Running-phase state in a Josephson washboard potential

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We investigate the dynamics of the phase variable of an ideal underdamped Josephson junction in switching current experiments. These experiments have provided the first evidence for macroscopic quantum tunneling in large Josephson junctions and are currently used for state readout of superconducting qubits. We calculate the shape of the resulting macroscopic wave packet and predict that the propagation of the wave packet long enough after a switching event leads to an average voltage increasing linearly with time.

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I. INTRODUCTION

The dynamics of a large underdamped Josephson junction characterized by a capacitance C and Josephson energy $E_J = \bar{\Phi}_0 I_{c0}$ can be described by the motion of a particle in a washboard potential $U(\gamma) = E_J(1 - \cos\gamma) + I\bar{\Phi}_0\gamma$. The particle has C as the mass, the flux $\bar{\Phi}_0\gamma$ as the coordinate, and the charge on the capacitor Q as the canonically conjugate momentum. Here, γ is the phase difference of the superconducting order parameter across the junction and $\bar{\Phi}_0 = \Phi_0/2\pi = \hbar/2e$ is the flux quanta divided by 2π. Much attention has been given since the discovery of the Josephson effect to the switching dynamics of the junction in the thermal activation regime and in the macroscopic tunneling (MQT) regime. Surprisingly, while the description of the state of the junction before a switching event and calculations of the corresponding probability have been topical issues for many decades, what happens with the quantum state of the junction after tunneling did not receive that much attention.

It is generally argued[1,2] that the junction ends up in a running-phase state, with the voltage Q/C increasing until it eventually becomes sufficiently large for quasiparticle transport to take over. This state is usually described classically: In the mechanical analogy between a pendulum and a Josephson junction, it corresponds to the situation in which the pendulum acquires enough energy to overcome the potential energy and rotate either clockwise or counterclockwise with a nonzero average angular momentum (a nonzero average voltage across the junction). The classical equations of motion for the phase can be solved numerically, and the results provide estimates and a good qualitative understanding for the physics of hysteresis and retrapping.[1,3] At relatively large temperatures, when switching occurs predominantly by thermal activation, treating the phase as a classical variable is surely justified; however, in the case of MQT, quantum mechanical effects play the essential role; therefore, a quantum-mechanical description is necessary.

II. RUNNING-PHASE STATE

In this paper, we present a quantum-mechanical description of the switching and the resulting running wave state; also, an application of the formalism is given to the case in which the junction serves as a qubit readout system (Appendix). The main result is an explicit formula for the macroscopic wave function of an ideally (strongly) underdamped junction after a MQT switching event. If the switching probability is exponential, which is the case for all the theoretical models and also confirmed experimentally, one expects[4] the following expression for the dynamics of the wave function:

$$|\Psi(t)\rangle = e^{-\Gamma t/2} e^{-i\omega_0 t}|\Psi_0\rangle + |\Psi_{out}(t)\rangle. \tag{1}$$

In this equation, Ψ_0 is the initial state, coresponding to a bound state inside one of the metastable wells, while $\Psi_{out}(t)$ is the wave function of the particle corresponding to states in the continuum, outside the well (Fig. 1). This expression gives indeed an exponentially decreasing probability for the particle to be inside the well, with lifetime Γ^{-1}. In the following, we are interested in the structure of $\Psi_{out}(t)$.

To solve this problem, the standard approach is to start with a wave packet localized initially in one of the metastable wells, and then expand it and evolve it in the eigenfunctions of the full Hamiltonian. This procedure works for simple potentials,[5] but even in these cases, the solutions are complicated. Fortunately, unlike problems in scattering theory, in condensed matter, the frequent situation is that we do not need an exact solution of the Schrödinger problem for tunneling, but rather we are interested in the most generic features of it. In most cases in solid state physics, tunneling is simply treated as a process that annihilates a particle on

