0i}=(1/L)\int dz\,\mathbf{r}_i(z)$ is the c.m. position of the ith flux line, while $\mathbf{u}_i(z)$ is the displacement of the ith flux line at height z with respect to $\mathbf{r}_{0i}$ and has the following decomposition into Rouse modes[17]:

$$\mathbf{u}_i(z)=\sum_{n\neq 0}\mathbf{u}_i(q_n)e^{iq_nz}, \tag{7}$$

with the Fourier coefficients

$$\mathbf{u}_i(q_n)=\frac{1}{L}\int dz\,\mathbf{u}_i(z)e^{-iq_nz}. \tag{8}$$

In keeping with the spirit of the calculation carried out in Ref. 9, in this section we shall perform a simple perturbative analysis of the physics encoded in the Hamiltonian (5) and expand $\bar{H}$ to quadratic order in the displacement field $\{\mathbf{u}\}$. The pure (disorder free) part of $\bar{H}$ yields

$$\bar{H}_{pure}=\bar{H}^{(0)}_{pure}+\bar{H}^{(1)}_{pure}, \tag{9}$$

where

$$\bar{H}^{(0)}_{pure}=\frac{1}{2}\sum_{a=1}^{p}\sum_{i\neq j}LV(\mathbf{r}_{0i}^a-\mathbf{r}_{0j}^a) \tag{10}$$

is the Hamiltonian of a system of perfectly straight flux lines interacting through the potential $V(\mathbf{r})$ and where

$$\bar{H}^{(1)}_{pure}=\sum_{a=1}^{p}\sum_{i=1}^{N}\int dz\,\frac{1}{2}\left[K\left(\frac{d\mathbf{u}_i^a}{dz}\right)^2+\mu^{(i)}_{\alpha\beta}u^a_{i\alpha}(z)u^a_{i\beta}(z)\right]+\sum_{a=1}^{p}\sum_{i=1}^{N}\sum_{j(\neq i)}\int dz\,\frac{1}{2}\mu^{(ij)}_{\alpha\beta}u^a_{i\alpha}(z)u^a_{j\beta}(z) \tag{11}$$

represents the contribution of internal modes to the "pure" part. In the above equation, the coefficients $\mu^{(i)}_{\alpha\beta}$ and $\mu^{(ij)}_{\alpha\beta}$ are given by

$$\mu^{(i)}_{\alpha\beta}=\sum_{j(\neq i)=1}^{N}\partial_\alpha\partial_\beta V(\mathbf{r}_{0i}^a-\mathbf{r}_{0j}^a), \tag{12}$$

$$\mu_{\alpha\beta}^{(ij)} = -\partial_\alpha \partial_\beta V(\mathbf{r}_{0i}^a - \mathbf{r}_{0j}^a). \tag{13}$$

In a similar fashion, a Taylor expansion of the disorder part of $\bar{H}$ to quadratic order in the displacements $\{\mathbf{u}_i^a\}$ gives a decomposition similar to the one in Eq. (9), namely:

$$\bar{H}_{dis} = \bar{H}_{dis}^{(0)} + \bar{H}_{dis}^{(1)}. \tag{14}$$

Here, $\bar{H}_{dis}^{(0)}$ is given by

$$\bar{H}_{dis}^{(0)} = -\frac{1}{2T}\sum_{a,b}\sum_{i,j} L\bar{\Delta}(\mathbf{r}_{0i}^a - \mathbf{r}_{0j}^b) \tag{15}$$

and is the disorder part of the disorder-averaged Hamiltonian of a system of perfectly straight flux lines in a Gaussian random potential with variance $\bar{\Delta}(\mathbf{r}) = \int_{-\infty}^{\infty} dz \Delta(\mathbf{r}, z)$. On the other hand, $\bar{H}_{dis}^{(1)}$ is given by

$$\begin{aligned} H_{dis}^{(1)} = &-\frac{1}{2T}\sum_{a=1}^{p}\sum_{i=1}^{N}\int dz \int dz' \frac{1}{2}[u_{i\alpha}^a(z) - u_{i\alpha}^a(z')][u_{i\beta}^a(z) \\ &- u_{i\beta}^a(z')]\partial_\alpha\partial_\beta\Delta(\mathbf{0}, z - z') \\ &-\frac{1}{2T}\sum_{a=1}^{p}\sum_{i\neq j}\int dz \int dz' \frac{1}{2}[u_{i\alpha}^a(z) - u_{j\alpha}^a(z')][u_{i\beta}^a(z) \\ &- u_{j\beta}^a(z')]\partial_\alpha\partial_\beta\Delta(\mathbf{r}_{0i}^a - \mathbf{r}_{0j}^a, z - z') \\ &-\frac{1}{2T}\sum_{a\neq b}\sum_{i,j}\int dz \int dz' \frac{1}{2}[u_{i\alpha}^a(z) - u_{j\alpha}^a(z')][u_{i\beta}^b(z) \\ &- u_{j\beta}^b(z')]\partial_\alpha\partial_\beta\Delta(\mathbf{r}_{0i}^a - \mathbf{r}_{0j}^b, z - z'). \end{aligned} \tag{16}$$

It is not difficult to see that the first term on the right-hand side of the above equation represents same replica, single-line contributions to the Hamiltonian of the internal modes of vortices, while the second and third terms represent contributions to $\bar{H}_{dis}^{(1)}$ coming from same replica, different flux lines and from different replicas, respectively.

Collecting all terms, it follows that $\bar{H}$ can be written in the form

$$\bar{H} = \bar{H}^{(0)} + \bar{H}^{(1)}, \tag{17}$$

where

$$\bar{H}^{(0)} = \bar{H}_{pure}^{(0)} + \bar{H}_{dis}^{(0)}, \tag{18a}$$

$$\bar{H}^{(1)} = \bar{H}_{pure}^{(1)} + \bar{H}_{dis}^{(1)}. \tag{18b}$$

In the spirit of Ref. 9, we shall derive an effective Hamiltonian for the internal modes of the flux lines by averaging $\bar{H}^{(1)}$ over the center-of-mass positions $\{\mathbf{r}_{0i}^a\}$:

$$H_u = \langle \bar{H}^{(1)} \rangle_0 = \langle \bar{H}_{pure}^{(1)} \rangle_0 + \langle \bar{H}_{dis}^{(1)} \rangle_0, \tag{19}$$

where the average is carried out with statistical weight $\exp(-\bar{H}^{(0)}/T)/Z_0$ [with $Z_0 = \mathrm{Tr}(e^{-\bar{H}^{(0)}/T})$]. The pure part of H_u has already been evaluated in Ref. 9, with the result

$$\begin{aligned} H_u^{pure} = &\sum_{a=1}^{p}\left\{\sum_{i=1}^{N}\int dz\, \frac{1}{2}\left[K\left(\frac{d\mathbf{u}_i^a}{dz}\right)^2 + \mu \mathbf{u}_i^a(z)\cdot\mathbf{u}_i^a(z)\right]\right. \\ &\left. + \sum_{i=1}^{N}\sum_{j(\neq i)}\int dz\, \frac{1}{2}\frac{\mu}{N-1}\mathbf{u}_i^a(z)\cdot\mathbf{u}_j^a(z)\right\}, \end{aligned} \tag{20}$$

where the "mass" coefficient μ is given by

$$\mu = \frac{\rho}{d_\perp}\int d\mathbf{r}\; g_0(\mathbf{r})\nabla_\perp^2 V(\mathbf{r}) \tag{21}$$

and where

$$g_0(\mathbf{r} - \mathbf{r}') = \frac{1}{\rho^2}\sum_{i=1}^{N}\sum_{j(\neq i)}\langle \delta(\mathbf{r} - \mathbf{r}_{0i}^a)\,\delta(\mathbf{r}' - \mathbf{r}_{0j}^a)\rangle_0 \tag{22}$$

is the pair distribution function of the two-dimensional liquid formed by the centers of mass of flux lines belonging to the same replica in the vortex liquid. In a similar fashion, we show in Appendix A that $H_u^{dis} = \langle \bar{H}_{dis}^{(1)} \rangle_0$ can be written in the form

$$\begin{aligned} H_u^{dis} = &\sum_{a=1}^{p}\sum_{i=1}^{N}\int dz\left[\frac{1}{2}\delta K(\partial_z \mathbf{u}_i^a)^2 + \frac{1}{2}\delta\mu(\mathbf{u}_i^a(z))^2\right] \\ &+ \sum_{a=1}^{p}\sum_{i\neq j}\int dz \int dz'\, \delta\mu_{ij}^{(a)}(z - z')\mathbf{u}_i^a(z)\cdot\mathbf{u}_j^a(z') \\ &+ \sum_{a\neq b}\sum_{i,j}\int dz \int dz'\, \delta\mu_{ij}^{(ab)}(z - z')\mathbf{u}_i^a(z)\cdot\mathbf{u}_j^b(z'). \end{aligned} \tag{23}$$

Here the long-wavelength disorder contribution δK to the tilt modulus of the flux lines is given by

$$\delta K = -\frac{1}{d_\perp T}\int_{-\infty}^{\infty} dz z^2 \nabla_\perp^2 \Delta(\mathbf{r}, z)\Big|_{\mathbf{r}=0}, \tag{24}$$

and the "mass" coefficients $\delta\mu$ are given by

$$\delta\mu = -\frac{\rho}{d_\perp T}\int d\mathbf{r}\sum_{b=1}^{p} g_{0,ab}(\mathbf{r})\nabla_\perp^2\bar{\Delta}(\mathbf{r}), \tag{25a}$$

$$\delta\mu_{ij}^{(a)}(z) = \frac{\rho}{(N-1)d_\perp T}\int d\mathbf{r} g_0(\mathbf{r})\nabla_\perp^2\Delta(\mathbf{r}, z), \tag{25b}$$

$$\delta\mu_{ij}^{(ab)}(z) = \frac{\rho}{N d_\perp T}\int d\mathbf{r} g_{0,a\neq b}(\mathbf{r})\nabla_\perp^2\Delta(\mathbf{r}, z). \tag{25c}$$

We immediately note that the two-body coefficients $\delta\mu_{ij}^{(a)}(z)$ and $\delta\mu_{ij}^{(ab)}(z)$ vanish in the thermodynamic ($N \to \infty$) limit. In Eq. (25a) above, the sum $\Sigma_{b=1}^{p} g_{0,ab}(\mathbf{r})$ denotes the quantity

$$\sum_{b=1}^{p} g_{0,ab}(\mathbf{r}) = g_0(\mathbf{r}) + \sum_{b(\neq a)=1}^{p} g_{0,a\neq b}(\mathbf{r}), \tag{26}$$

where

$$g_{0,a\neq b}(\mathbf{r}-\mathbf{r}') = \frac{1}{\rho^2}\sum_{i=1}^{N}\sum_{j=1}^{N}\langle \delta(\mathbf{r}-\mathbf{r}_{0i}^{a})\delta(\mathbf{r}'-\mathbf{r}_{0j}^{b})\rangle_0 \quad (27)$$

is the pair distribution function of the c.m. mode of flux lines from different replicas. Now, in Eq. (25a) and under the *assumption* of replica symmetry, all pair distribution functions $g_{0,a\neq b}(\mathbf{r})$ are equal to the same function $\tilde{g}_0(\mathbf{r})$, and we may replace the sum $\Sigma_{b\neq a}g_{0,a\neq b}$ by $(p-1)\tilde{g}_0$, with Eq. (25a) becoming

$$\delta\mu = -\frac{\rho}{d_\perp T}\int d\mathbf{r}[g_0(\mathbf{r})+(p-1)\tilde{g}_0(\mathbf{r})]\nabla_\perp^2\bar{\Delta}(\mathbf{r}),$$

which, in the limit $p\to 0$ reduces to

$$\delta\mu|_{p\to 0} = -\frac{\rho}{d_\perp T}\int d\mathbf{r}[g_0(\mathbf{r})-\tilde{g}_0(\mathbf{r})]\nabla_\perp^2\bar{\Delta}(\mathbf{r}). \quad (28)$$

In the replica-symmetric ground state considered here, the diagonal and off-diagonal (in replica space) pair distribution functions should be equal,

$$g_0(\mathbf{r}) = \tilde{g}_0(\mathbf{r}), \quad (29)$$

and we therefore obtain that the correction $\delta\mu$ identically vanishes, which shows that the bare mass μ generated by interactions between flux lines is unrenormalized by disorder.

We now can write the following expression for the effective Hamiltonian $H_u = H_u^{pure} + H_u^{dis}$ of the internal fluctuations of flux lines in a vortex liquid:

$$H_u \simeq \sum_{a=1}^{p}\sum_{i=1}^{N}\int dz\,\frac{1}{2}[K_R(\partial_z\mathbf{u}_i^a)^2+\mu(\mathbf{u}_i^a(z))^2], \quad (30)$$

where we discarded the terms proportional to $1/N$ [which vanish in the thermodynamic limit; see Eqs. (25b) and (25c)] and where the renormalized tilt modulus K is given by

$$K_R = K + \delta K, \quad (31)$$

with δK given by Eq. (24). At this stage, an explicit expression for the disorder correlator $\Delta(\mathbf{r},z)$ is called for. For point disorder such as oxygen vacancies in HTSC's, we shall take $\Delta(\mathbf{r},z)=\Delta_0\exp[-(r^2+z^2)/2\xi^2]$ (with the understanding that the disorder correlation length ξ is much smaller than the average intervortex distance $a=1/\sqrt{\rho}$), upon which we obtain the following expression for the effective tilt modulus:

$$K_R = K + \frac{\sqrt{2\pi}\Delta_0\xi}{d_\perp T}. \quad (32)$$

This expression shows that a flux-line liquid in the presence of a weak pinning potential is equivalent to an unpinned liquid but with a higher tilt modulus—i.e., that flux lines are stiffened by weak point disorder—which is what one would expect based on physical intuition.

For disorder that is correlated along the direction of flux lines—e.g., columnar pins[18]—the variance $\Delta(\mathbf{r},z)$ is z independent. Upon using for this case a disorder correlator of the form $\Delta(\mathbf{r},z)=\Delta_0\exp(-r^2/2\xi^2)$, we obtain

$$K_R = K + \frac{L^3\Delta_0}{3d_\perp T\xi^2}, \quad (33)$$

which shows that the tilt modulus is much strongly renormalized by correlated disorder than it is by ordinary point disorder, in agreement with the predictions of Refs. 19 and 20. The divergence of the right-hand side of the above equation, however, signals the breakdown of perturbation theory for correlated disorder, which is best treated with other, nonperturbative methods[19] that are better suited to strong pinning situations.

III. VARIATIONAL APPROACH

We now generalize the analysis of the previous section to take into account the effect of possible large fluctuations of flux-line trajectories by using the replica Gaussian variational approach for elastic manifolds.[14] For compactness, we shall only give the salient features of the calculation and refer the reader interested in more details to Ref. 10, where a similar calculation was done for the pure case, the generalization to the disordered case being straightforward.

We start by introducing the following variational Hamiltonian:

$$H_v = H_0[\{\mathbf{r}_{0i}\}] + H_1[\{\mathbf{u}_{0i}\}], \quad (34)$$

where H_0 and H_1 are trial Hamiltonians for the c.m. and internal modes, respectively, and are to be determined variationally. Although one can, in principle, use a very general trial Hamiltonian for the internal modes of flux lines of the form

$$H_1 = \sum_{a,b}\sum_{i,j}\int dz dz'[G^{-1}(q_n)]_{ij,ab}^{\alpha\beta}u_{ia}^{\alpha}(q_n)u_{jb}^{\beta}(-q_n), \quad (35)$$

the insight we gained from the perturbative solution suggests the following, simplified form:

$$H_1 = \sum_{a,i}\sum_{n\neq 0}G^{-1}(q_n)|\mathbf{u}_{ia}(q_n)|^2. \quad (36)$$

Variation of the trial free energy

$$F_1 = -T\ln Z_1 + \langle \bar{H}-H_v\rangle_1, \quad (37)$$

where $Z_1=\mathrm{Tr}[\exp(-H_1/T)]$ and $\langle\cdots\rangle_1$ denotes averaging with statistical weight $\exp(-H_1/T)/Z_1$, with respect to the c.m. Hamiltonian H_0, leads to the result that the optimal choice for H_0 is given by [we henceforth use the shorthand notation $\int_{\mathbf{q}}=\int d^{d_\perp}\mathbf{q}/(2\pi)^{d_\perp}$]

$$\tilde{H}_0 = \frac{L}{2}\sum_{i,j}\Bigg\{\sum_{n\neq 0}d_\perp pT[G_0^{-1}(q_n)-G^{-1}(q_n)]G(q_n) + \sum_{a=1}^{p}\int_{\mathbf{q}}V(\mathbf{q})e^{i\mathbf{q}\cdot(\mathbf{r}_{0i}^a-\mathbf{r}_{0j}^a)}e^{(-q^2/2d_\perp)\phi_{ij}(0)} - \frac{1}{T}\sum_{a,b}\int_{\mathbf{q}}\int dz\Delta(\mathbf{q},z)e^{i\mathbf{q}\cdot(\mathbf{r}_{0i}^a-\mathbf{r}_{0j}^b)}e^{(-q^2/2d_\perp)\phi_{ij}(z)}\Bigg\}, \quad (38)$$

where we defined $\phi_{ij}(z)=\langle[\mathbf{u}_i(z)-\mathbf{u}_j(0)]^2\rangle$. Further variation

of the resulting free energy $F_v=-T\ln Z_1|_{H_0=\tilde{H}_0}$ with respect to $G(q_n)$ leads to the following result:

$$\tilde{G}^{-1}(q_n)=q_n^2-\frac{\rho}{d_\perp}\int_{\mathbf{q}} q^2V(\mathbf{q})g_0(\mathbf{q})e^{-q^2G(0)} + \frac{1}{d_\perp T}\int_{\mathbf{q}} dz q^2\Delta(\mathbf{q},z)[1-\cos(q_nz)]e^{-q^2\phi(z)/2d_\perp} + \frac{\rho}{2}\int_{\mathbf{q}} dz q^2\Delta(\mathbf{q},z)[g_0(q)+(p-1)\tilde{g}_0(q)]e^{-q^2\phi(z)/2d_\perp}, \tag{39}$$

where now $\phi(z)$ denotes the relative displacement of internal modes within the same flux line, $\phi(z)=\langle[\mathbf{u}_i(z)-\mathbf{u}_i(0)]^2\rangle$, and where the tilde indicates that $\tilde{G}^{-1}(q_n)$ has been averaged over all possible configurations of the c.m. positions $\{\mathbf{r}_{0i}^a\}$ which are compatible with a liquid structure.[9,10] Under the assumption of replica symmetry [which implies that $\tilde{g}_0(q)=g_0(q)$], the last term vanishes again in the limit $p\to 0$, and we obtain

$$\tilde{G}^{-1}(q_n)=q_n^2-\frac{\rho}{d_\perp}\int_{\mathbf{q}} q^2V(\mathbf{q})g_0(\mathbf{q})e^{-q^2\langle u^2\rangle/d_\perp} + \frac{1}{d_\perp T}\int_{\mathbf{q}} dz\, q^2\Delta(\mathbf{q},z)[1-\cos(q_nz)]e^{(-q^2/2d_\perp)\phi(z)}. \tag{40}$$

The last term on the right-hand side of the above expression leads to the following renormalized value of the tilt modulus of flux lines in the long-wavelength limit:

$$K_R=K+\frac{1}{d_\perp T}\int_{\mathbf{q}}\int dz\, q^2z^2\Delta(\mathbf{q},z)e^{-q^2\phi(z)/2d_\perp}. \tag{41}$$

We thus see that the inverse propagator for the elastic distortions of flux lines in the vortex liquid is given by the generic form

$$\tilde{G}^{-1}(q_n)=L(K_Rq_n^2+\mu_R), \tag{42}$$

where the effective "mass" coefficient of internal modes fluctuations μ_R is identical to the quantity derived in Ref. 10:

$$\mu_R=-\frac{\rho}{d_\perp}\int_{\mathbf{q}} q^2V(\mathbf{q})g_0(\mathbf{q})e^{-q^2\langle u^2\rangle/d_\perp}, \tag{43}$$

except that it now depends on K_R (through $\langle u^2\rangle$) and hence on the disorder strength Δ_0.