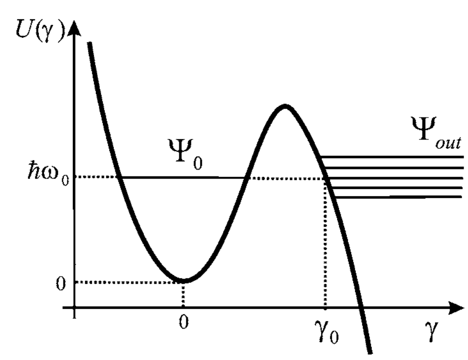

FIG. 1. Tunneling out of one of the metastable wells of the washboard potential.

some mode of a solid and creates one on another mode. We will approach our problem in the same spirit.[6] A good approximation in MQT is that no other state within the well is involved with the exception of the state with energy ω_0 in which the system is prepared, $|\Psi_0\rangle$; therefore, one can write a reduced Hamiltonian of the form

$$H = \hbar\omega_0|\Psi_0\rangle\langle\Psi_0| + \int \hbar\epsilon|\psi_\epsilon\rangle\langle\psi_\epsilon| + \int d\epsilon[k(\omega_0,\epsilon)|\Psi_0\rangle\langle\psi_\epsilon| + k(\epsilon,\omega_0)|\psi_\epsilon\rangle\langle\Psi_0|], \tag{2}$$

where by $\{\psi_\epsilon\}$ we denote the continuum of eigenvectors outside the barrier. We then write the wave function in the form

$$|\Psi(t)\rangle = a(t)e^{-i\omega_0 t}|\Psi_0\rangle + \int d\epsilon b(\epsilon,t)e^{-i\epsilon t}|\psi_\epsilon\rangle \tag{3}$$

with $a(0)=1$, $b(0)=0$. Inserting this expression in the Schrödinger equation, we get an integro-differential equation for $a(t)$. The Laplace transform of this equation reads

$$\mathcal{L}[a](s) = \frac{1}{s+\mathcal{L}[\chi](s)}, \tag{4}$$

where

$$\chi(t) = \frac{1}{\hbar^2}\int d\epsilon|k(\epsilon,\omega_0)|^2 e^{i(\omega_0-\epsilon)t}. \tag{5}$$

In general, the tunneling matrix element $k(\omega_0,\epsilon)$ depends on the energies ϵ, and they are determined by the overlap of the left and right wave functions under the barrier.[6] We notice that since typically the lifetime of the metastable states is much larger than the oscillation period in the well (in other words, the last term in the Hamiltonian is a perturbation), the states $\{|\epsilon\rangle\}$ which contribute effectively to tunneling are located in a relatively small energy interval compared to the plasma oscillation frequency; therefore, the shape of these states under the barrier is approximately identical. We can then take the tunneling matrix element as being a complex constant; since we will be interested exclusively in the outgoing component, the relative phase between the wave function inside the well and that outside will not play any role. We have confirmed this assumption also by expanding the initial wave function in terms of the WKB solution of the washboard potential calculated in Ref. 7. Therefore, we take $k=\hbar\sqrt{\Gamma/2\pi}$ to be real; we obtain $\mathcal{L}[\chi](s)=\Gamma/2=$constant, which turns out to be the decay probability of the system. Indeed, the inverse Laplace transform of Eq. (4) gives precisely the classical exponential decay law

$$a(t) = e^{-\Gamma t/2}. \tag{6}$$

The outgoing wave packet becomes

$$|\Psi_{out}(t)\rangle = -i\sqrt{\frac{\Gamma}{2\pi}}\int d\epsilon\frac{e^{(-\Gamma/2+i\omega_0)t}-e^{-i\epsilon t}}{\Gamma/2+i(\omega_0-\epsilon)}|\psi_\epsilon^\pm\rangle. \tag{7}$$

To conclude this derivation, we find that with the identification $k=\hbar\sqrt{\Gamma/2\pi}$, the Hamiltonian [Eq. (2)] becomes a model Hamiltonian for decay in the continuum which can be solved exactly with the solution given by Eqs. (1) and (7). A similar type of model Hamiltonian has been obtained in Refs. 8 and 9. One can show, using the properties of the Lorentz distribution, that these wave functions are correctly normalized, as explained above.

Let us now single out one component of the wave $\Psi_{out}(t)$, namely

$$\Psi_{out}^{\rightarrow} = i\sqrt{\frac{\Gamma}{2\pi}}\int d\epsilon\frac{e^{-i\epsilon t}}{\Gamma/2+i(\omega_0-\epsilon)}|\psi_\epsilon^+\rangle. \tag{8}$$

We first notice that the normalization of the total function $|\Psi_{out}\rangle$ is such that $\langle\Psi_{out}|\Psi_{out}\rangle=1-\exp(-\Gamma t)$, which reflects correctly the fact that the probability of finding the particle outside comes from an exponential decay law, while that of $|\Psi_{out}^{\rightarrow}\rangle$ is such that $\langle\Psi_{out}^{\rightarrow}|\Psi_{out}^{\rightarrow}\rangle=1$. In the following we will see that $|\Psi_{out}^{\rightarrow}\rangle$, indeed, plays a special role. To move on, we notice that part of the expression for the outgoing phase contains a term which decays exponentially on a time scale Γ^{-1}. These terms are associated with the fast components of the localized wave function which would escape first. Although a calculation that includes these terms is no doubt interesting, especially for the problem of nonexponential decay rates,[5] in what follows we will regard them as transient oscillatory effects whose presence will be difficult to assess experimentally anyway, and we will neglect their contribution. In the WKB approximation, far enough from the classical turning point, the eigenvalues $\{|\psi_\epsilon\rangle\}$ have the form (up to a normalization factor and constant phase factors due to matching to the region left of the classical turning point) of incoming and outgoing scattering states

$$\psi_\epsilon^\pm(\gamma) \approx \sqrt{\frac{e}{C\bar{\Phi}_0 V_\epsilon(\gamma)}}\exp\left[\pm i\frac{C}{e}\int_{\gamma_0}^{\gamma}V_\epsilon(\varphi)d\varphi\right], \tag{9}$$

where

$$V_\epsilon(\gamma) = \frac{\hbar\omega_p}{\sqrt{2}e}\sqrt{\frac{2e\epsilon+I\gamma}{I_{c0}}+1-\cos\gamma}. \tag{10}$$

The physical meaning of this voltage is that it corresponds to the (classical) energy accumulated on the capacitor when the phase difference across the junction is γ and the initial energy of the system is ϵ; indeed, $CV_\epsilon^2(\gamma)/2=\hbar\epsilon-U(\gamma)$. We now use the fact that for values of γ outside the well and far enough from the classical turning point, the inequality $|\hbar\epsilon-\hbar\omega_0| \underset{\text{not}}{\ll} \hbar\omega_0-U(\gamma)$ holds. Therefore, we can take $V_\epsilon(\gamma)=V_{\omega_0}(\gamma)=V_0(\gamma)$ in the denominator of Eq. (9) and approximate the exponent as

$$V_\epsilon(\gamma) \approx V_0(\gamma)\left[1+\frac{\epsilon-\omega_0}{2[\omega_0-\hbar^{-1}U(\gamma)]}\right]. \tag{11}$$

With these approximations, using Eqs. (7) and (9), we can perform the integral over the angular frequencies ϵ; as a result, the contribution of the ingoing scattering states is zero, while the outgoing scattering states build up a wave packet of the form

$$\Psi_{out}^{\rightarrow}(\gamma,t) = \frac{\mathcal{N}}{\sqrt{V_0(\gamma)}} \exp\left[\frac{i}{\hbar}\int_{\gamma_0}^{\gamma} C\widetilde{V}(\varphi)\bar{\Phi}_0 d\varphi - \left(i\omega_0 + \frac{\Gamma}{2}\right) \times \left(t - \int_{\gamma_0}^{\gamma}\frac{\bar{\Phi}_0 d\varphi}{V_0(\varphi)}\right)\right]\Theta\left[t - \int_{\gamma_0}^{\gamma}\frac{\bar{\Phi}_0 d\varphi}{V_0(\varphi)}\right]. \quad (12)$$