In Eq. (41), the relative displacement of the internal modes of a given flux line,

$$\phi(z)=d_\perp T\sum_{n\neq 0}\tilde{G}(q_n)[1-\cos(q_nz)] \tag{44a}$$

$$=\frac{d_\perp T}{\sqrt{\mu K_R}}[1-\exp(-\sqrt{\mu/K_R}|z|)], \tag{44b}$$

depends on K_R, and we therefore see that Eq. (41) is in fact a self-consistent equation for the effective tilt modulus. For the explicit evaluation of $\mu_R(T)$ and K_R, we shall make use of the analytical ansatz of Ref. 9 for the pair correlation function $g_0(\mathbf{r})$, which is given by

$$g_0(\mathbf{r})=1-\eta\exp(-\alpha r^2/a^2), \tag{45}$$

where α is a constant of order unity and $0<\eta<1$. The numerical constant η quantifies the degree of correlation between c.m. positions of flux lines. It is close to unity when flux lines are strongly anticorrelated due to the repulsive interactions between their surrounding supercurrents and close to zero in situations where there is considerable cutting and crossing of flux lines. Using the above ansatz for $g_0(r)$, we obtain the following expression for the "mass" μ_R of the internal modes as a function of T, (Ref. 10):

$$\frac{\mu_R(T)}{\mu_0}=\left[\sqrt{1+\left(\frac{\alpha T}{4a^2\sqrt{K_R\mu_0}}\right)^2}-\left(\frac{\alpha T}{4a^2\sqrt{K_R\mu_0}}\right)\right]^2, \tag{46}$$

where $\mu_0=\mu_R(T=0)=2\eta\pi\rho\varepsilon_0/d_\perp$. Since μ_R is a monotonically decreasing function[10] of the parameter $\nu=(\alpha T/4a^2\sqrt{K_R\mu_0})$, we arrive at the important conclusion that pinning disorder (which increases the value of the tilt modulus from its bare value K to the effective value $K_R>K$) *increases* the value of μ, thereby reducing even further (than the sole increase in K) the thermal wandering of flux lines. This effect, which did not appear in the elementary treatment of Sec. II is expected to be rather weak for the weak disorder considered in this work, but may nevertheless reveal itself in actual experiments.

Going back to the effective tilt modulus of the pinned flux line liquid, we see that due to the highly nonlinear character of the self-consistency equation (41), it is not possible to solve this equation for K_R and obtain a general expression for the effective tilt modulus in closed analytic form. For weak disorder, however, such that $\Delta_0\ll KT/\xi$, one can evaluate the second term in Eq. (41) perturbatively in Δ_0, by using for $\phi(z)$ its expression in the absence of disorder, Eq. (44b), with K instead of K_R. In the limit of small correlation lenght ξ, we obtain

$$K_R=K\left(1+\frac{\sqrt{\pi}\Delta_0K\xi^3}{2\sqrt{2}d_\perp T^2}\right). \tag{47}$$

Comparison with the results of the previous section shows that the correction to the bare tilt modulus K goes to zero at high temperature more rapidly than in the simple Taylor result of Eq. (32) ($1/T^2$ as opposed to $1/T$). This discrepancy is due to the fact that the Taylor expansion of Sec. II does not take proper account of thermal fluctuations of the internal modes of flux lines, as opposed to the self-consistent approach of this section which in fact can be shown[14] to become exact in the limit $d_\perp\to\infty$.

IV. ACTION FORMULATION OF THE DYNAMICS OF FLUX-LINE LIQUIDS

We now turn our attention to the dynamics of vortex liquids. The general dynamical behavior of flux-line assemblies (solids and liquids) in a random pinning environment has

attracted a lot of attention in recent years due, on the one hand, to the considerable technological implications of understanding the flow of vortices in HTSC's and, on the other hand, because of the fundamental theoretical questions and variety of physical regimes displayed by these systems. Most of the above-mentioned attention has focused on the dynamics of flux solids, with questions about the degree of crystalline and temporal order in the driven regime, and glassiness in the absence of external drive, at the forefront of theoretical issues that have been addressed. Although there have been a number of studies of the dynamics of disordered flux liquids, these studies were either done within hydrodynamic approaches, which, by definition (since they involve a coarse graining procedure), are unable to give information about actual flux-line trajectories inside the superconducting sample, or used qualitative arguments to separate length and time scales in the plastic regime near the melting point.[21] While the latter approach is very valuable in that it helps draw a good qualitative picture of the physics of driven flux liquids, it assumes that the vortex liquid is very viscous and hence only applies very close to the melting point. As we mentioned in the Introduction, here our goal is to go beyond these previous treatments and establish a general framework for a systematic perturbative study of driven flux-line liquids in presence of weak disorder.

We shall assume that the motion of flux lines in the liquid state in presence of an external driving force $\mathbf{F}$ is governed by the following, overdamped Langevin equation:

$$\gamma\partial_t \mathbf{r}_i(z,t) = -\frac{\delta H}{\delta \mathbf{r}_i(z,t)} + \mathbf{F} + \zeta_i(z,t). \tag{48}$$

For simplicity, the distribution of the thermal noise $\zeta_i(z,t)$ will be taken to be Gaussian, with zero mean and correlations (we use units such that $k_B=1$)

$$\langle \zeta_{i\alpha}(z,t)\zeta_{j\beta}(z',t')\rangle = 2\gamma T\delta_{ij}\delta_{\alpha\beta}\delta(z-z')\delta(t-t'). \tag{49}$$

In the above equations, the parameter γ is the microscopic friction coefficient characteristic of the interaction of the system with the degrees of freedom of the surrounding heat bath. In our case of flux lines in a type-II superconductor, γ describes the drag on a flux line due to the interactions of the electrons in the normal vortex cores with the underlying solid and is given by the Bardeen-Stephen expression[16,22]

$$\gamma = \frac{\rho h^2}{8\pi e^2 \xi_{ab}^2}\sigma_n, \tag{50}$$

where ρ is the average density of flux lines, ξ_{ab} is the superconducting coherence length in the (ab) planes, and σ_n is the normal-state conductivity. (Throughout this paper, we assume that the average direction of the flux-lines is parallel to the $\hat{\mathbf{c}}$ axis of the HTSC.)

The dynamics represented by Eq. (48) is best studied using the action formulation of Martin, Siggia, and Rose[23–26] (MSR), whereby disorder-averaged observables are obtained from the following, disorder-averaged generating functional:[8]

$$\mathcal{Z} = \int \prod_{i=1}^{N} [d\mathbf{r}_i(z,t)][d\,\tilde{\mathbf{r}}_i(z,t)]\mathrm{e}^{-\mathcal{A}}. \tag{51}$$

Here, the MSR dynamical "action" can be written in the form

$$\mathcal{A} = \mathcal{A}_{free} + \mathcal{A}_{int} + \mathcal{A}_{dis}, \tag{52}$$

where the "free" part

$$\mathcal{A}_{free} = \sum_{i=1}^{N} \int dzdt \left\{ \frac{1}{2}(2\gamma T)\tilde{\mathbf{r}}_i^2(z,t) + i\tilde{\mathbf{r}}_i(z,t)\cdot[\gamma\partial_t \mathbf{r}_i(z,t) - K\partial_z^2 \mathbf{r}_i(z,t)] \right\} \tag{53}$$

corresponds to an "ideal gas" of noninteracting flux lines and

$$\mathcal{A}_{int} = \sum_{i\neq j} \int dzdt\; i\tilde{\mathbf{r}}_i(z,t)\cdot\boldsymbol{\nabla} V(\mathbf{r}_i(z,t)-\mathbf{r}_j(z,t)) \tag{54}$$

is the part of the action describing the interactions between vortices. The last term in Eq. (52) decribes the pinning of flux lines by the underlying disorder potential and is given by

$$\mathcal{A}_{dis} = -\frac{1}{2}\sum_{n,m} \int dzdt \int dz'dt' \tilde{\mathbf{r}}_{n\alpha}(z,t)\tilde{\mathbf{r}}_{m\beta}(z',t') \times \partial_\alpha\partial_\beta \Delta(\mathbf{r}_n(z,t)-\mathbf{r}_m(z',t')). \tag{55}$$

(In the above equations and in what follows, $\Sigma_{i\neq j}$ stands for the double summation $\Sigma_{i=1}^{N}\Sigma_{j(\neq i)}$.) Our main goal in the next two sections will be to find a way to calculate expectation values of dynamical observables in the liquid phase (like, for example, the dynamic structure factor of the flux-line liquid) by integrating directly over the conformation variables $\{\tilde{\mathbf{r}}_i(z,t),\mathbf{r}_i(z,t)\}$ using the MSR generating functional of Eq. (51), and not by integrating over the density operators, as is done in the (static) boson analogy[2] and other hydrodynamic approaches.[8] This means that we have to avoid writing the dynamical action $\mathcal{A}$ in terms of the density operator $\hat{\rho}(\mathbf{r},z;t)=\Sigma_{i=1}^{N}\delta(\mathbf{r}-\mathbf{r}_i(z,t))$ and instead keep the conformation variables $\{\tilde{\mathbf{r}}_i(z,t),\mathbf{r}_i(z,t)\}$ as the true and only dynamical variables in the problem. Our strategy will be very similar to the strategy adopted in the previous two sections, which consists in separating out the center of mass from the internal modes of flux lines and trying to find a decoupled approximation to the dynamic action $\mathcal{A}$,

$$\mathcal{A} = \mathcal{A}^{(0)} + \mathcal{A}^{(u)}, \tag{56}$$

such that $\mathcal{A}^{(0)}$ and $\mathcal{A}^{(u)}$ depend only on the c.m. and on the internal modes, respectively. Since our ability to perform functional integrations is limited to Gaussians, for a decomposition of the form (56) to be useful at all we will need to write $\mathcal{A}^{(u)}$ as a bilinear form in the displacement fields $\{\tilde{\mathbf{u}}_i(z,t),\mathbf{u}_i(z,t)\}$ fields. Such a decomposition will allow us to evaluate averages of the form

$$\langle \hat{\rho}(\mathbf{r},z;t)\hat{\rho}(\mathbf{r}',z';t')\rangle = \sum_{n,m} \langle \delta(\mathbf{r}-\mathbf{r}_{0n}(t)-\mathbf{u}_n(z,t)) \times \delta(\mathbf{r}'-\mathbf{r}_{0m}(t')-\mathbf{u}_m(z',t'))\rangle \quad (57)$$

by integrating over the c.m. modes $\{\mathbf{r}_{0i}(t),\tilde{\mathbf{r}}_{0i}(t)\}$ and internal conformation variables $\{\tilde{\mathbf{u}}_i(z,t),\mathbf{u}_i(z,t)\}$ of flux lines, which are the true dynamical variables of the system, rather than the averaged density $\rho(\mathbf{r},z;t)=\langle\hat{\rho}(\mathbf{r},z;t)\rangle$, which is a coarse-grained density with no detailed information on these conformation variables. We shall first start by considering the case of a pure (disorder free) flux liquid at equilibrium, which will serve as a starting point to our treatment of the disordered case to be considered in Sec. VI and needs therefore to be as accurately specified and perfectly understood as possible.

V. EQUILIBRIUM DYNAMICS OF PURE FLUX LIQUIDS

In the free part $\mathcal{A}_{free}$ of the action, Eq. (53), we rewrite the flux-line trajectories $\{\mathbf{r}_i(z,t)\}$ and conjugate fileds $\{\tilde{\mathbf{r}}_i(z,t)\}$ in the form

$$\mathbf{r}_i(z,t)=\mathbf{r}_{0i}(t)+\mathbf{u}_i(z,t), \quad (58a)$$

$$\tilde{\mathbf{r}}_i(z,t)=\tilde{\mathbf{r}}_{0i}(t)+\tilde{\mathbf{u}}_i(z,t). \quad (58b)$$

Using the fact that $\int dz\mathbf{u}_i(z,t)=\int dz\tilde{\mathbf{u}}_i(z,t)=0$, we easily obtain that the free part of the action $\mathcal{A}_{free}$ can be written as the sum

$$\mathcal{A}_{free}=\mathcal{A}^{(0)}_{free}+\mathcal{A}^{(u)}_{free}, \quad (59)$$

where

$$\mathcal{A}^{(0)}_{free}=\sum_{i=1}^{N}\int dt\left\{\frac{1}{2}(2L\gamma T)\tilde{\mathbf{r}}^2_{0i}(t)+i\tilde{\mathbf{r}}_{0i}(t)L\gamma\partial_t\mathbf{r}_{0i}(t)\right\} \quad (60)$$

depends only on c.m. variables, while

$$\mathcal{A}^{(u)}_{free}=\sum_{i=1}^{N}\int dzdt\left\{\frac{1}{2}(2\gamma T)\tilde{\mathbf{u}}^2_i(z,t)+i\tilde{\mathbf{u}}_i(z,t)[\gamma\partial_t\mathbf{u}_i(z,t)-K\partial_z^2\mathbf{u}_i(z,t)]\right\}$$

is the free action for the internal modes of the flux lines. From Eq. (60), it is not difficult to see that the c.m. mode of flux lines is characterized by a friction coefficient $\gamma_0=L\gamma$ and, hence, that the diffusion constant D_0 of a free flux line is inversely proportional to the thickness L of the sample, as already pointed out in Refs. 8 and 27:

$$D_0=\frac{1}{L\gamma}. \quad (61)$$

In the interaction part of the action $\mathcal{A}_{int}$, we expand the interaction potential $V(\mathbf{r}_i(z,t)-\mathbf{r}_j(z,t))$ to linear order in the displacement field:

$$V(\mathbf{r}_i(z,t)-\mathbf{r}_j(z,t))=V(\mathbf{r}_{0i}(t)-\mathbf{r}_{0j}(t))+[\mathbf{u}_i(z,t)-\mathbf{u}_j(z,t)]\cdot\boldsymbol{\nabla}V(\mathbf{r}_{0i}(t)-\mathbf{r}_{0j}(t)).$$

The interaction part $\mathcal{A}_{int}$ can then be written in the form

$$\mathcal{A}_{int}=\mathcal{A}^{(0)}_{int}+\mathcal{A}^{(1)}_{int}, \quad (62)$$

with

$$\mathcal{A}^{(0)}_{int}=\sum_{i\neq j}\int dt\; iL\tilde{\mathbf{r}}_{0i}(t)\cdot\boldsymbol{\nabla}V(\mathbf{r}_{0i}(t)-\mathbf{r}_{0j}(t)), \quad (63a)$$

$$\mathcal{A}^{(1)}_{int}=\sum_{i\neq j}\int dzdt\; i\tilde{u}_{i\alpha}(z,t)[u_{i\beta}(z,t)-u_{j\beta}(z,t)] \times\partial_\alpha\partial_\beta V(\mathbf{r}_{0i}(t)-\mathbf{r}_{0j}(t)). \quad (63b)$$

Combining Eqs. (59) and (62), we see that we can already write the total action $\mathcal{A}$ in the form $\mathcal{A}=\mathcal{A}^{(0)}+\mathcal{A}^{(1)}$, where

$$\mathcal{A}^{(0)}=\mathcal{A}^{(0)}_{free}+\mathcal{A}^{(0)}_{int}=\sum_{i=1}^{N}\int dt\left\{\frac{1}{2}(2L\gamma T)\tilde{\mathbf{r}}^2_{0i}(t)+i\tilde{\mathbf{r}}_{0i}(t)\cdot[L\gamma\partial_t\mathbf{r}_{0i}(t)+L\boldsymbol{\nabla}V(\mathbf{r}_{0i}(t)-\mathbf{r}_{0j}(t))]\right\} \quad (64)$$

depends exclusively on c.m. variables and can be thought of as the dynamical MSR action of a liquid of hard rods of length L interacting through the potential $V_0(\mathbf{r})=LV(\mathbf{r})$. The effective action $\mathcal{A}^{(1)}$ is, on the other hand, given by

$$\mathcal{A}^{(1)}=\sum_{i=1}^{N}\int dzdt\left\{\frac{1}{2}(2\gamma T)\tilde{\mathbf{u}}_i(z,t)+i\tilde{\mathbf{u}}_i(z,t)\cdot[\gamma\partial_t\mathbf{u}_i(z,t)-K\partial_z^2\mathbf{u}_i(z,t)]+i\tilde{u}_{i\alpha}(z,t)\left[\left(\sum_{k(\neq i)}\partial_\alpha\partial_\beta V(\mathbf{r}_{0i}(t)-\mathbf{r}_{0k}(t))\right)\delta_{ij}-\sum_{j(\neq i)}\partial_\alpha\partial_\beta V(\mathbf{r}_{0i}(t)-\mathbf{r}_{0j}(t))\right]u_{j\beta}(z,t)\right\} \quad (65)$$

and describes the internal fluctuations of the flux lines. As it stands, however, $\mathcal{A}^{(1)}$ still contains c.m. dynamical variables. In order to obtain an effective action which depends only on the internal modes, we need to take the average of $\mathcal{A}^{(1)}$ over all configurations of the c.m. coordinates $\{\mathbf{r}_{0i}(t)\}$ which are compatible with a liquid structure. In the spirit of a cumulant expansion,[9] we shall write $\mathcal{A}^{(u)}=\langle\mathcal{A}^{(1)}\rangle_0$, where the average here is taken with statistical weight $\exp(-\mathcal{A}^{(0)})$, $\mathcal{A}^{(0)}$ being the dynamical action for the c.m. mode [Eq. (64)]. Performing the above Gaussian average, we obtain

$$\mathcal{A}^{(u)} = \sum_{i=1}^{N} \int dz dt \Bigg\{ \frac{1}{2}(2\gamma T)\tilde{\mathbf{u}}_i^2(z,t) + i\tilde{u}_{i\alpha}(z,t) \times \Bigg[[(\gamma\partial_t - K\partial_z^2)\delta_{\alpha\beta} + \mu_{\alpha\beta}^{(i)}] u_{i\beta}(z,t) + \sum_{j(\neq i)} \mu_{\alpha\beta}^{(ij)} u_{j\beta}(z,t) \Bigg] \Bigg\}, \tag{66}$$

where we defined[9]

$$\mu_{\alpha\beta}^{(i)} = \left\langle \sum_{i\neq j} \partial_\alpha \partial_\beta V(\mathbf{r}_{0i}(t) - \mathbf{r}_{0j}(t)) \right\rangle_0, \tag{67a}$$

$$\mu_{\alpha\beta}^{(ij)} = -\langle \partial_\alpha \partial_\beta V(\mathbf{r}_{0i}(t) - \mathbf{r}_{0j}(t)) \rangle_0. \tag{67b}$$

It is easy to see that

$$\mu_{\alpha\beta}^{(i)} = \int d\mathbf{r} d\mathbf{r}' \partial_\alpha \partial_\beta V(\mathbf{r}-\mathbf{r}') \left\langle \sum_{i\neq j} \delta(\mathbf{r}-\mathbf{r}_{0i}(t)) \times \delta(\mathbf{r}'-\mathbf{r}_{0j}(t)) \right\rangle_0 = \rho^2 \int d\mathbf{r} d\mathbf{r}' \partial_\alpha \partial_\beta V(\mathbf{r}-\mathbf{r}') g_0(\mathbf{r}-\mathbf{r}'), \tag{68}$$

where

$$g_0(\mathbf{r}-\mathbf{r}') = \frac{1}{\rho^2} \left\langle \sum_{i\neq j} \delta(\mathbf{r}-\mathbf{r}_{0i}(t)) \delta(\mathbf{r}'-\mathbf{r}_{0j}(t)) \right\rangle_0 \tag{69}$$

is the (equal-time) equilibrium pair distribution function of the c.m. mode of the flux-line liquid. Using the rotational symmetry of both $g_0(\mathbf{r})$ and $V(\mathbf{r})$, we obtain that $\mu_{\alpha\beta}^{(i)} = \mu\delta_{\alpha\beta}$ with[9]

$$\mu = \frac{\rho}{d_\perp} \int d\mathbf{r} g_0(r) \nabla^2 V(r), \tag{70}$$

which is the result (21) of Sec. II, and similarly that[9]

$$\mu_{\alpha\beta}^{(ij)} = -\frac{\mu\delta_{\alpha\beta}}{N-1}. \tag{71}$$

The above results lead to the following expression for the effective action of the internal modes of flux lines:

$$\mathcal{A}^{(u)} = \sum_{i=1}^{N} \int dz dt \Bigg\{ \frac{1}{2}(2\gamma T)\tilde{\mathbf{u}}_i^2(z,t) + \sum_{j=1}^{N} i\tilde{u}_{i\alpha}(z,t) \Bigg[\Bigg(\gamma\partial_t - K\partial_z^2 + \frac{N}{N-1}\mu \Bigg) \delta_{ij} - \frac{\mu}{N-1} \Bigg] u_{j\alpha}(z,t) \Bigg\}. \tag{72}$$

The above effective action can be written in Fourier space in the form

$$\mathcal{A}^{(u)} = \sum_{i,j} \sum_{n\neq 0} \int_\omega \Bigg[\frac{1}{2} \tilde{\mathbf{u}}_i(q_n,\omega) \tilde{\Gamma}_{ij}(q_n,\omega) \cdot \tilde{\mathbf{u}}_j(-q_n,-\omega) + i\tilde{\mathbf{u}}_i(q_n,\omega) \Gamma_{ij}(q_n,\omega) \cdot \mathbf{u}_j(-q_n,-\omega) \Bigg], \tag{73}$$

where the shorthand $\int_\omega$ stands for $\int_{-\infty}^{\infty} d\omega/2\pi$ and where the dynamical kernels $\tilde{\Gamma}_{ij}(q_n,\omega)$ and $\Gamma_{ij}(q_n,\omega)$ are given by

$$\tilde{\Gamma}_{ij}(q_n,\omega) = 2\gamma TL\delta_{ij}, \tag{74a}$$

$$\Gamma_{ij}(q_n,\omega) = L\Bigg[\Bigg(-i\gamma\omega + Kq_n^2 + \frac{N}{N-1}\mu \Bigg) \delta_{ij} - \frac{\mu}{N-1} \Bigg]. \tag{74b}$$

In the thermodynamic limit $N\to\infty$, the kernel $\Gamma_{ij}(q_n,\omega)$ reduces to the diagonal form

$$\Gamma_{ij}(q_n,\omega) \simeq L(-i\gamma\omega + Kq_n^2 + \mu)\delta_{ij}. \tag{75}$$

We hence obtain in our perturbative approach that the internal modes of different flux lines are effectively decoupled: in the thermodynamic $N\to\infty$ limit, the effect of the interactions between vortices on their *internal* fluctuations is entirely encoded in the μ term, which acts as a quadratic confining potential $(\sim \frac{1}{2}\mu u_i^2)$ for the *internal* modes of individual flux lines (in total agreement with the findings of the static approach of Ref. 9).

Knowledge of the propagator $\Gamma_{ij}(q_n,\omega)$ allows us to find the reponse and correlation functions $R_{ij}(q_n,\omega)$ and $C_{ij}(q_n,\omega)$, respectively, which we define as follows:

$$R_{ij}(z-z',t-t') = \frac{\delta\langle u_{i\alpha}(z,t)\rangle}{\delta\zeta_{j\alpha}(z',t')} = \frac{1}{d_\perp}\langle \mathbf{u}_i(z,t)\cdot i\tilde{\mathbf{u}}_j(z',t')\rangle, \tag{76a}$$

$$C_{ij}(z-z',t-t') = \frac{1}{d_\perp}\langle \mathbf{u}_i(z,t)\cdot \mathbf{u}_j(z',t')\rangle. \tag{76b}$$

Within the mean-field approach of this section and in the thermodynamic limit $N\to\infty$, the above functions are both diagonal in the vortex labels i,j. If we denote by $\Gamma(z,t)$, $R(z,t)$, and $C(z,t)$ the diagonal parts of the vertex, response, and correlation functions, respectively, in such a way that $\Gamma_{ij}(z,t)=\Gamma(z,t)\delta_{ij}$, $R_{ij}(z-z',t-t')=R(z,t)\delta_{ij}$, and $C_{ij}(z-z',t-t')=C(z,t)\delta_{ij}$, then one can easily verify that[11–13]

$$R(q_n,\omega) = \frac{1}{\Gamma(q_n,\omega)}, \tag{77a}$$

$$C(q_n,\omega) = \frac{\tilde{\Gamma}(q_n,\omega)}{|\Gamma(q_n,\omega)|^2}. \tag{77b}$$

In the thermodynamic $(N\to\infty)$ limit, we obtain from Eqs. (75) and (77a) that the response function $R(q_n,\omega)$ is given by

$$R(q_n,\omega) \simeq \frac{1}{L(-i\gamma\omega + Kq_n^2 + \mu)}, \quad q_n \neq 0. \tag{78}$$

Performing a partial Fourier transform back to the variable t, we obtain

$$R(q_n,t) = \frac{\theta(t)}{\gamma L} e^{-(\mu + Kq_n^2)t/\gamma}, \tag{79}$$

where θ is Heaviside's unit-step function. On the other hand, from Eq. (77b), we readily obtain for the correlation function $C(q_n,t)$ the following expression:

$$C(q_n,t) = \frac{T}{L(Kq_n^2+\mu)} e^{-(\mu+Kq_n^2)|t|/\gamma}, \quad q_n \neq 0. \tag{80}$$

It is easy to verify that the fluctuation-dissipation relation

$$\theta(t)\partial_t C(q_n,t) = -TR(q_n,t), \quad q_n \neq 0, \tag{81}$$

holds for the internal modes of flux lines, which indicates that these modes will eventually reach thermal equilibrium at long enough times. We however should emphasize that the above expressions of the response and correlation functions are only valid for the internal modes of the flux lines. The c.m. mode of vortices, as described by the effective action (64), is still diffusive, although with a diffusion constant D which we expect to be reduced by the interactions to a value that is smaller[28] than the bare diffusion constant of free, noninteracting vortices $D_0 = 1/(\gamma L)$ of Eq. (61). Indeed, from Ref. 28, one can estimate the value of the interacting diffusion constant D in the absence of disorder to be of order (here ξ_{ab} is the superconducting coherence length):

$$D \approx \frac{D_0}{1 + 2\pi\rho\xi_{ab}^2 g_0(2a)}, \tag{82}$$

which is smaller than the bare diffusion constant D_0 for all values of applied magnetic fields smaller than the upper critical field $H_{c2} = \phi_0/2\pi\xi_{ab}^2$.

Having derived the equilbrium dynamics of pure flux liquids, we now turn our attention to the more general case of driven vortex liquids in the presence of a random pinning potential. It should be pointed out at this stage that the "mass" coefficient μ given in Eq. (70) can be generalized to take into account large vortex distortions, as was done for the static case in Ref. 10. This generalization is performed within a dynamic Hartree approximation in Appendix B.

VI. DYNAMICS OF DRIVEN FLUX-LINE LIQUIDS: PERTURBATION THEORY

A. Coarse-grained effective action for flux-line dynamics

Following Refs. 11–13, we decompose the internal modes of flux lines into short- and long-wavelength parts,

$$\mathbf{u}_i(z,t) = \mathbf{u}_i^<(z,t) + \mathbf{u}_i^>(z,t), \tag{83}$$

where (we here for convenience adopt a continuous notation for the q_n summations)

$$\mathbf{u}_i^<(z,t) = \int_{q<\Lambda_<} \mathbf{u}_i(q_z,t) e^{iq_z z}, \tag{84a}$$

$$\mathbf{u}_i^>(z,t) = \int_{\Lambda_<<q<\Lambda} \mathbf{u}_i(q_z,t) e^{iq_z z}. \tag{84b}$$

In the above equations, Λ and $\Lambda_<$ are high- and low-momentum cutoffs, respectively. The ultraviolet cutoff Λ is given in terms of the superconducting coherence length ξ_c along the direction of the flux lines by $\Lambda = \pi/\xi_c$. Inserting the above decomposition, Eq. (83), and a similar decomposition for the response field $\tilde{\mathbf{u}}_i(z,t)$, into the dynamical action, we find after integrating out the short-wavelength modes $\{\mathbf{u}_i^>, \tilde{\mathbf{u}}_i^>\}$ that the long-wavelength effective action, to first-order perturbation theory, is given by

$$\mathcal{A}_{eff} = \mathcal{A}_{pure} + \langle \mathcal{A}_{dis}[\tilde{\mathbf{u}}^< + \tilde{\mathbf{u}}^>, \mathbf{u}^< + \mathbf{u}^>] \rangle_>, \tag{85}$$

where

$$\langle \mathcal{A}_{dis} \rangle_> = \frac{1}{2}\sum_{i,j} \int dz dt \int dz' dt' \tilde{r}_{i\alpha}^<(z,t)\, \delta\tilde{\Gamma}_{ij}^{\alpha\beta}(z,t;z',t')\, \tilde{r}_{j\beta}^<(z',t') + \sum_i \int dz dt\; i\tilde{\mathbf{r}}_{i\alpha}^<(z,t)\, \delta\Gamma_{i\alpha}(z,t), \tag{86}$$

where we defined

$$\delta\tilde{\Gamma}_{ij}^{\alpha\beta}(z,t;z',t') = \int_{\mathbf{q}} iq_\alpha iq_\beta \Delta(\mathbf{q}, z-z') \times e^{i\mathbf{q}\cdot\left[\mathbf{r}_i^<(z,t) - \mathbf{r}_j^<(z',t')\right]} e^{-q_\alpha q_\beta \phi_{ij}^{\alpha\beta}(z,t;z',t')/2}, \tag{87a}$$

$$\delta\Gamma_{i\alpha}(z,t;z',t') = (-i)\sum_j \int dz' dt' R_{ij}(z,t;z',t') \times \int_{\mathbf{q}} q_\alpha q^2 \Delta(\mathbf{q}, z-z') e^{i\mathbf{q}\cdot[\mathbf{r}_i^<(z,t) - \mathbf{r}_j^<(z',t')]} \times e^{-q_\alpha q_\beta \phi_{ij}^{\alpha\beta}(z,t;z',t')/2}, \tag{87b}$$

with the definition $\phi_{ij}^{\alpha\beta}(z,t;z',t') = \langle [u_{i\alpha}(z,t) - u_{i\alpha}(z',t')] \times [u_{j\beta}(z,t) - u_{j\beta}(z',t')] \rangle$. In perturbation theory, valid at large driving forces, it is convenient to take the limit $\Lambda_< \to 0$ and use for the response and correlation functions the expressions (79) and (80), which are spatially invariant and diagonal in vortex indices, in which case expressions (87a) and (87b) are greatly simplified and become

$$\delta\tilde{\Gamma}_{ij}^{\alpha\beta}(z,t;z',t') = \int_{\mathbf{q}} iq_\alpha iq_\beta \Delta(\mathbf{q}, z-z') \times e^{i\mathbf{q}\cdot[\mathbf{r}_i^<(z,t) - \mathbf{r}_j^<(z',t')]} e^{(-q^2/2d_\perp)\Phi(z-z',t-t')}, \tag{88a}$$

$$\delta\Gamma_i^{\alpha}(z,t;z',t') = \frac{-i}{d_\perp}\int dz'dt' R(z-z';t-t') \times \int_{\mathbf{q}} iq_\alpha q^2 \Delta(\mathbf{q},z-z') e^{i\mathbf{q}\cdot[\mathbf{r}_i^{<}(z,t)-\mathbf{r}_j^{<}(z',t')]} \times e^{(-q^2/2d_\perp)\Phi(z-z';t-t')}. \quad (88b)$$

In the above expression, $\Phi(z,t)$ denotes the quantity

$$\Phi(z,t) = \phi_0(t) + \phi(z,t), \quad (89)$$

where $\phi_0(t) = \langle[\mathbf{r}_0(t) - \mathbf{r}_0(0)]^2\rangle_0$ is the relative displacement of the c.m. mode, while

$$\phi(z,t) = \langle[\mathbf{u}_i(z,t) - \mathbf{u}_i(0,0)]^2\rangle = \frac{2Td_\perp}{L}\sum_{n=1}^{\infty}\frac{1-\cos(q_n z)e^{-(Kq_n^2+\mu)|t|/\gamma}}{Kq_n^2+\mu} \quad (90)$$

is the relative displacement of the internal mode $\mathbf{u}$ of a given flux line in the vortex liquid.

Using Eqs. (85), (86), and (88b), effective equations of motion for the c.m. and internal modes of flux lines can be derived in a standard way.[11–13] For the internal modes, we obtain

$$\tilde{\gamma}_{\alpha\beta}\partial_t u_{i\beta}(z,t) = (-\mu\delta_{\alpha\beta} + \tilde{K}_{\alpha\beta}\partial_z^2)u_{i\beta} - \frac{1}{2}\lambda_{\alpha\beta\gamma}\partial_z u_{i\beta}\partial_z u_{i\gamma} + \zeta_i(z,t), \quad (91)$$

where now, in addition to the usual (but renormalized) elastic tension term, new non-linear Kardar-Parisi-Zhang (KPZ) terms have appeared. The c.m. mode, on the other hand, obeys the following equation of motion:

$$\tilde{\gamma}_{\alpha\beta}\partial_t r_{0i\beta}(t) = F_\alpha - F_{fr,\alpha}, \quad (92)$$

where the friction force $\mathbf{F}_{fr}$ arises from the additional drag experienced by the vortex liquid as a result of the presence of the random pinning potential. In the following subsection, we outline the main steps of the coarse-graining procedure which leads to the above equation of motion, before moving on in Sec. VII to calculating the dynamic structure factor of the flux-line liquid in the presence of disorder.

B. Derivation of renormalized quantities

1. Friction force

The first-order correction to the friction force is extracted from the $\sim\mathcal{O}(\tilde{\mathbf{r}}_{0i})$ terms of the effective action of Eq. (85):

$$\mathcal{A}_{eff}[\tilde{\mathbf{r}}_{0i}] = \frac{1}{d_\perp}\sum_i \int dt\, L\tilde{\mathbf{r}}_{0i\alpha}(t)\int dz\int_{\mathbf{q}} q_\alpha q^2\Delta(\mathbf{q},z) \times R(z,t)e^{i\mathbf{q}\mathbf{v}t-(q^2/2d_\perp)\Phi(z,t)}. \quad (93)$$

This term is of the form

$$\mathcal{A}_{eff}[\tilde{\mathbf{r}}_{0i}] = \sum_i \int dt\, iL\tilde{\mathbf{r}}_{0i\alpha}(t)F_{fr,\alpha}, \quad (94)$$

with the effective friction force

$$F_{fr,\alpha} = \int dzdt\int_{\mathbf{q}}\frac{q_\alpha q^2}{d_\perp}\Delta(\mathbf{q},z)R(z,t)\sin(\mathbf{q}\cdot\mathbf{v}t)e^{(-q^2/2d_\perp)\Phi(z,t)}. \quad (95)$$

This expression is identical to previously derived expressions[21] for the friction force on individual flux lines in the presence of disorder, except that here the response and correlation functions to be used for an explicit evaluation of $F_{fr,\alpha}$ should be ones that are relevant to a flux liquid—e.g., Eqs. (79) and (80), respectively.