Here $\mathcal{N}$ is a normalization factor which can be obtained through $\int_{-\infty}^{\infty} d(\bar{\Phi}_0\gamma)|\Psi_{out}^{\rightarrow}(\gamma,t)|^2=1$ with the mention that we make a negligible error by extending the integral to $-\infty$, i.e., before the well region (where the actual values are exponentially small). The voltage $\widetilde{V}(\gamma)$ is defined as $\widetilde{V}(\gamma)=V_0(\gamma) -\hbar\omega_0/CV_0(\gamma)\approx V_0(\gamma)$. It is interesting to see also what happens with the rest of the components of $|\Psi_{out}\rangle$. Although they do contribute to the normalization as discussed before, they are decaying both in time and away from the barrier as $\exp[(-\Gamma/2)\int_{\gamma_0}^{\gamma} V_0^{-1}(\varphi)\bar{\Phi}_0 d\varphi]$ which, as we will see below, would give far from the barrier a factor of $\exp[-\Gamma/2(t+\sqrt{2\gamma}/\omega_p)]$. It is clear that these terms can be neglected starting roughly from a time Γ^{-1}. The wave function, Eq. (12), contains all the information about the dynamical evolution of the state of the circuit containing the Josephson junction and is the main result of this paper.

III. PREDICTIONS OF THE MODEL

In a typical experiment, the voltage across the junction is monitored by a voltmeter at room temperature. A fundamental issue is to find a microscopic mechanism for the junction-voltmeter interaction and a suitable theory of quantum measurement that would model the collapse of the wave function; however, this is beyond the scope of this paper. Still, Eq. (12) gives a quite clear qualitative picture of what happens: The particle rolls down the washboard potential with a quasiclassical speed given by energy conservation $CV_0^2(\gamma)/2=\hbar\omega_0-U(\gamma)$. Quantum mechanics enters in the picture through the tunneling rate; we expect the results of the measurements to have a spread determined by Γ. One can assume that the measurement projects the outgoing state onto eigenvalues of the voltage operator; therefore, the probability of recording the value V at the moment t will be given by the standard quantum mechanics recipe

$$P(V,t) = \frac{1}{2\pi\hbar}\left|\int_{-\infty}^{\infty} d(\bar{\Phi}_0\gamma)\Psi_{out}^{*}(\gamma,t)\exp(iVC\gamma/2e)\right|^2. \quad (13)$$

As an example, had the outside of the well potential U been 0, we would have gotten for the charge CV, by performing the integration in Eq. (13), a standard Cauchy-Breit-Wigner distribution centered around CV_0 and full width at half maximum $\Gamma\hbar/V_0$

$$P(CV) = \frac{1}{\pi}\frac{\Gamma\hbar}{2V_0}\left[(CV-CV_0)^2 + \left(\frac{\Gamma\hbar}{2V_0}\right)^2\right]^{-1}. \quad (14)$$

Considering again the case of a junction with a washboard potential $U(\gamma)$, we notice that a good approximation is $U(\gamma)\approx E_J\gamma$. This comes from the fact that switching is typically observed at values of the bias current close to the critical current of the junction, as well as from the observation that for times larger than Γ^{-1} the wave packet is concentrated at large values of $\gamma \gg 1$, in which case, the $\cos\gamma$ term in the potential is negligible. In other words, the particle gets fast so soon that the "speed bumps" created by the Josephson effect are not slowing it down significantly. This can be checked *a posteriori*. A first observation is that the relevant quantity for the dynamics of the center of the wave packet is the argument of the Θ function; the condition that this argument vanishes sets the maximum value of $|\Psi_{out}|^2$ and gives a phase $\omega_p t/2 \gg 1$ for t larger than Γ^{-1}. A legitimate concern is whether the wave function does not spread faster than it moves downward. This is not the case, as we will see below: The spread of the wave function increases linearly with time, while the average coordinate (phase) is advancing as t^2. With these observations, the normalization constant can be calculated, and the outgoing wavepacket becomes

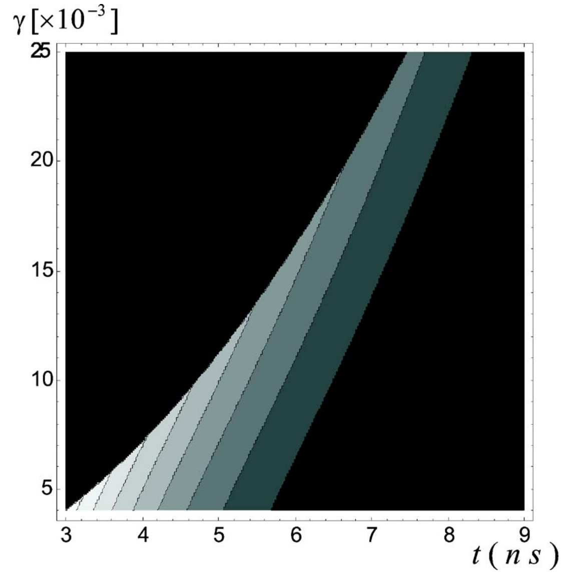

FIG. 2. (Color online) Contour plot (γ,t) of the modulus of the running-phase wave function Eq. (15) for $\omega_p=60\Gamma=30$ GHz. Black regions correspond to a zero wave function, gray-to-white shadings show larger, nonzero values for the modulus.

$$\psi_{out}^{\rightarrow}(\gamma,t) = \sqrt{\frac{\Gamma}{\sqrt{2\gamma}\bar{\Phi}_0\omega_p}}\exp\left[i\frac{\hbar\omega_p\gamma^{3/2}}{6\sqrt{2}E_c} - \left(i\frac{\omega_0}{\omega_p} + \frac{\Gamma}{2\omega_p}\right) \times (t\omega_p - \sqrt{2\gamma})\right]\Theta(t\omega_p - \sqrt{2\gamma}). \quad (15)$$

A plot of the wave function is given in Fig. 2.

The average phase (flux) corresponding to this wave packet can be obtained

$$\langle\gamma\rangle(t) = \omega_p^2(t^2/2 - \Gamma^{-1}t + \Gamma^{-2}), \quad (16)$$

and we notice that the dominant term is quadratic in t. The spread of the flux variable is given by (we keep only the dominant term here)

$$\sqrt{\langle\gamma^2\rangle(t)-\langle\gamma\rangle^2(t)}=\omega_p^2\Gamma^{-1}t. \tag{17}$$

To get the average voltage we can use Ehrenfest theorem; we obtain

$$\langle V(t)\rangle=\bar{\Phi}_0\omega_p^2(t-\Gamma^{-1}). \tag{18}$$

The dominant term for the voltage is linear in time and satisfies the classical energy conservation $C\langle V\rangle^2(t)/2=\hbar\omega_0-U(\langle\gamma\rangle(t))\approx I_{c0}\bar{\Phi}_0\langle\gamma\rangle(t)$.

We now analyze what happens in typical switching current experiments, as they are done now in the context of superconducting qubits:[10,11] the bias current of the junction is increased fast to a value that allows tunneling; it is kept there for a time $0<\tau<\Gamma^{-1}$, then it is lowered to a value that suppresses tunneling. This value has to be large enough so that the experimentalist can get a reliable reading of voltage on the quasiparticle branch if the junction has switched; in practice, it satisfies $I\approx I_{c0}$. Although the change of the bias current has a major effect with respect to tunneling through the barrier, where the the tunneling rate decreases exponentially with the height of the barrier, from the point of view of the structure of the running-phase state, it amounts only to a modification of the parameter Γ. Finally, the current is put to zero and, after waiting long enough for retrapping to occur, the whole cycle can be repeated. In our model, the essential physics is that after the time τ, the tunneling matrix element t is zero; therefore, the system evolves only under the action of H_0. The wave function is "cut" into two separate pieces, one which is (almost) the bound state Ψ_0 inside the well, the other being the wave packet in the continuum which evolves as