2. Friction coefficient

The disorder-correction to the friction coefficient is extracted from the effective action as follows. In the expression (88b) of the effective kernel $\delta\Gamma_{i\alpha}$, we make use of the expansion

$$e^{i\mathbf{q}\cdot[\mathbf{u}_i(z,t)-\mathbf{u}_j(z',t')]} \simeq 1 + iq_\alpha[u_{i\alpha}(z,t) - u_{j\alpha}(z',t')] - \frac{1}{2}q_\alpha q_\beta[u_{i\alpha}-u_{j\alpha}][u_{i\beta}-u_{j\beta}] + \cdots. \quad (96)$$

We further shall assume that the disorder is weak, so that the internal displacements of flux lines vary slowly on the scale of the equilibrium kernel $\Gamma(z,t)$—i.e., on the scale of $\sqrt{\mu/K}$. In this case, one can make use of the following gradient expansion:

$$u_{i\alpha}(z,t) - u_{j\alpha}(z',t') \simeq (t-t')\partial_t u_{i\alpha}(z,t) + (z-z')\partial_z u_{i\alpha} - \frac{1}{2}(z-z')^2\partial_z^2 u_{i\alpha}. \quad (97)$$

Using both expansions (96) and (97) in Eq. (88b), we obtain that the effective action $\mathcal{A}_{eff}$ contains a term of the form

$$\mathcal{A}_{eff} = \sum_i \int dzdt\; i\tilde{r}_{i\alpha}^{<}(z,t)[\delta\gamma_{\alpha\beta}\partial_t u_{i\beta}(z,t)] \quad (98)$$

and, hence, that the effective friction coefficient $\tilde{\gamma}_{\alpha\beta}$ in Eq. (91) is given by

$$\tilde{\gamma}_{\alpha\beta} = \gamma\delta_{\alpha\beta} + \delta\gamma_{\alpha\beta}, \quad (99)$$

with

$$\delta\gamma_{\alpha\beta} = \int dzdt\int_{\mathbf{q}}\frac{q_\alpha q_\beta q^2}{d_\perp}\Delta(\mathbf{q},z)tR(z,t)e^{i\mathbf{q}\cdot\mathbf{v}t-(q^2/2d_\perp)\Phi(z,t)}. \quad (100)$$

Note that in the absence of drive ($v=0$), $\tilde{\gamma}_{\alpha\beta}$ is isotropic, $\tilde{\gamma}_{\alpha\beta} = (\gamma+\delta\gamma)\delta_{\alpha\beta}$, where now

$$\delta\gamma = \frac{1}{d_\perp^2}\int dzdt\int_{\mathbf{q}} q^4\Delta(\mathbf{q},z)tR(z,t)e^{(-q^2/2d_\perp)\Phi(z,t)}. \quad (101)$$

3. Elastic dispersion

It also follows from Eqs. (96) and (97) that the elastic coefficients appearing in Eq. (91) can be written in the form

$$\tilde{K}_{\alpha\beta} = K\delta_{\alpha\beta} + \delta K_{\alpha\beta}, \tag{102}$$

where the disorder-dependent correction $\delta K_{\alpha\beta}$ is given by

$$\delta K_{\alpha\beta} = \frac{1}{2d_\perp}\int dzdt \int_{\mathbf{q}} q_\alpha q_\beta q^2 z^2 \Delta(\mathbf{q},z) R(z,t) \times e^{i\mathbf{q}\cdot\mathbf{v}t-(q^2/2d_\perp)\Phi(z,t)}. \tag{103}$$

This equation can be rewritten, using the fluctuation-dissipation theorem, Eq. (81), in the form

$$\delta K_{\alpha\beta} = -\int dz \int_{\mathbf{q}} q_\alpha q_\beta z^2 \Delta(\mathbf{q},z) \times \int_0^\infty dt\, e^{i\mathbf{q}\cdot\mathbf{v}t-(q^2/2d_\perp)\phi_0(t)} \partial_t e^{(-q^2/2d_\perp)\phi(z,t)}. \tag{104}$$

In the static ($v=0$) limit, the above correction to the tilt modulus becomes isotropic, $\delta K_{\alpha\beta} = \delta K \delta_{\alpha\beta}$, with

$$\delta K = \frac{1}{d_\perp T}\int dz \int_{\mathbf{q}} q^2 z^2 \Delta(\mathbf{q},z) e^{(-q^2/2d_\perp)\Phi(z,0)} + \mathcal{O}(1/L), \tag{105}$$

which is the result (41) that we obtained in Sec. III within a static replica approach.

The Taylor expansion (97) also yields convective terms of the form $\chi_{\alpha\beta}\partial_z u_{i\beta}(z,t)$ on the right-hand side of Eq. (91). It is easy to see, however, that the coefficients of these convective terms,

$$\chi_{\alpha\beta} = \frac{1}{2d_\perp}\int dzdt \int_{\mathbf{q}} q_\alpha q_\beta q^2 z \Delta(\mathbf{q},z) R(z,t) \times e^{i\mathbf{q}\cdot\mathbf{v}t} e^{(-q^2/2d_\perp)\Phi(z,t)}, \tag{106}$$

vanish identically by virtue of the fact that the integrand is odd in the integration variable z [provided that the disorder correlator is even in z—i.e., $\Delta(\mathbf{r},-z)=\Delta(\mathbf{r},z)$].

4. KPZ nonlinearity

The effective action (85) contains an additional contribution of the form

$$\mathcal{A}_{eff}[\tilde{u}u^2] = \sum_i \int dzdt\; i\tilde{r}^{<}_{i\alpha}(z,t)\left\{-\frac{1}{2}\lambda_{\alpha\gamma\rho}[\partial_z u_\delta(z,t)] \times [\partial_z u_\rho(z,t)]\right\}, \tag{107}$$

with

$$\lambda_{\alpha\beta\gamma} = \frac{-i}{d_\perp}\int dzdt \int_{\mathbf{q}} q_\alpha q_\beta q_\gamma q^2 z^2 \Delta(\mathbf{q},z) R(z,t) \times e^{i\mathbf{q}\cdot\mathbf{v}t} e^{(-q^2/2d_\perp)\Phi(z,t)}. \tag{108}$$

This means that disorder induces a KPZ nonlinearity in the driven state, much as it does for driven vortex solids. In fact, the emergence of nonlinear KPZ terms in driven flux-line liquids was predicted a long time ago within a macroscopic approach by Hwa,[29] who found that these terms affect the dynamics of the flux liquid on long length scales, with the vortices forming a smooth, laminar phase at small drives and a rough, turbulent phase at large values of the applied force. We shall not study the effect of the KPZ terms in any more detail here and instead refer the reader to the above-mentioned paper for more details on this particular topic.

VII. DYNAMIC STRUCTURE FACTOR OF DISORDERED FLUX-LINE LIQUIDS

We are now in a position to calculate the dynamic structure factor $S(\mathbf{r},z;t)$ of our flux-line liquid. By definition,

$$S(\mathbf{r},z;t) = \langle \hat{\rho}(\mathbf{r},z;t)\hat{\rho}(\mathbf{0},0;0)\rangle, \tag{109}$$

where $\langle\cdots\rangle$ now stands for averaging over both c.m. and internal conformation variables of vortices and where space translational invariance of the flux liquid has been assumed. In what follows, it will be convenient to consider the partial Fourier transform

$$S(\mathbf{q},z;t) = \frac{1}{L_\perp^{d_\perp}}\langle \hat{\rho}(\mathbf{q},z;t)\hat{\rho}(-\mathbf{q},0;0)\rangle, \tag{110}$$

where $L_\perp$ is the size of the system in the plane perpendicular to flux lines. Using the definition of the density operator at time t,

$$\hat{\rho}(\mathbf{r},z;t) = \sum_{i=1}^{N} \delta(\mathbf{r}-\mathbf{r}_i(z,t)), \tag{111}$$

we readily obtain that the Fourier transform $S(\mathbf{q},z;t)$ is given by

$$S(\mathbf{q},z;t) = \frac{1}{L_\perp^{d_\perp}}\sum_{i=1}^{N}\sum_{j=1}^{N}\langle e^{-i\mathbf{q}\cdot[\mathbf{r}_i(z,t)-\mathbf{r}_j(0,0)]}\rangle. \tag{112}$$

We now separate the c.m. mode from the internal modes of the flux lines and assume that the dynamical action $\mathcal{A}$ has been written in the decoupled form

$$\mathcal{A} = \mathcal{A}^{(0)} + \mathcal{A}^{(u)}, \tag{113}$$

where $\mathcal{A}^{(0)}$ depends only on the c.m. variables $\{\mathbf{r}_{0i}(t)\}$, while $\mathcal{A}^{(u)}$ only depends on the internal modes $\{\mathbf{u}_i(z,t)\}$. It then follows that the average on the right-hand side of Eq. (112) can be written in the form

$$\langle e^{-i\mathbf{q}\cdot[\mathbf{r}_i(z,t)-\mathbf{r}_j(z',t')]}\rangle = \langle e^{-i\mathbf{q}\cdot[\mathbf{r}_{0i}(t)-\mathbf{r}_{0j}(t')]}\rangle_0 \times e^{(-1/2d_\perp)q^2\langle[\mathbf{u}_i(z,t)-\mathbf{u}_j(z',t')]^2\rangle_u}. \tag{114}$$

In the above expressions, $\langle\cdots\rangle_0$ and $\langle\cdots\rangle_u$ denote averages with statistical weights $\exp(-\mathcal{A}^{(0)})$ and $\exp(-\mathcal{A}^{(u)})$, respectively. Now, in the approaches of Secs. IV and VI, the internal modes of different flux lines are decoupled, which implies that, for $i\neq j$,

$$\langle [\mathbf{u}_i(z,t) - \mathbf{u}_j(z',t')]^2 \rangle = 2\langle u^2(z,t) \rangle, \tag{115}$$

and so expression (112) of $S(\mathbf{q},q_z;t)$ becomes

$$S(\mathbf{q},z;t) = \frac{1}{L_\perp^{d_\perp}} \left\{ \sum_{i=1}^{N} \langle e^{-i\mathbf{q}\cdot[\mathbf{r}_{0i}(t)-\mathbf{r}_{0i}(0)]} \rangle_0 e^{(-q^2/d_\perp)\phi(z,t)} + \sum_{i=1}^{N} \sum_{j\neq i} \langle e^{-i\mathbf{q}\cdot[\mathbf{r}_{0i}(t)-\mathbf{r}_{0j}(0)]} \rangle_0 e^{(-q^2/d_\perp)\langle u^2(z,t)\rangle} \right\}. \tag{116}$$

Given that all vortices in the flux liquid are equivalent to each other in our mean-field approach, we see that the first term on the right-hand side of Eq. (116) involves the sum of N identical terms, which we can simply write as $NF(\mathbf{q},t)$, with[30]

$$F(\mathbf{q},t) = \frac{1}{N} \sum_{i=1}^{N} \langle e^{i\mathbf{q}\cdot[\mathbf{r}_{0i}(t)-\mathbf{r}_{0i}(0)]} \rangle_0 \simeq e^{(-q^2/2d_\perp)\phi_0(t)}. \tag{117}$$

On the other hand, it is easy to verify that

$$\sum_{i\neq j} \langle e^{-i\mathbf{q}\cdot[\mathbf{r}_{0i}(t)-\mathbf{r}_{0j}(t')]} \rangle_0 = L_\perp^{d_\perp} \rho^2 g_0(\mathbf{q},t-t'), \tag{118}$$

where $g_0(\mathbf{q};t-t')$ is the Fourier transform of the time-dependent version of the pair distribution function of standard liquid-state theory, which, in real space, is given by

$$g_0(\mathbf{r}-\mathbf{r}',t-t') = \frac{1}{\rho^2} \sum_{i\neq j} \langle \delta(\mathbf{r}-\mathbf{r}_{0i}(t)) \delta(\mathbf{r}'-\mathbf{r}_{0j}(t')) \rangle_0. \tag{119}$$

Using the fact that $N=L_\perp^{d_\perp}\rho$, we finally obtain

$$S(\mathbf{q},z,t) = \rho e^{(-1/2d_\perp)q^2\Phi(z,t)} + \rho^2 g_0(\mathbf{q},t-t') e^{(-1/d_\perp)q^2\langle u^2(z,t)\rangle}. \tag{120}$$

A principal difficulty with the expression above is to find a good approximation for the time-dependent pair correlation function $g(\mathbf{q},t-t')$. The simplest such approximation is the so-called "convolution approximation," which was introduced many years ago by Vineyard[31] and which consists in writing for $g_0(\mathbf{q},t)$ the following expression:

$$g_0(\mathbf{q},t) = g_0(\mathbf{q}) F(\mathbf{q},t). \tag{121}$$

Here $g_0(\mathbf{q})=g_0(\mathbf{q},t=0)$ is the ordinary (equal-time) pair distribution function of static liquid-state theory and $F(\mathbf{q},t)$ is the quantity defined in Eq. (117). Using this approximation in Eq. (120) above leads to the following result:

$$S(\mathbf{q},z;t) = \rho e^{(-q^2/2d_\perp)\phi_0(t)} \{ \rho g_0(\mathbf{q}) e^{(-q^2/d_\perp)\langle u^2\rangle} + e^{(-q^2/2d_\perp)\phi(z,t)} \}, \tag{122}$$

where we used Eq. (117) to express $F(\mathbf{q},t)$ in terms of $\phi_0(t)$. In the case where the c.m. mode of the flux lines obeys a simple diffusion law of the form

$$\phi_0(t) = 2d_\perp TD|t|, \tag{123}$$

we obtain the following expression for the structure factor $S(\mathbf{q},z;t)$:

$$S(\mathbf{q},z;t) = \rho e^{-DT|t|q^2} [\rho g_0(\mathbf{q}) e^{(-1/d_\perp)q^2\langle u^2\rangle} + e^{(-q^2/2d_\perp)\phi(z,t)}]. \tag{124}$$

In the following, last section, we shall discuss some limiting cases and compare our theoretical prediction for the static structure factor to experimental results.

VIII. DISCUSSION AND CONCLUSIONS

We now discuss the meaning and phsical implications of our results. We shall start by addressing the nature of the unusual massive phonon mode of the internal fluctuations of flux lines, which seems to violate translational invariance. The first occurrence of this massive mode is in Eq. (11), and it is very easy to verify that this equation is translationally invariant (even though this is not obvious from the way it is written). In Eq. (30), the off-diagonal μ_{ij} terms, which are necessary to maintain translational invariance, were discarded for simplicity. Keeping these off-diagonal terms only leads to corrections of order 1/N to the elastic propagator (as was shown in detail in Ref. 9) which vanish in the thermodynamic limit. In fact, even though a massive term may seem unusual, it is very well known that massive phonon modes *do* appear in ordinary crystal lattices if the lattice has a basis (these are the so-called[32] "optical phonons"). In the case of flux liquids, the internal modes of the flux lines are the analogs of atoms belonging to the basis, and the μ terms may be thought of as the analogues of "optical phonons" in crystals.

We now turn our attention to the static structure factor $S(\mathbf{r},z)$ of the flux line liquid, which is defined as

$$S(\mathbf{r},z) = \langle \hat{\rho}(\mathbf{r},z) \hat{\rho}(0,0) \rangle. \tag{125}$$

In Ref. 10, we have shown that for an effective Hamiltonian H_u of the general form

$$H_u = \frac{1}{2} \sum_{i=1}^{N} \sum_{n\neq 0} G^{-1}(q_n) |\mathbf{u}_i(q_n)|^2, \tag{126}$$

the structure factor is given by

$$S(\mathbf{q},z) = \rho^2 g_0(\mathbf{q}) e^{-q^2\langle u^2\rangle/d_\perp} + \rho e^{(-1/2d_\perp)q^2\phi(z)}, \tag{127}$$

where the mean-square relative displacement $\phi(z)$ has already been evaluated in Eq. (44b) and where the mean projected area $\langle u^2\rangle$ of a given flux line is given in terms of the elastic propagator G by

$$\langle u^2 \rangle = d_\perp T \sum_{n\neq 0} G(q_n). \tag{128}$$

In the present case of a disordered vortex liquid with the inverse elastic propagator of Eq. (42), the above expression for $\langle u^2\rangle$ yields

$$\langle u^2 \rangle = \frac{d_\perp T}{2\sqrt{\mu_R K_R}}. \tag{129}$$

We thus obtain for the structure factor the following expression:

$$S(\mathbf{q},z)=\rho^2 g_0(\mathbf{q})\exp\left(-\frac{Tq^2}{2\sqrt{\mu_R K_R}}\right)+\rho\exp\left\{-\frac{Tq^2}{2\sqrt{\mu_R K_R}}[1-e^{-|z|\sqrt{\mu_R/K_R}}]\right\}. \quad (130)$$

The above result for the structure factor is very different from the result obtained by previous authors in the hydrodynamic formulations of Refs. 2–4, 6, and 8 which is given by

$$S(\mathbf{q},q_z)=\frac{\rho Tq^2/K}{q_z^2+\varepsilon^2(\mathbf{q})/T^2}+\Delta(\mathbf{q},q_z)\left[\frac{\rho q^2/K}{q_z^2+\varepsilon^2(\mathbf{q})/T^2}\right]^2, \quad (131)$$

where the excitation spectrum has the usual bosonic[33] form

$$\frac{\varepsilon(\mathbf{q})}{T}=\left[\left(\frac{Tq^2}{2K}\right)^2+\frac{\rho V(q)q^2}{K}\right]^{1/2}. \quad (132)$$

In Ref. 10 we made a number of comments on the structure factor of interacting, but otherwise disorder-free flux-line liquids derived in the boson mapping approach, Eq. (131) and revealed that it had a number of quite disturbing inconsistencies. We here make the similar observation that while the form (127) follows from the very general assumption that a decomposition of the form (34) [with an arbitrary elastic propagator $G(q_n)$ in Eq. (36)] can be written for the Hamiltonian of the vortex liquid, the structure factor of Eq. (131) does not correspond to *any* choice of elastic propagator $G(q_n)$ and cannot be derived from a microscopic approach like ours. In Appendix C we show that an expression for the structure factor that is similar to Eq. (131) (with an additive contribution proportional to disorder) can be derived for the correlations of the density of the *center-of-mass* coordinates of flux lines. One may therefore conclude that the discrepancy between our results and those of Refs. 2–4 and 6 are due to the fact that in these previous studies the nondiffusive internal modes of vortices were treated on equal footing with the c.m. mode (a diffusive mode for which hydrodynamics is naturally expected to be valid). A more detailed discussion of the formulation of Gaussian hydrodynamics of continuous media and of the shortcomings of previous attempts to formulate the hydrodynamics of flux-line liquids can be found in Appendix C.

We now consider some limiting cases. For a "perfect gas" of noninteracting flux lines, $g_0(r)=1$, and Eq. (122) reduces to

$$S(\mathbf{q},z;t)=(2\pi)^2\rho^2\delta(\mathbf{q})+\rho e^{-D_0 T|t|q^2}e^{(-q^2/2d_\perp)\phi(z,t)}, \quad (133)$$

with the diffusion constant of the noninteracting c.m. mode $D_0=1/(\gamma L)$. For noninteracting flux lines, the correlation function $\phi(z,t)$ is given by

$$\phi(z,t)=\frac{2Td_\perp}{L}\sum_{n=1}^{\infty}\frac{1}{Kq_n^2}\left[1-\cos(q_n z)e^{-Kq_n^2|t|/\gamma}\right]. \quad (134)$$

At times shorter than the characteristic Rouse time $t_{Rouse}=\gamma L^2/K$, the sum can be transformed into an integral, with the result[8]

$$\phi(t)=\frac{2Td_\perp}{K}|z|f\left(\frac{K|t|}{\gamma z^2}\right), \quad (135)$$

where f is the function given by [here $\Gamma(a,x)$ is the incomplete gamma function[16]]

$$f(u)=\frac{1}{\pi}\int_0^\infty\frac{dx}{x^2}[1-e^{-ux^2}\cos x]=\frac{1}{2}+\frac{1}{4\sqrt{\pi}}\Gamma\left(-\frac{1}{2},\frac{1}{4u}\right), \quad (136)$$

with the limiting behavior $f(u)\simeq 1/2$ for $u\to 0$ and $f(u)\simeq\sqrt{u}/\pi$ for $u\gg 1$. On the other hand, at long enough times $t>t_{Rouse}$, the sum in Eq. (134) is rapidly converging to the limiting value

$$\phi(t\gg t_{Rouse})=\frac{2Td_\perp}{LK}\sum_{n=1}^{\infty}\frac{1}{q_n^2}=\frac{LTd_\perp}{12K}. \quad (137)$$

It thus follows that, at long times ($t>t_{Rouse}$), the structure factor of noninteracting flux lines can be written in the form

$$S(\mathbf{q},z;t)=\rho e^{-D_0 T|t|q^2}[1+(2\pi)^2\rho\delta(\mathbf{q})]e^{-LTq^2/24K},\quad \approx(2\pi)^2\rho^2\delta(\mathbf{q}). \quad (138)$$

Let us now consider the other limiting case of a liquid of infinitely rigid flux lines. If we formally let $K\to\infty$ in our equations, it is not difficult to verify that $\langle u^2\rangle=0$ and $\phi(z,t)=\langle[\mathbf{u}_i(z,t)-\mathbf{u}_i(0,0)]^2\rangle=0$, and thus Eq. (122) reduces to the appropriate expression for a liquid of point particles, in this case the liquid formed by the centers of mass of the different vortices:

$$S(\mathbf{q},z;t)=\rho e^{(-1/2d_\perp)q^2\phi_0(t)}[1+\rho g_0(\mathbf{q})]. \quad (139)$$

Note that the z dependence has dropped from this last equation. Now, since by definition $\phi_0(0)=0$, we see that the equal-time structure factors $S(q,z;t=0)$ of Eqs. (122), (124), and (139) reduce to the corresponding quantities obtained in Refs. 9 and 10. In particular, for a liquid of rigid flux lines, Eq. (139) reduces to the correct expression of the static structure factor of standard liquid-state theory,

$$S(\mathbf{q},z)=\rho[1+\rho g_0(\mathbf{q})], \quad (140)$$

an expression which cannot be reproduced using boson mapping and other hydrodynamic methods.

We now turn our attention to the interacting structure factor of Eq. (122). By contrast to the case of noninteracting flux lines treated above, we see here that due to the presence of the confining term μ, the relaxation of the internal modes is extremely fast: on time scales larger than the characteristic time $t_\mu=\gamma/\mu$, the correlation function $\phi(z,t)$ of Eq. (90) reaches its limiting value

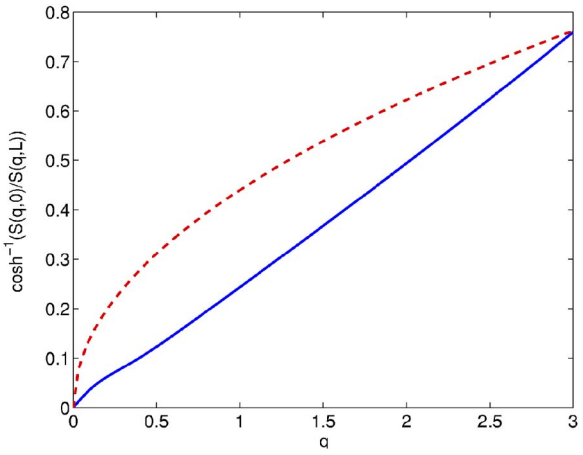

FIG. 1. (Color online) Solid line: plot of $\cosh^{-1}[S(q,0)/S(q,L)]$ vs q, using our result for the static structure factor, Eq. (130). Dashed line: approximate experimental curve, from Ref. 35. As in this last reference, the wave vector q is measured in units of $q_{BZ}=(4\pi/\sqrt{3}a)$.

$$\phi(z,t>t_\mu)=\frac{2Td_\perp}{\pi}\int_0^\infty\frac{dq}{Kq^2+\mu}=\frac{Td_\perp}{\sqrt{K\mu}}, \tag{141}$$

which implies that the low-frequency behavior of the correlation function $S(q,z;\omega)$ is given by

$$S(\mathbf{q},z;\omega)=\frac{2\rho DTq^2}{\omega^2+DT^2q^4}[1+\rho g_0(\mathbf{q})]e^{-Tq^2/\sqrt{K\mu}}. \tag{142}$$

By taking the limit $K\to\infty$, we again see that the low-frequency behavior of the dynamic structure factor of an interacting liquid of hard rods is identical to the frequency behavior of the corresponding noninteracting system, provided the bare diffusion constant D_0 is replaced by the renormalized quantity D and that the static structure of the liquid is taken into account through the factor $[1+\rho g_0(\mathbf{q})]$. For finite K, the only effect of the internal fluctuations of the vortices on the structure factor on long time scales is to introduce the additional "Debye-Waller" factor $\exp(-Tq^2/\sqrt{K\mu})$. In the limit of noninteracting flux lines, where $\mu\to 0$ and $D\to D_0$, the exponent $(Tq^2/\sqrt{K\mu})$ in this last factor goes to its upper bound $(LTq^2/24K)$, the pair distribution function $g_0(\mathbf{q})\to\delta(\mathbf{q})$, and we recover the result (138) of an ideal gas of vortices.

Going beyond the above limiting cases, we here would like to comment on the experiments of Yao *et al.*[34] and Yoon *et al.*[35] who measured the structure factor of the vortex liquid in $Bi_2Sr_2CaCu_2O_8$ (BSCCO) and attempt a quantitative fit of experimental data using our theoretical prediction. For the extremely dilute vortex liquid studied in these experiments, the tilt modulus K is given by the single-vortex value $K\approx\varepsilon_0$, which has the numerical value[21] $\varepsilon_0(\mathrm{K/\AA})=1.964\times 10^8/[\lambda(\mathrm{\AA})]^2$. Figure 1 shows a plot of $\cosh^{-1}[S(q,z=0)/S(q,z=L)]$ using the experimental parameters of Ref. 35: namely,

$$a\simeq 1.5\times 10^4\ \mathrm{\AA}, \tag{143a}$$

$$\lambda\simeq 0.62\times 10^4\ \mathrm{\AA}, \tag{143b}$$

$$L\simeq 0.2\ \mathrm{mm}, \tag{143c}$$

$$T=80\ \mathrm{K}, \tag{143d}$$

and with the fit parameters $\alpha=1$ and $\eta=0.2$. Comparing our plot with the experimental curves (Fig. 2 of Ref. 35), we see that our mean-field approach is able to produce a reasonably good qualitative fit of the data, which is quite surprising, given the rather simplified form of our model Hamiltonian, Eq. (1), and of our analytic ansatz for the pair distribution function $g_0(r)$, Eq. (45).

It is worth nothing at this point that the confining coefficient μ can be reproduced (up to numerical factors of order unity) by taking the short-wavelength limit $q\to q_{BZ}$ [short-wavelength fluctuations being the dominant ones in a liquid and $q_{BZ}=(4\pi/\sqrt{3}a)$ being the wave vector at the Brillouin zone boundary of a solid at the same density] of the compression modulus $c_{11}(q)=B^2/4\pi(1+\lambda^2q^2)$ of usual elasticity theory.[36] That we are able to fit the experimental data with a value of the compression modulus that qualitatively agrees with elasticity theory is rather reassuring and strongly supports our claim that an approach based on conformation variables of vortices is more adequate to describe flux-line liquids than boson mapping methods which by contrast, when used to fit the data of Yoon *et al.*,[35] give a result for c_{11} which is smaller than the theoretical prediction by three orders of magnitude.[35,37]

The failure of hydrodynamic approaches to describe experimental data in a way that is consistent with standard elasticity theory is further indication of the importance of separating the internal modes and the c.m. variables of the vortices in the liquid state. Indeed, there is a very important qualitative distinction between the c.m. mode on the one hand and the internal modes on the other, for while the c.m. mode is diffusive, internal modes of continuous media are nondiffusive (due to the elastic restoring forces and this regardless of whether the confining mass μ is zero or not). The results of the present study outline the importance of separating the nondiffusive modes from the diffusive ones that can be studied using hydrodynamic treatments.

Another quantity which deserves attention is the friction force experienced by the flux liquid driven in presence of disorder. For a flux liquid in presence of point disorder with correlations $\Delta(\mathbf{r},z)=\Delta_0\exp(-r^2/2\xi^2)\delta(z)/2\pi\xi^2$ we obtain, in the large-v limit and in $d=3$ dimensions,

$$F_{fr}\sim\Delta_0\frac{\sqrt{\gamma/K}}{\tilde{\xi}^{9/2}}\frac{1}{\sqrt{v}}, \tag{144}$$

where factors of order unity have been dropped and where we defined the length $\tilde{\xi}$ such that $\tilde{\xi}^2=\xi^2+\langle u^2\rangle$. We see that the friction force vanishes at large drives in agreement with previous predictions for vortex lattices and liquids in the plastic regime near the melting temperature.[21]

In summary, in this paper, we have extended the approach developed in Refs. 9 and 10 to the case of a flux liquid in the presence of a random pinning potential. This approach,

which is based on the use of the conformation variables $\{\mathbf{r}_n(z)\}$ as the *true* dynamical variables in terms of which a Gaussian approximation is taken, gives physically more reasonable results[9,10] than the boson mapping[3] or other hydrodynamic approaches[8] which, instead, use the density as the basic dynamical variable of the vortex liquid. Within our approach, we find that the only effect of the pinning potential in the static equilibrium case is to renormalize the tilt modulus and the confining potential of the internal modes of the flux lines, increasing their stiffness and reducing their thermal wandering. In a similar fashion, we find that in presence of pinning, apart from the appearance of nonlinear KPZ terms and standard renormalization of the coefficients, the equation of motion of flux lines keeps the same form as in an unpinned vortex liquid. As a consequence, and unlike the hydrodynamic approximations,[2–4,8] we find that the structure factor $S(\mathbf{r},z)$ has the same functional form as in a liquid of interacting but unpinned flux lines, with suitably renormalized parameters. Our formulation of the equilibrium dynamics of vortex liquids is in full agreement with the standard dynamical theory of classical liquids and, through the pair distribution function $g_0(r)$ of the c.m. mode, takes into account nontrivial correlations in the positions of flux lines. In particular, we find that the long-time dynamics of a liquid of interacting flux lines is qualitatively similar to the dynamics of an ordinary classical liquid of hard rods with a renormalized diffusion coefficient D (which is reduced by the interactions with respect to the free value D_0), the only effect of internal fluctuations of flux lines at long-time scales being to reduce the structure factor of the vortex liquid through the introduction of a Debye-Waller-like thermal smearing term.

ACKNOWLEDGMENTS

The author acknowledges stimulating interaction with Professor Leo Radzihovsky, Professor Cristina Marchetti, and Dr. George Crabtree. Most of this work was done while the author was still at the University of Colorado at Boulder, where he was supported in part by the David and Lucile Packard Foundation. The final stages were done in Gainesville, and the author wishes to thank the Department of Physics, University of Florida, for financial support.

APPENDIX A: EFFECTIVE HAMILTONIAN OF THE INTERNAL MODES OF FLUX LINES

In this appendix, we show details of how we perform the average $\langle H_{dis}^{(1)}\rangle_0$, where $H_{dis}^{(1)}$ is the disorder part of the Hamiltonian of the internal modes of the flux lines and where the average is taken with statistical weight $\exp(-\bar{H}^{(0)}/T)/\mathrm{Tr}(\exp-\bar{H}^{(0)}/T)$. As can be seen from Eq. (16), the first term of the Hamiltonian $H_{dis}^{(1)}$ does not depend on the c.m. coordinates. Assuming long-wavelength elastic distorsions of the flux lines, we shall write[38]

$$\mathbf{u}_i^a(z)-\mathbf{u}_i^a(z')\simeq(z-z')\partial_z\mathbf{u}_i(z), \tag{A1}$$

upon which one can see that the first term on the right-hand side of Eq. (16) can be written in the form

$$\sum_{a=1}^{p}\sum_{i=1}^{N}\int dz\,\frac{1}{2}\delta K(\partial_z\mathbf{u}_i^a(z))^2, \tag{A2}$$

where the long-wavelength disorder contribution δK to the tilt modulus of the flux lines is given by

$$\delta K=-\frac{1}{d_\perp T}\int_{-\infty}^{\infty}dz\,z^2\nabla_\perp^2\,\Delta(\mathbf{r},z)\big|_{\mathbf{r}=0} \tag{A3}$$

and vanishes for an ultralocal (in z) disorder, with a variance $\Delta(\mathbf{r},z)\propto\delta(z)$.

Now, for the second and third terms of $H_{dis}^{(1)}$, we need to evaluate averages of the form

$$\langle f_{ij}^{ab}\rangle_0=\langle f(\mathbf{r}_{0i}^a-\mathbf{r}_{0j}^b)\rangle_0, \tag{A4}$$

where $f(\mathbf{r})$ is an arbitrary function of the space variable $\mathbf{r}$. We have

$$\langle f_{ij}^{ab}\rangle_0=\bar{Z}^{-1}\int\prod_{c,k}d\mathbf{r}_{oi}^c\,f(\mathbf{r}_{0i}^a-\mathbf{r}_{0j}^b)\mathrm{e}^{-\beta\bar{H}}, \tag{A5}$$

where $\bar{Z}=\mathrm{Tr}[\exp(-\beta\bar{H})]$. Separating out the integrations over $\mathbf{r}_{0i}^a$ and $\mathbf{r}_{0j}^b$, we obtain

$$\langle f_{ij}^{ab}\rangle_0=\int d\mathbf{r}_{0i}^a d\mathbf{r}_{0j}^b\,f(\mathbf{r}_{0i}^a-\mathbf{r}_{0j}^b)\bar{Z}^{-1}\int{\prod_{c,k}}'d\mathbf{r}_{0k}^c\mathrm{e}^{-\beta\bar{H}}, \tag{A6}$$

where the prime on the product indicates that the variables $\mathbf{r}_{0i}^a$ and $\mathbf{r}_{0j}^b$ do not appear in the integration measure. Now, two cases have to be distinguished.

(i) If $a=b$, i.e., $\mathbf{r}_{0i}^a$ and $\mathbf{r}_{0j}^b$ belong to the same replica, then the quantity in the second line on the right-hand side of Eq. (A6) is given by (see also Ref. 9)

$$\frac{\rho^2}{N(N-1)}g_0(\mathbf{r}_{0i}^a-\mathbf{r}_{0j}^a), \tag{A7}$$

where $g_0(\mathbf{r})$ is the pair distribution function of the c.m. of a given replica in the flux liquid. This leads to the following result for $\langle f_{ij}^{aa}\rangle_0$:

$$\langle f_{ij}^{aa}\rangle_0=\frac{\rho}{N-1}\int d\mathbf{r}\,f(\mathbf{r})g_0(\mathbf{r}). \tag{A8}$$

(ii) If $a\neq b$ ($\mathbf{r}_{0i}^a$ and $\mathbf{r}_{0j}^b$ belong to different replicas), then the quantity on the second line of Eq. (A6) is now given by

$$\frac{\rho^2}{N^2}\tilde{g}_0(\mathbf{r}_{0i}^a-\mathbf{r}_{0j}^b), \tag{A9}$$

where $\tilde{g}_0(\mathbf{r})$ is the pair distribution function of particles belonging to different replicas, Eq. (27). This leads to the result

$$\langle f_{ij}^{ab}\rangle_0=\frac{\rho}{N}\int d\mathbf{r}\,f(\mathbf{r})\tilde{g}_0(\mathbf{r}). \tag{A10}$$

Using the results (A8)–(A10) to take the average of the second and third terms on the right-hand side of Eq. (16) and rearranging the resulting sums, we obtain, after a few manipulations, the result (23) of the text.

APPENDIX B: HARTREE APPROXIMATION FOR THE EQUILIBRIUM DYNAMICS OF UNDISORDERED VORTEX LIQUIDS

In this appendix, we show how the result of Sec. V for the effective "mass" term μ, where we used a simple Taylor expansion in the flux-line displacements $\{\mathbf{u}\}$, can be generalized to take into account the potentially large excursions of the vortices around their c.m. positions which are possible in a vortex liquid. Here we shall use a Hartree approximation, which is analogous to the approach introduced a long time ago in the context of spin glasses[39] and was extended more recently to the spherical p-spin model[40] and to fluctuating manifolds in random media.[41] We begin by rewriting the interaction part $\mathcal{A}_{int}$ of Eq. (54) in the form

$$\mathcal{A}_{int} = \mathcal{A}_{int}^{(0)} + \mathcal{A}_{int}^{(u)}, \tag{B1}$$

where

$$\mathcal{A}_{int}^{(0)} = \sum_{i\neq j} \int dz dt \int_{\mathbf{q}} V(\mathbf{q}) i q_\alpha i\tilde{r}_{0i\alpha}(t) e^{i\mathbf{q}\cdot[\mathbf{r}_{0i}(t)-\mathbf{r}_{0j}(t)]} \times e^{i\mathbf{q}\cdot[\mathbf{u}_i(z,t)-\mathbf{u}_j(z,t)]} \tag{B2}$$

is the c.m. part of the interacting action and where

$$\mathcal{A}_{int}^{(u)} = \sum_{i\neq j} \int dz dt \int_{\mathbf{q}} V(\mathbf{q}) i q_\alpha i\tilde{u}_{i\alpha}(z,t) e^{i\mathbf{q}\cdot[\mathbf{r}_i(z,t)-\mathbf{r}_j(z,t)]} = \sum_{i\neq j} \int dz dt \int_{\mathbf{q}} V(\mathbf{q}) i q_\alpha i\tilde{u}_{i\alpha}(z,t) e^{i\mathbf{q}\cdot[\mathbf{r}_{0i}(t)-\mathbf{r}_{0j}(t)]} \times e^{i\mathbf{q}\cdot[\mathbf{u}_i(z,t)-\mathbf{u}_j(z,t)]} \tag{B3}$$

is the internal modes contribution to $\mathcal{A}_{int}$. In the c.m. piece, we get rid of the $\{\mathbf{u}\}$ dependence by making the replacement

$$e^{i\mathbf{q}\cdot[\mathbf{u}_i(z,t)-\mathbf{u}_j(z,t)]} \rightarrow e^{-q_\alpha q_\beta \phi_{ij}^{\alpha\beta}(z,t;z,t)/2}, \tag{B4}$$

where we defined the correlation function $\phi_{i,j}^{\alpha\beta}(z,t;z',t') = \langle [u_{i\alpha}(z,t)-u_{j\alpha}(z',t')][u_{i\beta}(z,t)-u_{j\beta}(z',t')]\rangle$. This yields, for the c.m. part $\mathcal{A}_{int}^{(0)}$,

$$\mathcal{A}_{int}^{(0)} \simeq \sum_{i\neq j} \int dt \int_{\mathbf{q}} \tilde{V}(\mathbf{q}) i q_\alpha iL\tilde{r}_{0i\alpha}(t) e^{i\mathbf{q}\cdot[\mathbf{r}_{0i}(t)-\mathbf{r}_{0j}(t)]}, \tag{B5}$$

with the effective interaction potential *per unit length* between vortices,

$$\tilde{V}(\mathbf{q}) = V(\mathbf{q}) e^{-q_\alpha q_\beta \phi_{ij}^{\alpha\beta}(z,t;z,t)/2}, \tag{B6}$$

which is smeared with respect to the original potential $V(q)$ by thermal fluctuations of the internal modes of flux lines.