$$|\Psi_{out}(t)\rangle=-i\sqrt{\frac{\Gamma}{2\pi}}\int d\epsilon\frac{e^{(-\Gamma/2+i\omega_0)\tau}e^{-i\epsilon(t-\tau)}-e^{-i\epsilon t}}{\Gamma/2+i(\omega_0-\epsilon)}|\psi_\epsilon^{\pm}\rangle, \tag{19}$$

with normalization $\langle\Psi_{out}(t)|\Psi_{out}(t)\rangle=1-e^{-\Gamma\tau}$. Now, for $t-\tau>\Gamma^{-1}$, we can see that the outgoing function consists of two consecutive (separated by the time τ) and dephased (with $\omega_0\tau$) outgoing wave packets with the structure of $|\Psi_{out}^{\rightarrow}(t)\rangle$ which propagate at the same speed across the phase coordinate γ. The second wave packet, which has a probability amplitude smaller by a factor of $e^{-\Gamma\tau/2}$, results from the waves localized near the barrier during the time τ when tunneling was in progress. After integration over energy, we get

$$|\Psi_{out}(t)\rangle=e^{-(\Gamma/2+i\omega_0)\tau}|\Psi_{out}^{\rightarrow}(t-\tau)\rangle-|\Psi_{out}^{\rightarrow}(t)\rangle, \tag{20}$$

where $|\Psi_{out}^{\rightarrow}(t)\rangle$ is given by Eqs. (12) and (15). To check that the normalization $\langle\Psi_{out}(t)|\Psi_{out}(t)\rangle=1-e^{-\Gamma\tau}$ remains valid, we notice that in the region of overlap of the two wave packets, which coincides with the domain where $|\Psi_{out}^{\rightarrow}(t-\tau)\rangle$ is finite $t-\tau>\int_{\gamma_0}^{\gamma}\bar{\Phi}_0V_0^{-1}(\varphi)d\varphi$, there exists a very simple relation between them: $|\Psi_{out}^{\rightarrow}(t)\rangle=\exp[-i\omega_0\tau-\Gamma\tau/2]|\Psi_{out}^{\rightarrow}(t-\tau)\rangle$. Using this property and the previous expressions, Eqs. (16) and (18), we can calculate the average phase and voltage on the state Eq. (20)

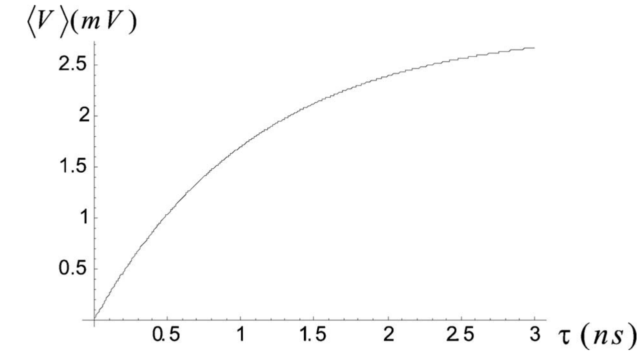

FIG. 3. The average voltage as a function of the time τ for $\omega_p=30$ GHz, $\Gamma=1$ ns, and $t=10$ ns.

$$\langle\gamma\rangle(t)=\frac{1}{2}\omega_p^2[1-e^{-\Gamma\tau}]t^2-e^{-\Gamma\tau}\omega_p^2\tau t-\frac{1}{2}e^{-\Gamma\tau}\omega_p^2, \tag{21}$$

and

$$\langle V\rangle(t)=\bar{\Phi}_0\omega_p^2t(1-e^{-\Gamma\tau})-\bar{\Phi}_0\omega_p^2e^{-\Gamma\tau}\tau. \tag{22}$$

In Fig. 3, we present a plot of the average voltage as a function of the time τ. We see that for values of τ of the same order or larger than the lifetime Γ^{-1}, the average voltage at t becomes flat, reflecting the fact that the junction has switched, as in the case of Eq. (18).