We now turn our attention to the relatively more involved task of constructing a self-consistent Gaussian approximation for $\mathcal{A}_{int}^{(u)}$. Following Scheidl (who derived a similar self consistent approach for a flux-line lattice pinned by disorder[42]), we expand the exponential with respect to the displacements and contract the fields in all possible ways pairwise until one or two fields remain uncontracted. For even and odd terms in the displacement, this yields

$$i\tilde{r}_{i\alpha}(z,t)\frac{1}{(2n)!}\{i\mathbf{q}\cdot[\mathbf{u}_i(z,t)-\mathbf{u}_j(z,t)]\}^{2n} \rightarrow i\tilde{r}_{i\alpha}(z,t)\frac{1}{n!}\left\{-\frac{1}{2}q_\alpha q_\beta \phi_{i,j}^{\alpha\beta}(z,t;z,t)\right\}^n,$$

$$i\tilde{r}_{i\alpha}(z,t)\frac{1}{(2n+1)!}\{i\mathbf{q}\cdot[\mathbf{u}_i(z,t)-\mathbf{u}_j(z,t)]\}^{2n+1} \rightarrow i\tilde{r}_{i\alpha}(z,t)\{i\mathbf{q}\cdot[\mathbf{u}_i(z,t)-\mathbf{u}_j(z,t)]\} \times\frac{1}{n!}\left\{-\frac{1}{2}q_\alpha q_\beta \phi_{i,j}^{\alpha\beta}(z,t;z,t)\right\}^n.$$

Resummation yields

$$\mathcal{A}_{int}^{(u)} \simeq \sum_{i\neq j} \int dz dt i\tilde{r}_i^\alpha(z,t)\{V_{i,j}^\alpha(z,t;z,t) + [u_{i\beta}(z,t)-u_{j\beta}(z,t)]V_{i,j}^{\alpha,\beta}(z,t;z,t)\}, \tag{B7}$$

where we defined, for convenience,

$$V_{i,j}^\alpha(z,t;z,t) = \int_{\mathbf{q}} V(\mathbf{q}) i q_\alpha e^{i\mathbf{q}\cdot[\mathbf{r}_{0i}(t)-\mathbf{r}_{0j}(t)]} e^{-q_\alpha q_\beta \phi_{i,j}^{\alpha\beta}(z,t;z,t)/2}, \tag{B8a}$$

$$V_{i,j}^{\alpha,\beta}(z,t;z,t) = \int_{\mathbf{q}} V(\mathbf{q}) i q_\alpha i q_\beta e^{i\mathbf{q}\cdot[\mathbf{r}_{0i}(t)-\mathbf{r}_{0j}(t)]} e^{-q_\alpha q_\beta \phi_{i,j}^{\alpha\beta}(z,t;z,t)/2}. \tag{B8b}$$

The term $V_{i,j}^{\alpha,\beta}(t)$ represents a force acting on particle i and arising from particle j, which is proportional to the small displacement u_j^α.

With hindsight from the results of Sec. V, we shall assume that $\langle u_{i\alpha}(z,t)u_{j\beta}(z,t)\rangle = \delta_{ij}\delta_{\alpha\beta}\langle u_{i\alpha}^2\rangle$, which implies that, for $i \neq j$,

$$\phi_{i,j}^{\alpha\beta}(z,t;z,t) = \frac{2}{d_\perp}\delta_{\alpha\beta}\langle u^2\rangle. \tag{B9}$$

The effective smeared interaction potential of Eq. (B6) becomes

$$\tilde{V}(\mathbf{q}) = V(\mathbf{q}) e^{(-1/d_\perp)\langle u^2\rangle}. \tag{B10}$$

The quantities V_{ij}^α and $V_{ij}^{\alpha\beta}$ in Eqs. (B8a) and (B8b), on the other hand, are now given by

$$V_{i,j}^\alpha(z,t;z,t) = \int_{\mathbf{q}} V(\mathbf{q}) i q_\alpha e^{i\mathbf{q}\cdot[\mathbf{r}_{0i}(t)-\mathbf{r}_{0j}(t)]-(q^2/2d_\perp)\langle u^2\rangle},$$

$$V_{i,j}^{\alpha,\beta}(z,t;z,t) = \int_{\mathbf{q}} V(\mathbf{q}) i q_\alpha i q_\beta e^{i\mathbf{q}\cdot[\mathbf{r}_{0i}(t)-\mathbf{r}_{0j}(t)]-(q^2/2d_\perp)\langle u^2\rangle}. \tag{B11}$$

Taking the average of $\mathcal{A}_{int}$ over the center-of-mass positions with statistical weight $e^{-\mathcal{A}_0}$, we obtain the following effective interaction action:

$$\mathcal{A}_{int}^{(u)} \simeq \sum_{i\neq j} \int dz dt i\tilde{u}_i^{\alpha}(z,t)\{\tilde{V}_{i,j}^{\alpha}(z,t;z,t) + [u_{i\beta}(z,t) - u_{j\beta}(z,t)]\tilde{V}_{i,j}^{\alpha,\beta}(z,t;z,t)\}, \quad \text{(B12)}$$

where now

$$V_{i,j}^{\alpha}(z,t;z,t) = \int_{\mathbf{q}} V(\mathbf{q}) i q_{\alpha} g_0(\mathbf{q}) \mathrm{e}^{(-q^2/2d_{\perp})\langle u^2\rangle}, \quad \text{(B13a)}$$

$$V_{i,j}^{\alpha,\beta}(z,t;z,t) = \int_{\mathbf{q}} V(\mathbf{q}) i q_{\alpha} i q_{\beta} g_0(\mathbf{q}) \mathrm{e}^{(-q^2/2d_{\perp})\langle u^2\rangle}, \quad \text{(B13b)}$$

and where we used the fact that

$$\langle \mathrm{e}^{i\mathbf{q}\cdot[\mathbf{r}_{0i}(t)-\mathbf{r}_{0j}(t)]}\rangle_0 = \frac{\rho}{(N-1)} g_0(\mathbf{q}). \quad \text{(B14)}$$

It is not difficult to see that $\tilde{V}^{\alpha}$ vanishes due to spherical symmetry of the interaction potential, $V(\mathbf{q})=V(q)$. Putting together all terms in Eq. (B12), one obtains that the effective action for the internal modes of flux lines can be again rewritten in the quadratic form of Eqs. (72), (73), (74a), and (74b), with now the mass coefficient μ given by the self-consistent equation

$$\mu = -\frac{\rho}{d_{\perp}} \int_{\mathbf{q}} q^2 V(\mathbf{q}) g_0(\mathbf{q}) \mathrm{e}^{-q^2\langle u^2\rangle/d_{\perp}}, \quad \text{(B15)}$$

which is identical to the result obtained in Ref. 10 using a static variational approach.

APPENDIX C: GAUSSIAN HYDRODYNAMICS OF FLUX LINES REVISITED

In this section, we revisit the Gaussian hydronamic formulation of the statistical mechanics of flux liquids, which will make it easier for us to compare the results of our microscopic approach to the results of previous publications,[2–4,6] which were mostly based on macroscopic coarse-graining methods. We shall first consider the case of a liquid of rigid flux lines, before considering the general case of a liquid of flexible vortices.

1. Hydrodynamics of rigid flux lines

Let us consider a system of rigid flux lines, described by the following Hamiltonian:

$$H = \frac{1}{2}\sum_{i,j} V_0(\mathbf{r}_i - \mathbf{r}_j) + \sum_i V_{ext}(\mathbf{r}_i), \quad \text{(C1)}$$

where $V_0(\mathbf{r})=LV(\mathbf{r})$ and where we introduced a one-body external potential $V_{ext}(\mathbf{r})$. The hydrodynamics of the liquid described by the above Hamiltonian is constructed in a standard way[43] as follows. First, one introduces a variational Hamiltonian, consisting of the sum of noninteracting single-vortex Hamiltonians:

$$H_v = \sum_{i=1}^{N} H_1(\mathbf{r}_i), \quad \text{(C2)}$$

with the Hamiltonian H_1 to be determined by minimization of the variational free energy F_{ρ} (the significance of the subscript ρ will become clear shortly), which is given by

$$F_{\rho} = -T \ln Z_v + \langle H - H_v\rangle_v. \quad \text{(C3)}$$

In the above expression, $\langle\cdots\rangle_v$ denotes averaging with statistical weight $e^{-\beta H_v}/Z_v$ and Z_v is the partition function associated with the Hamiltonian H_v and is given by

$$Z_v = \int d\mathbf{r}_1 \cdots d\mathbf{r}_N \exp(-\beta \sum_{i=1}^{N} H_1(\mathbf{r}_i)) = \left(\int d\mathbf{r} e^{-\beta H_1(\mathbf{r})}\right)^N. \quad \text{(C4)}$$

In the same way, it is easy to show that

$$\langle H_v\rangle_v = \frac{N\int d\mathbf{r}\ H_1(\mathbf{r}) e^{-\beta H_1(\mathbf{r})}}{\int d\mathbf{r}\ e^{-\beta H_1(\mathbf{r})}}. \quad \text{(C5)}$$

Now, the variational average of the density operator $\rho(\mathbf{r}) = \langle \Sigma_{i=1}^{N} \delta(\mathbf{r}-\mathbf{r}_i)\rangle_v$ is given by

$$\rho(\mathbf{r}) = \frac{N e^{-\beta H_1(\mathbf{r})}}{\int d\mathbf{r}\ e^{-\beta H_1(\mathbf{r})}}, \quad \text{(C6)}$$

so that

$$\ln\left(\frac{\rho(\mathbf{r})}{N}\right) = -\beta H_1(\mathbf{r}) - \ln\left(\int d\mathbf{r}\ e^{-\beta H_1(\mathbf{r})}\right). \quad \text{(C7)}$$

Hence

$$\int d\mathbf{r} T\rho(\mathbf{r}) \ln\left(\frac{\rho(\mathbf{r})}{N}\right) = -\frac{N\int d\mathbf{r}\ H_1(\mathbf{r}) e^{-\beta H_1(\mathbf{r})}}{\int d\mathbf{r}\ e^{-\beta H_1(\mathbf{r})}} - NT \ln\left(\int d\mathbf{r}\ e^{-\beta H_1(\mathbf{r})}\right). \quad \text{(C8)}$$

Comparing the above equation with Eqs. (C4) and (C5), we see that

$$\int d\mathbf{r}\ T\rho(\mathbf{r}) \ln\left(\frac{\rho(\mathbf{r})}{N}\right) = -T \ln Z_v - \langle H_v\rangle_v. \quad \text{(C9)}$$

On the other hand, it is not difficult to show that the variational average of the original Hamiltonian H is given by

$$\langle H\rangle_v = \frac{1}{2}\int d\mathbf{r} d\mathbf{r}' \rho(\mathbf{r}) V_0(\mathbf{r}-\mathbf{r}')\rho(\mathbf{r}') + \int d\mathbf{r} V_{ext}(\mathbf{r})\rho(\mathbf{r}). \quad \text{(C10)}$$

Collecting together the results (C9) and (C10), it follows that the variational free energy of Eq. (C3) can be written as a functional of the averaged density $\rho(\mathbf{r})$ (hence the subscript ρ in F_ρ) and is given by

$$F_\rho = \int d\mathbf{r}\ T\rho(\mathbf{r})\ln\left(\frac{\rho(\mathbf{r})}{N}\right) + \int d\mathbf{r} V_{ext}(\mathbf{r})\rho(\mathbf{r}) + \frac{1}{2}\int d\mathbf{r}d\mathbf{r}'\rho(\mathbf{r})V_0(\mathbf{r}-\mathbf{r}')\rho(\mathbf{r}'). \tag{C11}$$

The first term on the right-hand side of the above equation is the entropic contribution to the free energy of the liquid, which is qualitatively important [we shall see below that this term modifies the temperature dependence of the density response and correlation functions—see Eqs. (C18) and (C20)], but which has been systematically overlooked in previous studies of flux-line liquids.[2–4,6] We now need to minimize the variational free energy F_ρ with respect to the density $\rho(\mathbf{r})$, which may be thought of as an "order parameter," with the constraint that the total number of particles N is held fixed. This amounts to minimizing the variational version of the grand potential,

$$\Omega_\rho = F_\rho - \mu\int d\mathbf{r}\ \rho(\mathbf{r}), \tag{C12}$$

with respect to variations in the density $\rho(\mathbf{r})$, using[44]

$$\frac{\delta\Omega_\rho}{\delta\rho(\mathbf{r})} = 0, \tag{C13}$$

with the Lagrange multiplier μ (chemical potential) in Eq. (C12) fixed by the condition $\int d\mathbf{r}\rho(\mathbf{r})=N$. This minimization procedure leads[43] to an expression for the one-body density $\rho(\mathbf{r})$ which is identical to Eq. (C6) above, with H_1 given by

$$H_1(\mathbf{r}) = V_{ext}(\mathbf{r}) + \int d\mathbf{r}' V_0(\mathbf{r}-\mathbf{r}')\rho(\mathbf{r}'). \tag{C14}$$

Now, the density response function is given by[43]

$$\chi(\mathbf{r},\mathbf{r}') = -\frac{\delta\rho(\mathbf{r})}{\delta V_{ext}(\mathbf{r}')}. \tag{C15}$$

Taking the functional derivative of $\rho(\mathbf{r})$ in Eq. (C6) with respect to $V_{ext}(\mathbf{r}')$, one can easily show that

$$\chi(\mathbf{r},\mathbf{r}') = \beta\rho(\mathbf{r})\frac{\delta H_1(\mathbf{r})}{\delta V_{ext}(\mathbf{r}')} - \frac{\beta\rho(\mathbf{r})}{N}\int d\mathbf{r}_1\rho(\mathbf{r}_1)\frac{\delta H_1(\mathbf{r}_1)}{\delta V_{ext}(\mathbf{r}')} \simeq \beta\rho(\mathbf{r})\frac{\delta H_1(\mathbf{r})}{\delta V_{ext}(\mathbf{r}')}, \tag{C16}$$

where, in going from the first to the second equality, we discarded a term proportional to $1/N$ which vanishes in the thermodynamic $N\to\infty$ limit. Now, using the result (C14) for H_1 into Eq. (C16), we obtain

$$\chi(\mathbf{r},\mathbf{r}') = \beta\rho(\mathbf{r})\left[\delta(\mathbf{r}-\mathbf{r}') - \int d\mathbf{r}'' V_0(\mathbf{r}-\mathbf{r}'')\chi(\mathbf{r}'',\mathbf{r}')\right]. \tag{C17}$$

In the absence of an external one-body potential (i.e., when $V_{ext}=0$), the system is translationally invariant and $\chi(\mathbf{r},\mathbf{r}') = \chi(\mathbf{r}-\mathbf{r}')$. Taking the Fourier transform of Eq. (C17) then leads to the result

$$\chi(\mathbf{q}) = \frac{1}{V_0(\mathbf{q}) + T/\rho}. \tag{C18}$$

The T/ρ term in the denominator originates from the $\rho\ln\rho$ entropic term in expression (C11) of the free energy. As we mentioned above, this term, which determines the temperature variation of the density response function, was totally ignored in previous studies of flux liquids.[2–4,6] From the above response function, the Ursell function $S_{nn}(\mathbf{r}) = \langle\rho(\mathbf{r})\rho(\mathbf{0})\rangle - \rho^2$ (we use the terminology and notation of Ref. 43) is given by

$$S_{nn}(\mathbf{r}) = T\chi(\mathbf{r}), \tag{C19}$$

which gives, in Fourier space,

$$S_{nn}(\mathbf{q}) = T\chi(\mathbf{q}) = \frac{T}{V_0(\mathbf{q}) + T/\rho}. \tag{C20}$$

The above Ursell function can be obtained directly from Eq. (C11) by writing the density $\rho(\mathbf{r})$ as $\rho(\mathbf{r})=\rho+\delta\rho(\mathbf{r})$ and expanding F_ρ up to quadratic order in the density fluctuation $\delta\rho(\mathbf{r})$. Taking the Gaussian average of the product $\delta\rho(\mathbf{r})\delta\rho(\mathbf{0})$ by integrating directly over the density fluctuation field $\delta\rho(\mathbf{r})$,

$$\langle\delta\rho(\mathbf{r},z)\delta\rho(\mathbf{0},0)\rangle = \frac{\int[d(\delta\rho(\mathbf{r},z))]\delta\rho(\mathbf{r},z)\delta\rho(\mathbf{0},0)e^{-\beta F_\rho}}{\int[d(\delta\rho(\mathbf{r},z))]e^{-\beta F_\rho}}, \tag{C21}$$

leads directly to the result (C20). Note however that this last procedure to obtain the Ursell function is an *ad hoc* one [which is valid only because Ω_ρ is stationary with respect to variations of the density $\rho(\mathbf{r})$ around its equilibrium value ρ] and that the most systematic and justifiable way to obtain S_{nn} in this variational formulation of hydrodynamics is through extremizing the grand potential Ω_ρ with respect to the density (a step that is rigourosouly exact[44]) and then using the resulting variational Hamiltonian to find the density response function and, hence, $S_{nn}(\mathbf{q})$, as we did in Eqs. (C11)–(C20). We insist that in the latter method no integration over density variables [as in Eq. (C21)] is performed, and we thus see that Eq. (C21) is by no means a necessary step to obtain the Ursell function.

2. Hydrodynamics of flexible flux lines

We now want to generalize the approach of the above subsection to flexible flux lines. For the sake of homogeneity with the rest of the paper, we shall use the same Hamiltonian as in the text, namely:

$$H = \sum_{i=1}^{N} \int dz \left[\frac{1}{2} K \left(\frac{d\mathbf{r}_i}{dz} \right)^2 + V_{ext}(\mathbf{r}_i(z), z) \right] + \frac{1}{2} \sum_{i,j} \int dz V(\mathbf{r}_i(z) - \mathbf{r}_j(z)), \tag{C22}$$

where we again, by analogy with the previous subsection, have introduced a one-body external potential $V_{ext}(\mathbf{r}, z)$. In an obvious generalization of Eq. (C2) to continuous systems, we shall use the following variational Hamiltonian:

$$H_v = \sum_{i=1}^{N} H_1[\mathbf{r}_i(z)], \tag{C23}$$

where H_1 now is a functional of the entire trajectory $\mathbf{r}_i(z)$ of the ith flux line. The corresponding partition function is given by

$$Z_v = \int [d\mathbf{r}_1(z)] \cdots [d\mathbf{r}_N(z)] \exp\left(-\beta \sum_{i=1}^{N} H_1[\mathbf{r}_i(z)] \right) = \left(\int [d\mathbf{r}(z)] e^{-\beta H_1[\mathbf{r}(z)]} \right)^N. \tag{C24}$$

The variational average of H_v on the other hand is given by

$$\langle H_v \rangle_v = \frac{N \int [d\mathbf{r}(z)] H_1[\mathbf{r}(z)] e^{-\beta H_1[\mathbf{r}(z)]}}{\int [d\mathbf{r}(z)] e^{-\beta H_1[\mathbf{r}(z)]}}, \tag{C25}$$

in total analogy with Eq. (C5). Let us now define the operator

$$\hat{\rho}[\mathbf{r}(z)] = \sum_{i=1}^{N} \prod_{z} \delta(\mathbf{r}(z) - \mathbf{r}_i(z)), \tag{C26}$$

whose statistical average $\langle \hat{\rho}[\mathbf{r}(z)] \rangle$ may be interpreted as the average probability density for finding a vortex with a specific trajectory $\mathbf{r}(z)$ in the system. Now, the variational average $\rho[\mathbf{r}(z)] = \langle \hat{\rho}[\mathbf{r}(z)] \rangle_v$ of this trajectory-density operator is given by

$$\rho[\mathbf{r}(z)] = \frac{N e^{-\beta H_1[\mathbf{r}(z)]}}{\int [d\mathbf{r}(z)] e^{-\beta H_1[\mathbf{r}(z)]}}. \tag{C27}$$

Forming $T\rho[\mathbf{r}(z)] \ln(\rho[\mathbf{r}(z)]/N)$ and then taking the functional integral over the trajectory $\mathbf{r}(z)$, one can again easily show that

$$\int [d\mathbf{r}(z)] \, T\rho[\mathbf{r}(z)] \ln\left(\frac{\rho[\mathbf{r}(z)]}{N} \right) = -T \ln Z_v - \langle H_v \rangle_v. \tag{C28}$$

Now, the variational average of H can be expressed in terms of the trajectory density $\rho[\mathbf{r}(z)]$ and is given by

$$\langle H \rangle_v = \int dz \int [d\mathbf{r}(z)] \left\{ \left[\frac{1}{2} K \left(\frac{d\mathbf{r}}{dz} \right)^2 + V_{ext}(\mathbf{r}(z), z) \right] \rho[\mathbf{r}(z)] + \rho[\mathbf{r}(z)] V(\mathbf{r}(z) - \mathbf{r}'(z)) \rho[\mathbf{r}'(z)] \right\}, \tag{C29}$$

and hence we obtain that the variational free energy $F_\rho = -T \ln Z_v + \langle H - H_v \rangle_v$ can be written in the form

$$F_\rho = \int dz \left\{ \int [d\mathbf{r}(z)] T\rho[\mathbf{r}(z)] \ln\left(\frac{\rho[\mathbf{r}(z)]}{N} \right) + \int [d\mathbf{r}(z)] \left[\frac{1}{2} K \left(\frac{d\mathbf{r}}{dz} \right)^2 + V_{ext}(\mathbf{r}(z)) \right] \rho[\mathbf{r}(z)] + \int [d\mathbf{r}(z)][d\mathbf{r}'(z)] \rho[\mathbf{r}(z)] V(\mathbf{r}(z) - \mathbf{r}'(z)) \rho[\mathbf{r}'(z)] \right\} \tag{C30}$$

and is a generalized functional of the averaged trajectory density $\rho[\mathbf{r}(z)]$. In the subsequent steps of the hydrodynamic method, which involve minimizing F_ρ with respect to the density $\rho[\mathbf{r}(z)]$, it is convenient to discretize the z axis, with unit step d, and think of $\rho[\mathbf{r}(z)]$ as a function $\rho(\mathbf{r}(0), \mathbf{r}(d), \ldots, \mathbf{r}(Md))$ of the positions $(\mathbf{r}(0), \mathbf{r}(d), \ldots, \mathbf{r}(Md))$ of $M+1$ particles interacting with each other harmonically, with a spring constant K/d. The integration measure in this case can be defined as

$$[d\mathbf{r}(z)] = \prod_{m=0}^{M} d\mathbf{r}(md). \tag{C31}$$

Now, from Eq. (C27) we see that

$$\int [d\mathbf{r}(z)] \rho[\mathbf{r}(z)] = N, \tag{C32}$$

and hence we see that the grand potential $\Omega_\rho = F_\rho - \mu N$ can be written in the form

$$\Omega_\rho = F_\rho - \mu \int [d\mathbf{r}(z)] \rho[\mathbf{r}(z)]. \tag{C33}$$

In the same way as in the previous subsection, extremization of Ω_ρ with respect to $\rho[\mathbf{r}(z)]=\rho(\mathbf{r}(0),\mathbf{r}(d),\ldots,\mathbf{r}(Md))$ (thought of as a "generalized" order parameter) under the constraint (C32) leads to the following result for the variational Hamiltonian $H_1[\mathbf{r}(z)]$:

$$H_1[\mathbf{r}(z)]=\int dz\left\{\left[\frac{1}{2}K\left(\frac{d\mathbf{r}}{dz}\right)^2+V_{ext}(\mathbf{r}(z))\right]+\int[d\mathbf{r}'(z)]V(\mathbf{r}(z)-\mathbf{r}'(z))\rho[\mathbf{r}'(z)]\right\}. \tag{C34}$$

Like we did in the text, we now decompose the trajectory $\mathbf{r}(z)$ into c.m. and internal coordinates, $\mathbf{r}(z)=\mathbf{r}_0+\mathbf{u}(z)$, with $\mathbf{r}_0=\int_0^L dz\mathbf{r}(z)$ being the c.m. coordinate of the trajectory $\mathbf{r}(z)$, and Taylor expand the interaction part in the small displacement $\mathbf{u}(\mathbf{r})$:

$$V(\mathbf{r}(z)-\mathbf{r}'(z))=V(\mathbf{r}_0-\mathbf{r}'(z))+u_\alpha(z)\partial_\alpha V(\mathbf{r}_0-\mathbf{r}'(z))+\frac{1}{2}u_\alpha(z)u_\beta(z)\partial_\alpha\partial_\beta V(\mathbf{r}_0-\mathbf{r}'(z)). \tag{C35}$$

The Hamiltonian H_1 becomes

$$H_1[\mathbf{r}(z)]=\int dz\left[\frac{1}{2}K\left(\frac{d\mathbf{u}}{dz}\right)^2+V_{ext}(\mathbf{r}(z))\right]+\int dz\int[d\mathbf{r}'(z)]V(\mathbf{r}_0-\mathbf{r}'(z))\rho[\mathbf{r}'(z)]+\int dz\,u_\alpha(z)\int[d\mathbf{r}'(z)]\partial_\alpha V(\mathbf{r}_0-\mathbf{r}'(z))\rho[\mathbf{r}'(z)]+\frac{1}{2}\int dz\,u_\alpha(z)u_\beta(z)\int[d\mathbf{r}'(z)]\partial_\alpha\partial_\beta V(\mathbf{r}_0-\mathbf{r}'(z))\rho[\mathbf{r}'(z)]. \tag{C36}$$

Using the fact that $\int_0^L dz\mathbf{u}(z)=0$, we see that the third term on the right-hand side of the above equation vanishes and, therefore, that $H_1[\mathbf{r}(z)]$ in the absence of an external potential ($V_{ext}=0$) can be written in the form

$$H_1[\mathbf{r}(z)]=H_{1,c.m.}(\mathbf{r}_0)+H_{1,u}[\mathbf{u}(z)], \tag{C37}$$

where

$$H_{1,c.m.}(\mathbf{r}_0)=\int dz\int[d\mathbf{r}'(z)]V(\mathbf{r}_0-\mathbf{r}'(z))\rho[\mathbf{r}'(z)], \tag{C38a}$$

$$H_{1,u}[\mathbf{u}(z)]=\int dz\left[\frac{1}{2}K\left(\frac{d\mathbf{u}}{dz}\right)^2+\frac{1}{2}\mu_{\alpha\beta}u_\alpha(z)u_\beta(z)\right]. \tag{C38b}$$

In the above expression of $H_{1,u}[\mathbf{u}(z)]$, we defined the confining "mass" tensor

$$\mu_{\alpha\beta}=\int[d\mathbf{r}'(z)]\partial_\alpha\partial_\beta V(\mathbf{r}_0-\mathbf{r}'(z))\rho[\mathbf{r}'(z)]. \tag{C39}$$

As defined above, the quantities $\mu_{\alpha\beta}$ depend on $\mathbf{r}_0$. We shall, however, verify *a posteriori* that in the homogeneous liquid state this dependence drops out and the $\mu_{\alpha\beta}$'s reduce to ordinary constants (and are in fact all equal to zero in the hydrodynamic limit).

An immediate consequence of the decomposition (C37) is that the density matrix also can be written in a decoupled form

$$\rho[\mathbf{r}(z)]=\rho_{c.m.}(\mathbf{r}_0)\rho_u[\mathbf{u}(z)], \tag{C40}$$

with

$$\rho_{c.m.}(\mathbf{r}_0)=\frac{e^{-\beta H_{1,c.m.}(\mathbf{r}_0)}}{\int d\mathbf{r}_0\,e^{-\beta H_{1,c.m.}(\mathbf{r}_0)}}, \tag{C41}$$

$$\rho_u[\mathbf{u}(z)]=\frac{e^{-\beta H_{1,u}[\mathbf{u}(z)]}}{\int[d\mathbf{u}(z)]e^{-\beta H_{1,u}[\mathbf{u}(z)]}}. \tag{C42}$$

Now, if we replace the density matrix $\rho[\mathbf{r}(z)]$ by the decoupled form (C40) back into expression (C38a) of $H_{1,c.m.}$, we obtain

$$H_{1,c.m.}(\mathbf{r}_0)=\int d\mathbf{r}_0'\,L\tilde{V}(\mathbf{r}_0-\mathbf{r}_0')\rho_{c.m.}(\mathbf{r}_0'), \tag{C43}$$

where we defined

$$\tilde{V}(\mathbf{r})=\int[d\mathbf{u}'(z)]V(\mathbf{r}-\mathbf{u}'(z))\rho_u[\mathbf{u}'(z)] \tag{C44a}$$

$$=\int_{\mathbf{k}}e^{i\mathbf{k}\cdot\mathbf{r}-k^2\langle u^2\rangle/2}V(\mathbf{k}), \tag{C44b}$$

and where, in going from the first to the second line of the last equation, we assumed that the density matrix $\rho_u[\mathbf{u}'(z)]$ represents an isotropic Gaussian distribution for the displacement field $\{\mathbf{u}'(z)\}$. In Eq. (C44a), the integration measure $[d\mathbf{u}(z)]$ stands for

$$[d\mathbf{u}(z)] = \prod_{m=0}^{M-1} d\mathbf{u}(md). \tag{C45}$$

Note that the transformation from the variables $\{\mathbf{r}(md)\}(m=0,\ldots,M)$ to the variables $\{\mathbf{r}_0,\mathbf{u}(md)\}(m=0,\ldots,M-1)$ being linear, it has a constant Jacobian, which we shall henceforth ignore for simplicity.

Now, if we insert the decoupled form (C40) of $\rho[\mathbf{r}(z)]$ into expression (C39), we obtain

$$\begin{aligned}\mu_{\alpha\beta} &= \int d\mathbf{r}_0' \rho_{c.m.}(\mathbf{r}_0) \int [d\mathbf{u}'(z)] \partial_\alpha \partial_\beta V(\mathbf{r}_0 - \mathbf{r}_0' - \mathbf{u}'(z)) \\ &\quad \times \rho_u[\mathbf{u}'(z)] \\ &= \int d\mathbf{r}_0' \partial_\alpha \partial_\beta \tilde{V}(\mathbf{r}_0 - \mathbf{r}_0') \rho_{c.m.}(\mathbf{r}_0'). \end{aligned} \tag{C46}$$

In a uniform flux liquid, the averaged c.m. density reduces to a constant $\rho_{c.m.}(\mathbf{r}_0')=\rho$ and the above integral vanishes. This is a direct consequence of the coarse-graining procedure, for if instead of the *averaged* c.m. density $\rho_{c.m.}(\mathbf{r}_0')$, we were still dealing with the c.m. density *operator* $\hat{\rho}_{c.m.}(\mathbf{r}_0)=\Sigma_{i=1}^N \delta(\mathbf{r}_0 - \mathbf{r}_{0i})$, then Eq. (C46) would give us:

$$\mu_{\alpha\beta} = \sum_{i=1}^{N} V(\mathbf{r}_0 - \mathbf{r}_{0i}), \tag{C47}$$

which is very similar to the undisordered version of Eq. (12), which then would yield finite and isotropic "mass" coefficients $\mu_{\alpha\beta}=\mu\delta_{\alpha\beta}$.

Now, if we calculate the structure factor $S(\mathbf{r},z)$ by directly taking the average of $\langle\hat{\rho}(\mathbf{r},z)\hat{\rho}(\mathbf{0},0)\rangle$ as in Eq. (57) and using the decoupled density matrix of Eq. (C40), one can easily show that the structure factor $S(\mathbf{q},z)$ of the flux-line liquid has the form given in Eqs. (127) and (130) of the text, with $\mu=0$, namely:

$$S(\mathbf{q},z) = \rho^2 g_0(\mathbf{q}) e^{-q^2\langle u^2\rangle/d_\perp} + \rho \exp\left(-\frac{Tq^2}{2K}|z|\right). \tag{C48}$$

We have

$$\rho^2 g_0(\mathbf{q}) = (2\pi)^2 \rho^2 \delta(\mathbf{q}) + S_{0,nn}(\mathbf{q}) - \rho, \tag{C49}$$

where $S_{0,nn}$ is the Ursell function of the c.m. mode $[S_{0,nn}(\mathbf{r})=\langle\rho_{c.m.}(\mathbf{r})\rho_{c.m.}(\mathbf{0})\rangle-\rho^2]$, which can be easily derived from the c.m. effective Hamiltonian (C43), following the same steps as in the previous subsection, with the result

$$S_{0,nn}(\mathbf{q}) = \frac{T}{L\tilde{V}(\mathbf{q}) + T/\rho}. \tag{C50}$$

Using the result (C49), Eq. (C48) can be rewritten in the form

$$\begin{aligned}S(\mathbf{q},z) &= (2\pi)^2\rho^2\delta(\mathbf{q}) + [S_{0,nn}(\mathbf{q}) - \rho] e^{-q^2\langle u^2\rangle/d_\perp} \\ &\quad + \rho \exp\left(-\frac{Tq^2}{2K}|z|\right). \end{aligned} \tag{C51}$$

Since $\mu=0$ in the hydrodynamic limit, the mean-squared displacement $\langle u^2\rangle \propto L$, and hence $e^{-q^2\langle u^2\rangle/d_\perp}$ is exponentially small for practically all values of q such that $0<q\leqslant 1/a$. Hence the above expression of the structure factor becomes

$$S(\mathbf{q},z) \simeq (2\pi)^2\rho^2\delta(\mathbf{q}) + \rho \exp\left(-\frac{Tq^2}{2K}|z|\right), \tag{C52}$$

which implies that the Ursell function of the vortex liquid is given by

$$S_{nn}(\mathbf{q},z) = \rho \exp\left(-\frac{Tq^2}{2K}|z|\right) \tag{C53}$$

and is identical, in the hydrodynamic limit, with the Ursell function of an ideal gas of noninteracting flux lines. We thus see that, while the present formulation of Gaussian hydrodynamics yields a structure factor that is of the correct general functional form, Eq. (C48) (unlike previous formulations of Refs. 2–4 and 6), it fails to produce a nonzero value for the confining "mass" term, with the consequence that the ensuing strong fluctuations of flux lines completely smear out the effect of interactions between flux lines [the second term on the right-hand side of Eq. (C51)].