Let us now note that an approximate form of the result Eq. (22) can be, to a large extent, inferred using classical arguments. In a deterministic picture of the running-wave state,[1,3] the phase γ is treated as the classical angle of a rotating Josephson pendulum: for strongly underdamped junctions, the equation of motion for the phase is $C\bar{\Phi}_0(d^2/dt^2)\gamma+I_{c0}\sin\gamma=I$. With the usual approximation $I\approx I_{c0}$ and for times $t\gg\tau$, $t\gg\Gamma^{-1}$, the solution of this equation gives a classical time-dependent voltage $\approx\bar{\Phi}_0\omega_p^2t$. Now, assuming a switching rate Γ, we observe that during the interval τ there is a (classical) probability of obtaining a nonzero voltage only in a $1-\exp(-\Gamma\tau)$ fraction of cases. Therefore, the statistical average of the voltage on a large enough number of switching events will be $\bar{\Phi}_0\omega_p^2t(1-e^{-\Gamma\tau})$, the same result as Eq. (22), with the last term neglected (this last term represents the quantum correction due to the spatial spread of the wave function). This quantitative agreement is somewhat surprising, because the outgoing wave packet has a relatively complicated shape and tunneling is a genuine quantum-mechanical process. The explanation is related to the fact that most of the "weight" of the wave packet is still concentrated close to the coordinates satisfying $t\omega_p=\sqrt{2\gamma}$ [see Eq. (15)]; the "tail" (whose size is determined by Γ) is indeed a reminder of the quantum-mechanical tunneling event in the past, and, in the first approximation, can be neglected.

IV. EXPERIMENTAL ISSUES

For designing an experiment to test these predictions, several remarks should be made. In the case of real junctions, the Josephson energy and the plasma frequency can be reduced by using a superconducting quantum interference device (SQUID) configuration and by adding capacitors in parallel with the junctions. This makes the time evolution of the switching state slower and, therefore, easier to detect. An important limitation on time comes from the fact that as soon as the voltage reaches the quasiparticle branch (at twice the value of the gap), our analysis is not valid. The other limitation is technological: Even with a good dilution refrigerator, thermalizing the junction is very difficult at low temperatures. With a good high-power refrigerator with base temperature of about 5 mK, we assume an optimistic value of 10 mK for the effective temperature of the electrons. This temperature corresponds to a crossover angular frequency of 8.66 GHz between the MQT and the thermal activation transition. A plasma frequency of $\omega_p=30$ GHz (zero bias current) will thus keep us safely in the MQT regime when the current is raised up to about half a percent close to the critical current, according to the formula that gives the plasma oscillation frequency at a finite bias current.[1,2] For Nb, with gap of 1.4 meV, this corresponds to a time of approximately 10 ns, as given by Eq. (18). A voltage increase on this time scale can be detected with standard experimental techniques. Suppose now that we choose to work at currents about 5% less than the critical current. We still have to satisfy the condition $t > \Gamma^{-1}$; an inspection of the formula that gives the tunneling rate for underdamped junctions (e.g., see Ref. 2) shows that switching rates of about 500 MHz and more (with the restriction $\Gamma \ll \omega_p$) can be achieved for E_J/ω_p of the order of 30, values which can be obtained easily with large junctions.

V. CONCLUSION

We have investigated the quantum mechanics of the phase of an underdamped Josephson junction after a MQT event. We derived an analytical form of the macroscopic wave function which has the shape of a wave front followed by an exponentially decaying tail. We calculate the resulting voltage across the junction and find that, in the first order of approximation, it should increase linearly in time. The results also confirm that classical considerations can be regarded as a good approximation in characterizing the running-phase state.

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APPENDIX: CASE STUDY

As a simple application of the formalism presented above, let us consider the problem of the measurement of a superconducting quantum bit using the switching characteristic of a readout junction. The idea, both in the case of a flux qubit measured by a SQUID and in the case of a charge qubit coupled to a large junction, is that the two states of the qubit are distinguishable by a small extra magnetic field or current that either increases or decreases the effective critical current. It has not been so clear though what type of measurement this procedure leads to and what information about the state of the qubit can be obtained in the case of an initial superposition. Let us consider the case of the Quantronium,[10] in which a charge qubit is coupled to a large current-biased junction through two smaller junctions of Josephson energy E_J^q each. During the measurement, to minimize the noise, the qubit is kept at the "sweet spot" in which the charging energy of the states $|n=0\rangle$ and $|n=1\rangle$ is degenerate (n counts the excess number of Cooper pairs on the island). The qubit states are of the spin-1/2 type, $|\uparrow\rangle = 1/\sqrt{2}(|0\rangle + |1\rangle)$ and $|\downarrow\rangle = 1/\sqrt{2}(|0\rangle - |1\rangle)$; in this basis, the Hamiltonian is