Now, one might argue that the decomposition given in Eq. (C23), which approximates the N-body Hamiltonian of the system by a sum of N noninteracting one-body Hamiltonians, is not the only possible choice for the variational Hamiltonian H_v of the hydrodynamic method. Indeed, another (nonequivalent) choice, which has been implicitely used in previous hydrodynamic treatments of this problem consists, in the discretized scheme where the z axis is cut into $M+1$ equidistant slices, in using the ansatz

$$H_v = \sum_{i=1}^{N} \sum_{m=0}^{M} h(\mathbf{r}(md),md), \tag{C54}$$

with the Hamiltonian of a single-vortex "element" $h(\mathbf{r}(md),md)$ at height $z=md$ to be determined variationally. The above ansatz amounts to assuming for the Hamiltonian $H_1[\mathbf{r}(z)]$ of the preceding paragraphs the following form:

$$H_1[\mathbf{r}(z)] = \sum_{m=0}^{M} h(\mathbf{r}(md),md). \tag{C55}$$

The variational partition function Z_v is now given by

$$Z_v = \prod_{m=0}^{M} \left(\int d\mathbf{r} e^{-\beta h(\mathbf{r},md)} \right)^N, \tag{C56}$$

while the variational average $\langle H_v\rangle_v$ is given by

$$\langle H_v\rangle_v = \frac{N\sum_{m=0}^{M} \int d\mathbf{r}\ h(\mathbf{r},md) e^{-\beta h(\mathbf{r},md)}}{\int d\mathbf{r}\ e^{-\beta h(\mathbf{r},md)}}. \tag{C57}$$

If we define the density operator at height $z=md$, $\hat{\rho}(\mathbf{r},md)$, by

$$\hat{\rho}(\mathbf{r},md) = \sum_{i=1}^{N} \delta(\mathbf{r} - \mathbf{r}_i(md)), \tag{C58}$$

then it follows that the averaged density $\rho(\mathbf{r},md) = \langle \hat{\rho}(\mathbf{r},md) \rangle_v$ is given by

$$\rho(\mathbf{r},md) = \frac{Ne^{-\beta h(\mathbf{r},md)}}{\int d\mathbf{r} e^{-\beta h(\mathbf{r},md)}}. \tag{C59}$$

Thus, here again we can write

$$\int d\mathbf{r}\ T\rho(\mathbf{r},md)\ln\left(\frac{\rho(\mathbf{r},md)}{N}\right) = -\frac{N\int d\mathbf{r}\ h(\mathbf{r},md)e^{-\beta h(\mathbf{r},md)}}{\int d\mathbf{r}\ e^{-\beta h(\mathbf{r},md)}} - TN\ln\left(\int d\mathbf{r}\ e^{-\beta h(\mathbf{r},md)}\right). \tag{C60}$$

Summing over m, we obtain

$$\sum_{m=0}^{M} \int d\mathbf{r}\ T\rho(\mathbf{r},md)\ln\left(\frac{\rho(\mathbf{r},md)}{N}\right) = -T\ln Z_v - \langle H_v \rangle_v. \tag{C61}$$

Taking the variational average $\langle H \rangle_v$ of the original Hamiltonian H, we finally find that the variational free energy $F_\rho = -T\ln Z_v + \langle H - H_v \rangle_v$ can be written in the form (we now switch back to a continuum notation)

$$F_\rho = \int d\mathbf{r}dz\left[\frac{T}{d}\rho(\mathbf{r},z)\ln\left(\frac{\rho(\mathbf{r},z)}{N}\right) + \frac{1}{2}K_1\mathbf{t}^2(\mathbf{r},z)\right] + \frac{1}{2}\int d\mathbf{r}d\mathbf{r}'\int dz\rho(\mathbf{r},z)V(\mathbf{r}-\mathbf{r}')\rho(\mathbf{r}',z) + \int d\mathbf{r}dzV_{ext}(\mathbf{r},z)\rho(\mathbf{r},z), \tag{C62}$$

where $K_1 = K/\rho$ and where we defined the "tilt" field operator[3,4]

$$\hat{\mathbf{t}}(\mathbf{r},z) = \sum_{i=1}^{N} \frac{d\mathbf{r}_i}{dz}\delta(\mathbf{r} - \mathbf{r}_i(z)). \tag{C63}$$

The tilting field $\mathbf{t}(\mathbf{r},z)$ and the density field $\rho(\mathbf{r},z)$ are not independent, but are related to each other by the continuity equation[3,4,6]

$$\partial_z\rho(\mathbf{r},z) + \boldsymbol{\nabla}_\perp \cdot \mathbf{t}(\mathbf{r},z) = 0. \tag{C64}$$

Nelson and Le Doussal[3] and Benetatos and Marchetti[6] obtain density and tilt correlation functions by expanding F_ρ of Eq. (C62) to quadratic order in $\delta\rho(\mathbf{r},z) = \rho(\mathbf{r},z) - \rho$ (omitting, for some unstated reason, the entropic $\rho\ln\rho$ term) and calculating statistical averages in the manner of Eq. (C21), with the constraint (C64) enforced. For example, for the Ursell function $S_{nn}(\mathbf{r},z) = \langle \delta\rho(\mathbf{r},z)\,\delta\rho(\mathbf{0},0)\rangle$, these authors write

$$S_{nn}(\mathbf{r},z) = \frac{\int [d(\delta\rho(\mathbf{r},z))]\int [d\mathbf{t}(\mathbf{r},z)]\delta(\partial_z\rho(\mathbf{r},z) + \boldsymbol{\nabla}_\perp\cdot\mathbf{t}(\mathbf{r},z))\,\delta\rho(\mathbf{r},z)\,\delta\rho(\mathbf{0},0)e^{-\beta F_\rho}}{\int [d(\delta\rho(\mathbf{r},z))]\int [d\mathbf{t}(\mathbf{r},z)]\delta(\partial_z\rho(\mathbf{r},z) + \boldsymbol{\nabla}_\perp\cdot\mathbf{t}(\mathbf{r},z))e^{-\beta F_\rho}}. \tag{C65}$$

with the result

$$S_{nn}(\mathbf{q},q_z) = \frac{Tq_\perp^2}{[V(\mathbf{q}) + T/(\rho d)]q_\perp^2 + K_1 q_z^2}. \tag{C66}$$

(As we mentioned earlier, in previous treatments the $T/\rho d$ term in the denominator of the above expression, which comes from the entropic $\rho\ln\rho$ term in F_ρ, is missing.) Unfortunately, the Gaussian integration in Eq. (C65) is *not* justifiable from the point of view of a variational approach. In other words, correlation functions obtained by using Eq. (C65) cannot be reproduced by a standard variational method, which here would consist in extremizing the free energy F_ρ with respect to the fields $\delta\rho(\mathbf{r},z)$ and $\mathbf{t}(\mathbf{r},z)$, with steps similar to those of Eqs. (C11)–(C20) of the previous subsection. Indeed, as we have seen in the end of Appendix C1 above, the rationale behind the Gaussian averaging in Eq. (C65) is that the free energy is (presumably) a functional of $\delta\rho(\mathbf{r},z)$ and $\mathbf{t}(\mathbf{r},z)$ that is stationary at thermal equilibrium [with the constraint (C64) enforced].

Technically, the standard way to implement the constraint (C64) in a variational procedure is through the introduction of a (functional) Lagrange multiplier $\lambda(\mathbf{r},z)$, whereby one defines the following, modified grand potential (the last term in this equation is simply $-\mu N$):

$$\tilde{\Omega}_\rho = F_\rho + \int d\mathbf{r}dz\ \lambda(\mathbf{r},z)[\partial_z\rho(\mathbf{r},z) + \boldsymbol{\nabla}_\perp\cdot\mathbf{t}(\mathbf{r},z)] - \frac{\mu}{L}\int d\mathbf{r}dz\ \rho(\mathbf{r},z). \tag{C67}$$

Extremizing $\tilde{\Omega}_\rho$ with respect to $\rho(\mathbf{r},z)$ and $\mathbf{t}(\mathbf{r},z)$ leads to the following coupled equations:

$$\partial_z \lambda(\mathbf{r},z) = \frac{T}{d}\left[\ln\left(\frac{\rho(\mathbf{r},z)}{N}\right)+1\right] + V_{ext}(\mathbf{r},z) + \int d\mathbf{r}' V(\mathbf{r}-\mathbf{r}')\rho(\mathbf{r}',z) + \frac{\mu}{L}, \tag{C68a}$$

$$\boldsymbol{\nabla}_\perp \lambda(\mathbf{r},z) = K_1 \mathbf{t}(\mathbf{r},z). \tag{C68b}$$

Equation (C68b) is a vector equation of standard form, which amounts to finding the "electric potential" $\lambda(\mathbf{r},z)$ associated with the planar "electric field" $-K_1\mathbf{t}(\mathbf{r},z)$ and has a well-defined solution in λ if and only if $\boldsymbol{\nabla}_\perp \times \mathbf{t}=0$. Since $\mathbf{t}(\mathbf{r},z)$ is a randomly fluctuating field that does *not* necessarily satisfy this last condition, we arrive at the very important conclusion that it is *not* legitimate to extremize the grand potential with respect to the pair of vector fields $\{\rho(\mathbf{r},z),\mathbf{t}(\mathbf{r},z)\}$ and hence that it is *not* legitimate to calculate statistical averages using the procedure examplified in Eq. (C65).

We thus see that the previous formulations of Gaussian hydrodynamics which lead to expressions for the Ursell function of the form given in Eq. (C66) correspond to a convoluted and *ad hoc* attempt, without any rationale other than hand-waving symmetry considerations, to generalize the variational Gaussian hydrodynamics of point particles to continuous systems. In fact, even if we ignore this lack of rationale and accept the use of Eq. (C65), one other source of inaccuracy of the previous formulations of Gaussian hydrodynamics lies the underlying Hamiltonians of Eqs. (C54) and (C55), which these theories are all implicitly based on. These Hamiltonians indeed represent a very crude approximation to the Hamiltonian of a single flux line in the first place, since $H_1[\mathbf{r}(z)]$ involves relatively strong (harmonic) interactions between flux-line segments while Eq. (C55) models a single flux line as a superposition of noninteracting elements. In fact, a necessary condition for the applicability of the Gaussian hydrodynamic approach is that the interactions be weak.[43] For continuous systems with constituent parts interacting strongly, a correct formulation of Gaussian hydrodynamics must take these strong elastic interactions into account as exactly as possible, as we did in Eqs. (C23)–(C34), for otherwise one may obtain abnormal behavior for z correlations in the system [for example, the Ursell function $S_{nn}(\mathbf{q},z)$ obtained from Eq. (C66) decays more rapidly than the corresponding quantity for an ideal gas of noninteracting flux lines, which is very surprizing, as discussed in detail in Ref. 10].

3. Gaussian hydrodynamics of flexible flux lines in the presence of disorder

We now generalize the formulation of Gaussian hydrodynamics that we developed in the previous subsection to flexible flux lines in the presence of disorder. Since the disordered case involves only minor technical modifications of the undisordered hydrodynamics, we shall only give the salient features of the calculation, leaving out the (obvious) technical details. Our starting point is the replicated Hamiltonian of Eq. (5), which we rewrite here for clarity (we remind the reader that p denotes the total number of replicas):

$$\bar{H} = \sum_{a=1}^{p}\sum_{i=1}^{N}\int dz \frac{1}{2}\left\{K\left(\frac{d\mathbf{r}_i^a}{dz}\right)^2 + \sum_{j(\neq i)} V(\mathbf{r}_i^a(z)-\mathbf{r}_j^a(z))\right\} - \frac{1}{2T}\sum_{a,b=1}^{p}\sum_{i,j=1}^{N}\int dz dz' \Delta(\mathbf{r}_i^a(z)-\mathbf{r}_j^b(z');z-z'). \tag{C69}$$

Rewriting the above Hamiltonian in terms of the trajectory density operator of Eq. (C26) and using a variational ansatz for the total Hamiltonian of the system of the form

$$H_v = \sum_{a=1}^{p}\sum_{i=1}^{N} H_1[\mathbf{r}_i^a(z)], \tag{C70}$$

it is not difficult to show that the variational free energy $F_\rho = -T\ln Z_v + \langle H - H_v\rangle_v$ is given by

$$F_\rho = \sum_{a=1}^{p}\int dz \int [d\mathbf{r}(z)]\left\{T\rho^a[\mathbf{r}(z)]\ln\left(\frac{\rho^a[\mathbf{r}(z)]}{N}\right) + \left[\frac{1}{2}K\left(\frac{d\mathbf{r}}{dz}\right)^2 + V_{ext}(\mathbf{r}(z))\right]\rho^a[\mathbf{r}(z)]\right\} + \frac{1}{2}\sum_{a=1}^{p}\int dz\int[d\mathbf{r}(z)] \times [d\mathbf{r}'(z)]\rho^a[\mathbf{r}(z)]V(\mathbf{r}(z)-\mathbf{r}'(z))\rho^a[\mathbf{r}'(z)] - \frac{1}{2T}\sum_{a,b}\int dz dz' \int [d\mathbf{r}(z)][d\mathbf{r}'(z')]\rho^a[\mathbf{r}(z)]\Delta(\mathbf{r}(z)-\mathbf{r}'(z'),z-z')\rho^b[\mathbf{r}'(z')]. \tag{C71}$$

Extremizing F_ρ with respect to $\rho^a[\mathbf{r}(z)]=\rho^a(\mathbf{r}(0),\ldots,\mathbf{r}(Md))$ leads to an expression similar to the right-hand side of Eq. (C27) for $\rho^a[\mathbf{r}(z)]$, with the following expression for the effective Hamiltonian H_1:

$$H_1[\mathbf{r}(z)] = \int dz\left\{\left[\frac{1}{2}K\left(\frac{d\mathbf{r}}{dz}\right)^2 + V_{ext}(\mathbf{r}(z))\right] + \int[d\mathbf{r}'(z)]V(\mathbf{r}(z)-\mathbf{r}'(z))\rho^a[\mathbf{r}'(z)] - \frac{1}{T}\sum_b\int dz'\int[d\mathbf{r}'(z')]\Delta(\mathbf{r}(z)-\mathbf{r}'(z'),z-z')\rho^b[\mathbf{r}'(z)]\right\}. \tag{C72}$$

We now write $\mathbf{r}(z)=\mathbf{r}_0+\mathbf{u}(z)$ and Taylor expand the above Hamiltonian up to quadratic order in the small displacement field

$\mathbf{u}(\mathbf{r},z)$. Here again we find that, to order $\mathcal{O}(u^2)$, H_1 can be written in the decoupled form (C37), with [here $\bar{\Delta}(\mathbf{r}) = \int_{-\infty}^{\infty} dz \Delta(\mathbf{r},z)$]

$$H_{1,c.m.}(\mathbf{r}_0) = \int dz \int [d\mathbf{r}'(z)] V(\mathbf{r}_0 - \mathbf{r}'(z)) \rho^a[\mathbf{r}'(z)] - \frac{1}{T}\sum_b \int dz' \int [d\mathbf{r}'(z')] \bar{\Delta}(\mathbf{r}_0 - \mathbf{r}'(z'), z - z') \rho^b[\mathbf{r}'(z')], \quad \text{(C73a)}$$

$$H_{1,u}[\mathbf{u}(z)] = \int dz \left[\frac{1}{2} K \left(\frac{d\mathbf{u}}{dz} \right)^2 + \frac{1}{2} \mu_{\alpha\beta} u_\alpha(z) u_\beta(z) \right]. \quad \text{(C73b)}$$

In the above expression of $H_{1,u}[\mathbf{u}(z)]$, the confining "mass" tensor is given by

$$\mu_{\alpha\beta} = \int [d\mathbf{r}'(z)] \partial_\alpha \partial_\beta V(\mathbf{r}_0 - \mathbf{r}'(z)) \rho^a[\mathbf{r}'(z)] - \frac{1}{T}\sum_b \int dz' \int [d\mathbf{r}'(z')] \partial_\alpha \partial_\beta \Delta(\mathbf{r}_0 - \mathbf{r}'(z'), z - z') \rho^b[\mathbf{r}'(z')] \quad \text{(C74)}$$

and vanishes in hydrodynamics for the same reason as in the undisordered case. Also, like in the pure case, the density matrix decouples, $\rho[\mathbf{r}(z)] = \rho_{c.m.}[\mathbf{r}_0] \rho_u[\mathbf{u}(z)]$, and hence the internal modes $\{\mathbf{u}'\}$ in the expression of $H_{1,c.m.}(\mathbf{r}_0)$ can be integrated out, with the result

$$H_{1,c.m.}(\mathbf{r}_0) = V_e^a(\mathbf{r}_0) + \sum_b \int d\mathbf{r}_0' \Gamma_{ab}(\mathbf{r}_0 - \mathbf{r}_0') \rho_{c.m.}^b(\mathbf{r}'), \quad \text{(C75)}$$

where we introduced an external "source" potential V_e^a that depends only on the c.m. position $\mathbf{r}_0$ and where the kernel Γ_{ab} is given by

$$\Gamma_{ab}(\mathbf{r}_0) = L\left(\tilde{V}(\mathbf{r}_0) \delta_{ab} - \frac{\tilde{\Delta}(\mathbf{r}_0)}{T} \right). \quad \text{(C76)}$$

In the above equation, the potential $\tilde{V}$ is given by Eq. (C44b), while $\tilde{\Delta}$ is similarly given by

$$\tilde{\Delta}(\mathbf{r}) = \int [d\mathbf{u}'(z)] \bar{\Delta}(\mathbf{r} - \mathbf{u}'(z)) \rho_u^a[\mathbf{u}'(z)] \quad \text{(C77)}$$

$$= \int_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r} - k^2\langle u^2 \rangle/2} \bar{\Delta}(\mathbf{k}). \quad \text{(C78)}$$

By analogy with Eqs. (C15)–(C17), the density response function $\chi^{ab}(\mathbf{r},\mathbf{r}') = -\delta\rho_{c.m.}^a(\mathbf{r}_0)/\delta V_e^b(\mathbf{r}_0')$ satisfies the following equation:

$$\chi^{ab}(\mathbf{r}_0,\mathbf{r}_0') = \beta\rho^a(\mathbf{r}) \left[\delta_{ab}\delta(\mathbf{r} - \mathbf{r}') - \sum_c \int d\mathbf{r}_0'' \, \Gamma_{ac}(\mathbf{r}_0 - \mathbf{r}_0'') \chi^{cb}(\mathbf{r}_0'', \mathbf{r}_0') \right]. \quad \text{(C79)}$$

For a homogeneous liquid [$V_e = 0$, $\rho(\mathbf{r}) = \rho$], $\chi^{ab}(\mathbf{r}_0,\mathbf{r}_0')$ is translationally invariant, and the above equation can be cast, in Fourier space, into the following matricial form:

$$\sum_c \tilde{\Gamma}^{ac}(\mathbf{q}) \chi^{cb}(\mathbf{q}) = \delta_{ab}, \quad \text{(C80)}$$

with

$$\tilde{\Gamma}_{ac}(\mathbf{q}) = \left(\frac{T}{\rho} + L\tilde{V}(\mathbf{q}) \right) \delta_{ac} - \frac{L\tilde{\Delta}(\mathbf{q})}{T}. \quad \text{(C81)}$$

The matrix $\tilde{\Gamma}$ can easily be inverted using an identity for inverting $p \times p$ matrices of the form

$$(A^{-1})_{ij} = a\delta_{ij} + b, \quad \text{(C82)}$$

namely,

$$A_{ij} = \frac{1}{a}\delta_{ij} - \frac{b}{a(a + pb)}. \quad \text{(C83)}$$

For the response function $\chi^{ab}(\mathbf{q})$, this gives in the limit $p \to 0$ the result

$$\chi^{ab}(\mathbf{q}) = \frac{1}{L\tilde{V}(\mathbf{q}) + T/\rho} \delta_{ab} + \frac{L\tilde{\Delta}(\mathbf{q})}{T[L\tilde{V}(\mathbf{q}) + T/\rho]^2}, \quad \text{(C84)}$$

and hence the diagonal (in replica space) Ursell function for the c.m. mode is given by

$$S_{0,nn}(\mathbf{q}) = \frac{T}{L\tilde{V}(\mathbf{q}) + T/\rho} + \frac{L\tilde{\Delta}(\mathbf{q})}{[L\tilde{V}(\mathbf{q}) + T/\rho]^2}. \quad \text{(C85)}$$

We thus see that disorder produces a Lorentzian-squared correction to the Ursell function fo the c.m. mode in the hydrodynamic limit. The Ursell function of the flux-line liquid is, however, unchanged with respect to the pure case (since the confining mass μ is still zero) and is given by Eq. (C53), in contrast to the results of Refs. 3 and 4. Note, however, that if a finite mass coefficient μ is introduced, then the Ursell function of the flux liquid in the hydrodynamic limit is given by Eq. (C51), which in the present context becomes

$$S_{nn}(\mathbf{q},z)=\left[\frac{T}{L\tilde{V}(\mathbf{q})+T/\rho}+\frac{L\tilde{\Delta}(\mathbf{q})}{[L\tilde{V}(\mathbf{q})+T/\rho]^2}-\rho\right]\times e^{-q^2\langle u^2\rangle/d_\perp}+\rho\exp\left(-\frac{Tq^2}{2K}|z|\right). \quad \text{(C86)}$$

Equations (C85) and (C86) show that quenched disorder contributes an additive term to the Ursell function of the c.m. mode describing fluctuations in the average density of vortices. The terms of S_{nn} that describe the internal modes of flux-lines are only affected by disorder through the renormalization of the microscopic coefficients (K and μ) of the model.

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