$$H_Q = -E_J^q \sigma_z \cos\frac{\gamma}{2} - E_J \cos\gamma + I\bar{\Phi}_0\gamma + \frac{Q^2}{2C}. \quad \text{(A1)}$$

The macroscopic wave function is then spin dependent

$$\Psi(\gamma,t) = \Psi_\uparrow(\gamma,t)|\uparrow\rangle + \Psi_\downarrow(\gamma,t)|\downarrow\rangle, \quad \text{(A2)}$$

where each component evolves according to

$$i\hbar\frac{\partial}{\partial t}\psi_\sigma(\gamma,t) = \left[-\frac{\hbar^2\partial^2}{\partial\gamma^2} + U_\sigma(\gamma)\right]\psi_\sigma(\gamma,t), \quad \text{(A3)}$$

with $\sigma=\{\uparrow,\downarrow\}$ being the spin index and $U_\sigma(\gamma) = -E_J\cos\gamma + I\bar{\Phi}_0\gamma \mp E_J\cos\gamma/2$, with minus/plus corresponding to $\uparrow/\downarrow$, respectively. The Hamiltonian is then of the form Eq. (2), with ω_0^σ and tunneling matrix now spin dependent, leading to two tunneling rates Γ_σ which can be calculated from $U_\sigma(\gamma)$ by standard methods. Consider now the situation in which the qubit has been prepared in a superposition of the form $\alpha|\uparrow\rangle + \beta|\downarrow\rangle$, and then the current is increased close to the critical value. The evolution will be then

$$\Psi(\gamma,t) = \alpha[e^{-\Gamma_\uparrow t/2}e^{-i\omega_0^\uparrow t}\Psi_0^\uparrow(\gamma) + \Psi_{out}^\uparrow(\gamma,t)]|\uparrow\rangle + \beta[e^{-\Gamma_\downarrow t/2}e^{-i\omega_0^\downarrow t}\Psi_0^\downarrow(\gamma) + \Psi_{out}^\downarrow(\gamma,t)]|\downarrow\rangle. \quad \text{(A4)}$$

The switching probability within a time interval τ is then determined by the condition that a voltage has been recorded; according to the standard rules of quantum mechanics, from Eq. (A4), this is

$$P^{out}(\tau) = 1 - |\alpha|^2 e^{-\Gamma_\uparrow\tau} - |\beta|^2 e^{-\Gamma_\downarrow\tau}. \quad \text{(A5)}$$

The coefficients α and β can be controlled experimentally by adjusting the duration of the microwave radiation pulses. Therefore, Eq. (A5) can be readily tested experimentally with the present experimental setups.

It should be mentioned that the description above of the measurement process offers a perfect illustration of the well-known paradox of Schrödinger: to find the state of the qubit, one entangles it with the measuring apparatus, which has two classically distinguishable states (corresponding to the voltmeter across the large measuring junction indicating zero or some finite value). It is difficult then to ascribe any "element of reality" to the superposition Eq. (A4) as well as to the notion of collapse of the wave function. One cannot seriously believe that during this experiment, the macroscopic needle of the voltmeter (and together with it the experimentalist) has been in a such a superposition and also it is impossible to say at what stage only one of the possible outcomes has been selected. Surely, this is not a consequence of an incomplete or approximate description of the interaction between the qubit and the measurement system or of a lack of understanding of the functioning of the latter. No matter how long and how detailed we want to make the description of the chain qubit-to-experimentalist registering the result, this fundamental paradox of quantum mechanics is here to stay.

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